Breuillard 1 (I+II) 1

$\left\{\begin{array}{l}\text{Kesten's Thm} \\ \text{Tits Alternative} \\ \text{Powers } (C^*\text{-simplicity})\end{array}\right.$

Kesten's Thm (1950's)

Γ : finitely generated group
μ : probability measure on Γ
symmetric ($\mu(x^{-1}) = \mu(x)$)

Def μ is adapted to Γ if $\langle \mathrm{supp}(\mu) \rangle = \Gamma$.

$\mu \rightsquigarrow$ operator on $\ell^2\Gamma$ by convolution $f \mapsto \mu * f$

$\lambda = \lambda_\Gamma$ = left regular representation of Γ

$[\lambda(\gamma) f](x) = f(\gamma^{-1}x)$

$$\lambda(\mu) = \sum_{\gamma \in \Gamma} \lambda(\gamma)\mu(\gamma)$$

$$\mu^n := \underbrace{\mu * \mu * \cdots * \mu}_{n} \in \mathrm{Prob}(\Gamma)$$

μ^n is the distribution at time n of the random product

$S_n = X_1 \cdot \cdots \cdot X_n$ $\quad (X_i)_{i \geq 1}$ are IDD random variables with values in Γ.

$P(X_i = x) = \mu(x)$

Fact
- $\mu(yx^{-1}) = \mathrm{prob}(x \rightsquigarrow y) = \langle \lambda(\mu)\delta_x, \delta_y \rangle_{\ell^2\Gamma}$

  $\delta_x \in \ell^2\Gamma \quad \delta_x(y) = \begin{cases} 1 & (y = x) \\ 0 & (y \neq x) \end{cases}$

- $\mu^n(x) = P(S_n = x)$ $\qquad \lambda(\mu) = \lambda(\mu)^*$ [bounded self-adjoint] (μ is symmetric)

- $\mu^{2(n+1)}(e) \leq \mu^{2n}(e)$ $\qquad \lambda(\mu^n) = \lambda(\mu)^n$

- $\mu^{2n}(x) \leqslant \mu^{2n}(e) \quad \forall x \in \Gamma$.

<pf>. $\mu^{2n+2}(e) = \langle \lambda(\mu)^{2n+2}\delta_e, \delta_e \rangle$

$$= \langle \lambda(\mu)^{n+1}\delta_e, \lambda(\mu)^{n+1}\delta_e \rangle$$

$$= \| \lambda(\mu)^{n+1}\delta_e \|^2$$

$$\leq \| \lambda(\mu)^{n}\delta_e \|^2 = \mu^{2n}(e)$$

since $\| \lambda(\mu) \| \leq 1$.

- $\mu^{2n}(x) = \langle \lambda(\mu)^{2n}\delta_e, \delta_x \rangle = \langle \lambda(\mu)^{n}\delta_e, \lambda(\mu)^{n}\delta_x \rangle$

$$\leq \| \lambda(\mu)^{n}\delta_e \| \, \| \lambda(\mu)^{n}\delta_x \|$$

However $\| \lambda(\mu)^n \delta_x \|^2 = \langle \lambda(\mu)^{2n}\delta_x, \delta_x \rangle$

$$= \mu^{2n}(x^{-1}x) = \mu^{2n}(e)$$

$$= \| \lambda(\mu)^n \delta_e \|^2$$

So $\mu^{2n}(x) \leq \| \lambda(\mu)^n \delta_e \|^2 = \mu^{2n}(e)$. □

Def The spectral radius of the random walk is defined as the spectral radius of $\lambda(\mu)$ viewed as a bounded operator on $\ell^2\Gamma$ $\rho(\mu) = \| \lambda(\mu) \|$

= norm of μ where μ is viewed as an element of the reduced C^*-alg $C^*_\lambda(\Gamma)$.

= norm closure of span $\{ \lambda(\gamma) \mid \gamma \in \Gamma \}$ in $\mathbb{B}(\ell^2\Gamma)$.

Thm (Kesten)

$\| \lambda(\mu) \| = \lim_{n \to \infty} \mu^{2n}(e)^{1/2n}$ $\qquad \mu^{2n}(e) = P(S_{2n} = e)$

Remarks

(1) $n \mapsto \dfrac{1}{\mu^{2n}(e)}$ is submultiplicative

i.e. $\qquad \mu^{2m+2n}(e) \geq \mu^{2m}(e)\, \mu^{2n}(e) \qquad \forall n, \forall m$

$$P(S_{2m+2n} = e) \geq P(S_{2m} = e \wedge S_{2n} = e)$$

$$= P(S_{2m} = e) \cdot P(S_{2n} = e)$$

∵ The subadditivity lemma shows that

$$\lim_{n\to\infty} \mu^{2n}(e)^{1/2n} \text{ exists.}$$

(2) $$\underset{\| \lambda(\mu)^n \delta_e \|^2}{\underset{\|}{\mu^{2n}(e)}} \leq \underset{\text{Kesten's bound.}}{\underset{\uparrow}{\|\lambda(\mu)\|^{2n}}} \quad \forall n.$$

Proof of Kesten's Thm

$\forall \varepsilon > 0, \quad \exists f \in \mathbb{C}\Gamma$ (finitely supported function)

s.t. $\|f\|_2 = 1$

$$\langle \lambda(\mu) f, f \rangle \geq \|\lambda(\mu)\| - \varepsilon$$

Spectral Thm

$\nu_f \in \mathrm{Prob}([-1,1]) \quad \langle \lambda(\mu) f, f \rangle = \int_{-1}^{1} t \, d\nu_f(t)$

$$\left(\langle \lambda(\mu)^{2n} f, f \rangle \right)^{1/2n} = \left(\int_{-1}^{1} t^{2n} \, d\nu_f(t) \right)^{1/2n}$$

$\downarrow n \to \infty$

$$\max \{ |t| \mid t \in \mathrm{supp}(\nu_f) \} \geq \int_{-1}^{1} t \, \nu_f(dt)$$

$$= \langle \lambda(\mu) f, f \rangle \geq \|\lambda(\mu)\| - \varepsilon$$

$$\langle \lambda(\mu)^{2n} f, f \rangle = \sum \underbrace{\mu^{2n}(x)}_{\leq \mu^{2n}(e)} f(x^{-1}y) \bar{f}(y)$$

$$\limsup_{n\to\infty} \langle \lambda(\mu)^{2n} f, f \rangle^{1/2n} \leq \lim_{n\to\infty} \left(\mu^{2n}(e) \right)^{1/2n}$$

$$\vee\!|$$

$$\|\lambda(\mu)\| - \varepsilon.$$

$\varepsilon > 0$ arbitrary, done $\square$

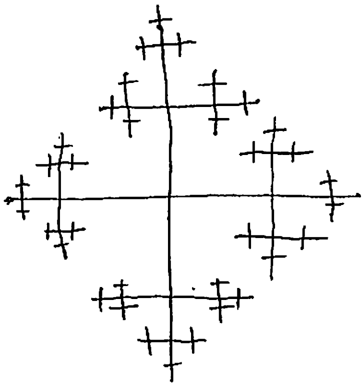
Thm (Kesten 2) If Γ is a free group on k-generators,

$$\Gamma = \langle \underbrace{s_1, \cdots, s_k}_{S_k} \rangle$$

Then $\| \lambda(\mu_{S_k}) \| = \sqrt{\dfrac{2k-1}{k}}$

$$\mu_{S_k} = \frac{1}{2k} \sum_{i=1}^{k} (\delta_{s_i} + \delta_{s_i^{-1}})$$

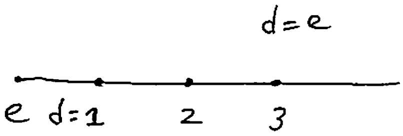
indeed By Kesten 1, $\mu^{2n}(e) = \dfrac{|\text{loops of length } 2n \text{ around } e|}{|\text{paths of length } 2n \text{ from } e|}$



$2k+1$ regular tree.

$q+1 = 2k$

$\dfrac{q}{q+1}$ to go right



$\dfrac{q}{q+1}$ to go left

$$|\text{loops around } e| \simeq \binom{2n}{n} q^n$$

$$|\text{paths from } e| = (q+1)^{2n}$$

$$\mu^{2n}(e) \approx 2^{2n} \left(\frac{\sqrt{q}}{q+1} \right)^{2n}$$

$$\lambda(\mu) = \frac{2\sqrt{q}}{q+1} = \frac{\sqrt{2k-1}}{k} \quad \square$$

Thm K 3

Γ is amenable $\iff$ $\rho(\mu) = 1$ (μ adapted and symmetric)

If $H \triangleleft \Gamma$ then H is amenable

$$\iff \|\lambda_\Gamma(\mu)\| = \|\lambda_{\Gamma/H}(\mu)\|$$

Remark

$\|\lambda_\Gamma(\mu)\| = 1 \iff \exists$ almost Γ- invariant sequence in $\ell^2\Gamma$

i.e. iff Γ is amenable.

$$\left[\begin{array}{l} \exists f_n \in \ell^2\Gamma \quad \|f_n\|_2 = 1 \\ \|\lambda_\Gamma(\mu) f_n\|_2 \to 1 \\ \leadsto \|\lambda_\Gamma(\gamma) f_n - f_n\|_2 \to 0 \quad \forall \gamma \in \Gamma. \end{array}\right]$$

$$\Gamma \text{ amenable} \iff \mu^{2n}(e)^{1/2n} \xrightarrow{n\to\infty} 1$$

- $\forall H \leq \Gamma \qquad \|\lambda_{\Gamma/H}(\mu)\| \geq \|\lambda_\Gamma(\mu)\|$

  indeed by Kesten 1.

- H is amenable $\quad \lambda_{\Gamma/H} \succ \lambda_\Gamma$ weak containment (see Bekka - de la Harpe - Valette)

  $\lambda_{\Gamma/H}$: quasi-regular representation relative to H

II. Tits Alternative

1972

Thm (Tits)

(If Γ is a finitely generated linear group. Then either Γ has a free subgroup non-abelian or Γ is virtually solvable.)

linear group: subgroup of some $GL_n(k)$, k : field, $n \in \mathbb{N}$

Cor Γ : finitely generated linear group

(Then Γ is amenable $\iff$ Γ is virtually solvable.)

Y. Shalom 1999. See Lang, algebra $|\cdot|_k \longrightarrow$ uniquely extends to the algebraic closure

Ping - pong k : local field. ($\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, and finite extensions $\mathbb{F}_p((t))$, and finite extensions.)

Def

$\gamma \in GL_d(k)$ is called proximal if it has a unique eigenvalue of maximal modulus. $|\lambda_1(k)| > |\lambda_2(k)| \geqslant \cdots \geqslant |\lambda_d(k)|$

A linear rep ρ of Γ is proximal if $\rho(\Gamma)$ contains a proximal element.

$\rho : \Gamma \longrightarrow GL_n(k)$ is I-P if $\begin{cases} \rho \text{ is proximal} \\ \rho \text{ is strongly irreducible} \end{cases}$

strongly irr = no finite union of proper subspaces is preserved by $\rho(\Gamma)$.

$\left(\Longleftrightarrow \left(\overline{\rho(\Gamma)}^{\text{zariski}}\right)^{0} \text{ is irreducible} \right)$

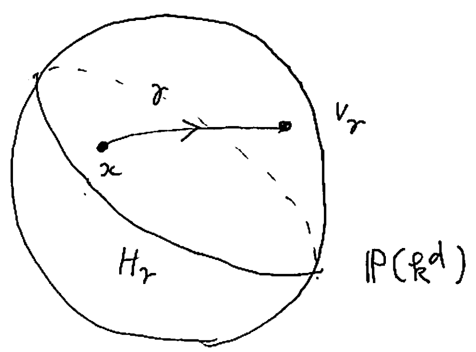
Remark

If $\gamma \in GL_d(k)$ is proximal, then γ^n $(n \geqslant 1)$ behave like "contractions" in $\mathbb{P}(k^d)$

Let V_γ = eigen direction corresponding to the max eigenvalue $\in \mathbb{P}(k^d)$

H_γ = sum of remaining generalized eigenspaces.

Then $\forall x \notin H_\gamma$ $\gamma^n x \xrightarrow{n \to \infty} V_\gamma$



Main Lemma of Tits

If Γ is not virtually solvable, then $\exists\, k$ a local field $\exists\, V$ a k- vector space (finite - $\dim_k$)
and $\exists\, \rho : \Gamma \longrightarrow GL(V)$ I-P.

ideas
- Choose k s.t. $\Gamma \le GL_d(k)$ unbounded.
- pass to a wedge power
  $\Lambda^m k^d$ then it becomes proximal
  but may be not irr.
- pass to irreducible quotient.

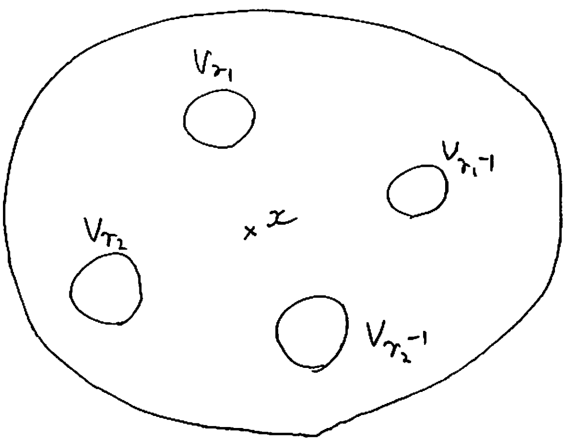
ping-pong Lemma

Suppose $\gamma_1, \gamma_2 \in GL_d(k)$ $\gamma_1, \gamma_2, \gamma_1^{-1}, \gamma_2^{-1}$ proximal.

and assume $V_{\gamma_1^{\pm 1}} \notin H_{\gamma_1} \cup H_{\gamma_2^{-1}}$

$V_{\gamma_2^{\pm 1}} \notin H_{\gamma_1} \cup H_{\gamma_2^{-1}}$

Then $\langle \gamma_1^n, \gamma_2^n \rangle$ is free for n large.



$w(\gamma_1^n, \gamma_2^n) = \gamma_1^n \gamma_2^{-n} (\gamma_1^n)^3$

$w \cdot x \neq x$

$\Downarrow$

$w \neq 1$

$\Downarrow$

$\langle \gamma_1^n, \gamma_2^n \rangle$ is free.

Q ? Can one get an estimate on the spectral radius

$\| \lambda_\Gamma(\mu) \|$?

$\mu \in \mathrm{Prob}(\Gamma)$

(Kesten's bound

$\lambda(\mu_{S_k}) = \frac{\sqrt{2k-1}}{k} < 1$

$k \to \infty$

$\longrightarrow 0$)

Thm (Uniform spectral gap for linear groups 'o8)

(Given $d, k \geq 1$. $\exists \beta = \beta(k, d) < 1$

s.t. $\forall \gamma_1, \dots, \forall \gamma_k \in GL_d$ (some field)

- either $\langle \gamma_1, \dots, \gamma_k \rangle$ is amenable
- or $\| \lambda_\Gamma \left(\frac{1}{2k} \sum_{k=1}^{n} (\delta_{\gamma_k} + \delta_{\gamma_k^{-1}}) \right) \| < \beta$)

Deduced from

Thm (Uniform Tits Alternative)

(Given d, $\exists N = N(d) \in \mathbb{N}$

s.t. $\forall F \subseteq GL_d$ (Some field) $F = F^{-1}, 1 \in F$

- either $\langle F \rangle$ is virtually solvable
- or $(F)^N$ contains generators of a non-abelian free subgroups.)

cf

connected to

Eskin-Mozes-Oh

Γ F^N

$N = N(\Gamma)$

books
{ Bougerol
Benoist - Quint

Random walks on GL_d

$\mu \in \mathrm{Prob}(GL_d(k))$

$S_n = X_1 \cdot \cdots \cdot X_n$ i.i.d $\sim \mu$.

Guivarch Pos ?

If $\rho : \Gamma \to GL(V)$ is I-P then $\rho(S_n)$ is proximal with probability 1, as $n \to \infty$.

Thm (Aoun 2011)

If $(S_n^1), \dots, (S_n^k)$ are k-independent adapted RW on a non-amenable linear group, then

$\langle S_n^1, \dots, S_n^k \rangle$ is free of rank k.

III Powers

C^*-simplicity of groups.

Γ: discrete group is C^*-simple if $C^*_\lambda(\Gamma)$ is simple.

$\Leftrightarrow \left(\forall \pi \text{ unitary repr } \pi \prec \lambda_\Gamma \Rightarrow \pi \sim \lambda_\Gamma \right)$

If $N \triangleleft \Gamma$, $N \neq \{1\}$, N is amenable

Then Γ not C^*-simple

$\lambda_{\Gamma/N} \prec \lambda_\Gamma$

but $\lambda_{\Gamma/N} \not\sim \lambda_\Gamma$ $\quad \langle \lambda_\Gamma(\gamma)\delta_e, \delta_e \rangle$

Open Problem

Is this the only obstruction ?

Def

$\mathrm{Rad}(\Gamma) = \langle N \triangleleft G \mid N \text{ amenable} \rangle$

(·· amenable radical)

Powers 1975

Free groups are C^*-simple.

- Bekka - Cowling - de la Harpe '90s
  Zariski-dense subgroups of center-free semi-simple algebraic groups.

- Poznansky any linear group w/ $\mathrm{Rad}(\Gamma) = 1$
  is C^*-simple

- Gromov hyperbolic groups, Baumslag-Solitar groups
  Free Burnside groups.
  Osin, Olshanskii.

Thm (Kalantar-Kennedy)

Γ is C^*-simple $\Longleftrightarrow$ Γ has a topologically free boundary action.

Powers Lemma Γ discrete group. Assume $\forall \varepsilon > 0$ $\forall F \overset{\text{finite}}{\subseteq} \Gamma \setminus \{e\}$

$\exists g_1, \cdots, \exists g_k \in \Gamma$ s.t.

$$\left\| \lambda_\Gamma \left(\frac{1}{k} \sum_{i=1}^{k} \delta_{g_i x g_i^{-1}} \right) \right\| < \varepsilon$$

$\forall x \in F$

Then Γ is C^*-simple (and has a unique trace).

Kesten's Thm $\to$ This estimate is satisfied for free groups.