Kennedy 1

Motivating Question

G : discrete group.

$C_r^*(G)$ reduced group C^*-alg.

Thm (1974, Powers)

($C_r^*(\mathbb{F}_n)$ is simple, and has a unique trace.)

Q: (When is $C_r^*(G)$ simple?
When does $C_r^*(G)$ have a unique trace?

Injectivity

Defn

Let $\mathcal{C}$ be a category of objects and morphisms with a notion of embedding. An object $I \in \mathcal{C}$ is <u>injective</u> if for objects $X, Y \in \mathcal{C}$, w/ an embedding $\iota : X \longrightarrow Y$ and a morphism $\varphi : X \longrightarrow I$, then there is a morphism $\psi : Y \longrightarrow I$ s.t. $\varphi = \psi \circ \iota$

$$\begin{tikzcd} Y \arrow[dr, "\psi"] & \\ X \arrow[u, hook, "\iota"] \arrow[r, "\varphi"'] & I \end{tikzcd}$$

Typically we will take $X \subseteq Y$, so ι is the inclusion map

In this case $\psi|_X = \varphi$. Then

1 ψ is a morphism in $\mathcal{C}$.

2 $\operatorname{ran}(\psi) \subseteq I$

Thm (Hahn-Banach)

(The space of complex numbers $\mathbb{C}$ is injective in the category of Banach spaces with bounded (or contractive) linear maps. Embedding is inclusion.)

An operator system is a unital, self-adjoint subspace of a C^*-algebra.

Let X and Y be operator systems. A map $\varphi: X \to Y$ is positive if $\varphi(x) \geq 0$ whenever $x \in X$, $x \geq 0$.

We say φ is completely positive if $\varphi_n := id_n \otimes \varphi$ on $M_n \otimes X$ is positive for all $n \in \mathbb{N}$.

$$\varphi_n((x_{ij})) = (\varphi(x_{ij})) \quad (x_{ij}) \in M_n(X) = M_n \otimes X$$

Completely positive (CP) maps are much nicer than positive maps.

<u>Reason</u> #1 Stinespring's Thm.

CP maps $\leftrightarrow$ pieces of $*$-homs.

#2 Arveson's extension Thm

<u>Thm</u> For a Hilbert space H, $\mathbb{B}(H)$, the algebra of all bounded operators on H, is injective in the category of operator systems w/ cp maps. (or unital cp maps.)

In category terms a category has sufficiently many objects if any objects embeds into an injective object.

<u>Cor</u> The category of op-systems / ucp maps has sufficiently many injections.

<u>Pf</u> GNS Thm.

Let G be a discrete group. An operator system X is a G-operator system if there is a map (group-hom) from G into the group of order automorphisms of X, i.e., the unital, complete isometries. $X \to X$.
(for each n, $\varphi_n : M_n(X) \to M_n(X)$ is isometric)

G acts on $\ell^\infty(G)$ by $sf(t) = f(s^{-1}t)$ $s,t \in G$, $f \in \ell^\infty(G)$.

A cp map $\varphi: X \to Y$ for G-op systems. X, Y is G-equivariant if $\quad s\varphi(x) = \varphi(sx) \quad s \in G,\ x \in X.$

Can Consider the category of G-op systems w/ G-equivariant cp maps.

An embedding is, in addition, completely isometric.

(In general $\mathbb{B}(H)$ is not injective in the category of G-op systems.
Want to show : injective objects exist in the category of G-op systems.)

Need to know : we have enough injections.

<u>Thm</u> (The op system $\ell^\infty(G, X)$ is injective in the category of G-op systems, if X is injective in the category of op-systems.

The $G \curvearrowright \ell^\infty(G, X)$ by $sf(t) = f(s^{-1}t)$)

<Proof> Let X be injective.

Suppose $Y \overset{\iota}{\hookrightarrow} Z$ w/ ι : an embedding
Y, Z : G-op systems.

$$\varphi : Y \longrightarrow \ell^\infty(G, X)$$

<u>Define</u> $\xi : Y \longrightarrow X$ by $\xi(y) := \varphi(y)(e)$

ξ is a ucp map.

By injectivity of X, get an $\overset{\text{ucp}}{\text{extension}}$ $\eta : Z \longrightarrow X$

Now define $\psi : Z \to \ell^\infty(G, X)$

by $\psi(z)(t) = \eta(t^{-1}z)$

ψ is G-equivariant and ucp.

Hence $\ell^\infty(G, X)$ is injective. $\square$

<u>Cor</u> (Category of G-op system has sufficiently many injections.)

Q : Are there any other injective objects ?

One answer: Connes classified injective vN factors

— the semidiscrete ones.

e.g. Take A = C^*-alg, that is nuclear

e.g. $C_r^*(G)$ G: amenable.

then A^{**} is injective.

Another answer: Hamana there are lots of injective objects can approximate an arbitrary op-systems arbitrarily well by injective ones.

$A \subseteq I$