Overview

C^*-algebras associated to C^*-correspondences and applications to noncommutative geometry.

Overview of the presentation.

- C^*-algebras associated to C^*-correspondences
- Restricted direct sum C^*correspondences and pullbacks
- Even dimensional mirror quantum spheres
- Labelled graph algebras

Overview

C^*-algebras associated to C^*-correspondences and applications to noncommutative geometry.

Overview of the presentation

- C^*-algebras associated to C^*-correspondences
- Restricted direct sum C^*correspondences and pullbacks
- Even dimensional mirror quantum spheres
- Labelled graph algebras

C^*-correspondences

C^*-correspondences

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

Definition

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

for all $\xi, \eta \in X$, $a \in A$.

.

C^*-correspondences

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

Definition

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

- $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle \cdot a$

for all $\xi, \eta \in X$, $a \in A$.

.

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

Definition

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

- $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle \cdot a$
- $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$

for all $\xi, \eta \in X$, $a \in A$.

.

C^*-correspondences

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

Definition

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

- $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle \cdot a$
- $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$
- $\langle \xi, \xi \rangle \geq 0$ and $\|\xi\|_X = \sqrt{\|\langle \xi, \xi \rangle\|_A}$.

for all $\xi, \eta \in X$, $a \in A$.

C^*-correspondences

C^*-correspondences generalise the theory of Hilbert spaces by replacing the field of scalars $\mathbb{C}$ with an arbitrary C^*-algebra A.

We begin with the definition of a right Hilbert-module.

Definition

Let X be a Banach space and A be a C^*-algebra. Suppose we have a right action $X \times A \to X$ of A on X and an A valued inner-product $\langle \cdot, \cdot \rangle : X \times X \to A$ that satisfies

- $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle \cdot a$
- $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$
- $\langle \xi, \xi \rangle \geq 0$ and $\|\xi\|_X = \sqrt{\|\langle \xi, \xi \rangle\|_A}$.

for all $\xi, \eta \in X$, $a \in A$.
Then we say X is a *right Hilbert A-module*.

Adjointable and compact operators

Adjointable and compact operators

We say a linear operator $T : X \to X$ is *adjointable* if there exists an operator $T^* : X \to X$ such that

$$\langle T(\xi), \eta \rangle = \langle \xi T^*(\eta) \rangle$$

for all $\xi, \eta \in X$.

Adjointable and compact operators

We say a linear operator $T : X \to X$ is *adjointable* if there exists an operator $T^* : X \to X$ such that

$$\langle T(\xi), \eta \rangle = \langle \xi T^*(\eta) \rangle$$

for all $\xi, \eta \in X$.

We write $\mathcal{L}(X)$ for the collection of all adjointable operators $T : X \to X$.

Adjointable and compact operators

We say a linear operator $T : X \to X$ is *adjointable* if there exists an operator $T^* : X \to X$ such that

$$\langle T(\xi), \eta \rangle = \langle \xi T^*(\eta) \rangle$$

for all $\xi, \eta \in X$.

We write $\mathcal{L}(X)$ for the collection of all adjointable operators $T : X \to X$.

Then $\mathcal{L}(X)$ is a C^*-algebra.

Adjointable and compact operators

Adjointable and compact operators

For $\xi, \eta \in X$, define $\theta_{\xi,\eta}$ to be the operator satisfying

$$\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle.$$

for all $\zeta \in X$.

Adjointable and compact operators

For $\xi, \eta \in X$, define $\theta_{\xi,\eta}$ to be the operator satisfying

$$\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle.$$

for all $\zeta \in X$.

This is an adjointable operator with $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi}$. We call

$$\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi,\eta} : \xi, \eta \in X\}$$

the *compact* operators.

Adjointable and compact operators

For $\xi, \eta \in X$, define $\theta_{\xi,\eta}$ to be the operator satisfying

$$\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle.$$

for all $\zeta \in X$.

This is an adjointable operator with $(\theta_{\xi,\eta})^* = \theta_{\eta,\xi}$. We call

$$\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi,\eta} : \xi, \eta \in X\}$$

the *compact* operators.

Then $\mathcal{K}(X)$ is a closed two-sided ideal in $\mathcal{L}(X)$.

C^*-correspondences

Definition

A *C^*-correspondence* is a pair (X, A) where X is a Hilbert A-module, equipped with a $*$-homomorphism

$$\phi_X : A \to \mathcal{L}(X).$$

Definition

A *C^*-correspondence* is a pair (X, A) where X is a Hilbert A-module, equipped with a $*$-homomorphism

$$\phi_X : A \to \mathcal{L}(X).$$

We call ϕ_X the *left action* of A on X.

Representations

We define the C^*-algebra associated to a C^*-correspondence (X, A) as a universal object associated to representations of (X, A).

Representations

We define the C^*-algebra associated to a C^*-correspondence (X, A) as a universal object associated to representations of (X, A).

Definition

Let (X, A) be a C^*-correspondence and let B be a C^*-algebra. We say a pair (π, t) is a *representation of* (X, A) *on* B if $\pi : A \to B$ is a $*$-homomorphism and $t : X \to B$ is a linear map satisfying

Representations

We define the C^*-algebra associated to a C^*-correspondence (X, A) as a universal object associated to representations of (X, A).

Definition

Let (X, A) be a C^*-correspondence and let B be a C^*-algebra. We say a pair (π, t) is a *representation of* (X, A) *on* B if $\pi : A \to B$ is a $*$-homomorphism and $t : X \to B$ is a linear map satisfying

- $t(\phi_X(a)\xi) = \pi(a)t(\xi)$ for all $a \in A, \xi \in X$

Representations

We define the C^*-algebra associated to a C^*-correspondence (X, A) as a universal object associated to representations of (X, A).

Definition

Let (X, A) be a C^*-correspondence and let B be a C^*-algebra. We say a pair (π, t) is a *representation of* (X, A) *on* B if $\pi : A \to B$ is a $*$-homomorphism and $t : X \to B$ is a linear map satisfying

- $t(\phi_X(a)\xi) = \pi(a)t(\xi)$ for all $a \in A, \xi \in X$
- $\pi(\langle \xi, \eta \rangle) = t(\xi)^* t(\eta)$ for all $\xi, \eta \in X$.

Covariance

Covariance

Definition (Katsura 2003)

Define an ideal J_X of A by

$$J_X := \{a \in A : \phi_X(a) \in \mathcal{K}(X) \text{ and } a \cdot b = 0 \text{ for all } b \in \ker \phi_X\}$$

Covariance

Definition (Katsura 2003)

Define an ideal J_X of A by

$$J_X := \{a \in A : \phi_X(a) \in \mathcal{K}(X) \text{ and } a \cdot b = 0 \text{ for all } b \in \ker \phi_X\}$$

Definition (Katsura 2003)

We say a representation (π, t) of (X, A) on B is *covariant* if for all $a \in J_X$ we have

$$\pi(a) = \psi_t(\phi_X(a))$$

where $\psi_t : \mathcal{K}(X) \to B$ satisfies $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$.

The algebra

So we are ready to define the C^*-algebra associated to a C^*-correspondence (X, A).

The algebra

So we are ready to define the C^*-algebra associated to a C^*-correspondence (X, A).

Definition (Katsura, 2003)

For a C^*-correspondence (X, A) define $\mathcal{O}_X$ to be the C^*-algebra generated by the images of X and A under the universal covariant representation (π_X, t_X).

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Definition

Given two C^*-correspondences (X, A) and (Y, B), a pair (ψ_X, ψ_A) where $\psi_X : X \to Y$ is a linear map and $\psi_A : A \to B$ is a C^*-homomorphism, is called a *morphism of C^*-correspondences* if it satisfies

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Definition

Given two C^*-correspondences (X, A) and (Y, B), a pair (ψ_X, ψ_A) where $\psi_X : X \to Y$ is a linear map and $\psi_A : A \to B$ is a C^*-homomorphism, is called a *morphism of C^*-correspondences* if it satisfies

- $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Definition

Given two C^*-correspondences (X, A) and (Y, B), a pair (ψ_X, ψ_A) where $\psi_X : X \to Y$ is a linear map and $\psi_A : A \to B$ is a C^*-homomorphism, is called a *morphism of C^*-correspondences* if it satisfies

- $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,
- $\psi_X(\phi_X(a)\xi) = \phi_Y(\psi_A(a))\psi_X(\xi)$ for all $\xi \in X$ and $a \in A$, and

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Definition

Given two C^*-correspondences (X, A) and (Y, B), a pair (ψ_X, ψ_A) where $\psi_X : X \to Y$ is a linear map and $\psi_A : A \to B$ is a C^*-homomorphism, is called a *morphism of C^*-correspondences* if it satisfies

- $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,
- $\psi_X(\phi_X(a)\xi) = \phi_Y(\psi_A(a))\psi_X(\xi)$ for all $\xi \in X$ and $a \in A$, and
- $\psi_A(J_X) \subset J_Y$ and

Categorical viewpoint

By defining appropriate morphisms between C^*-correspondences, they fit nicely into the language of categories.

Definition

Given two C^*-correspondences (X, A) and (Y, B), a pair (ψ_X, ψ_A) where $\psi_X : X \to Y$ is a linear map and $\psi_A : A \to B$ is a C^*-homomorphism, is called a *morphism of C^*-correspondences* if it satisfies

- $\langle \psi_X(\xi), \psi_X(\eta) \rangle = \psi_A(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in X$,
- $\psi_X(\phi_X(a)\xi) = \phi_Y(\psi_A(a))\psi_X(\xi)$ for all $\xi \in X$ and $a \in A$, and
- $\psi_A(J_X) \subset J_Y$ and
- for all $a \in J_X$ we have $\phi_Y(\psi_A(a)) = \psi_X^+(\phi_X(a))$ where $\psi_X^+ : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies $\psi_X^+(\theta_{\xi,\eta}) = \theta_{\psi_X(\xi),\psi_X(\eta)}$.

Functors

There is a functor F from the category of C^*-correspondences to the category of C^*-algebras such that

Functors

There is a functor F from the category of C^*-correspondences to the category of C^*-algebras such that

- $F(X, A) = \mathcal{O}_X$

Functors

There is a functor F from the category of C^*-correspondences to the category of C^*-algebras such that

- $F(X, A) = \mathcal{O}_X$
- $\Psi = F(\psi_X, \psi_A) : \mathcal{O}_X \to \mathcal{O}_Y$ is a C^*-homomorphism satisfying

$$\Psi(\pi_X(a)) = \pi_Y(\psi_A(a)) \quad \text{and} \quad \Psi(t_X(\xi)) = t_Y(\psi_X(\xi))$$

  for all $a \in A$ and $\xi \in X$.

Functors

There is a functor F from the category of C^*-correspondences to the category of C^*-algebras such that

- $F(X, A) = \mathcal{O}_X$
- $\Psi = F(\psi_X, \psi_A) : \mathcal{O}_X \to \mathcal{O}_Y$ is a C^*-homomorphism satisfying

  $$\Psi(\pi_X(a)) = \pi_Y(\psi_A(a)) \text{ and } \Psi(t_X(\xi)) = t_Y(\psi_X(\xi))$$

  for all $a \in A$ and $\xi \in X$.

Not all homomorphisms $\varphi : \mathcal{O}_X \to \mathcal{O}_Y$ arise this way.

Overview

C^*-algebras associated to C^*-correspondences and applications to noncommutative geometry.

Overview of the presentation

- C^*-algebras associated to C^*-correspondences
- Restricted direct sum C^*correspondences and pullbacks: Our main theorem
- Even dimensional mirror quantum spheres
- Labelled graph algebras

Restricted direct sums

Restricted direct sums of C^*-correspondences are a generalisation of pullbacks of C^*-algebras.

Restricted direct sums

Restricted direct sums of C^*-correspondences are a generalisation of pullbacks of C^*-algebras.

Definition (Bakić, Guljăs (2003))

Given C^*-correspondences $(X, A), (Y, B)$ and (Z, C), and morphisms of C^*-correspondences $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$, define the *restricted direct sum*

Restricted direct sums

Restricted direct sums of C^*-correspondences are a generalisation of pullbacks of C^*-algebras.

Definition (Bakić, Guljăs (2003))

Given C^*-correspondences $(X, A), (Y, B)$ and (Z, C), and morphisms of C^*-correspondences $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$, define the *restricted direct sum*

$$X \oplus_Z Y := \{(\xi, \eta) \in X \oplus Y : \psi_X(\xi) = \omega_Y(\eta)\}.$$

Restricted direct sums

Restricted direct sums of C^*-correspondences are a generalisation of pullbacks of C^*-algebras.

Definition (Bakić, Guljǎs (2003))

Given C^*-correspondences $(X, A), (Y, B)$ and (Z, C), and morphisms of C^*-correspondences $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$, define the *restricted direct sum*

$$X \oplus_Z Y := \{(\xi, \eta) \in X \oplus Y : \psi_X(\xi) = \omega_Y(\eta)\}.$$

Proposition

The restricted direct sum $X \oplus_Z Y$ is a C^*-correspondence over the C^*-algebra $A \oplus_C B$ defined to be the pullback C^*-algebra of A and B along ψ_A and ω_B.

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Theorem

Let $(X, A), (Y, B)$ *and* (Z, C) *be* C^**-correspondences fix morphisms of* C^**-correspondences* $(\psi_X, \psi_A) : (X, A) \to (Z, C)$*,*
$(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$ *satisfying*

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Theorem

Let $(X, A), (Y, B)$ *and* (Z, C) *be* C^**-correspondences fix morphisms of* C^**-correspondences* $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$ *satisfying*

- $\psi_X(X) = \omega_Y(Y)$

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Theorem

Let $(X, A), (Y, B)$ *and* (Z, C) *be* C^**-correspondences fix morphisms of* C^**-correspondences* $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$ *satisfying*

- $\psi_X(X) = \omega_Y(Y)$
- $\psi_A(A) = \omega_B(B)$, *and*

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Theorem

Let $(X, A), (Y, B)$ and (Z, C) be C^*-correspondences fix morphisms of C^*-correspondences $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$ satisfying

- $\psi_X(X) = \omega_Y(Y)$
- $\psi_A(A) = \omega_B(B)$, *and*
- $\psi_A(\ker(\phi_X)) = \omega_B(\ker(\phi_Y))$.

Gluing C^*-correspondences

Our main result says that the process of taking restricted direct sums on the level of C^*-correspondences lifts to the process of taking pull-backs on the level of induced C^*-algebras via the functor F.

Theorem

Let $(X, A), (Y, B)$ and (Z, C) be C^*-correspondences fix morphisms of C^*-correspondences $(\psi_X, \psi_A) : (X, A) \to (Z, C)$, $(\omega_Y, \omega_B) : (Y, B) \to (Z, C)$ satisfying

- $\psi_X(X) = \omega_Y(Y)$
- $\psi_A(A) = \omega_B(B)$, *and*
- $\psi_A(\ker(\phi_X)) = \omega_B(\ker(\phi_Y))$.

Then

$$\mathcal{O}_{X \oplus_Z Y} \cong \mathcal{O}_X \oplus_{\mathcal{O}_Z} \mathcal{O}_Y$$

where $\mathcal{O}_X \oplus_{\mathcal{O}_Z} \mathcal{O}_Y$ is the pullback C^*-algebra of $\mathcal{O}_X$ and $\mathcal{O}_Y$ along $\Psi = F(\psi_X, \psi_A)$ and $\Omega = F(\omega_Y, \omega_B)$.

Noncommutative spaces

We can use this to construct new examples of noncommutative spaces.

C^*-algebras associated to C^*-correspondences and applications to noncommutative geometry.

Overview of the presentation

- C^*-algebras associated to C^*-correspondences
- Restricted direct sum C^*correspondences and pullbacks
- Even dimensional mirror quantum spheres
- Labelled graph algebras

Even dimensional mirror quantum spheres

The motivating examples for this research are the even-dimensional mirror quantum spheres, first defined for dimension 2 by Hajac, Matthes and Szymanski in 2006, and generalised to higher dimension by Hong and Szymanski in 2008.

Even dimensional mirror quantum spheres

The motivating examples for this research are the even-dimensional mirror quantum spheres, first defined for dimension 2 by Hajac, Matthes and Szymanski in 2006, and generalised to higher dimension by Hong and Szymanski in 2008.

For $n \in \mathbb{N}$, the $2n$-dimensional mirror quantum sphere is defined as the pullback of the following diagram

$$\begin{array}{ccc} & & C(\mathbb{D}_q^{2n}) \\ & & \downarrow{\scriptstyle \beta \circ \pi} \\ C(\mathbb{D}_q^{2n}) & \xrightarrow{\quad \pi \quad} & C(S_q^{2n-1}) \end{array}$$

where $\pi : C(\mathbb{D}_q^{2n}) \to C(S_q^{2n-1})$ is the natural surjection and $\beta \in \mathrm{Aut}(C(S_q^{2n-1}))$.

Mirror quantum spheres as C^*-correspondences

Hong and Szymanski showed that the algebras $C(\mathbb{D}_q^{2n})$ and $C(S_q^{2n-1})$ are graph algebras, so we can easily find C^*-correspondences for these algebras

Mirror quantum spheres as C^*-correspondences

Hong and Szymanski showed that the algebras $C(\mathbb{D}_q^{2n})$ and $C(S_q^{2n-1})$ are graph algebras, so we can easily find C^*-correspondences for these algebras

$$(X, A) \text{ such that } \mathcal{O}_X \cong C(\mathbb{D}_q^{2n})$$

$$(Z, C) \text{ such that } \mathcal{O}_Z \cong C(S_q^{2n-1})$$

Mirror quantum spheres as C^*-correspondences

Hong and Szymanski showed that the algebras $C(\mathbb{D}_q^{2n})$ and $C(S_q^{2n-1})$ are graph algebras, so we can easily find C^*-correspondences for these algebras

$$(X, A) \text{ such that } \mathcal{O}_X \cong C(\mathbb{D}_q^{2n})$$

$$(Z, C) \text{ such that } \mathcal{O}_Z \cong C(S_q^{2n-1})$$

There is a morphism of C^*-correspondences $(\sigma_X, \sigma_A) : (X, A) \to (Z, C)$ such that $\Sigma = F(\sigma_X, \sigma_A) : \mathcal{O}_X \to \mathcal{O}_Z$ and $\pi : C(\mathbb{D}_q^{2n}) \to C(S_q^{2n-1})$ are the same map.

However, there is no morphism of C^*-correspondences $(\rho_X, \rho_A) : (X, A) \to (Z, C)$ such that $F(\rho_X, \rho_A) = \pi \circ \beta$.

Mirror quantum spheres as C^*-correspondences

However, there is no morphism of C^*-correspondences $(\rho_X, \rho_A) : (X, A) \to (Z, C)$ such that $F(\rho_X, \rho_A) = \pi \circ \beta$.

There is another C^*-correspondence (Y, B) and morphism $(\rho_Y, \rho_B) : (Y, B) \to (Z, C)$ such that

However, there is no morphism of C^*-correspondences $(\rho_X, \rho_A) : (X, A) \to (Z, C)$ such that $F(\rho_X, \rho_A) = \pi \circ \beta$.

There is another C^*-correspondence (Y, B) and morphism $(\rho_Y, \rho_B) : (Y, B) \to (Z, C)$ such that

- $\mathcal{O}_Y \cong C(\mathbb{D}_q^{2n})$

Mirror quantum spheres as C^*-correspondences

However, there is no morphism of C^*-correspondences $(\rho_X, \rho_A) : (X, A) \to (Z, C)$ such that $F(\rho_X, \rho_A) = \pi \circ \beta$.

There is another C^*-correspondence (Y, B) and morphism $(\rho_Y, \rho_B) : (Y, B) \to (Z, C)$ such that

- $\mathcal{O}_Y \cong C(\mathbb{D}_q^{2n})$
- $F(\rho_Y, \rho_B) = \pi \circ \beta$

However, there is no morphism of C^*-correspondences $(\rho_X, \rho_A) : (X, A) \to (Z, C)$ such that $F(\rho_X, \rho_A) = \pi \circ \beta$.

There is another C^*-correspondence (Y, B) and morphism $(\rho_Y, \rho_B) : (Y, B) \to (Z, C)$ such that

- $\mathcal{O}_Y \cong C(\mathbb{D}_q^{2n})$
- $F(\rho_Y, \rho_B) = \pi \circ \beta$

But the C^*-correspondence (Y, B) no longer comes from a directed graph.

C^*-algebras associated to C^*-correspondences and applications to noncommutative geometry.

Overview of the presentation

- C^*-algebras associated to C^*-correspondences
- Restricted direct sum C^*correspondences and pullbacks
- Graph algebras
- Even dimensional mirror quantum spheres
- Labelled graph algebras

Labelled graphs

Labelled graphs are a generalisation of directed graphs, where two or more edges may carry the same label, and the range and sources of edges become sets of vertices.

Labelled graphs

Labelled graphs are a generalisation of directed graphs, where two or more edges may carry the same label, and the range and sources of edges become sets of vertices.

Definition (Bates, Pask (2007))

A labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ is a directed graph E together with a surjective labelling map $\mathcal{L} : E^1 \to \mathcal{A}$ which assigns to each edge $e \in E^1$ a label $a \in \mathcal{A}$.

Labelled graphs

Labelled graphs are a generalisation of directed graphs, where two or more edges may carry the same label, and the range and sources of edges become sets of vertices.

Definition (Bates, Pask (2007))

A labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ is a directed graph E together with a surjective labelling map $\mathcal{L} : E^1 \to \mathcal{A}$ which assigns to each edge $e \in E^1$ a label $a \in \mathcal{A}$.

The range and source maps then become $r, s : \mathcal{A} \to \mathcal{P}(E^0)$ satisfying

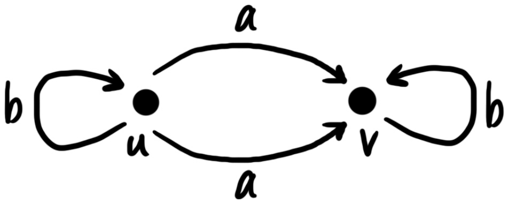
$$s(a) = \{s(e) : \mathcal{L}(e) = a\} \text{ and } r(a) = \{r(e) : \mathcal{L}(e) = a\}$$

Example

Example of a labelled graph $(E, \mathcal{L})$.

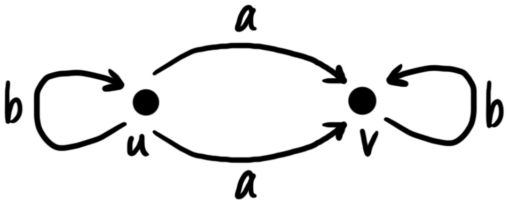
Example

Example of a labelled graph $(E, \mathcal{L})$.



Example

Example of a labelled graph $(E, \mathcal{L})$.



Then we have

$$s(a) = \{u\} \quad r(a) = \{v\}$$

$$s(b) = \{u, v\} = r(b)$$

We associate C^*-algebras to *labelled spaces* $(E, \mathcal{L}, \mathcal{B})$ where $(E, \mathcal{L})$ is a labelled graph and $\mathcal{B} \subset 2^{E^0}$.

C^*-algebra

We associate C^*-algebras to *labelled spaces* $(E, \mathcal{L}, \mathcal{B})$ where $(E, \mathcal{L})$ is a labelled graph and $\mathcal{B} \subset 2^{E^0}$.

The C^*-algebra is generated by a collection of partial isometries associated to the labels on E^1, and projections associated to the sets of vertices $A \in \mathcal{B}$.

C^*-algebra

We associate C^*-algebras to *labelled spaces* $(E, \mathcal{L}, \mathcal{B})$ where $(E, \mathcal{L})$ is a labelled graph and $\mathcal{B} \subset 2^{E^0}$.

The C^*-algebra is generated by a collection of partial isometries associated to the labels on E^1, and projections associated to the sets of vertices $A \in \mathcal{B}$.

Not all labelled graphs admit a suitable set $\mathcal{B}$ in order to associate a C^*-algebra. When $\mathcal{B}$ exists we say $\mathcal{B}$ is *accommodating* for $(E, \mathcal{L})$.

Definition (Bates, Pask (2003))

Let $(E, \mathcal{L})$ be a labelled graph, B an accommodating set for $(E, \mathcal{L})$. A representation of $(E, \mathcal{L})$ is a collection $\{p_A : A \in \mathcal{B}\}$ of projections and a collection $\{s_a : a \in \mathcal{L}(E^1)\}$ of partial isometries such that:

C^*-algebra

Definition (Bates, Pask (2003))

Let $(E, \mathcal{L})$ be a labelled graph, B an accommodating set for $(E, \mathcal{L})$. A representation of $(E, \mathcal{L})$ is a collection $\{p_A : A \in \mathcal{B}\}$ of projections and a collection $\{s_a : a \in \mathcal{L}(E^1)\}$ of partial isometries such that:

- For $A, B \in \mathcal{B}$, we have $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ where $p_\emptyset = 0$

C^*-algebra

Definition (Bates, Pask (2003))

Let $(E, \mathcal{L})$ be a labelled graph, B an accommodating set for $(E, \mathcal{L})$. A representation of $(E, \mathcal{L})$ is a collection $\{p_A : A \in \mathcal{B}\}$ of projections and a collection $\{s_a : a \in \mathcal{L}(E^1)\}$ of partial isometries such that:

- For $A, B \in \mathcal{B}$, we have $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ where $p_\emptyset = 0$
- For $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$, we have $p_A s_a = s_a p_{r(A,a)}$ where $r(A, a) = \{r(e) : s(e) \in A, \mathcal{L}(e)) = a\}$

Definition (Bates, Pask (2003))

Let $(E, \mathcal{L})$ be a labelled graph, B an accommodating set for $(E, \mathcal{L})$. A representation of $(E, \mathcal{L})$ is a collection $\{p_A : A \in \mathcal{B}\}$ of projections and a collection $\{s_a : a \in \mathcal{L}(E^1)\}$ of partial isometries such that:

- For $A, B \in \mathcal{B}$, we have $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ where $p_\emptyset = 0$
- For $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$, we have $p_A s_a = s_a p_{r(A,a)}$ where $r(A, a) = \{r(e) : s(e) \in A, \mathcal{L}(e)) = a\}$
- For $a, b \in \mathcal{L}(E^1)$, we have $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$

C^*-algebra

Definition (Bates, Pask (2003))

Let $(E, \mathcal{L})$ be a labelled graph, B an accommodating set for $(E, \mathcal{L})$. A representation of $(E, \mathcal{L})$ is a collection $\{p_A : A \in \mathcal{B}\}$ of projections and a collection $\{s_a : a \in \mathcal{L}(E^1)\}$ of partial isometries such that:

- For $A, B \in \mathcal{B}$, we have $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ where $p_\emptyset = 0$
- For $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$, we have $p_A s_a = s_a p_{r(A,a)}$ where $r(A, a) = \{r(e) : s(e) \in A, \mathcal{L}(e)) = a\}$
- For $a, b \in \mathcal{L}(E^1)$, we have $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$
- For $A \in \mathcal{B}$ define $L^1(A) := \{a \in \mathcal{L}(E^1) : s(a) \cap A \neq \emptyset\}$. Then if $L^1(A)$ is finite and non-empty, we have

$$p_A = \sum_{a\in L^1(A)} s_a p_{r(A,a)} s_a^* + \sum_{v\in A:v \text{ is a sink}} p_{\{v\}}.$$

Even dimensional mirror quantum sphere

Using the representation of the even dimensional mirror quantum sphere as a C^* algebra associated to a C^*-correspondence, we can prove that it is in fact a labelled graph algebra.

Even dimensional mirror quantum sphere

Using the representation of the even dimensional mirror quantum sphere as a C^* algebra associated to a C^*-correspondence, we can prove that it is in fact a labelled graph algebra.

Even dimensional mirror quantum sphere

Using the representation of the even dimensional mirror quantum sphere as a C^* algebra associated to a C^*-correspondence, we can prove that it is in fact a labelled graph algebra.

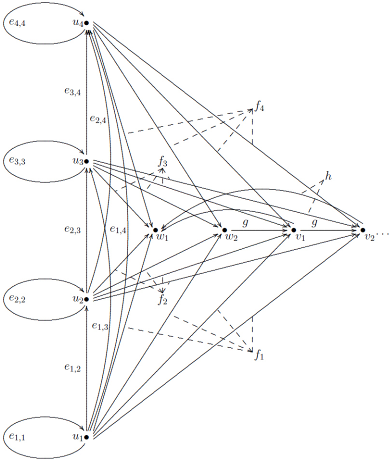


Figure: Labelled graph for $C(S^{10}_{q,\beta})$.