Torsion in the Elliott invariant and dimension theories of C*-algebras.

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19.November 2009

Overview

The Elliott conjecture

Conjecture 1.1 (The Elliott conjecture)

Let A, B be simple, nuclear, separable C-algebras. Then A and B are isomorphic if and only if* $\mathrm{Ell}(A)$ *and* $\mathrm{Ell}(B)$ *are isomorphic.*

it does not hold at its boldest, so we need to restrict to classes of "nice" C*-algebras (i.e. with some regularity properties, like $\mathcal{Z}$-stability)

besides proving the conjecture, there are other interesting questions:

- What is the range of the invariant?
- How do we detect properties of the algebra in its invariant?

Case distinction

Definition 1.2

A C-algebra A is:*

- **stably projectionless** $:\Leftrightarrow$ $A \otimes \mathbb{K}$ *contains no projection*
- **stably unital** $:\Leftrightarrow$ $A \otimes \mathbb{K}$ *contains an approximate unit of projections*
- **stably finite** $:\Leftrightarrow$ $A \otimes \mathbb{K}$ *contains no infinite projection*

Proposition 1.3

Let A be a simple, nuclear C-algebra. Then there are three disjoint (and exhaustive) possibilities:*

(F_0) $K_0^+ = 0$ *and* $T(A) \neq 0$

(F_1) $K_0^+ \cap -K_0^+ = 0$, $K_0^+ - K_0^+ = K_0 \neq 0$ *and* $T(A) \neq 0$

(Inf) $K_0^+ = K_0$ *and* $T(A) = 0$

The range of the Elliott invariant - necessary conditions

Let A be a simple, stable, stably finite, nuclear, separable C*-algebra. Then its Elliott invariant $\mathrm{Ell}(A) = (G_0, G_1, C, <.,.>)$ has the following properties:

- $G_0 = (G_0, G_0^+)$ is a countable, simple, pre-ordered, abelian group
- G_1 is a countable, abelian group
- $C \neq \emptyset$ is a topological convex cone with a compact, convex base that is a metrizable Choquet simplex
- $\rho : G_0 \to \mathrm{Aff}_0(C)$ is an order-homomorphism
- $r : C \to \mathrm{Pos}(G_0)$ is a continuous, affine map
- If $G_0^+ \neq 0$, then r is assumed surjective

We will call such an invariant **admissible** (and stable).

The range of the Elliott invariant

Theorem 1.4 (Elliott 1996)

For every weakly unperforated, admissible, stable Elliott invariant $\mathcal{E}$ exists a simple, stable ASH-algebra A with $\mathrm{Ell}(A) = \mathcal{E}$.

Definition 1.5 (weak unperforation)

The pairing $\rho : G_0 \to \mathrm{Aff}_0(C)$ is **weakly unperforated** *if $\rho(g) \gg 0$ implies $g > 0$ for all $g \in G_0$.*
An ordered group is **weakly unperforated** *if $ng > 0$ implies $g > 0$.*

- The pairing is weakly unperforated $\Leftrightarrow$ the order on G is determined by the map $\rho : G \to \mathrm{Aff}_0(C)$, i.e. $G_0^{++} = \rho^{-1}(\mathrm{Aff}_0(C)^{++})$
- If A is stably unital, then the two definitions agree.
- By using a weakly unperforated pairing we can treat the cases (F_0) and (F_1) at once.

Non-commutative dimension theories

Definition 2.1

A non-commutative dimension theory assigns to each C-algebra A (in some class) a value* $d(A) \in \mathbb{N} \cup \{\infty\}$ *such that:*

(i) $d(I), d(A/I) \leq d(A)$ *whenever* $I \lhd A$ *is an ideal in A*

(ii) $d(\lim_k A_k) \leq \underline{\lim}_k d(A_k)$ *whenever* $A = \varinjlim_k A_k$ *is a countable limit*

(iii) $d(A \oplus B) = \max\{d(A), d(B))$

Example 2.2

The following are dimension theories:

- The real and stable rank (for all C*-algebras)
- The decomposition rank and nuclear dimension (for separable C*-algebras)

The topological dimension

Definition 2.3 (locally Hausdorff space)

A topological space X is called **locally Hausdorff** *if every closed subset $F \neq \emptyset$ contains a relatively open Hausdorff subset $\emptyset \neq F \cap G$*

Definition 2.4 (Brown, Pederson 2007)

Let A be a C-algbera. If $\mathrm{Prim}(A)$ is locally Hausdorff, then the* **topological dimension** *of A is*

$$\mathrm{topdim}(A) = \sup_K \dim(K)$$

where the supremum runs over all locally closed, compact, Hausdorff subsets $K \subset \mathrm{Prim}(A)$.

Remark 2.5

If A is type I, then $\mathrm{Prim}(A)$ is locally Hausdorff. The topological dimension is a dimension theory for σ-unital, type I C*-algebras.

Definition 2.6 (Pedersen 1999)

A **NCCW-complex** *is a C^*-algebra $A = A_l$ which is obtained as an iterated pullback*

$$\begin{array}{ccc} A_k & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow \gamma_k \\ F_k \otimes C(D^n) & \xrightarrow{\partial_k} & F_k \otimes C(S^{n-1}) \end{array}$$

(for $k = 1, \ldots, l$) where $A_0 = F_0, F_1, \ldots, F_k$ have finite vector-space dimension.

Theorem 2.7 (Eilers-Loring-Pedersen 1998)

Every NCCW-complex of dimension ≤ 1 is semiprojective.

The AH- and ASH-dimension

Definition 2.8

We define classes of separable C-algebras:*

- $\underline{\mathrm{H}}(n)$:= *all homogeneous* A *with* $\mathrm{topdim}(A) \leq n$
- $\underline{\mathrm{SH}}(n)$:= *all subhomogeneous* A *with* $\mathrm{topdim}(A) \leq n$
- $\underline{\mathrm{SH}}(n)'$:= *all NCCW-complexes with* $\mathrm{topdim}(A) \leq n$

Let $\underline{\mathrm{AH}}(n)$, $\underline{\mathrm{ASH}}(n)$, $\underline{\mathrm{ASH}}(n)'$ *denote the classes of countable limits of such algebras.*

Example 2.9

$SH(0)' = F \subset SH(0) \subset AF$, $AH(0) = ASH(0)' = ASH(0) = AF$.

Definition 2.10

We let $\dim_{AH}(A) \leq n :\Leftrightarrow A \in \underline{\mathrm{AH}}(n)$,
and similarly for $\dim_{ASH}(A)$ *and* $\dim_{ASH'}(A)$

Remark 2.11

$\mathrm{dr}(A) \leq \dim_{ASH}(A) \leq \dim_{ASH'}(A) \leq \dim_{AH}(A)$

- Dadarlat-Eilers: There exists a (non-simple) algebra which is a limit of $\underline{\mathrm{AH}}(3)$-algebras, but not an *AH*-algebra itself.
- This implies that the AH-dimension is not a dimension theory (in the above sense) for all AH-algebras. It might be for simple algebras.
- Note however: a limit of $\underline{\mathrm{AH}}(k)$-algebras is again in $\underline{\mathrm{AH}}(k)$ for $k = 0, 1$, and similarly for $\underline{\mathrm{ASH}}(k)'$.
- The situation for $\underline{\mathrm{ASH}}(1)$ seems to be open (is $\underline{\mathrm{AASH}}(1) = \underline{\mathrm{ASH}}(1)$?). It might be that $\underline{\mathrm{ASH}}(1) = \underline{\mathrm{ASH}}(1)'$.
- Also, for $\underline{\mathrm{AH}}(2)$ the situation is unclear (is $\underline{\mathrm{AAH}}(2) = \underline{\mathrm{AH}}(2)$?).

Section Detecting ASH-dimension in the Elliott invariant

The main result

Theorem 3.1 (Elliott 1996)

Let $\mathcal{E}$ be an admissible, stable, weakly unperforated invariant. Then there exists a simple, stable C-algebra A in* $\underline{\mathrm{ASH}}(2)'$ *such that* $\mathrm{Ell}(A) = \mathcal{E}$.

Theorem 3.2 (T)

Let $\mathcal{E}$ be an admissible, stable, weakly unperforated invariant with G_0 torsion-free. Then there exists a simple, stable C-algebra A in* $\underline{\mathrm{ASH}}(1)'$ *such that* $\mathrm{Ell}(A) = \mathcal{E}$.

Remark 3.3 (Unital version)

Let $\mathcal{E}$ be an admissible, unital, weakly unperforated invariant. Then there exists a simple, unital C*-algebra A in $\underline{\mathrm{ASH}}(2)'$ such that $\mathrm{Ell}(A) = \mathcal{E}$. If G_0 is torsion-free, we can find A in $\underline{\mathrm{ASH}}(1)'$.

These algebras all have $\mathrm{dr} < \infty$, and are thus $\mathcal{Z}$-stable.

Appications

For this slide assume EC is true for the class $\underline{\mathrm{C}}$ of simple, stably finite, $\mathcal{Z}$-stable, unital, nuclear, separable C*-algebras.

Corollary 3.4

Let A be in $\underline{\mathrm{C}}$. Then the following are equivalent:

- *A is in $ASH(1)'$*
- *A is in $ASH(1)$*
- *$K_0(A)$ is torsion-free*

Corollary 3.5

Let A be in $\underline{\mathrm{C}}$. Then $\dim_{ASH}(A) = \dim_{ASH'}(A) \leq 2$ and we can detect the exact ASH-dimension as follows:

1.) $\dim_{ASH}(A) = 0 \quad \Leftrightarrow K_0(A)$ *is a simple dimension group, $K_1(A) = 0$ and r_A is a homeomorphism*

2.) $\dim_{ASH}(A) \leq 1 \quad \Leftrightarrow K_0(A)$ *is torsion-free*

Proposition 3.6 (T)

Let A be a separable, type I C^*-algebra with $\mathrm{sr}(A) = 1$*. Then* $K_0(A)$ *is torsion-free.*

Theorem 3.7 (T)

Let A be a separable, type I C^*-algebra. TFAE:

- $\mathrm{sr}(A) = 1$
- *A is residually stably finite, and* $\mathrm{topdim}(A) \leq 1$

Corollary 3.8

Let A be a separable, type I C^*-algebra with $\mathrm{dr}(A) \leq 1$*. Then* $\mathrm{sr}(A) = 1$*.*

Question 3.9

Does every (simple) C*-algebra with $\mathrm{dr}(A) \leq 1$ have torsion-free K_0-group? Does $\mathrm{sr}(A) = 1$ for a type I C*-algebra imply $\dim_{ASH}(A) \leq 1$ (or at least $\mathrm{dr}(A) \leq 1$)?

Section Ingredients of the proof

The integral Chern character for low-dimensional spaces.

The chern classes of vector bundles can be used to define homomorphisms

$$\mathrm{ch}^0 : K^0(X) \to H^{\mathrm{ev}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k}(X;\mathbb{Q})$$

$$\mathrm{ch}^1 : K^1(X) \to H^{\mathrm{odd}}(X;\mathbb{Q}) = \bigoplus_{k\geq 0} H^{2k+1}(X;\mathbb{Q})$$

which become isomorphisms after tensoring with $\mathbb{Q}$.

Theorem 4.1 (T)

Let X be a compact space of dimension ≤ 3. Then:

- $\chi^0 : K^0(X) \to H^0(X) \oplus H^2(X)$ *is an isomorphism*
- $\chi^1 : K^1(X) \to H^1(X) \oplus H^3(X)$ *is an isomorphism*

Corollary 4.2

Let X be a compact space.

- *If* $\dim(X) \leq 2$, *then* $K^1(X)$ *is torsion-free.*
- *If* $\dim(X) \leq 1$, *then* $K^0(X)$ *is torsion-free.*

Corollary 4.3

- *If* $\dim_{AH}(A) \leq 1$, *then* $K_0(A)$ *and* $K_1(A)$ *are torsion-free*
- *If* $\dim_{AH}(A) \leq 2$, *then* $K_1(A)$ *is torsion-free*

It is possible that the converses hold (within the class of simple AH-algebras of bounded dimension).

Strategy for constructing C*-algebras with prescribed invariant

To construct a (simple) C*-algebra with a prescribed Elliott invariant $\mathcal{E}$ we use roughly the following strategy (due to Elliott):

1. decompose $\mathcal{E}$ as a direct limit $\cong \varinjlim(\mathcal{E}^k, \theta_{k+1,k})$ where the $\mathcal{E}^k$ are basic
2. construct C*-algebras A_k (building blocks) and $*$-homomorphisms $\varphi_{k+1,k} : A_k \to A_{k+1}$ such that $\mathrm{Ell}(A_k) = \mathcal{E}_k$ and $\mathrm{Ell}(\varphi_{k+1,k}) = \theta_{k+1,k}$.
3. the limit $A := \varinjlim_k A_k$ already has $\mathrm{Ell}(A) = \mathcal{E}$, but is not necessarily simple. Deform the connecting maps $\varphi_{k+1,k}$ such that the limit gets simple (while the invariant is unchanged)

Theorem 4.4 (Effros-Handelman-Shen, Elliott, T)

Let G be a countable, ordered group. Then:

1.) *G is unperforated with Riesz interpolation*
$\Leftrightarrow G \cong \varinjlim_k G_k$ *and each* $G_k = \mathbb{Z}^{r_k} = \bigoplus_{i=1}^{r_k}(\mathbb{Z})$

2.) *G is weakly unperforated with Riesz interpolation*
$\Leftrightarrow G \cong \varinjlim_k G^k$ *and each* $G^k = \bigoplus_{i=1}^{r_k}(\mathbb{Z} \oplus \mathbb{Z}_{[k,i]})$ *(for some numbers* $[k,i] \geq 1$*)*

Let $G_* = G_0 \oplus G_1$ be a countable, graded, ordered group. Then:

3.) *G_* is weakly unperforated with Riesz interpolation*
$\Leftrightarrow G_* \cong \varinjlim_k G_*^k$ *and each*
$G_*^k = \bigoplus_{i=1}^{r_k}((\mathbb{Z} \oplus \mathbb{Z}_{[k,i]}) \oplus_{str} (\mathbb{Z} \oplus \mathbb{Z}_{[k,i]}))$

4.) *G_* is weakly unperforated with Riesz interpolation and G_0 is torsion-free*
$\Leftrightarrow G_* \cong \varinjlim_k G_*^k$ *and each* $G_*^k = \bigoplus_{i=1}^{r_k}((\mathbb{Z}) \oplus_{str} (\mathbb{Z}_{[k,i]}))$