Last time

Tucker-Drob (V+VI)

- $AC(G) = \langle \{ N \trianglelefteq G \mid G/C_G(N) \text{ is amenable} \} \rangle$
- $I(G) = \langle \{ N \trianglelefteq G \mid \underset{\text{amenable}}{N \rtimes G \curvearrowright N} \} \rangle$
- Dani's Lemma

Def ('73 Kegel-Wehrfritz)

(A group G is said to satisfy m.c.c. (minimal condition on centralizers) if $\{ C_G(B) \mid B \leq G \}$ satisfies D.C.C. (descending chain condition) i.e., for any $B \subseteq G$, $\exists B_0 \subseteq B$ finite with $C_G(B_0) = C_G(B)$)

Prop 13

(Linear groups satisfy m.c.c.)

<pf> If $H \leq G$ then $C_H(B) = C_G(B) \cap H$

so m.c.c. passes to subgroups.

So suffices to show m.c.c. for $GL_n(F)$

If $B \subseteq GL_n(F)$

$$C_{M_n(F)}(B) = \{ x \in M_n(F) \mid xb = bx \ \ \forall x \in B \}$$

is exactly the set of solutions to the system of linear equations

$(xb - bx = 0, \ b \in B)$

By linear alg / Hilbert-basis theorem, $\exists B_0 \subseteq B$ finite

s.t. $C_{M_n(F)}(B) = C_{M_n(F)}(B_0)$

Thus $C_{GL_n(F)}(B) = C_{GL_n(F)}(B_0)$ $\square$

Remark

$BS(m,n)$ not m.c.c. when $|m|, |n| > 1$ & $|m| \neq |n|$

(so they are not linear)

Lem 14

Suppose G is m.c.c. Then

(i) $AC(G) = I(G)$.

(ii) $G/C_G(AC(G))$ is amenable.

(iii) Every conjugation invariant mean on G lives on $AC(G)$.

<pf> We'll show

(iii)' Every conj-inv mean m on G

$\exists N \lhd G$ $\quad G/C_G(N)$ amenable, s.t. $m(N) = 1$

(this implies (iii)).

Consider conjugation action $G \overset{conj}{\curvearrowright} G$

$\{G_B \mid B \subseteq G\} = \{C_G(B) \mid B \subseteq G\}$

satisfies D.C.C.

So by Dani's Lemma, G/G_0 is amenable,

where $G_0 = \{g \in G \mid m(C_G(g)) = 1\}$

Let $N = C_G(G_0)$ By m.c.c. $\exists F \overset{finite}{\subseteq} G_0$

with $N = C_G(F) = \bigcap_{g \in F} C_G(g)$

by finite additivity $m(N) = 1$.

Also $C_G(N) = G_0$ so $G/C_G(N)$ is amenable.

(i) & (ii) By Yesterday

$\exists\, m \in M(I(G))$ which is $I(G) \rtimes G$ - inv.

By (iii′) $\exists\, N \lhd G$ $G/C_G(N)$ amenable & $m(N) = 1$

Then $N \leq AC(G) \leq I(G)$

m: left- invariant under $I(G)$, $m(N) = 1$

$\Rightarrow N = AC(G) = I(G)$ $\square$

Thm Let G be m.c.c. Then TFAE

(1) G is inner - amenable.

(2) $AC(G) = I(G)$ is infinite.

(3) $\exists$ short exact sequence

$$1 \to N \to G \to K \to 1$$

K is amenable, and either $Z(N)$ is infinite or

$N = LM$, where $L, M \lhd G$ commuting, $L \cap M$ is finite, M: infinite amenable.

<Pf> (3) $\Rightarrow$ (2) $\Rightarrow$ (1) ✓

(1) $\Rightarrow$ (2) If m is atomless, conj – inv on G

Then $m(AC(G)) = 1$

$\Rightarrow$ $AC(G)$ is infinite.

(2) $\Rightarrow$ (3) Let $N = C_G(AC(G))AC(G)$

Then $K = G/N$ is amenable.

Case 1: $\underbrace{C_G(AC(G)) \cap AC(G)}_{= Z(N)}$ is infinite.

Case 2 $\underbrace{C_G(AC(G))}_{L} \cap \underbrace{AC(G)}_{M.}$ is finite

infinite, amenable. $\square$

Cost of actions (Levitt)

Let $G \curvearrowright (X, \mu)$ be a probability measure preserving (pmp) action. A measurable graph $\mathcal{G}$ on X is a graphing of the action $G \curvearrowright (X, \mu)$ if the connected components of $\mathcal{G}$ are precisely the orbits of the action.

undirected
no self-loop

$(\mathcal{G} \subseteq X \times X$: measurable subset.$)$

The cost of $\mathcal{G}$ is

$$\mathrm{Cost}(\mathcal{G}) = \frac{1}{2}\int_X \deg_{\mathcal{G}}(x)\, d\mu(x)$$

$$\mathrm{Cost}(G \curvearrowright (X,\mu)) = \inf\{\mathrm{Cost}(\mathcal{G}) \mid \mathcal{G} \text{ is a graphing of } G \curvearrowright (X,\mu)\}$$

G is said to have fixed price if

$$\mathrm{Cost}(G \curvearrowright (X,\mu)) = \mathrm{Cost}(G \curvearrowright (Y, \nu))$$

for any two free pmp actions of G.

Fixed price conjecture

Every (countable) group has fixed price.

This is known to hold for many groups.

- Infinite Amenable groups (f.p. $= 1$) (Levitt '93) Ornstein-Weiss.
- Finite groups (f.p. $= 1 - \frac{1}{|G|}$)
- Free groups F_n (f.p. $= n$) (Gaboriau '00)

- $Z(G)$ infinite (f.p. $= 1$)
- $H \times K$ K : infinite amenable group. (f.p. $= 1$)

} Gaboriau

<u>Thm (T-D)</u>

(Inner-amenable groups have fixed price $= 1$.)

$$\beta_1^{(2)}(G) \leq \mathrm{Cost}(G) \left[= \inf \left(\mathrm{Cost}(G \curvearrowright X) \right) \right] - 1 \quad (\text{Gaboriau})$$

Open : is this an equality ?

$\rightsquigarrow$ $\beta_1^{(2)}(\text{inner-amen}) = 0$ (T-D, Chifan-Sinclair-Udrea, Ozawa ?)

<u>Def (Popa)</u>

A subgroup $H \leq G$ is called <u>q-normal</u> if the set

$$\{ g \in G \mid gHg^{-1} \cap H \text{ is infinite} \}$$

generates G.

($H \leq_q G$ $\quad H \curvearrowright 2^H$ cocycle superrigid $\Downarrow$ $\Gamma \curvearrowright 2^\Gamma$ ")

<u>q-normality Lemma</u> (Gaboriau - Furman)

(If H is q-normal in G then

$$\mathrm{Cost}(G \curvearrowright (X,\mu)) \leq \mathrm{Cost}(H \curvearrowright (X,\mu))$$)

<u>Lemma</u> If M is a finite normal subgroup of G, then

$$\sup_{G \curvearrowright (X,\mu)} \{ \mathrm{Cost}(G \curvearrowright X) \} \leq \sup_{G/M \curvearrowright (X,\mu)} \left\{ 1 + \frac{\mathrm{Cost}(G/M \curvearrowright X)}{|M|} \right\}$$

$$|M| (\mathrm{Cost}(G) - 1) \leq \mathrm{Cost}(G/M) - 1$$

(**Prop** Let G be inner-amenable. Let H be a non-amenable subgroup of G. Then $\exists K \leq G$ with $H \lhd_g K \lhd_g G$.)

<pf> Important Lemma

$G \curvearrowright X$ amenable G_x amenable $\forall x \in X$

$\Rightarrow G$: amenable.

Improvement G: non-amenable, $G \curvearrowright X$ amenable with invariant mean $m \in M(X)$.

$$\Rightarrow m(\underbrace{\{x \in X \mid G_x \neq \text{amenable}\}}_{X_0}) = 1$$

Assume $m(X_0) < 1$ then $Y := X \setminus X_0$

$G \curvearrowright Y$ amenable G_y amenable $\forall y \in Y$

$\rightsquigarrow G$ amenable, contradiction.

<pf> (of Prop) Fix m: G-conj inv, atom-less

Then $H \overset{\text{conj}}{\curvearrowright} G$ is amenable w/ inv-mean m.

By improvement : $\{g \mid C_H(g)$ is non-amenable$\}$ has measure 1

Let $K = \langle H, \{g \in G \mid C_H(g) \text{ is non-amenable}\} \rangle$

Then $H \lhd_g K$ since $gHg^{-1} \cap H \geq C_H(g)$ is infinite for all $g \in G$ s.t. $C_H(g) \neq$ amenable.

$m(K) = 1 \Rightarrow m(gKg^{-1}) = 1$

$\Rightarrow m(K \cap gKg^{-1}) = 1$

m: atom-less

so $K \cap gKg^{-1}$ is infinite.

$\therefore$

Goal now is to find $(H_n)_{n=1}^{\infty}$ of non-amenable subgroups of G with $\sup \mathrm{Cost}(H_n) \xrightarrow{n\to\infty} 1$

Since then by g-normality Lemma + previous Prop we get

$\sup \mathrm{Cost}(G) \leq 1$

<u>Prop</u>

Let G be a non-amenable, inner-amenable group.

Then either

1) $\exists$ infinite amenable subgroup $K \subseteq G$ with $C_G(K)$ non-amenable.

or 2) $\exists$ sequence of finite subgroups M_n $(n \geq 1)$ with $|M_n| \to +\infty$ and $C_G(M_n)$ is nonamenable for all n.

<u>Proof</u>

Fix atomless conj-invariant mean m on G

By Improvement, (since G is non-amenable)

$G \overset{conj}{\curvearrowright} G$ amenable

m : atomless

$\exists g_1 \in G \setminus \{1\}$ with $\underset{H_1}{\overset{\parallel}{C_G(g_1)}}$ is nonamenable.

If $\langle g_1 \rangle$ is infinite then (1) holds and we're done.

Otherwise, $m(\underbrace{\langle g_1 \rangle}_{\text{finite set}}) = 0$.

So non-amenability of $C_G(g_1) = H_1$ and the improvement, we can find $g_2 \in H_1 \setminus \underbrace{\langle g_1 \rangle}_{M_1}$ with $C_{H_1}(g_2)$ non-amenable.

$\parallel$

$C_G(\underbrace{\langle g_1, g_2 \rangle}_{M_2})$

Keep going until we get $|M_n| = \infty$ for some n or $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots\cdots$ $|M_n| \to \infty$

$C_G(M_i)$ non-amenable.

If $\forall$ M_n is finite, this procedure never stops, and we set $(M_n)_{n=1}^{\infty}$ as in (2).

If $|M_n| = \infty$ at some n. and M_n : amenable subgroup.
Then (1).

Finally if some M_n is non-amenable then $\underset{\displaystyle H_n}{\overset{C_G(M_n)}{\|}}$ & M_n are commuting non-amenable subgroups,

$$H^{(1)} := H_n \, , \quad H^{(2)} := M_n.$$

By Improved Lemma

$$\left\{ g \;\middle|\; \begin{array}{l} C_{H^{(1)}}(g) \text{ is non-amenable} \\ C_{H^{(2)}}(g) \text{ is } \quad '' \end{array} \right\}$$

has measure 1

Fix some $g_1 \neq 1$ in this set, and let $M_1 = \langle g_1 \rangle$

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either something good happens, or $\forall n$ $\exists$ pairwise commuting non-amenable subgroups $H_1^{(n)}, \dots, H_n^{(n)}$

Define M_n by taking $g_1 \in H_1^{(n)} - \{1\}$

$$g_{i+1} \in H_i^{(n)} - \langle g_1, \dots, g_i \rangle$$

$$M_n = \langle g_1, \dots, g_{n-1} \rangle$$

Then $|M_n| \geqq 2^{n-1}$

and $\underset{\text{amenable}}{\overset{H^{(n)}}{\nparallel}} \subseteq C_G(M_n)$ so $C_G(M_n)$ non-amenable.

infinite

<u>Known</u>: $G \times H$

has some action with cost 1

- And $G \times H$ has fixed price $= 1$ whenever G or H has an infinite amenable subgroup.

<u>Open Problem</u>

Does $G \times H$ have fixed price $= 1$?