def: a type II_1 factor is a $*$-subalgebra of $B(H)$ s.t.

- M is WOT-closed and unital (von Neumann algebra) $M = M''$
- $Z(M) = \mathbb{C}1$ (factor)
  $\to$ simple
- $\exists\, \tau : M \to \mathbb{C}$ a trace (type II_1)
  - $\tau(1) = 1$
  - $\tau(x^*x) > 0$ if $x \neq 0$
  - WOT-continuous
  - $\tau(xy) = \tau(yx)$
  - $M \neq M_n(\mathbb{C})$

examples:

- $L\Gamma$: let Γ be a countable group
  $H = \ell^2\Gamma$
  $(\lambda_g(\xi))(h) = \xi(g^{-1}h)$
  - $L\Gamma = \{\lambda_g \mid g \in \Gamma\}''$
  - $\tau(x) = \langle \delta_e, x\delta_e \rangle$
  - $L\Gamma$ is a factor iff Γ is ICC
    infinite conjugacy classes
- $L^\infty(X) \rtimes \Gamma$: let $\Gamma \curvearrowright (X,\mu)$ be a p.m.p. action
  $H = L^2(X,\mu) \otimes \ell^2\Gamma = L^2(X \times \Gamma)$
  ~~$(x\,\xi)(g) = x\,\xi(g) \quad \forall x \in L^\infty(X)$~~
  $(a\,\xi)(x,g) = a(x)\,\xi(x,g) \quad \forall a \in L^\infty(X)$
  $(u_g\,\xi)(x,h) = \xi(g^{-1}x, g^{-1}h) \quad \forall g \in \Gamma$

  then $u_g a u_g^* = \sigma_g(a)$ $\qquad \sigma_g(a)(x) = a(g^{-1}x)$
  - $L^\infty(X,\mu) \rtimes \Gamma = \{a, u_g \mid a \in L^\infty(X), g \in \Gamma\}''$
  - $\tau(x) = \langle 1 \otimes \delta_e, x(1 \otimes \delta_e) \rangle$
  - $L^\infty(X,\mu) \rtimes \Gamma$ is a factor if $\Gamma \curvearrowright (X,\mu)$ is free and ergodic
    - free: $\forall g \in \Gamma, \mu(\mathrm{Fix}(g)) = 0$
    - ergodic: ~~if $\mu(U) \neq 0$, then $\mu(\Gamma U) = 1$~~
      Γ-invariant subset $U \subset X$ has $\mu(U) = 0, 1$

__questions__

- when ~~are~~ is $L\Gamma \cong L\Lambda$?

  $L^\infty(X)\rtimes\Gamma \cong L^\infty(Y)\rtimes\Lambda$?

- if isomorphic: what are the isomorphisms?

  in particular: $\mathrm{Aut}(M) = \{\psi : M \to M\}$?

  obvious automorphisms: $u \in \mathcal{U}(M) \leadsto \mathrm{Ad}_u : x \mapsto u x u^*$

  $\mathrm{Inn}(M) = \{ \mathrm{Ad}_u \mid u \in \mathcal{U}(M)\}$: normal subgroup

  $\mathrm{Out}(M) = \mathrm{Aut}(M) / \mathrm{Inn}(M)$ + $\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid pMp \cong qMq\}$

- what properties of Γ / $\Gamma \curvearrowright X$ are invariants of the type II_1 factor?

__~~Early~~ results:__

- ~~if Γ is amenable, then~~

  Γ, Λ amenable, ICC, then $L\Gamma \cong L\Lambda \cong L^\infty(X)\rtimes\Gamma = R$ (Connes, 197-)

  In fact: $L\Gamma \cong R$ iff Γ is ICC amenable

  . free, ergo $\Gamma \curvearrowright X$: $L^\infty(X)\rtimes\Gamma \cong R$ iff Γ is amenable.

- $L\mathbb{F}_n \not\cong R$ (M-vN, 1948)
- $L\mathbb{F}_n \cong L\mathbb{F}_m$? (open)
- $\mathrm{Out}(R)$ is huge (contains $\mathcal{U}_\infty$)
- $\mathcal{F}(R) = \mathbb{R}_+^*$

__Thm (Singer '55)__ Let $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ be free, ergodic pmp actions

then $\psi : \underset{\underset{L^\infty(X)}{\cup}}{L^\infty(X)\rtimes\Gamma} \xrightarrow{\cong} \underset{\underset{L^\infty(Y)}{\cup}}{L^\infty(Y)\rtimes\Lambda}$, $\psi(L^\infty(X)) = L^\infty(Y)$

iff $\Gamma \curvearrowright X$ is OE to $\Lambda \curvearrowright Y$

__Connes 1980:__

- property (T) for type II_1 factors

  $L\Gamma$ has (T) iff Γ has (T)

- $\mathrm{Out}(L\Gamma)$, $\mathcal{F}(L\Gamma)$ is countable if M has (T)
- there are at most countably many Γ_i s.t. $L\Gamma_i \cong M$ if M has (T)

conjecture: at most 1 group, $\mathrm{Out}(M) = \{id\}$, $\mathcal{F}(M) = \{1\}$

__Popa 2001+-__

subalgebra with (relative) (T) $\longleftrightarrow$ deformation of the algebra M

- ~~$L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes SL_2\mathbb{Z})$~~

  $\mathcal{F}(L\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}) = \{1\}$

  $L\mathbb{Z}^2 \subset L(\mathbb{Z}^2 \rtimes SL_2\mathbb{Z})$ has rel(T) $\Longleftrightarrow$ $SL_2\mathbb{Z}$ has (H)

- $\mathrm{Out}(M) = \{e\}$ $(\exists M)$ (IPP 05)
- $\exists \Gamma$ s.t. $\mathcal{L}\Lambda \cong \mathcal{L}\Gamma \Rightarrow \Lambda \cong \Gamma$ (IPV)

$\Gamma \curvearrowright X$ s.t. $L^\infty(Y) \rtimes \Lambda \cong L^\infty(X) \rtimes \Gamma \Rightarrow \Lambda \cong \Gamma$, $\Gamma \curvearrowright X \cong \Lambda \curvearrowright Y$ (PV

<u>link with ergodic theory:</u>

<u>def</u>: let $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ be free, p.m.p, ergo action

- $\Delta : X \to Y$, $\delta : \Gamma \to \Lambda$ is a conjugation if
$$\Delta(gx) = \delta(g)\Delta(x) \quad \text{a.e.}$$
$\rightsquigarrow$ actions are conjugate
- $\Delta : X \to Y$ is an O.E. if
$$\Delta(\Gamma x) = \Lambda \Delta(x) \quad \text{a.e.}$$
$\rightsquigarrow$ actions are orbit equivalent $\quad \Delta(gx) = \omega(g,x)\Delta(x) \rightsquigarrow \omega(gh,x) = \omega(g,hx)\omega(h,x)$

<u>thm (Singer '55)</u>

$\Delta : X \to Y$ is an O.E. iff $\Delta_*(a)(x) = a(\Delta^{-1}(x))$, $\Delta_* : L^\infty(X) \to L^\infty(Y)$

extends to an isom $L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$

so: $(L^\infty(X) \subset L^\infty(X) \rtimes \Gamma) \cong (L^\infty(Y) \subset L^\infty(Y) \rtimes \Lambda)$ iff $\Gamma \curvearrowright X \underset{OE}{\cong} \Lambda \curvearrowright Y$

<u>def</u>: [FM '77] ~~an~~ countable Borel equivalence relation $\mathcal{R}$ on a ~~measure~~ prob space (X, μ) is a Borel subset $\mathcal{R} \subset X \times X$ that is an equivalence s.t. all classes are countable

~~This p.m.p if every $\varphi : X \to X$ with $\varphi(x)\mathcal{R}x$ a.e. is p.m.p~~

- $[\mathcal{R}] = \{\Delta : X \to X \mid \Delta(x)\mathcal{R}x \text{ a.e.}\}$
- we say that $\mathcal{R}$ is p.m.p if every $\Delta \in [\mathcal{R}]$ is p.m.p.
- ergodic if $[\mathcal{R}] \curvearrowright X$ is ergodic

<u>ex</u>: - $\Gamma \curvearrowright X$ ~~a~~ p.m.p action

$\rightsquigarrow \mathcal{R}_\Gamma = \{(x, gx) \mid g \in \Gamma\}$ is a ctble Borel equiv

- every $\mathcal{R}$ is of this form (FM)
- but not for a free action

<u>construction</u>: on $\mathcal{R}$, define measure η by $\eta(A) = \int \#\{y \mid (x,y) \in A\} \, d\mu(x)$

on $L^2(\mathcal{R}, \eta)$, consider $(a\xi)(x,y) = a(x)\xi(x,y)$ $\quad \forall a \in L^\infty(X,\mu)$

$(u_\varphi \xi)(x,y) = \xi(\varphi(x), y)$ $\quad \forall \varphi \in [\mathcal{R}]$

$$\mathcal{L}\mathcal{R} = \{a, u_\varphi\}''$$

- $\tau(x) = \langle 1_\Delta, x 1_\Delta \rangle$ $\quad \Delta = \{(x,x) \mid x \in X\}$
- $\mathcal{L}\mathcal{R}$ is a factor iff $\mathcal{R}$ is ergodic

~~ex~~ if $\mathcal{R} = \mathcal{R}_{\not}(\Gamma \curvearrowright X)$, ~~then~~ and $\Gamma \curvearrowright X$ is free, then

$$L^\infty(X) \rtimes \Gamma \cong L\mathcal{R}$$

~~ex~~

def: a subalgebra $A \subset M$ of a II_1 factor is a Cartan subalgebra if it is

- max'l ab, i.e. $A' \cap M = A$
- regular, i.e. $N_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M

ex: . $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ if $\Gamma \curvearrowright X$ is free

. $L^\infty(X) \subset L\mathcal{R}$ ~~more general example~~

thm [FM] Every ~~regular~~ Cartan subalgebra is of the form $L^\infty(X) \subset L_\Omega\mathcal{R}$

strategy to answer our question:

- if ψ: ~~iso~~ $L^\infty(X) \rtimes \Gamma \to (L^\infty(Y) \rtimes \Lambda)_{\not}$ is an iso, show that $\psi(L^\infty(X)) = u L^\infty(Y) u^*$] Cyril
- study orbit equivalences $\Delta : X \to Y$] Damien

break

technical tool: Popa's intertwining by bimodules

~~goal is to prove $\psi(L$~~

$A, B \subset (M, \tau)$ $\rightsquigarrow$ goal is to prove $A = uBu^*$ for some $u \in \mathcal{U}(M)$

thm [Popa 01]: Let $A, B \subset (M, \tau)$ TFAE

(1) . $\exists n \in \mathbb{N}$, $p \in M_n(B)$, $v \in M_{1,n}(M)$, $\theta : A \to pM_n(B)p$ s.t.

$$x v = v\theta(x) \quad \forall x \in A$$

(2) . $\exists\, {}_A K_B \subset {}_A L^2(M, \tau)_B$, finitely generated as a B-module ~~for~~

(3) . $\nexists (u_k)_k$ in $\mathcal{U}(A)$ s.t. $\forall x, y \in M$: $\|E_B(x u_k y)\|_2 \to 0$ $\leftarrow$ means $A \prec_M B$

proof:

(1) $\Rightarrow$ (3): suppose that $\exists\, n, p, v, \alpha$ as in (1)

and $\exists (u_k)_k$ as in (3)

then $0 \neq \| E_{pM_n(B)p}(v^*v)\|_2 = \| E_{pM_n(B)p}(v^*v\theta(u_k))\|_2$

$$= \| E_{pM_n(B)p}(v^* u_k v)\|_2 \to 0$$

(3) $\Rightarrow$ (2):

Basic construction: on $L^2(M)$, consider $\mathcal{B}_B(L^2(M))$ on

e_B: proj onto $L^2(B) \subset L^2(M)$

then $\mathcal{B}_B(L^2(M)) \cong \mathcal{B}(\ell^2) \otimes B \rightsquigarrow$ trace $Tr_B = Tr \otimes \tau$, $e_B x e$

$\mathcal{B}_B(L^2(M)) = \langle M, e_B \rangle$, $Tr_B(e_B) = 1$, $e_B x e_B = e_B E_B(x)$ $\forall x \in M$

$K \subset L^2(M)$ is an A-B subbimodule iff $P_K \in \langle M, e_B \rangle \cap A'$

K is finite dimensional iff $Tr_B(P_K) < \infty$

$\Downarrow$

$\exists_{\neq 0} K_0 \subset K$ s.t. K_0 is finitely generated

if $\nexists u_k$ in $\mathcal{U}(A)$ s.t. $\forall x, y \in M: \|E_B(x u_k y)\|_2 \to 0$

then $\underset{\varepsilon > 0}{\exists} x_1, \dots, x_n, y_1, \dots, y_n$ in M s.t. $\forall u \in \mathcal{U}(A)$: $\|E_B(x_i u y_i)\|_2 \geq \varepsilon$ for some $i = 1, \dots, n$

~~we can assume that~~

write $\{a_1, \dots, a_m\} = \{x_1, \dots, x_n, y_1^*, \dots, y_n^*\}$, then $\forall u \in \mathcal{U}(A)$:

$$\sum_{i,j=1}^{m} \|E_B(a_i^* u a_j)\|_2^2 \geq \varepsilon$$

set $x = \sum_{i=1}^{m} a_i e_B a_i^* \in \langle M, e_B \rangle$, $Tr_B(x) < \infty$, $x > 0$.

$\rightsquigarrow x \in L^2(\langle M, e_B \rangle, Tr_B)$

$C = \overline{\mathrm{conv}}^{\text{norm}} \{u x u^* \mid u \in \mathcal{U}(A)\} \subset L^2(\langle M, e_B \rangle, Tr_B)$

all elts in C have finite trace.

- Take $w \in C$ with minimal $\|\cdot\|_{Tr_B, 2}$ $\rightsquigarrow$ $u w u^* = w$
- $w \neq 0$ because

$$\langle x, u x u^* \rangle_{Tr_B} = Tr(x u x u^*) = \sum_{i,j} Tr(a_i e_B a_i^* u^* a_j e_B a_j^* u)$$

$$= \sum_{i,j} Tr(\underbrace{e_B a_j^* u a_i e_B}_{E_B(a_j^* u a_i) e_B} e_B a_i^* u^* a_j e_B)$$

$$= \sum_{i,j} \|E_B(a_j^* u a_i)\|_2^2 \geq \varepsilon$$

take a spectral projection P of w $\rightsquigarrow_{\neq 0} P \in \langle M, e_B \rangle \cap A'$ $\rightsquigarrow P(L^2(M)) = K$.

2 ⇒ 1)

$K_B \cong p(\mathbb{C}^n \otimes L^2(B))_B$, say $\mathcal{U} : p(\mathbb{C}^n \otimes L^2(B))_B \to K_B$, $p \in M_n(B)$
is a unitary

set $\xi \in M_{1,n}(L^2(M))$

$\xi = [\mathcal{U}(\delta_1 \otimes 1), \ldots, \mathcal{U}(\delta_n \otimes 1)]$

$\theta(x) = \mathcal{U}^* (x) \mathcal{U} \in p M_n(B) p \quad \forall x \in A$

observe $x \xi = \xi \theta(x)$

consider $T_\xi : M(L^2(M)) \to M_{1,n}(L^2(M))$
$\hat{x} \mapsto \xi x$

closable unbounded operator satisfying $a T_\xi x = T_\xi \theta(a) x \quad \forall x \in M_n(M)$

⇝ polar decomposition gives $v \in M_{1,n}(M)$ s.t. $a v = v \theta(a)$

~~application: $\mathcal{F}(L\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}) = \{1\}$~~

~~· $SL_2\mathbb{Z}$ has Haagerup property so $\exists \varphi_n : SL_2\mathbb{Z} \to \mathbb{C}$ s.t.~~

~~· φ_n is posdef~~
~~· $\varphi_n(g) \to 1$ pointwise~~
~~· φ_n is c_0~~
~~· $\varphi_n(1) = 1$~~

define ψ

lemma [Popa 01] If A, B are Cartan subalgebras and $A \prec_M B$, then $A = u B u^*$ for some $u \in \mathcal{U}(M)$.

application: $\mathcal{F}(L\mathbb{Z}^2 \rtimes SL_2\mathbb{Z}) = \{1\}$

· ———

set $\psi_n(u_{(x,g)}) = \varphi_n(g) u_{(x,g)}$ ucp, tends to id pointwise $\|\cdot\|_2$

let $\alpha : pMp \to pMp$ be a $*$-iso

⇝ $\alpha(p L\mathbb{Z}^2 p) \subset pMp$ has rel (T)

⇝ $\psi_n \to id$ uniformly on $(\alpha(p L\mathbb{Z}^2 p))_1$

suppose $\alpha(pL\mathbb{Z}^2 p) \not\prec_M L\mathbb{Z}^2$, then $\exists (w_k)_k$ in $\mathcal{U}(pL\mathbb{Z}^2 p)$ s.t.

$$\forall x, y \in M : \| E_{L\mathbb{Z}^2}(x \alpha(w_k) y) \|_2 \to 0$$

in particular, $\|E_{\ell\mathbb{Z}^2}(u_g^a d(v_k))\|_2 \to 0 \quad \forall g \in SL_2\mathbb{Z}$

~~[illegible]~~

we know that $\|d(v_k)\|_2^2 = 1 \quad \forall k$

we show that this is not the case

~~[illegible]~~

Take $n \in \mathbb{N}$ s.t. $\|\psi_n(d(v_k)) - d(v_k)\|_2^2 < \frac{1}{4} \quad \forall k$

but φ_n ~~[illegible]~~ is c_0. Take $F \subset SL_2\mathbb{Z}$ finite s.t. $|\varphi_n(g)|^2 < \frac{1}{8} \quad \forall g \notin F$

take k s.t. $\|E_{\ell\mathbb{Z}^2}(u_g^* d(v_k))\|_2^2 < \frac{1}{8|F|} \quad \forall g \in F$

write $d(v_k) = \sum_g u_g v_{k,g}$

$\Rightarrow \|d(v_k)\|_2 \leq \|\psi_n(d(v_k))\|_2 + \|\psi_n(d(v_k)) - d(v_k)\|_2$

$$\|\psi_n(d(v_k))\|_2^2 = \sum_g \varphi_n(g)^2 \|v_{k,g}\|_2^2 \leq \sum_{g \in F} \ldots + \sum_{g \notin F}$$

$$\leq \frac{1}{8} + \frac{1}{8} \underbrace{\sum_{g \notin F} \|v_{k,g}\|_2^2}_{\leq 1}$$

$\Rightarrow \|\psi_n(d(v_k))\|_2 < \frac{1}{4} \quad \Rightarrow 1 < \frac{1}{2} \quad \Rightarrow\!\Leftarrow$