A noncommutative pointwise ergodic theorem for amenable groups

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The starting point

\((\Omega, \mu)\) a measure space
\(T : \Omega \to \Omega\) measure preserving

**Theorem (Birkhoff)**

Let \( p \in [1, \infty) \) and \( f \in L_p(\Omega) \) then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} f(T^n\omega) = \hat{f}(\omega) \quad \text{a.e.}
\]

where \( \hat{f} \) is a \( T \)-invariant function.

- \((\Omega, \mu) \leadsto (\mathcal{M}, \tau)\) - \(\mathcal{M}\) a vNa, \(\tau\) a normal semifinite faithful positive trace
- \(T \leadsto G\) a locally compact **amenable** group of transformations acting by positive trace preserving contractions on \((\mathcal{M}, \tau)\)
Pointwise convergence

Almost uniform convergence

Let $p \in (0, \infty]$. Let $(x_n)_{n \geq 0} \in L_p(\mathcal{M})$ and $x \in L_p(\mathcal{M})$. We say that $(x_n)$ converges **almost uniformly** (a.u.) to $x$ if for any $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that

$$\tau(1 - e) \leq \varepsilon \quad \text{and} \quad \|e(x_n - x)\|_{\infty} \to 0.$$ 

We say that $(x_n)$ converges **bilaterally almost uniformly** (b.a.u.) if

$$\tau(1 - e) \leq \varepsilon \quad \text{and} \quad \|e(x_n - x)e\|_{\infty} \to 0.$$ 

Theorem (Lance ’76)

Let $T$ be a $\ast$-automorphism of a von Neumann algebra $\mathcal{M}$ preserving a normal faithful state $\rho$. Then for any $x \in \mathcal{M}$

$$\frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \to \hat{x} \quad \text{a.u.}$$
Noncommutative $L_p$-spaces

Let $p \in (0, \infty)$, $x \in \mathcal{M}$

$$\|x\|_p = \tau(|x|^p)^{1/p} \quad L_p(\mathcal{M}, \tau) = \left\{ x \in \mathcal{M} : \|x\|_p < \infty \right\}$$

$L_1(\mathcal{M}) = \mathcal{M}_*$, $L_\infty(\mathcal{M}) = \mathcal{M}$

Theorem (Junge, Xu ’07)

Let $T$ be a positive trace preserving contraction on $\mathcal{M}$. Then for any $p \in [1, \infty)$ and $x \in L_p(\mathcal{M})$

$$\frac{1}{n} \sum_{i=1}^{n-1} T^i(x) \rightarrow \widehat{x} \quad \text{b.a.u for } p \leq 2 \text{ and a.u. for } p > 2.$$
Ergodic averages

$G$ - locally compact second countable amenable group
$m$ - right invariant Radon measure on $G$
$G \curvearrowright \mathcal{M}$ - a weakly continuous action

Ergodic averages

Let $(F_n)_{n \geq 0}$ be a sequence of compact subsets of $G$ and $x \in L_p(\mathcal{M})$, $p \in [1, \infty)$. Define for any $n \geq 0$,

$$A_n^\alpha(x) = \frac{1}{m(F_n)} \int_{F_n} \alpha_g(x) dm(g)$$

Question

Given an amenable group $G$, can we find a Følner sequence $(F_n)_{n \geq 0}$ such that $A_n^\alpha(x)$ converges a.u. or b.a.u. for any action $\alpha$ and any $x \in L_p(\mathcal{M})$?

- We can if $\mathcal{M}$ is commutative (Lindenstrauss '01)
- We can if $G$ is of polynomial growth (Hong, Liao, Wang '21)
Key ingredient: maximal inequalities

$F_n$ - sequence of compact subsets of $G$
$A_n$ - associated averaging operators

**Weak type $(1, 1)$ maximal inequality**

- **Commutative case:** $f \in L^1(\Omega)$ and $\lambda > 0$
  \[
  \mu \left( \left\{ \sup_{n \geq 0} |A_n(f)| > \lambda \right\} \right) \leq C \frac{\|f\|_1}{\lambda}
  \]

- **Noncommutative case:** $x \in L^1(M)$ and $\lambda > 0$, there exists a projection $e$ such that
  \[
  \tau(1 - e) \leq C \frac{\|x\|_1}{\lambda} \quad \text{and} \quad \|eA_n(x)e\|_{\infty} \leq \lambda \quad \forall n \geq 0.
  \]

**Link:** $e = \left\{ \sup_{n \geq 0} |A_n(f)| \leq \lambda \right\}$
How to prove an ergodic theorem
in 5 simple steps

1. Show that there is uniform convergence of $A_n^\alpha(x)$ for $x$ in a dense subset of $L_p(\mathcal{M})$
   known techniques apply

2. Show that $\int +$ maximal inequality in $L_p(\mathcal{M}) \Rightarrow$ (bilaterally) almost uniform convergence
   techniques of Junge-Xu apply

3. Transference: maximal inequality for $\pi \Rightarrow$ maximal inequality for any action
   where $\pi$ the action of $G$ by translation on $L_\infty(G)\overline{\otimes}\mathcal{M}$
   proved in Hong-Liao-Wang

4. Interpolation: weak type $(1, 1)$ maximal inequality $\Rightarrow$ maximal inequality in $L_p$
   main technical result of Junge-Xu

5. Prove a weak type $(1, 1)$ inequality for $\pi$
   proved in Hong-Liao-Wang for groups of polynomial growth
Main result

From now on, \( A_n := A_n^\pi \)

Theorem (C, Wang)

Assume that \((F_n)_{n \geq 0}\) is a regular filtered f\ölner sequence. Let \( x \in L_1(\mathcal{N}) \) and \( \lambda > 0 \). There exists a projection \( e \in \mathcal{N} \) such that

\[
\tau(1 - e) \leq C \frac{\|x\|_1}{\lambda} \quad \text{and} \quad \|eA_n(x)e\|_\infty \leq \lambda \ \forall n \geq 0.
\]

Every second countable amenable group admits a regular filtered f\ölner sequence.

“covering lemmas” used in the commutative setting do not have noncommutative equivalents yet

usual noncommutative strategy: compare ergodic averages and martingale averages
The difference operator

The dyadic filtration on $\mathbb{R}^d$

- $G = \mathbb{R}^d$, $\mathcal{N} = L_\infty(\mathbb{R}^d) \otimes \mathcal{M}$, $F_n = B(0, 2^n)$
- for $n \geq 0$ define $\mathcal{P}_n = \{2^n[0, 1)^d + 2^n\nu : \nu \in \mathbb{Z}^d\}$, $(\mathcal{P}_n)_{n \geq 0}$ form a sequence of nested partitions of $\mathbb{R}^d$
- define $\mathcal{N}_n$ to be the subalgebra of $\mathcal{N}$ of functions constant on cubes of $\mathcal{P}_n$
- $E_n$ the associated conditional expectation

Theorem (Hong, Xu ’18)

For any $x$ in $L_1(\mathcal{N})$, we have

$$\| (A_n(x) - E_n(x))_{n \geq 0} \|_{RC(1, \infty)} \leq C \| x \|_1.$$ 

- We have a weak type $(1, 1)$ maximal inequality for $(E_n)_{n \geq 0}$ (Cuculescu ’71)
- The theorem above enables to transfer this inequality to $(A_n)_{n \geq 0}$
- Proof based on noncommutative Calderón-Zygmund decomposition
Beyond the dyadic filtration

$G$ - amenable group

**Completely regular filtered Følner sequence**

A completely regular filtered Følner sequence is a pair $((F_n)_{n \geq 0}, (\mathcal{P}_k)_{k \geq 0})$ such that

- $(F_n)$ is a Følner sequence
- $(\mathcal{P}_k)$ is a sequence of nested partitions of $G$
- for $n \geq k$ and $Q \in \mathcal{P}_k$, $F_n$ is $(2^{k-n}, Q)$-invariant
- for $k > n$ and $Q \in \mathcal{P}_k$, $Q$ is $(2^{n-k}, F_n)$-invariant.

$D$ is $(\varepsilon, K)$-invariant $\approx m(D \cdot K \setminus D) \leq \varepsilon m(D)$

- define conditional expectations $E_n$ like in the dyadic case
- if $((F_n)_{n \geq 0}, (\mathcal{P}_k)_{k \geq 0})$ is completely regular $(A_n - D_n)_{n \geq 0}$ of weak type $(1, 1)$
- uses noncommutative nondoubling Calderón-Zygmund decomposition
Finding regular filtered Følner sequences

It reduces to showing the following condition

**Tilability**

We say that a group $G$ is *tilable* if for any $\varepsilon > 0$ and $K \subset G$ compact, there exists a partition $\mathcal{P}$ of $G$ and a compact set $B \subset G$ such that

- every $Q \in \mathcal{P}$ is $(\varepsilon, K)$-invariant
- for every $Q \in \mathcal{P}$, there exists $g \in G$ such that $Q \subset g \cdot B$

- discrete groups are tilable (Downarowicz, Huczek, Zhang ’19)
- beyond discrete group, we can also find suitable partitions by imposing less restrictive conditions
- in both cases, the construction is based on the quasi-tilings of Ornstein and Weiss ’89