Amenability and approximations of C*-algebras

Wilhelm Winter
WWU Münster

Richard Kadison and his mathematical legacy
Copenhagen, November 2019
A system of completely positive approximations for a $C^*$-algebra $A$ is a net

$$(A \xrightarrow{\varphi_\lambda} F_\lambda \xrightarrow{\psi_\lambda} A)_{\lambda \in \Lambda}$$

such that

- the $F_\lambda$ are finite dimensional $C^*$-algebras
- the $\varphi_\lambda$ and the $\psi_\lambda$ are completely positive maps with a uniform norm bound
- $\varphi_\lambda \psi_\lambda \xrightarrow{\text{point-norm topology}} \text{id}_A$

Nuclear $C^*$-algebras are precisely those admitting such systems. They form a remarkably robust class of $C^*$-algebras.

Let us recall some sample results.
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In particular, they satisfy Kadison’s similarity property:
Every (bounded) representation is similar to a *-representation.
THEOREM [Christensen–Sinclair–Smith–White–W]

Separable nuclear C*-algebras satisfy the Kadison–Kastler conjecture: They are stable under small perturbations.
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More precisely, if two nuclear C*-algebras act on the same Hilbert space, if one of them is separable and nuclear, and if their unit balls are within $\frac{1}{420000}$ of each other, then they are isomorphic.
For $A \neq M_r(C)$ a separable, simple, unital, nuclear $C^\star$-algebra, the following are equivalent:

(i) $A$ has finite nuclear dimension.

(ii) $A \cong A \otimes \mathbb{Z}$.

I will define nuclear dimension in a moment. Think of the Jiang–Su algebra $\mathbb{Z}$ as the smallest possible $C^\star$-version of the hyperfinite $II_1$ factor.

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THEOREM [many hands]
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\[ \{ A \otimes \mathcal{Z} \mid A \text{ separable, simple, unital, nuclear, with UCT} \} \]

is classified by the Elliott invariant

\[ \left( K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A), r_A : T(A) \to S(K_0(A)) \right) . \]
For all of these results, understanding completely positive approximations is key. Let us look at these in more detail now.
A C ⋆-algebra $A$ has nuclear dimension at most $d$, $\dim_{\text{nuc}} A \leq d$, if there is a system $(A \psi_\lambda \to F_\lambda \phi_\lambda \to A)_{\lambda \in \Lambda}$ of completely positive approximations such that

▶ the $\psi_\lambda$ are contractions
▶ for each $\lambda$, $F_\lambda = F(0) \oplus \ldots \oplus F(d)$ and for each $k \in \{0, \ldots, d\}$, $\phi_\lambda|_{F(k)}$ is contractive and has order zero, i.e., preserves orthogonality.

In this situation one can arrange that

▶ the $\psi_\lambda$ are approximately order zero: $[a b] = 0 \Rightarrow [\psi_\lambda(a) \psi_\lambda(b) \to 0]$.

If the maps $\phi_\lambda$ may be chosen to be contractive, $A$ has decomposition rank $\leq d$. In this case can even arrange that

▶ the $\psi_\lambda$ are approximately multiplicative: $\psi_\lambda(a) \psi_\lambda(b) - \psi_\lambda(a b) \to 0$. 
DEFINITION [W–Zacharias, Kirchberg–W]

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Even without finite nuclear dimension, one can find approximations with similar properties.

**Theorem**

[Hirshberg–Kirchberg–White, Carrión–Hirshberg–White]

Any nuclear $C^\star$-algebra $A$ has a system $(\phi_\lambda \rightarrow F_\lambda \psi_\lambda \rightarrow A)_{\lambda \in \Lambda}$ of completely positive approximations such that

- the $\phi_\lambda$ are convex combinations of contractive order zero maps
- the $\psi_\lambda$ are contractive and approximately order zero.

If $A$ and all its tracial states are quasidiagonal, one can in addition arrange that

- the $\psi_\lambda$ are approximately multiplicative.

Note that, even for $C([0,1])$, such approximations are not entirely straightforward to write down.
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Note that, even for $C([0, 1])$, such approximations are not entirely straightforward to write down.
From now on we will assume $A$ to be separable, and we will restrict to systems of approximations with $\Lambda = \mathbb{N}$:

$$(A \xrightarrow{\psi_k} F_k \xrightarrow{\psi_k} A)_{k \in \mathbb{N}}$$

**Questions**

- When does a system $$(F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \ldots)$$ come from a $C^\ast$-algebra?
- Under which conditions on the system $$(F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \ldots)$$ can we recover $A$?
- How to read off information on $A$ (K-theory, traces, ... ) from $$(F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \ldots)$$?
- What structure does the inductive limit of $$(F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \ldots)$$ have?
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What structure does the inductive limit of $(F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots)$ have?
DEFINITION [Blackadar–Kirchberg, Courtney–W]

Consider an inductive system \( F_0 \varrho_0 \rightarrow F_1 \varrho_1 \rightarrow \ldots \) with finite dimensional \( C^\star \)-algebras \( F_k \) and completely positive contractive \( \varrho_k \), \( k + 1 \).

The system is asymptotically multiplicative, if the following holds: For every \( K \in \mathbb{N} \), \( x, y \in F_K \) and \( \epsilon > 0 \) there is \( K \leq M \in \mathbb{N} \) such that for every \( M \leq m < n \) we have
\[
\| \varrho_m, n (\varrho_K, m (x)) \varrho_K, n (y) \|_{F_n} < \epsilon.
\]

The system is asymptotically order zero, if the following holds: For every \( K \in \mathbb{N} \), \( x, y \in F_K \) and \( \epsilon > 0 \) there are \( K \leq M \in \mathbb{N} \) and \( \delta > 0 \) such that for every \( M \leq m < n \) we have
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THEOREM [Courtney–W, using Blackadar–Kirchberg, Carrión–Hirshberg–White]

Let $A$ be a separable nuclear $C\star$-algebra. Then, there exists a system of completely positive approximations such that the associated inductive system $F_0 \xrightarrow{\varrho_0} F_1 \xrightarrow{\varrho_1} \ldots$ has the following properties:

- it is asymptotically order zero
- the induced map $\Psi : A \to X := \lim_{\to} (F_k, \varrho_k, k+1) \subset \prod_{N} F_k / \bigoplus_{N} F_k$ is an orthogonality preserving complete order isomorphism.

If $A$ and all its traces are quasidiagonal, one can arrange:

- the system is asymptotically multiplicative
- $\Psi$ is an injective $\star$-homomorphism
- the maps $\varrho_k, k+1$ induce affine maps $T \leq 1 (F_k) \leftarrow T \leq 1 (F_{k+1})$ between the simplices of positive trace functionals such that $T \leq 1 (A) \approx \lim_{\to} T \leq 1 (F_k)$.
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- it is asymptotically order zero
- the induced map $\psi : A \rightarrow X := \lim_{\rightarrow} (F_k, q_{k,k+1}) \subset \prod_{\mathbb{N}} F_k / \bigoplus_{\mathbb{N}} F_k$ is an orthogonality preserving complete order isomorphism.
THEOREM [Courtney–W, using Blackadar–Kirchberg, Carrión–Hirshberg–White]

Let $A$ be a separable nuclear $C^*$-algebra. Then, there exists a system of completely positive approximations such that the associated inductive system $F_0 \xrightarrow{\varrho_{0,1}} F_1 \xrightarrow{\varrho_{1,2}} \ldots$ has the following properties:

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If $A$ and all its traces are quasidiagonal, one can arrange:

- the system is asymptotically multiplicative
- $\Psi$ is an injective $\star$-homomorphism
- the maps $\varrho_{k,k+1}$ induce affine maps $T_{\leq 1}(F_k) \leftarrow T_{\leq 1}(F_{k+1})$ between the simplices of positive trace functionals such that $T_{\leq 1}(A) \approx \lim_{\rightarrow} T_{\leq 1}(F_k)$.
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Upshot:

For a separable, unital, nuclear and sufficiently quasidiagonal C\(^\star\)-algebra \(A\), we now know how to read its trace space \(T(A)\) from a suitable system of completely positive approximations. We also know how to recover its multiplicative structure via \(\Psi : A \rightarrow X\), so we can describe projections, unitaries, partial isometries in terms of \(X\). Since our constructions are compatible with matrix amplification, we can describe K-theory in this context.
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Similar identities hold for unitaries and partial isometries, and so one can always describe K-theory for nuclear C\(^*\)-algebras in terms of their approximating systems.

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Along these lines, we can analyse a nuclear $C^*$-algebra in terms of asymptotically order zero inductive systems of finite dimensional $C^*$-algebras.

But given such an inductive system, when is there a $C^*$-algebra behind it? In other words, given $F_0 \to F_1 \to \cdots \to X$, when can $X$ be equipped with a $C^*$-algebra structure?
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THEOREM [Courtney–W]

Let $B$ be a $C^*$-algebra and let $X \subset B$ be a self-adjoint subspace. Suppose there is $0 \leq e \in X \cap X'$ such that $e$ is an order unit for $X_{sa}$ and $e^2 = e$. Then, $X$ is a $\star$-algebra with the unique product $\cdot : X \times X \to X$ satisfying $(x \cdot y) e = xy \in B$. Moreover, $X$ carries a norm which 'looks like' $\|x\|_\cdot = \|x e - 1\|_B$. With this product and norm, $X$ becomes a unital pre-$C^*$-algebra.
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