Connections between actions on Banach spaces, hyperbolic geometry and median geometry

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Properties to have in mind

We study hyperbolic and median geometry with two properties (and their variations) in mind:

- Kazhdan's property (T);
- a-T-menability.

  Affine isometric definitions:

- Property (T): every action by affine isometries on a Hilbert space has a global fixed point.
- a-T-menability: there exists a proper action by affine isometries on a Hilbert space.
- In both definitions, 'Hilbert' can be replaced by 'L^p–space', for $p \in [1, 2]$ fixed.

Examples of groups to have in mind

- Let X be a symmetric space of non-compact type (i.e. sectional curvature ≤ 0, no Euclidean factors), irreducible (i.e. $X \neq X_1 \times X_2$).
- rank of X = maximal k s.t. $\mathbb{R}^k$ isometrically embedded in X.
- rank one = $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{K}}$ with $\mathbb{K} = \mathbb{C}$, field of quaternions,$\mathbb{H}^2_{\mathbb{K}}$ with $\mathbb{K}$ field of octaves.
- higher rank = for instance $SL(n, \mathbb{R})/SO(n)$, of rank $n-1$.
- Consider $L = \mathrm{Isom}_0(X)$ and lattices: discrete subgroups G in L s.t. L/G has a finite measure induced by the Haar measure.
- $L = \mathrm{Isom}_0(X)$ (and its lattices) is a-T-menable for $X = \mathbb{H}^n_{\mathbb{R}}$ or $\mathbb{H}^n_{\mathbb{C}}$.
- $L = \mathrm{Isom}_0(X)$ (and lattices) have property (T) for $X = \mathbb{H}^n_{\mathbb{H}}$ or $\mathbb{H}^2_{\mathbb{K}}$ with $\mathbb{K}$ field of octaves.
- $L = \mathrm{Isom}_0(X)$ (and lattices) has property (T) for X of rank ≥ 2.

General belief: this is an order of increasing strength of property (T), decreasing strength of a-T-menability.

$\mathrm{Isom}_0(X)$ for X of rank ≥ 2 has 'a more robust property (T)'

Reformulated: Gromov hyperbolic groups have 'a less robust property (T).'

At the opposite end: $\mathrm{Isom}_0\left(\mathbb{H}^n_{\mathbb{R}}\right)$ is 'more a-T-menable' than $\mathrm{Isom}_0\left(\mathbb{H}^n_{\mathbb{C}}\right)$.

Shalom's conjecture: Every hyperbolic group admits a proper uniformly Lipschitz affine action on a Hilbert space.

Higher rank $\mathrm{Isom}_0(X)$ have fixed point properties also for uniformly Lipschitz affine actions on a Hilbert space (V. Lafforgue, T. de Laat- M. de la Salle).

A way of measuring the 'strength of a-T-menability/property (T)' is by measuring the degree of compatibility with the median geometry.

Strongest degree of compatibility with median geometry

A group is said to be cubulable if it acts properly discontinuously cocompactly on a CAT(0) cube complex.

cube complex= complex obtained by glueing unit cubes.

CAT(0) = an equivalent in the setting of metric spaces of "nonpositive sectional curvature".

Theorem (Bergeron-Wise, Kahn-Markovic)

Every fundamental group of a hyperbolic 3-manifold is cubulable.

Using this and the theory of special cube complexes (Haglund-Wise), Agol proved the virtual Haken conjecture.

Why is the geometry of a CAT(0) cube complex median

Chepoi, Guerasimov: a graph $\Gamma = (V, E)$ is the 1-skeleton of a CAT(0) cube complex $\Leftrightarrow$ V with the simplicial distance is median.

A median space = a metric space (X, d) such that every triple of points $x_1, x_2, x_3 \in X$ admits a unique median point $m \in X$ satisfying

$$d(x_i, m) + d(m, x_j) = d(x_i, x_j)$$

for all $i, j \in \{1, 2, 3\}, i \neq j$.

Other examples

1. *Real trees;*
2. $\mathbb{R}^n$ *with the norm* ℓ^1;
3. $L^1(X, \mu)$.

- (Assouad) Every median space embeds isometrically into an L^1–space.
- (Verheul) A complete median normed space is linearly isometric to an L^1–space.

Median spaces: non-discrete versions of CAT(0)-c.c.

Median spaces = non-discrete versions of 0-skeleta of CAT(0) cube complexes.

In the same way in which real trees = non-discrete versions of simplicial trees.

Bowditch: The metric of a complete connected finite rank median metric space has a bi-Lipschitz equivariant deformation that is CAT(0).

The rank of a median metric space X = the supremum over the set of integers k such that X contains an isometric copy of the set of vertices $\{-a, a\}^k$ of the cube of edge length $2a$, for some $a > 0$.

Convention

From now on we assume all median spaces to be complete and connected (hence geodesic).

Interest of the median geometry

1. G locally compact second countable has property (T) $\Leftrightarrow$ any continuous action by isometries on a median space has bounded orbits (Chatterji -D. - Haglund).
2. G a-(T)-menable $\Leftrightarrow$ it admits a proper continuous action by isometries on a median space (Chatterji -D. - Haglund).
3. connection with CAT(0)-geometry.
4. relevance of median graphs and their geometry in graph theory, computer science, optimization theory.

Degrees of compatibility with median geometry

A group is said to be

- cubulable if it acts properly discontinuously cocompactly on a CAT(0) cubical complex;
- strongly medianizable if it acts properly discontinuously cocompactly on a median space of finite rank;
- medianizable if it acts properly discontinuously cocompactly on a median space of infinite rank.

If a finitely generated group is (strongly) medianizable then the median space on which it acts is also locally compact.

Cubulable $\Rightarrow$ strongly medianizable $\Rightarrow$ medianizable.

Median geometry versus amenability

Cubulable $\Rightarrow$ weakly amenable with Cowling-Haagerup constant 1 (Guentner-Higson).

Proper action on a median space $\Leftrightarrow$ a-T-menable.

- Amenable$= \exists$ a sequence of positive definite, compactly supported functions on G converging to 1 uniformly on compacts subsets;
- a-T-menable $= \exists$ a sequence of continuous positive definite functions on G, vanishing at infinity, converging to 1 uniformly on compact sets.
- weakly amenable$= \exists$ a sequence (ϕ_n) of continuous, compactly supported functions on G, converging to 1 uniformly on compact sets, with $\sup_n \|\phi_n\|_{M_0A(G)} < \infty$.
  Above, $\|\cdot\|_{M_0A(G)}$ is the completely bounded norm on the space $M_0A(G)$ of completely bounded multipliers of the Fourier algebra $A(G)$.
  The Cowling-Haagerup constant is the best possible $\kappa \geq \sup_n \|\phi_n\|_{M_0A(G)}$.

Strongly medianizable versus cubulable

Possibly strongly medianizable $\Rightarrow$ cubulable.

For the moment they cannot be distinguished *via* their known properties:

- Tits alternative, super-rigidity (Caprace-Sageev, Fioravanti);
- irreducible uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ with $k \geq 2$ act on every finite dimensional CAT(0)-c.c. with a global fixed point (Chatterji-Fernos-Iozzi)
- also true for actions on finite rank median spaces (Fioravanti).

Theorem (Chatterji-D.)

The above lattices (with $k \geq 1$) are medianizable.

In view of Fioravanti's result, this is the best that one can get for these lattices, in terms of compatibility with a median geometry.

Medianizable lattices

The result is interesting in the case of one factor ($k = 1$) too:

- not known if all arithmetic uniform lattices in $SO(n, 1)$, with n odd and larger than 3, are cubulable;
- in particular, there is an example of arithmetic uniform lattices in $SO(7, 1)$, constructed using octaves, that is thought not to be cubulable.

Rips-type Theorems for median spaces

The classical Rips-type Theorems answer the question:

when can one extract from an action on a real tree $T \neq \mathbb{R}$ (minimal non-trivial) an action on a simplicial tree ?

Stable action = the family of stabilizers of non-trivial arcs satisfies the ACC (Ascending Chain Condition).

- (Bestvina-Feighn) If G is finitely presented then from a stable G-action on a real tree T one constructs an action on a simplicial tree with stabilizers of edges = stabilizers of arcs-by-cyclic.
- (Sela) Same with f.p. replaced by "trivial stabilizers of tripods".

Rips-type Theorems for median spaces

Interest of a median version, i.e. "when can one extract from an action on a median space (minimal non-trivial) an action on a CAT(0)-cube complex?"

- it would relate the negation of property (T) ($\Leftrightarrow$ existence of an action with infinite orbits on a median space) to actions on CAT(0) cube complexes;
- it would provide an assumption under which a-T-menability implies weak amenability with Cowling-Haagerup constant 1.
- An assumption is needed: $H \wr F_2$ with H finite is a-T-menable (Cornulier-Stadler-Valette), but cannot be weakly amenable with Cowling-Haagerup constant 1 (Ozawa-Popa).

Rips-type Theorems for median spaces

Our theorem emphasizes that one cannot expect, for actions on median spaces, a theorem similar to Bestvina-Feighn:

- uniform lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ are finitely presented;
- they act properly, minimally and cocompactly on median spaces;
- they cannot act non-trivially cocompactly with amenable stabilizers on a CAT(0) cube complex (which would therefore have to be of finite dimension), by Chatterji-Fernos-Iozzi.
- Still possible to obtain Rips-type theorems for actions on median spaces of finite rank.
- Consistent with the case of real trees, since these are median spaces of rank one.

Why are lattices in $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ median?

Theorem (Chatterji-D.)

The real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ embeds isometrically and $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$–equivariantly into a proper (i.e. all balls are compact) median space at finite Hausdorff distance from the embedded $\mathbb{H}^n_{\mathbb{R}}$.

The embedding is constructed using another structure closely connected with the median geometry: measured walls (introduced by Cherix-Martin-Valette).

A key property: the metric on $\mathbb{H}^n_{\mathbb{R}}$ coincides with the metric induced by a structure of measured walls.

Complex hyperbolic space

- $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist})$ cannot be isometrically embedded into a median space. In particular, dist cannot be a wall metric.
- Faraut and Harzallah: $\mathbb{H}^n_{\mathbb{C}}$ equipped with $\mathrm{dist}^{\frac{1}{2}}$ has a structure of measured walls.
- $(\mathbb{H}^n_{\mathbb{C}}, \mathrm{dist}^{\frac{1}{2}})$ cannot be at bounded Hausdorff distance from a median space.

Acylindrically hyperbolic groups

Recall that G is acylindrically hyperbolic if it admits an acylindrical, non-elementary action on a Gromov hyperbolic space.

We say G acts (D, B)-acylindrically on X hyperbolic space if there are functions $D = D(\epsilon), B = B(\epsilon) > 0$ so that for any $\epsilon > 0$ if $x, y \in X$ with $d(x, y) \geq D(\epsilon)$ then

$$|\{[g] \in G : d_X(x, gx), d_X(y, gy) \leq \epsilon\}| \leq B(\epsilon).$$

Examples

1. *non-elementary hyperbolic groups;*
2. *mapping class groups that are not virtually abelian;*
3. *groups of outer automorphisms of free non-abelian groups, $Out(F_n)$, $n \geq 2$.*

Actions that acylindrically hyp. groups cannot have I

Acylindrically hyp. groups cannot act by affine isometries on L^p–spaces, $p > 1$, such that their orbits are infinite.

1. Minasyan and Osin proved that there exists a finitely generated acylindrically hyperbolic group AH that is quotient of all hyperbolic groups.
2. With John Mackay we proved that random groups have the fixed point property for larger and larger classes of L^p–spaces.
3. De Laat and de la Salle proved that random groups satisfy fixed point properties for larger and larger classes of uniformly curved Banach spaces.
4. This implies that the Minasyan-Osin example cannot act by affine uniformly bi-Lipschitz transformations on L^p–spaces such that their orbits are infinite.

Actions that acylindrically hyp. groups cannot have II

Acylindrically hyp. groups cannot act properly by affine uniformly bi-Lipschitz transformations on L^1–spaces.

1. This is due to the example of Gromov monsters, graphical small cancellation groups constructed by Gromov, containing families of expanders uniformly embedded into their Cayley graphs (Gromov, Arzhantseva-Delzant, Osajda).
2. Gromov monsters cannot embed uniformly into any L^p–space, for $p \in [1, 2]$.
3. Gromov monsters are acylindrically hyperbolic (Gruber-Sisto).

Actions acylindrically hyp. groups can have

Theorem (D.-Mackay)

Any acylindrically hyperbolic group admits a uniformly Lipschitz affine action on ℓ_1 with unbounded orbits.

Corollary

Every infinitely presented graphical Gr(7) small cancellation group and every cubical small cancellation group admits a uniformly Lipschitz action on ℓ_1 with unbounded orbits.

This follows from the result of Gruber and Sisto, and that of Arzhantseva-Hagen proving that cubical small cancellation groups are acylindrically hyperbolic.

Residually finite hyperbolic group

Theorem (D.-Mackay)

Every residually finite hyperbolic group admits a uniformly Lipschitz affine action on $\ell_1 = \ell_1(\mathbb{N})$ with quasi-etrically embedded orbits, and hence likewise for $L_1 = L_1([0,1])$.

Related questions and results:

1. Shalom's conjecture: Every hyperbolic group admits a uniformly Lipschitz affine action on a Hilbert space. Our theorem does not yield actions on Hilbert spaces.
2. Nishikawa: (Any lattice in) $\mathrm{Sp}(n,1)$ has a proper affine uniformly Lipschitz action on L_2. This does not imply the L_1 case.
3. Vergara: Any group acting properly on a product of quasi-trees has a proper affine uniformly Lipschitz action on a subspace of L_1. E.g. residually finite hyperbolic, mapping class groups. Does not imply an action on L_1 either.