# Spectra of C* algebras, classification.

Eberhard Kirchberg

HU Berlin

Lect.1, Copenhagen, 09

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# Conventions and Notations

- Considered C*-algebras $A, B, \ldots$ are separable, ...
- ... except multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$, ...
  as e.g., $\mathrm{Q}(\mathbb{R}_+, B) := \mathrm{C_b}(\mathbb{R}_+, B)/\,\mathrm{C_0}(\mathbb{R}_+, B) \subset \mathrm{Q}(SB)$.
- $\mathrm{T}_0$ spaces $X, Y, \ldots$ are second countable.
- $\mathbb{O}(X)$, $\mathcal{F}(X)$ denote the (distributive) lattices of open and of closed subsets of $X$.
- $\mathrm{Prim}(A)$ is the $\mathrm{T}_0$ space of primitive ideals with kernel-hull topology (Jacobson topology).
- $\mathcal{I}(A)$ means the lattice of closed ideals of $A$ (It is naturally isomorphic to $\mathbb{O}(\mathrm{Prim}(A))$).
- $\mathbb{Q}$ denotes the Hilbert cube (with its coordinate-wise order).

# Spectra of C*-algebras

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- $X \cong \mathrm{Prim}(A \otimes B)$ (naturally) for every simple exact $B$ (e.g. $B \in \{\mathcal{O}_2, \mathcal{O}_\infty, \mathcal{U}, \mathcal{Z}, \mathbb{K}, C^*_{reg}(F_2)\}$).

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- If $A$ is purely infinite then $(W(A), \leq, +)$ is naturally isomophic to $(\mathcal{I}(A), \subset, +) \cong (\mathbb{O}(X), \subset, \cup)$.
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- $X$ is $T_0$, sober (i.e., is point-complete), locally quasi-compact and is second countable (by separability of $A$).
  (The sobriety comes from the fact that $X$ has the Baire property, as an open and continuous image of a Polish space — the space of pure states on $A$ —.)

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3) Is there some uniqueness for the corresponding algebra $A$ with $\mathrm{Prim}(A) \cong X$ (coming from 2), e.g. if we tensor $A$ with $\mathcal{O}_2$?

## Strategies and partial results (1)

### Lemma

*If $R \subset P \times P$ is a (partial) order relation ($y \le x$ iff $(x,y) \in R$) on a locally compact Polish space $P$ such that the map $\pi_1 \colon (x,y) \in R \mapsto x \in P$ is open and $\overline{x}^R := \{y \in P\,;\ (x,y) \in R\}$ is closed for all $x \in P$, then there is non-degenerate *-monomorphism $H_0 \colon \mathrm{C}_0(P,\mathbb{K}) \to \mathcal{M}(\mathrm{C}_0(P,\mathbb{K}))$, such that $\delta_\infty \circ H_0$ is unitarily equivalent to $H_0$, and $(x,y) \in R$ if and only if the irreducible representation $\nu_y \otimes \mathrm{id}$ is weakly contained in $\mathcal{M}(\nu_x \otimes \mathrm{id}) \circ H_0$.*

*Idea of proof:* Bounded *-weakly cont. maps $x \in P \to \gamma(x) \in B^*_+$ and c.p. maps $V \colon B \to \mathrm{C_b}(P)$, are 1-1-related by $\nu_x \circ V = \gamma(x)$. If $x \in P \to F(x) \in \mathcal{F}(\mathrm{Prim}(B))$ is lower semi-cont. (e.g. $F(x) := \overline{x}^R$, $B := \mathrm{C}_0(P)$), then supports of the $\gamma(x)$ can be chosen in $F(x)$ and $\gamma(x_0) = f$ for $f \in B^*_+$ supported in $F(x_0)$ (by Michael selection).

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Now we can calculate the primitive ideal space of the Toeplitz algebra $\mathcal{T}_{\mathcal{H}}$ ($\cong \mathcal{O}_{\mathcal{H}}$, the Cuntz-Pimsner algebra), where $\mathcal{H} := \mathrm{C}_0(P, \mathbb{K})$ is the Hilbert $\mathrm{C}_0(P, \mathbb{K})$ bi-module that is defined by $H_0$ of Lemma 1.

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**Proposition (H.Harnisch,K.)**

*With above assumptions, $\mathcal{T}_{\mathcal{H}} \cong \mathcal{O}_{\mathcal{H}}$, and $\mathcal{T}_{\mathcal{H}}$ is a stable separable nuclear strongly purely infinite algebra. Its ideal lattice is isomorphic to $\mathbb{O}_R(P) \cong \mathbb{O}(P/\sim)$ and the natural embedding $\mathrm{C}_0(P, \mathbb{K}) \hookrightarrow \mathcal{T}_{\mathcal{H}}$ defines* KK*-equivalence in* $\mathrm{KK}(\mathbb{O}_R(P); \mathrm{C}_0(P, \mathbb{K}), \mathcal{T}_{\mathcal{H}})$.

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It leads to the problem to find — for given $A$ — a l.c. Polish space $P$ and a continuous map $\pi \colon P \to X := \mathrm{Prim}(A)$, such that the relation $(x, y) \in R \Leftrightarrow \pi(y) \in \overline{\{\pi(x)\}}$ satisfies the conditions of Lemma 1, and that $\pi(P)$ is "sufficiently dense" in $X$ in the sense that $\pi^{-1} \colon \mathbb{O}(X) \to \mathbb{O}(P)$ is injective: $X \cong \pi(P)^c$ in notation below.

# Functorial passage to sober T0 spaces:

If $X$ is any topological $T_0$ space then the lattice $\mathcal{F}(X)$ of closed subsets order anti-isomorphic to $\mathbb{O}(X)$ by $F \mapsto X \setminus F$. The set $\mathcal{F}(X)$ becomes a $T_0$ space with the topology generated by the complements $\mathcal{F}(X) \setminus [\emptyset, F]$ of the order intervals $[\emptyset, F]$ (for $F \in \mathcal{F}(X)$).

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The map $\eta \colon x \in X \mapsto \overline{\{x\}} \in \mathcal{F}(X)$ is a topological homeomorphism from $X$ onto $\eta(X)$. The image $\eta(X)$ is contained in the set $X^c$ of $\vee$-prime elements of $\mathcal{F}(X)$, and $X^c$ is a sober subspace of $\mathcal{F}(X)$, such that $\eta^{-1}$ defines a lattice isomorphism from $\mathbb{O}(X^c)$ onto $\mathbb{O}(X)$. If $X$ is sober, iff, $\eta(X) = X^c$.

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**Result**: The lattice $\mathbb{O}(X)$ and the top. space $X$ define each other up to isomorphisms in a natural (functorial) way, if and only if, $X$ is sober. The passage $X \to X^c$ is functorial.

# Strategies (2): Regular Abelian subalgebras

An **abelian** $C^*$-subalgebra $C \subset A$ is ***regular***, iff, for $J_1, J_2 \in \mathcal{I}(A)$,

- $C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)$ and,
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If $P := \mathrm{Prim}(C) = X(C)$ and $Y := \mathrm{Prim}(A)$, then $J \mapsto C \cap J$ defines maps $\Psi \colon \mathbb{O}(Y) \hookrightarrow \mathbb{O}(P)$ and $\pi \colon P \to Y$, with $\pi^{-1}|\mathbb{O}(Y) = \Psi$.

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There are *regular* $C \subset A$ in AH-algebras (AF if $A$ is AF). Regular comm. $C \subset A$ are in general not maximal, and $C \cap J$ does *not* necessarily contain an approximate unit of $J$.

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If $P := \mathrm{Prim}(C) = X(C)$ and $Y := \mathrm{Prim}(A)$, then $J \mapsto C \cap J$ defines maps $\Psi \colon \mathbb{O}(Y) \hookrightarrow \mathbb{O}(P)$ and $\pi \colon P \to Y$, with $\pi^{-1}|\mathbb{O}(Y) = \Psi$. The $\pi$ is *pseudo-open* (i.e., relation $(x, y) \in R \Leftrightarrow \pi(y) \in \overline{\{\pi(x)\}}$ satisfies the assumptions on $R$ in Lem. 1) and is *pseudo-epimorphic* (i.e., $U \subset V \in \mathbb{O}(Y)$ and $\pi(P) \cap (V \setminus U) = \emptyset$ imply $U = V$).

There are *regular* $C \subset A$ in AH-algebras (AF if $A$ is AF). Regular comm. $C \subset A$ are in general not maximal, and $C \cap J$ does *not* necessarily contain an approximate unit of $J$. For w.p.i. $B$ and separable $E \subset \mathrm{Q}(\mathbb{R}_+, B)$, there exists separable $E \subset A \subset \mathrm{Q}(\mathbb{R}_+, B)$ such that $EAE = A$ and $A$ contains a regular abelian subalgebra.

## Theorem (On Prim($A$), H.Harnisch,E.Kirchberg,M.Rørdam)

*Let $X$ a point-complete $T_0$-space. TFAE:*

(i) $X \cong \mathrm{Prim}(E)$ *for some exact C*-algebra $E$.*

(ii) *The lattice of open sets $\mathbb{O}(X)$ is isomorphic to an* sup–inf *invariant sub-lattice of $\mathbb{O}(P)$ for some l.c. Polish space $P$.*

(iii) *There is a locally compact Polish space $P$ and a pseudo-open and pseudo-epimorphic continuous map $\pi\colon P \to X$.*

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(iii) *There is a locally compact Polish space* $P$ *and a pseudo-open and pseudo-epimorphic continuous map* $\pi\colon P \to X$.

*If* $X$ *satisfies (i)–(iii), then there is a stable* ***nuclear*** *C*-algebra* $A$ *with* $A \cong A \otimes \mathcal{O}_2$, *and a homeomorphism* $\psi\colon X \to \mathrm{Prim}(A)$,
*such that,*
*for every nuclear stable* $B$ *with* $B \otimes \mathcal{O}_2 \cong B$ *and every homeomorphism* $\phi\colon X \to \mathrm{Prim}(B)$,
*there is an isomorphism* $\alpha\colon A \to B$ *with* $\alpha(\psi(x)) = \phi(x)$ *for* $x \in X$.
*This* $\alpha$ *is unique up to unitary homotopy.*

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$X \cong \mathrm{Prim}(A)$ iff $X$ is a sober $T_0$ space that has a base (!!) of its topology consisting of open quasi-compact sets.

Hochster has characterized 1969 the prime ideal space $X$ of countable locally unital commutative (algebraic) rings. The space $X$ is as in the case of AF algebras, but with the additional property that the intersection of any two open quasi-compact sets is again quasi-compact.

Above Theorem 3 answers Questions 2) partially and 3) (almost) completely. Bratteli and Elliott gave (an interior) characterization of the primitive ideal space $X$ of separable AF algebras $A$:
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The latter spaces are special cases of *coherent* spaces. A sober $\mathrm{T}_0$ space $X$ is called "coherent" if the intersection $C_1 \cap C_2$ of two *saturated* quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact. A subset $C$ of $X$ is "saturated" if $C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C)$ means the intersection of all $U \in \mathbb{O}(X)$ with $U \supset C$.

## Proposition

*The image $\eta(X) \cong X$ in $\mathcal{F}(X) \setminus \{\emptyset\}$ of a l.q-c. second countable sober $T_0$ space $X$ is **closed** in the Fell-Vietoris topology on $\mathcal{F}(X)$, if and only if, $X$ is **coherent**, if and only if, the set $\mathcal{D}(X)$ of Dini functions on $X$ is **convex**, if and only if, $\mathcal{D}(X)$ is* min-***closed**, if and only if, $\mathcal{D}(X)$ is **multiplicatively closed**.*

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## Corollary

*If there is a coherent sober l.c. space $X$ that is not homeomorphic to the primitive ideal space of an amenable C*-algebra, then there is $n \in \mathbb{N}$ and a finite union $Y$ of (Hausdorff-closed) cubes in $[0,1]^n$ such that $Y$ with induced order-topology is not the primitive ideal space of any amenable C*-algebra.*