Purely infinite crossed product C*-algebras

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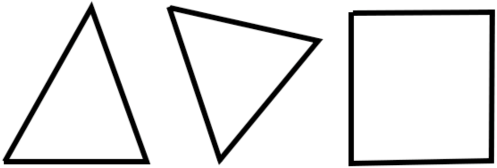
Classification



- Classification = the art of distinguishing different objects!

Classification of C*-algebras

Theorem (Kirchberg and Phillips, 90')

"Many" purely infinite C-algebras are classifiable.*

- $(a \precsim b)$ = there exists a sequence (r_n) such that $r_n^* b r_n \longrightarrow a$
- a properly infinite $= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \precsim \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$
- purely infinite = non-zero positive elements are properly infinite

Crossed product $A \rtimes_r G$

Goal

Contribute to classification of crossed products.

- $A =$ C*-algebra
- $G =$ countable discrete group
- $A \rtimes_r G =$ completion of $C_c(G, A)$ wrt. reduced norm

Non-amenable groups

Remark

- *For* $A = C(X)$*:*

$$G \text{ amenable} \Rightarrow A \rtimes_r G \text{ is not purely infinite.}$$

- G non-amenable $=$ There exist finite number of subsets of G that can be translated to cover G two times

Non-amenable groups

Proposition

Let X denote the Cantor set. The following are equivalent

- *G is non-amenable*
- *There exist a (minimal) action of G on X such that $C(X) \rtimes_r G$ has a properly infinite unit.*
- *There exist an action of G on X such that $C(X) \rtimes_r G$ admits no tracial state.*

Properly infinite $A \rtimes_r G$

Question

Suppose G is acting on a compact Hausdorff space X. Are the following two conditions equivalent:

- $C(X) \rtimes_r G$ *has a properly infinite unit*
- $C(X) \rtimes_r G$ *admits no tracial state*

Properly infinite $A \rtimes_r G$

Lemma

Suppose G is acting on a compact Hausdorff space X. Consider the properties

(i) *X is τ_X-paradoxical, i.e. X can be doubled up using open disjoint subsets of X.*

(ii) *$A \rtimes_r G$ has a properly infinite unit.*

Then $(i) \Rightarrow (ii)$.

Remark

For $X = \beta G$ - the Stone-Cech compactification of G - one can show that $(ii) \Rightarrow (i)$.

Properly infinite $A \rtimes_r G$

Lemma

Suppose G is acting on a compact Hausdorff space X. Consider the properties

(i) *X is τ_X-paradoxical, i.e. X can be doubled up using open disjoint subsets of X.*
$1 = x^*x = y^*y, xx^* \perp yy^* \leq 1$ *for some* $x, y \in C_c(G, A^+)$

(ii) *$A \rtimes_r G$ has a properly infinite unit.*
$1 = x^*x = y^*y, xx^* \perp yy^* \leq 1$ *for some* $x, y \in A \rtimes_r G$

Then (i) $\Rightarrow$ (ii).

Remark

For $X = \beta G$ - the Stone-Cech compactification of G - one can show that (ii) $\Rightarrow$ (i).

Ideal structure of $A \rtimes_r G$

Conjecture (implicitly stated by Renault, 80')

If the action of G on $\widehat{A}$, the spectrum of A, is essentially free then A separates the ideals in $A \rtimes_r G$.

- essentially free action on X = for every closed invariant set $Y \subseteq X$ the points in Y that are only fixed by $e \in G$ are dense in Y
- A separates ideals = two different ideals have different intersection with A.

Ideal structure of $A \rtimes_r G$

Theorem

If the action is exact and essentially free then A separates the ideals in $A \rtimes_r G$.

- exact = every invariant ideal I in A induces a short exact sequence at the level of reduced crossed products

Remark

Exactness is a necessary condition.

Ideal structure of $A \rtimes_r G$

Question

Can essential freeness be replaced with a weaker condition and still ensure A separates ideals in $A \rtimes_r G$?

Theorem (Pasnicu-Phillips)

Suppose A is unital and G is finite. Then A separates the ideals in $A \rtimes_r G$ if the action satisfies the Rokhlin property.

- Rokhlin property = there exist a projection $p \in A' \cap A_\infty$ s.t.

$$p \perp t.p, t \neq e, \quad \sum_t t.p = 1_{A_\infty}$$

Ideal structure of $A \rtimes_r G$

Theorem

A separates the ideals in $A \rtimes_r G$ if the action is exact and satisfies the residual Rokhlin property.*

- Rokhlin* property = there exist a projection $p \in A' \cap A_\infty^{**}$ s.t.

$$p \perp t.p, t \neq e, \quad \|a \sum_t t.p\| = \|a\|, a \in A$$

Remark

Essential freeness is in general stronger than residual Rokhlin property, for A abelian the properties coincide.*

Example

Example (Cuntz $B \rtimes_r Z \cong \mathcal{O}_n \otimes \mathcal{K}$)

- $B = M_{n^\infty} \otimes \mathcal{K}$, the stabilized UHF-algebra of type n^∞
- $(B, \mu_m \colon M_{n^\infty} \to B)$ is an inductive limit of $M_{n^\infty} \to^\lambda M_{n^\infty} \to^\lambda \cdots \to B, \quad \lambda(a) = a \otimes e_{11}$
- $\mathbb{Z}$ acts on $M_{n^\infty} \otimes \mathcal{K}$ via an automorphism $\bar{\lambda}$ that scales the trace be a factor $1/n$

$$p := (q_1, q_2, q_3 \dots) \in B_\infty,$$

$$q_k := \mu_k(\underbrace{1 \otimes \cdots \otimes 1}_{2k} \otimes (1 - e_{11}) \otimes \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{2k} \otimes 1 \otimes \dots), k \in \mathbb{N}$$

Purely infinite $A \rtimes_r G$

Theorem

Suppose the action is exact and has the residual Rokhlin property. Further assume that A is separable and has property (IP). TFAE*

- $A \rtimes_r G$ *is purely infinite*
- *All non-zero elements in* A^+ *are properly infinite (considered as elements in* $A \rtimes_r G$*)*

- (IP) = projections separates ideals

Purely infinite $A \rtimes_r G$

Corollary

Given $A = C(X)$ such that the action is exact and essentially free. If X has a basis of clopen τ_X-paradoxical sets then $A \rtimes_r G$ is purely infinite.

Theorem (Jolissaint-Robertson, $A = C(X)$)

Given $A = C(X)$ separable such that the action is properly outer. If X has no isolated points and the action is n-filling, then $A \rtimes_r G$ is purely infinite and simple.

Remark

If the action of G on a compact space X is n-filling then every open set is τ_X-paradoxical and the action is minimal and exact. Further essential freeness coincides with the notion of proper outerness.

Traceless $A \rtimes_r G$

Theorem

Suppose G is exact and acts essentially free on the Cantor set X. Let E denote the family of clopen subsets of X. Then the properties

(i) *The C*-algebra $C(X) \rtimes_r G$ is purely infinite.*

(ii) *The C*-algebra $C(X) \rtimes_r G$ is traceless.*

are equivalent provided that the semigroup $S(X, G, E)$ is almost unperforated.

- S(X,G,E) $=$ the type semigroup

$$\{\bigcup_{1}^{n} A_i \times \{i\} : A_i \in E, n \in \mathbb{N}\}/ \sim_S .$$

Thank you

Thank you for your attention