A noncommutative pointwise ergodic theorem for amenable groups

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The starting point

(Ω, μ) a measure space
$T : \Omega \to \Omega$ measure preserving

Theorem (Birkhoff)

Let $p \in [1, \infty)$ and $f \in L_p(\Omega)$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} f(T^n \omega) = \widehat{f}(\omega) \quad \text{a.e.}$$

where $\widehat{f}$ is a T-invariant function.

- $(\Omega, \mu) \rightsquigarrow (\mathcal{M}, \tau)$ - $\mathcal{M}$ a vNa, τ a normal semifinite faithful positive trace
- $T \rightsquigarrow G$ a locally compact **amenable** group of transformations acting by positive trace preserving contractions on $(\mathcal{M}, \tau)$

Pointwise convergence

Almost uniform convergence

Let $p \in (0, \infty]$. Let $(x_n)_{n \geq 0} \in L_p(\mathcal{M})$ and $x \in L_p(\mathcal{M})$. We say that (x_n) converges **almost uniformly** (a.u.) to x if for any $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that

$$\tau(1 - e) \leq \varepsilon \quad \text{and} \quad \|e(x_n - x)\|_\infty \to 0.$$

We say that (x_n) converges **bilaterally almost uniformly** (b.a.u.) if

$$\tau(1 - e) \leq \varepsilon \quad \text{and} \quad \|e(x_n - x)e\|_\infty \to 0.$$

Theorem (Lance '76)

Let T be a $*$-automorphism of a von Neumann algebra $\mathcal{M}$ preserving a normal faithful state ρ. Then for any $x \in \mathcal{M}$

$$\frac{1}{n} \sum_{i=0}^{n-1} T^i(x) \to \widehat{x} \quad \text{a.u.}$$

p-integrability

Noncommutative L_p-spaces

Let $p \in (0, \infty)$, $x \in \mathcal{M}$

$$\|x\|_p = \tau(|x|^p)^{1/p} \quad L_p(\mathcal{M}, \tau) = \overline{\left\{x \in \mathcal{M} : \|x\|_p < \infty\right\}}$$

$L_1(\mathcal{M}) = \mathcal{M}_*$, $L_\infty(\mathcal{M}) = \mathcal{M}$

Theorem (Junge, Xu '07)

Let T be a positive trace preserving contraction on $\mathcal{M}$. Then for any $p \in [1, \infty)$ and $x \in L_p(\mathcal{M})$

$$\frac{1}{n} \sum_{i=1}^{n-1} T^i(x) \to \widehat{x} \quad \text{b.a.u for } p \leq 2 \text{ and a.u. for } p > 2.$$

Ergodic averages

G - locally compact second countable amenable group
m - right invariant Radon measure on G
$G \curvearrowright^{\alpha} \mathcal{M}$ - a weakly continuous action

Ergodic averages

Let $(F_n)_{n\geq 0}$ be a sequence of compact subsets of G and $x \in L_p(\mathcal{M})$, $p \in [1,\infty)$. Define for any $n \geq 0$,

$$A_n^{\alpha}(x) = \frac{1}{m(F_n)} \int_{F_n} \alpha_g(x) dm(g)$$

Question

Given an amenable group G, can we find a Følner sequence $(F_n)_{n\geq 0}$ such that $A_n^{\alpha}(x)$ converges a.u. or b.a.u. for any action α and any $x \in L_p(\mathcal{M})$?

- We can if $\mathcal{M}$ is commutative (Lindenstrauss '01)
- We can if G is of polynomial growth (Hong, Liao, Wang '21)

Key ingredient: maximal inequalities

F_n - sequence of compact subsets of G
A_n - associated averaging operators

Weak type $(1,1)$ maximal inequality

- Commutative case: $f \in L_1(\Omega)$ and $\lambda > 0$

$$\mu\left(\left\{\sup_{n\geq 0} |A_n(f)| > \lambda\right\}\right) \leq C\frac{\|f\|_1}{\lambda}$$

- Noncommutative case: $x \in L_1(\mathcal{M})$ and $\lambda > 0$, there exists a projection e such that

$$\tau(1-e) \leq C\frac{\|x\|_1}{\lambda} \quad \text{and} \quad \|eA_n(x)e\|_\infty \leq \lambda \ \forall n \geq 0.$$

Link: $e = \left\{\sup_{n\geq 0} |A_n(f)| \leq \lambda\right\}$

How to prove an ergodic theorem

in 5 simple steps

1. Show that there is uniform convergence of $A_n^\alpha(x)$ for x in a dense subset of $L_p(\mathcal{M})$
   known techniques apply
2. Show that 1 + maximal inequality in $L_p(\mathcal{M}) \Rightarrow$ (bilaterally) almost uniform convergence
   techniques of Junge-Xu apply
3. **Transference**: maximal inequality for $\pi \Rightarrow$ maximal inequality for any action where π the action of G by translation on $L_\infty(G)\overline{\otimes}\mathcal{M}$
   proved in Hong-Liao-Wang
4. **Interpolation**: weak type $(1,1)$ maximal inequality $\Rightarrow$ maximal inequality in L_p
   main technical result of Junge-Xu
5. **Prove a weak type $(1,1)$ inequality for π**
   proved in Hong-Liao-Wang for groups of polynomial growth

Main result

From now on, $A_n := A_n^\pi$

Theorem (C, Wang)

- Assume that $(F_n)_{n\geq 0}$ is a *regular filtered følner sequence*. Let $x \in L_1(\mathcal{N})$ and $\lambda > 0$. There exists a projection $e \in \mathcal{N}$ such that

$$\tau(1-e) \leq C\frac{\|x\|_1}{\lambda} \text{ and } \|eA_n(x)e\|_\infty \leq \lambda \ \forall n \geq 0.$$

- Every second countable amenable group admits a regular filtered følner sequence.

- "covering lemmas" used in the commutative setting do not have noncommutative equivalents yet
- usual noncommutative strategy: compare ergodic averages and martingale averages

The difference operator

The dyadic filtration on $\mathbb{R}^d$

- $G = \mathbb{R}^d$, $\mathcal{N} = L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, $F_n = B(0, 2^n)$
- for $n \geq 0$ define $\mathcal{P}_n = \left\{2^n[0,1)^d + 2^n v : v \in \mathbb{Z}^d\right\}$, $(\mathcal{P}_n)_{n\geq 0}$ form a sequence of nested partitions of $\mathbb{R}^d$
- define $\mathcal{N}_n$ to be the subalgebra of $\mathcal{N}$ of functions constant on cubes of $\mathcal{P}_n$ E_n the associated conditional expectation

Theorem (Hong, Xu '18)

For any x in $L_1(\mathcal{N})$, we have

$$\left\|(A_n(x) - E_n(x))_{n\geq 0}\right\|_{RC(1,\infty)} \leq C \left\|x\right\|_1 .$$

- We have a weak type $(1,1)$ maximal inequality for $(E_n)_{n\geq 0}$ (Cuculescu '71)
- The theorem above enables to transfer this inequality to $(A_n)_{n\geq 0}$
- Proof based on noncommutative Calderón-Zygmund decomposition

Beyond the dyadic filtration

G - amenable group

Completely regular filtered Følner sequence

A completely regular filtered Følner sequence is a pair $((F_n)_{n\geq 0}, (\mathcal{P}_k)_{k\geq 0})$ such that

- (F_n) is a Følner sequence
- $(\mathcal{P}_k)$ is a sequence of nested partitions of G
- for $n \geq k$ and $Q \in \mathcal{P}_k$, F_n is $(2^{k-n}, Q)$-invariant
- for $k > n$ and $Q \in \mathcal{P}_k$, Q is $(2^{n-k}, F_n)$-invariant.

D is (ε, K)-invariant $\approx m(D \cdot K \setminus D) \leq \varepsilon m(D)$

- define conditional expectations E_n like in the dyadic case
- if $((F_n)_{n\geq 0}, (\mathcal{P}_k)_{k\geq 0})$ is completely regular $(A_n - D_n)_{n\geq 0}$ of weak type $(1, 1)$
- uses noncommutative nondoubling Calderón-Zygmund decomposition

Finding regular filtered Følner sequences

It reduces to showing the following condition

Tilability

We say that a group G is *tilable* if for any $\varepsilon > 0$ and $K \subset G$ compact, there exists a partition $\mathcal{P}$ of G and a compact set $B \subset G$ such that

- every $Q \in \mathcal{P}$ is (ε, K)-invariant
- for every $Q \in \mathcal{P}$, there exists $g \in G$ such that $Q \subset g \cdot B$

- discrete groups are tilable (Downarowicz, Huczek, Zhang '19)
- beyond discrete group, we can also find suitable partitions by imposing less restrictive conditions
- in both cases, the construction is based on the quasi-tilings of Ornstein and Weiss '89