Breuillard 2 (III+IV)

Remark Relation between spectral gap and exponential growth

Γ : discrete group $= \langle S \rangle$ $\quad S = S^{-1}$ finite $\quad \mu_S = \frac{1}{|S|} \sum_{s\in S} \delta_s$

$\| \lambda_\Gamma(\mu_S) \|$ $\quad \ni 1$ $\quad$ adopted to Γ.

$$\rho_S := \lim_{n\to\infty} |S^n|^{\frac{1}{n}} \qquad S^n = \underbrace{S \cdot \dots \cdot S}_{n \text{ times}}$$

Relation $\rho_S \geqslant \frac{1}{\| \lambda_\Gamma(\mu_S) \|}$

in particular

Γ non-amenable $\Rightarrow$ it has exponential growth.

why μ_S^n $\quad 1 = \sum_{x\in\Gamma} \mu_S^{2n}(x) \qquad \mathrm{supp}(\mu_S^n) \subseteq S^n$

$$= \sum_{x\in S^n} \mu_S^{2n}(x) \leqslant |S^{2n}| \, \mu^{2n}(e)$$

Kesten's Thm $\Rightarrow$ $\rho_S \geqslant \frac{1}{\| \lambda_\Gamma(\mu_S) \|}$

Powers Lemma Assume $\forall \varepsilon > 0$ $\forall F \overset{\text{finite}}{\subseteq} \Gamma \setminus \{1\}$

$$\left(\begin{array}{l} \exists g_1, \dots, \exists g_n \in \Gamma \\ \forall x \in F \ \left\| \lambda_\Gamma\left(\frac{1}{n} \sum_{i=1}^{n} \delta_{g_i x g_i^{-1}} \right) \right\| < \varepsilon \end{array} \right)$$

Then Γ is C^*-simple. and $C_\lambda^*(\Gamma)$ has unique trace.

Remark ① If one can find g_i's in such a way that $\{ g_i x g_i^{-1} \}_{1\leq i\leq k}$ are free generators of free group of rank k. then if k can be taken arbitrarily large then Powers condition holds

by Kesten's $\| \lambda_\Gamma(\ ) \| \leq \frac{\sqrt{2k-1}}{k}$

② This is the case when $\Gamma = F_2 = \langle a, b \rangle$

$\forall x \in F \subseteq \Gamma \setminus \langle a \rangle \quad a^n x a^{-n}$

Choose n large enough s.t. $a^n x a^{-n}$, $x \in F$

starts with a and ends with a^{-1}

$g_i = b^i a^n \quad i \geq 1$

$\longrightarrow g_1 \cdots g_k$

$(g_i x g_i^{-1})_{i=1}^{k}$ will be free.

<u>Open Problem</u> Γ is a linear group. Pick $\gamma \in \Gamma \setminus \mathrm{Rad}(\Gamma)$

① Can one find $x_1, \dots, x_k \in \mathrm{Conj}(\gamma)$

s.t. $\langle x_1, \dots, x_k \rangle = \langle x_1 \rangle * \cdots * \langle x_k \rangle$?

② If $S_n^1, \dots, S_n^k$ are independent RW in Γ

then is it true that

$$\langle \gamma^{S_n^1}, \dots, \gamma^{S_n^k} \rangle = \langle \gamma^{S_n^1} \rangle * \cdots * \langle \gamma^{S_n^k} \rangle$$

with prob. $\xrightarrow{n \to \infty} 1$

One would like to have a robust method to show that

$\| \lambda_\Gamma(\mu) \|$ is small.

<u>Pisier</u> Γ non-amenable. Can one find

$\forall \varepsilon$ a finite S_ε s.t. $\left\| \lambda_\Gamma \left(\frac{1}{|S_\varepsilon|} \sum_{s \in S_\varepsilon} \delta_s \right) \right\| < \varepsilon$?

Yes. (A. Thom)

<u>Problem</u> Suppose Γ is a linear group.

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$\forall \mu$ symmetric probability measure on Γ

s.t. $\| \lambda_\Gamma(\mu) \| > \varepsilon$

then $\exists H \leq \Gamma$ amenable s.t. $\mu(H) > \delta$.

1% thm(?) still open

Remark

① If H amenable then $\|\lambda_\Gamma(\mu)\| \geq \mu(H)$

(Kesten : $\mu = \mu(H) \frac{\mu(\cdot)}{\mu(H)} + (1-\mu(H)) \cdot \overset{\exists}{\nu}$)

② It follows from the uniform spectral thm of yesterday that given $d \in \mathbb{N}$ $\exists c(d) > 0$ s.t. $\forall \Gamma \subseteq GL_d$ (some field)

$\forall \underset{\text{symmetric}}{\mu \in \mathrm{Prob}(\Gamma)}$ if $\|\lambda_\Gamma(\mu)\| > 1-\varepsilon$

then $\exists H$ amenable $\leq \Gamma$

s.t. $\mu(H) \geq 1 - c(d)\varepsilon$. 99% thm

③ The 99% Thm fails for non-linear groups

J. Wilson built (2000) a finitely generated non-amenable group which is not of uniform exp growth.

$\Gamma = \langle a, b \rangle$ $\{a_n, b_n\}$ generating pairs

s.t. $\rho_{\{a_n^{\pm1}, b_n^{\pm1}\}} \xrightarrow{n\to\infty} 1$

$\mu_n = \frac{1}{4}(\delta_{a_n} + \delta_{a_n^{-1}} + \delta_{b_n} + \delta_{b_n^{-1}})$

$\|\lambda_\Gamma(\mu_n)\| \to 1$

Lemma (Ping-Pong) Suppose $\Gamma \curvearrowright X$ a set

Powers
Bozejko, Cowling, Harpe

Let $\gamma_1, \dots, \gamma_n \in \Gamma$ be s.t.

$\exists A_1, \dots, \exists A_n \subseteq X$ disjoint.

$\exists R_1, \dots, \exists R_n \subseteq X$ disjoint.

s.t. $\forall i = 1, \dots, n$ $\gamma_i(X \setminus R_i) \subseteq A_i$

then $\left\| \lambda_\Gamma \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{\gamma_i} \right) \right\| < \frac{2}{\sqrt{n}}$

<pf> Can assume that $X = \Gamma$

$$\left(\begin{array}{l} \text{Take } \pi : \Gamma \longrightarrow X \quad \text{For some } x_0 \in X. \\ \qquad\quad \gamma \longmapsto \gamma \cdot x_0 \\ \text{Replace } A_i \text{ by } \pi^{-1}(A_i) \\ \qquad\quad R_i \text{ by } \pi^{-1}(R_i) \end{array} \right)$$

$f \in \ell^2\Gamma.$ $\quad \lambda_\Gamma(r_i) f = \lambda(r_i)(f 1_{R_i} + f 1_{\Gamma \setminus R_i})$

$$= \lambda(r_i)(f 1_{R_i}) + 1_{A_i} \lambda(r_i)(f 1_{\Gamma \setminus R_i})$$

$$\left\| \sum_{i=1}^n \lambda(r_i) f \right\| \leq \underbrace{\left\| \sum_{i=1}^n \lambda(r_i)(1_{R_i} f) \right\|}_{① \ \leq \sqrt{n} \|f\|} + \underbrace{\left\| \sum_{i=1}^n 1_{A_i} (\lambda(r_i)(f 1_{\Gamma \setminus R_i})) \right\|}_{② \ \leq \sqrt{n} \|f\|}$$

$$①^2 = \sum_{\lambda \in \Gamma} \left| \sum_{i=1}^n (1_{R_i} f)(r_i^{-1} x) \right|^2 \overset{CS}{\leq} \sum_x n \sum_i |1_{R_i} f|^2 (r_i^{-1} x)$$

$$\leq n \|f\|^2$$

$$\sum_{i=1}^n 1_{R_i} \leq 1, \quad \sum_{i=1}^n 1_{A_i} \leq 1$$

$②^2$ similar. $\square$

<u>Question</u> Γ : non-amenable discrete group.

$S_n^1, \dots, S_n^k$ k independent RW associated to some adapted symmetric probability $\mu \in \mathrm{Prob}(\Gamma)$

Is it true that $\left\| \lambda_\Gamma \left(\frac{1}{2k} \sum_{i=1}^k (\delta_{S_n^i} + \delta_{(S_n^i)^{-1}}) \right) \right\| < \varepsilon(k)$

with prob $\xrightarrow{n \to \infty} 1$

for some function $\underset{\Gamma, \mu}{\varepsilon(k)} \xrightarrow{k \to \infty} 0$

(True for linear groups)

<u>Remark</u> The ping-pong type Lemma continues to hold if we assume only that every M R_i's have trivial intersection i.e., $\sum_{i=1}^{n} 1_{R_i}(x) \leq M \quad \forall x$

with the bound $\frac{2M}{\sqrt{n}}$ instead of $\frac{2}{\sqrt{n}}$ with prob $\xrightarrow{n\to\infty} 1$: $H_n^1 \cap \cdots \cap H_n^{m+1} = \{0\}$

Sketch of proof of Poznansky Thm

<u>Prop</u> Γ linear $\gamma \in \Gamma \setminus \mathrm{Rad}(\Gamma)$ $S_n^1, \dots, S_n^k$ k indep adapted RW on Γ then with probability $\xrightarrow{n\to\infty} 1$,

$$\left\| \lambda_\Gamma \left(\frac{1}{k} \sum_{i=1}^{k} \delta_{S_n^i \gamma (S_n^i)^{-1}} \right) \right\| \leq \frac{C(d)}{\sqrt{k}}$$

Apply Powers Lemma $\Rightarrow$ if Γ linear & $\mathrm{Rad}(\Gamma) = \{1\}$ then Γ is C^*-simple.

<u>Idea</u> • By Tits main lemma from yesterday,

$\exists \rho : \Gamma \to PGL_m(k)$, k local field

s.t. $\rho(\gamma) \neq 1$ and ρ is <u>I-P</u>.

• Guivarch with probability $\to 1$,

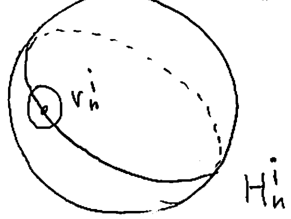
$\rho(S_n^i)$ is proximal.

$\to g_n^i = \rho(S_n^i \gamma (S_n^i)^{-1})$ is contractive.

i.e., $\exists H_n^i$ hyperplane in $\mathbb{P}(k^m)$ and V_n

s.t. $g_n^i \, \mathbb{P}(k^m) \setminus (H_n^i)_{e^{-\varepsilon n}} \subseteq (V_n^i)_{e^{-\varepsilon n}}$

$(A)_\delta$ = δ-neighborhood of A



<u>Aoun</u>'s thesis

$d(V_{S_n^i}, V_{S_n^j}) \geq e^{-\varepsilon n}$ with high prob.

IV : Spectral gaps for quasi-regular representations

$$\lambda_{\Gamma/H} = \mathrm{Ind}_H^\Gamma(1_H) \curvearrowright \ell^2(\Gamma/H)$$

Def

$H \leq \Gamma$ is co-amenable if $1_\Gamma \prec \lambda_{\Gamma/H}$ (i.e. $\lambda_{\Gamma/H}$ has almost-invariant vectors)

Thm (B'14)

Γ linear group, $\mu \in \mathrm{Prob}(\Gamma)$ symm adapted to Γ

Then $\exists\ \varepsilon := \varepsilon(\Gamma, \mu) > 0$

s.t. $\forall H \leq \Gamma$ either • H is co-amenable

or • $\|\lambda_{\Gamma/H}(\mu)\| \leq 1-\varepsilon$

Cor (Shalom–Hadiri) $H \leq \Gamma$ is co-amenable

$\Longleftrightarrow$ H and Γ has the same perfect core

Projection of Γ $= D^\infty((\overline{\Gamma}^{\text{Zariski}})^\circ)$ ← algebraic conn. comp.

$$D^\infty(G) = \bigcap_d D^d(G)$$

$$D^{n+1}(G) = [D^n(G), D^n(G)]$$

IV KK – criterion for C^*-simplicity

Thm (KK) Γ discrete group

Γ : C^*-simple $\Longleftrightarrow$ Γ has a topologically free boundary action.

<u>Prop</u> (BKKO) C^*-simplicity is stable under group extensions.

$$\left(1 \to N \to G \to M \to 1 \right)$$

<pf> is dynamical, just relies on the KK-criterion.

<u>Linear case</u>

Assume $\Gamma \subseteq PGL(V)$ V : vector space over a local field k
$\dim V \geq 2$

assume Γ is I-P.

<u>Def</u> (Limit set)

$$L_\Gamma = \overline{\{ V_\gamma \mid \gamma : \text{proximal in } \Gamma \}} \subseteq \mathbb{P}(V)$$

V_γ = attracting direction

<u>Facts</u>

(1) L_Γ is a Γ-boundary

(2) L_Γ is topologically free.

<u>Cor</u> Linear groups with no amenable radical are C^*-simple.

<pf> Tits main lemma : given Γ, $\exists \rho : \Gamma \to PGL(V)$
which is I-P $\dim V \geq 2$

by $\left(\begin{matrix} \text{above Facts} \\ \text{KK-criterion} \end{matrix} \right)$ $\rho(\Gamma)$ is C^*-simple.

If $\operatorname{Ker} \rho = \{1\}$, done

If $\operatorname{Ker} \rho \neq \{1\}$

$\dim \overline{\operatorname{Ker} \rho}^{z} < \dim \overline{\Gamma}^{z}$ induction using Prop above $\square$

Sketch of proof Fact (1)

L_Γ : minimal If $\gamma \in \Gamma$: proximal then V_γ, H_γ

$$\forall x \in \mathbb{P}(V) \setminus H_\gamma \quad \gamma^n x \longrightarrow V_\gamma.$$

V_{γ_1}, V_{γ_2} if $V_{\gamma_2} \notin H_{\gamma_1} \Rightarrow \gamma_1^n V_{\gamma_2} \to V_{\gamma_1}$

$$V_{\gamma_1} \in \overline{\Gamma V_{\gamma_2}}$$

if $V_{\gamma_2} \in H_{\gamma_1}$ irred $\exists g$ $gV_{\gamma_2} \notin H_{\gamma_1} \rightsquigarrow \gamma_1^n g V_{\gamma_2} \to V_{\gamma_1}$ ✓

L_Γ : strongly proximal : $\forall \mu \in \mathrm{Prob}(L_\Gamma)$

$\overline{\Gamma \cdot \mu}$ contains Dirac masses $\{\delta_x\}$

Pick μ, $\gamma \in \Gamma$ proximal.

V_γ, H_γ If $\mu(H_\gamma) = 0$ then $\gamma^n \mu \longrightarrow \delta_{V_\gamma}$

$$\mu = \mu(H_\gamma) \cdot \frac{\mu(\cdot)|_{H_\gamma}}{\mu(H_\gamma)} + (1 - \mu(H_\gamma)) \cdot \nu$$

$\overline{\Gamma \cdot \mu} \ni$ measure supported on $\{V_\gamma\} \cup H_\gamma$

Iterate with another γ' s.t. $H_\gamma \neq H_{\gamma'}$ (irreducibility)

(take suitable conjugate of γ)

$\overline{\Gamma \cdot \mu} \ni$ measure supp on a finite set.

$\longrightarrow$ take advantage of strong irreducibility to conclude.

Fact (2) L_Γ is top free. If $s \in \Gamma$ fixes pointwise some open set, then $s = 1$

Assume $\mathcal{U} \subseteq L_\Gamma$ is open $s \cdot x = x$ $\forall x \in \mathcal{U} \neq \emptyset$

Let $\mathcal{V}_\mathcal{U} = \overline{\mathcal{U}}^Z \subseteq \mathbb{P}(V)$ projective variety

Up to shrinking $\mathcal{U}$ if necessary can assume

$$\forall \mathcal{U}' \subseteq \mathcal{U} \text{ open} \quad \mathcal{V}_\mathcal{U} = \mathcal{V}_{\mathcal{U}'}$$

So this means that $\forall \gamma \in \Gamma$ if $\gamma \mathcal{U} \cap \mathcal{U} \neq \emptyset$

then $\mathcal{V}_{\mathcal{U}} = \mathcal{V}_{\mathcal{U} \cap \gamma \mathcal{U}} = \mathcal{V}_{\gamma^{-1}\mathcal{U} \cap \mathcal{U}}$

$$\Rightarrow \mathcal{V}_{\mathcal{U}} \subseteq \mathcal{V}_{\gamma \mathcal{U}} = \gamma \mathcal{U}_{\mathcal{U}}$$

$$\mathcal{V}_{\mathcal{U}} \subseteq \mathcal{V}_{\gamma^{-1}\mathcal{U}} = \gamma^{-1} \mathcal{V}_{\mathcal{U}}$$

$$\Rightarrow \gamma \mathcal{V}_{\mathcal{U}} = \mathcal{V}_{\mathcal{U}}$$

$$\Rightarrow \gamma \in \Gamma_{\mathcal{U}} = \mathrm{Stab}(\mathcal{V}_{\mathcal{U}})$$

However L_Γ is minimal cpt Γ- space

$$\Rightarrow \exists \gamma_1, \dots, \exists \gamma_n \in \Gamma \text{ s.t. } L_\Gamma = \bigcup_{i=1}^{n} \gamma_i \mathcal{U}$$

$$\Rightarrow [\Gamma : \Gamma_{\mathcal{U}}] < \infty$$

But $s \cdot x = x \quad \forall x \in \mathcal{U} \Rightarrow s \cdot x = x \quad \forall x \in \overline{\mathcal{U}}^{Z} = \mathcal{V}_{\mathcal{U}}$

$$\Rightarrow s \cdot \gamma x = \gamma x \quad \forall \gamma \in \Gamma_{\mathcal{U}}$$

[s- irreducible
$\Updownarrow$
$\forall$ finite index subgr acts irreducibly]

But $\Gamma_{\mathcal{U}}$ acts irreducibly on V

$$\Downarrow$$

$$s = 1$$