The Kadison-Singer Problem in Mathematics and Engineering
Lecture 6: The Proof of the Kadison-Singer Problem

Master Course on the Kadison-Singer Problem
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Recall

Notation:

If $v \in \mathbb{H}$, the operator vv^* on $\mathbb{H}$ is

$$vv^* u = \langle u, v \rangle v.$$

Marcus/Spielman/Srivastave

The main result:

Theorem (MSS)

If $\epsilon > 0$ *and* $u_1, u_2, \ldots, u_n$ *are independent random vectors in* $\mathbb{H}_d$ *and*

$$\sum_{i=1}^{n} \mathbb{E} u_i u_i^* = I_d,$$

and

$$\mathbb{E}\|u_i\|^2 \leq \epsilon \text{ for all } i,$$

than

$$\mathbb{P}\left[\left\|\sum_{i=1}^{n} u_i u_i^*\right\| \leq (1+\sqrt{\epsilon})^2\right] > 0.$$

The Main Consequence

Corollary

Let $u_1, u_2 \ldots, u_n$ *be column vectors in* $\mathbb{H}_d$ *with* $\|u_i\|^2 \le \alpha$ *for all* i *and*

$$\sum_{i=1}^{n} u_i u_i^* = I$$

I.e. (u_i) *is a Parseval frame.*
Then, there exists a partition of $\{1, 2, \ldots, n\}$ *into sets* S_1, S_2 *for that for* $j = 1, 2$ *we have*

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| \le \frac{(1 + \sqrt{2\alpha})^2}{2}.$$

Proof

Given $(u_i)_{i=1}^n$, let $(v_i)_{i=1}^n$ be independent random vectors with

$$\mathbb{P}\left(v_i = \begin{bmatrix} 0 \\ \sqrt{2}u_i \end{bmatrix}\right) = \frac{1}{2} \text{ and } \mathbb{P}\left(v_i = \begin{bmatrix} \sqrt{2}u_i \\ 0 \end{bmatrix}\right) = \frac{1}{2}.$$

Then

$$\mathbb{E}v_i v_i^* = \begin{bmatrix} u_i u_i^* & 0 \\ 0 & u_i u_i^* \end{bmatrix} \text{ and } \|v_i\|^2 = 2\|u_i\|^2 \leq 2\alpha.$$

Hence,

$$\sum_{i=1}^n \mathbb{E}v_i v_i^* = I.$$

Proof Continued

Applying the theorem with $\epsilon = 2\alpha$, we find a partition S_1, S_2 of $\{1, 2, \ldots, n\}$ so that

$$\left\| \sum_{i \in S_1} \begin{bmatrix} \sqrt{2}u_i \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}u_i \\ 0 \end{bmatrix}^* + \sum_{i \in S_2} \begin{bmatrix} 0 \\ \sqrt{2}u_i \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2}u_i \end{bmatrix}^* \right\| \le (1 + \sqrt{\epsilon})^2.$$

Hence, for $j = 1, 2$,

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\| = \left\| \sum_{i \in S_j} \begin{bmatrix} u_i \\ 0 \end{bmatrix} \begin{bmatrix} u_i \\ 0 \end{bmatrix}^* \right\| \le \frac{(1 + \sqrt{\epsilon})^2}{2}.$$

Recall: Weaver Conjecture KS_r

Weaver Conjecture KS_r for $r \in \mathbb{N}$

There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds.

Given $(\phi_i)_{i=1}^n$ vectors in ℓ_2^k with $\|\phi_i\| \leq 1$ for all i and suppose

$$\sum_{i=1}^n |\langle \phi, \phi_i \rangle|^2 = N, \text{ for all } \phi \in \ell_2^n.$$

Then there is a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A_j} |\langle \phi, \phi_i \rangle|^2 \leq N - \epsilon, \text{ for all } \phi \in \ell_2^n \text{ and all } j.$$

Consequences

Corollary

Weaver's conjecture KS_2 holds for $N = 18$ and $\epsilon = 2$.

Proof: Let $N = 18$ and $\epsilon = 2$ in KS_2 and let $\alpha = \frac{1}{18}$ in the corollary. Choose $(\phi_i)_{i=1}^n$ in $\mathbb{H}$ with $\|\phi_i\| \leq 1$ and

$$\sum_{i=1}^{n} |\langle \phi, \phi_i \rangle|^2 = N \text{ for all } \phi \in \mathbb{H}.$$

Let $u_i = \frac{\phi_i}{\sqrt{N}}$ for all $i = 1, 2, \ldots, n$.
Then $\|u_i\| \leq \alpha$ and

$$\sum_{i=1}^{n} u_i u_i^* = Id.$$

Proof Continued

So by the corollary, there is a partition S_1, S_2 of $\{1, 2, \ldots, n\}$ so that for $j = 1, 2$ and any $\|\phi\| = 1$, we have

$$\begin{aligned}
\sum_{i \in S_j} |\langle \phi, \phi_i \rangle|^2 &= \sum_{i \in S_j}^{n} \langle \phi_i \phi_i^* \phi, \phi \rangle \\
&= N \left\langle \left(\sum_{i=1}^{n} u_i u_i^* \right) \phi, \phi \right\rangle \\
&\leq N \| \sum_{i \in S_j}^{n} u_i u_i^* \| \\
&\leq 18 \frac{(1 + \sqrt{2/18})^2}{2} \\
&= 18 \frac{8}{9} = 16 = 18 - 2.
\end{aligned}$$

What they proved!

What MSS actually proved is:

Theorem

The C/Tremain Conjecture is true for $K = 18$ *and* $\epsilon = 2$.

In fact, every unit norm (18)-tight frame can be partitioned into two subsets each of which have frame bounds $2, 16$.

Problem: What is the smallest K that works?

How about $K = 3$?

Proof Outline of the Main Theorem

This is a monumental undertaking which relies heavily on a significant amount of earlier work on polynomial interlacing and bipartite graphs.

Step 1: Interlacing families of polynomials have the property that they always contain at least one polynomial whose largest root is at most the largest root of the sum of the polynomials in the family.

Step 2: Prove that the characteristic polynomials of the matrices in the theorem are such a family.

Step 3: To do this, consider the expected characteristic polynomials of a family of rank-1 semidefinite Hermitian matrices given by $u_i u_i^*$ called mixed characteristic polynomials.

Proof Outline Continued

Step 4: To show that these polynomials are an interlacing family, show that all mixed characteristic polynomials are real rooted.
This is done by using known methods, constructing multivariate real stable polynomials and then applying operators which preserve real stability - eventuality arriving at mixed characteristic polynomials.

Step 5: To bound the largest root of the expected characteristic polynomial, use a multivariate generalization of a barrier function argument originally done in the Spielman/Srivastava algorithm for proving the Restricted Invertibility Theorem and later refined by Batson/Spielman/Srivastava for bipartite graphs.

We need to know:

Problem: What is the paving number for projections with constant diagonal $1/2$?
It must be > 2.

Conjecture: There is a function $f(x)$ so that every unit norm $(2+\epsilon)$-tight frame can be partitioned into two subsets each of which has lower frame bound $f(\epsilon)$.

Problem: Find a proof of KS using Functional Analysis.

I.e. Replace the interlacing of zeros of polynomials argument with a interlacing of eigenvalues argument.

We need to know

Since this is all new, we have not examined the consequences of the fact that all these conjectures are now theorems.

This is because we all thought that KS had a negative answer so we did not spend a lot of time looking for consequences of a positive solution.

We also do not know exactly how the constants in one form of KS translate to the constants of another.

Towards a Simpler Solution

If $\|\phi_i\| = 1$ for $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} \phi_i \phi_i^* = K \cdot I,$$

and S_1, S_2 is a partition of $\{1, 2, \ldots, n\}$, let

$$T_j = \sum_{i \in S_j} \phi_i \phi_i^* \text{ for } j = 1, 2.$$

Then T_1 has eigenvectors $\{x_i\}_{i=1}^n$ with respective eigenvalues $\{\lambda_i\}_{i=1}^n$

$$\Leftrightarrow$$

T_2 has eigenvectors $\{x_i\}_{i=1}^n$ with respective eigenvalues $\{K - \lambda_i\}_{i=1}^n$.

Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava	$\Rightarrow$	Casazza/Tremain Conjecture
	and	Weaver Conjecture KS_r
	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Rightarrow$	R_ϵ-Conjecture
	$\Rightarrow$	Bourgain-Tzafriri Conjecture
	$\Rightarrow$	Feichtinger Conjecture
	$\Rightarrow$	Sundberg Problem

Finally:

Bourgain-Tzafriri Conjecture	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Leftrightarrow$	The Kadison-Singer Problem

The Major Players

Richard Kadison (1924-   )

Isadore Singer (1924-   )

Daniel Spielman

Adam Marcus

Nikhil Srivastava