The Kadison-Singer Problem in Mathematics and Engineering
Lecture 4: The Sundberg Problem, the Harmonic-Analysis Conjecture, KS in Number Theory and non-2-Pavable Projections

Master Course on the Kadison-Singer Problem
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Recall: The Bourgain-Tzafriri Conjecture

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(weak) Bourgain-Tzafriri Conjecture

$A = f(B)$

Recall: The Feichtinger Conjecture

Definition

$\{\phi_i\}_{i\in I}$ is a Riesz Basic Sequence in H if there exist Riesz basis bounds $A, B > 0$ so that for all scalars $(a_i)_{i\in I}$

$$A\sum_{i\in I}|a_i|^2 \le \left\|\sum_{i\in I} a_i\phi_i\right\|^2 \le B\sum_{i\in I}|a_i|^2$$

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Feichtinger Conjecture

Every unit norm frame a finite union of Riesz basic sequences.

BT implies FC

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Proof: Let (e_i) be the unit vector basis for ℓ_2.
Let (ϕ_i) be a unit norm frame for ℓ_2 with analysis operator T and synthesis operator $T^* : \ell_2 \to \ell_2$ with $T^* e_i = \phi_i$ for all $i = 1, 2, \ldots$.

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Fix $0 < A$ as in BT, and let $B = \|T\|^2$ and choose $r = r(B)$.

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Fix $0 < A$ as in BT, and let $B = \|T\|^2$ and choose $r = r(B)$.

For each $n = 1, 2, \ldots$, let $L_n : \ell_2^n \to \text{span } (\phi_i)_{i=1}^n$ be $Le_i = \phi_i$.

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There is another k so that for infinitely many of the above n, $2 \in A_k^n$.

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CONTINUE.

The Sundberg Problem

Sundberg Problem

Can every unit norm Bessel sequence be partitioned into a finite number of non-spanning sets?

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If $(e_i)_{i=1}^\infty$ is an orthonormal basis for ℓ_2 then $(e_i) \cup (\phi_i)$ is a unit norm frame for ℓ_2.

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By FC, we can partition this set (and hence we can partition (ϕ_i)) into a finite number of Riesz basic sequences say $(\phi_i)_{i \in A_j}$ for $j = 1, 2, \ldots, r$.

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But if we remove one vector from each family $(\phi_i)_{i \in A_j}$ then the resulting sets do not span.

End Proof

KS in Harmonic Analysis

Historical Note:

Jean Baptiste Joseph Fourier is credited with the discovery in 1824 that gases in the atmosphere might increase the surface temperature of the earth. Today, we call this the greenhouse effect.

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Much work was done in 1980's to solve PC for Laurant Operators by:

Bourgain/Tzafriri

Halpern/Kaftal/Weiss

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$$(1-\epsilon)(b-a)\|f\|^2 \le \|P_E f\|^2 \le (1+\epsilon)(b-a)\|f\|^2$$

$P_E f = \chi_E \cdot f$

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If we replace $1 \pm \epsilon$ by universal $0 < A < 1 < B < \infty$, we call this **weak H.A.**.

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(B) Weak HA is equivalent to FC for Laurant operators.

KS in Number Theory

Van der Waerden's Theorem:

Given a partition of the integers $(A_j)_{j=1}^r$, there is an $1 \leq i \leq r$ so that A_i has arbitrarily long arithmetic progressions.

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Question:

Does there exist a quantitative version of Van der Waerden's theorem?

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Theorem: [Gowers]

Let $0 < \gamma \leq 1/2$, let k be a positive integer, let

$$P \geq 2 \uparrow 2 \uparrow \gamma^{-1} \uparrow 2 \uparrow 2 \uparrow (k+9),$$

and let A be a subset of $\{1, 2, \ldots, P\}$ of size γP.

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and let A be a subset of $\{1, 2, \ldots, P\}$ of size γP.

Then A contains an arithmetic progression of length k.

Quantative Arithmetic Progressions

Definition

Let $g : \mathbb{N} \to [0, \infty)$. We say that $A \subset \mathbb{Z}$ satisfies the $g(N)$ arithmetic progression condition if for every $\delta > 0$ there exists $M \in \mathbb{Z}$ and $n, \ell \in \mathbb{N}$ such that
(i) $\ell < \delta g(N)$
and
(ii) $\{M, M + \ell, M + 2\ell, \ldots, M + N\ell\} \subset A$.

Bownik and Speegle

Theorem (Bownik/Speegle)

There exists a set $U \subset [0,1]$ *such that if* $A \subset \mathbb{Z}$ *satisfies the* $g(N) = N^{1/2} log^{-3} N$ *arithmetic condition,*

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Remark:

This means that there is no quantative van der Waerden theorem with sets of size $N^{1/2} log^{-3} N$.

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$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\|, \quad \text{for all } j = 1, 2, \ldots, r.$$

Q_{A_j} the orthogonal projection onto span $(e_i)_{i \in A_j}$

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Important: r depends only on ϵ and not on n or T.

Two Paving Fails

[Discrete Fourier Transform - DFT_n]

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$$DFT_n = \left(\omega^{ij}\right)_{i,j=1}^{n}.$$

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Then

$$\frac{1}{\sqrt{n}} DFT_n, \text{ is a unitary matrix.}$$

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Now form

$$B_n = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

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3. The square sum of the coefficients of every row equals 1.
4. Hence, the rows of this matrix form a unit norm two-tight frame,

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   I.e. This is the matrix of a rank 2n projection on $\mathbb{C}^{4n}$ with constant diagonal $1/2$.

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Choose $(a_i)_{i\in A}$ with $\sum_{i\in A} |a_i|^2 = 1$ and so that

$$P_{n-1}\left(\sum_{i\in A} a_i f_i\right) = 0.$$

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Letting $n \to \infty$ we have that this class of matrices is not $(\delta, 2)$-Riesable for any $\delta > 0$.

Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava	$\Rightarrow$	Casazza/Tremain Conjecture
	and	Weaver Conjecture KS_r
	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Rightarrow$	R_ϵ-Conjecture
	$\Rightarrow$	Bourgain-Tzafriri Conjecture
	$\Rightarrow$	Feichtinger Conjecture
	$\Rightarrow$	Sundberg Problem

Finally:

Bourgain-Tzafriri Conjecture	$\Rightarrow$	Weaver Conjecture
	$\Leftrightarrow$	Paving Conjecture
	$\Leftrightarrow$	The Kadison-Singer Problem