Noncommutative ergodic theory of higher rank lattices

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International Congress of Mathematicians 2022
Operator algebras, dynamics and groups
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Introduction
Let $k$ be any local field i.e. $k$ is a nondiscrete locally compact field.

**Definition**

Let $G$ be any simply connected semisimple connected linear algebraic $k$-group. We assume that

1. $G$ is **absolutely almost simple** (resp. **almost $k$-simple**) i.e. the only proper normal algebraic (resp. $k$-closed) subgroups are finite.

2. $G$ is **$k$-isotropic** i.e. $G$ has a nontrivial $k$-split torus.

Denote by $rk_k(G)$ the dimension of a maximal $k$-split torus in $G$. 
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**Example**

For every $n \geq 2$, $SL_n$ is a $k$-isotropic absolutely almost simple algebraic $k$-group. Moreover, we have $rk_k(SL_n) = n - 1$. 

Cyril HOUDAYER (Paris-Saclay & IUF)  
Noncommutative ergodic theory of higher rank lattices
Higher rank lattices

Let \( I = \{1, \ldots, d\} \) be any finite index set with \( d \geq 1 \).

For every \( i \in I \), let \( k_i \) be any local field, \( \mathbb{G}_i \) any \( k_i \)-isotropic almost \( k_i \)-simple algebraic \( k_i \)-group and set \( G_i = \mathbb{G}_i(k_i) \).
Higher rank lattices

Let \( l = \{1, \ldots, d\} \) be any finite index set with \( d \geq 1 \).

For every \( i \in l \), let \( k_i \) be any local field, \( G_i \) any \( k_i \)-isotropic almost \( k_i \)-simple algebraic \( k_i \)-group and set \( G_i = G_i(k_i) \).

**Definition (Higher rank lattice)**

Set \( G = \prod_{i=1}^{d} G_i \). We say that \( \Gamma < G \) is a **higher rank lattice** if

1. \( \Gamma < G \) is a discrete subgroup with finite covolume;
2. If \( d \geq 2 \), then \( \Gamma < G \) is with dense projections i.e. for every \( j \in \{1, \ldots, d\} \), the projection of \( \Gamma \) in \( \prod_{i \neq j} G_i \) is dense;
3. \( \sum_{i=1}^{d} \text{rk}_{k_i}(G_i) \geq 2 \).
Higher rank lattices

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3. $\sum_{i=1}^{d} \text{rk}_{k_i}(G_i) \geq 2$.

The higher rank assumption $\sum_{i=1}^{d} \text{rk}_{k_i}(G_i) \geq 2$ implies that:

- Either $d = 1$, $k = k_1$, $G = G_1$ and $\text{rk}_k(G) \geq 2$ (**simple** case).
- Or $d \geq 2$ (**semisimple** or **product** case).
Examples of higher rank lattices

Denote by $\mathcal{P}$ the set of all prime numbers.

**Examples (Borel–Harish-Chandra, Behr, Harder)**

Higher rank lattices in simple algebraic groups (case $d = 1$):

- $\text{SL}_n(\mathbb{Z}) < \text{SL}_n(\mathbb{R})$, $n \geq 3$
- $\text{SL}_n(\mathbb{F}_q[t^{-1}]) < \text{SL}_n(\mathbb{F}_q((t)))$, $n \geq 3$ and $q = p^r$ with $p \in \mathcal{P}$
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Higher rank lattices in **semisimple** algebraic groups (case $d \geq 2$):

- $\text{SL}_n(\mathbb{Z}[\sqrt{a}]) < \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{R})$, $n \geq 2$, $a \geq 2$ square free
- $\text{SL}_n(\mathbb{Z}[S^{-1}]) < \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \text{SL}_n(\mathbb{Q}_{p_k})$, $n \geq 2$, $p_1, \ldots, p_k \in \mathcal{P}$ and $S = p_1 \cdots p_k$
- $\text{SL}_n(\mathbb{F}_q[t, t^{-1}]) < \text{SL}_n(\mathbb{F}_q((t))) \times \text{SL}_n(\mathbb{F}_q((t^{-1})))$, $n \geq 2$
Let $\Gamma < G$ be any higher rank lattice.

**Margulis’ Normal Subgroup Theorem (1978)**

Any normal subgroup $N \triangleleft \Gamma$ is either finite or has finite index.
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Margulis’ strategy: assuming that $N \triangleleft \Gamma$ is an infinite normal subgroup, to prove that $\Gamma/N$ is a finite group, he showed that

1. $\Gamma/N$ has **property (T)** (Functional Analysis).
2. $\Gamma/N$ is **amenable** (Ergodic Theory). This follows from:
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**Margulis’ Factor Theorem (1978)**

*For every $i \in \{1, \ldots, d\}$, let $P_i < G_i$ be any minimal parabolic $k_i$-subgroup and set $P_i = P_i(k_i)$. Set $P = \prod_{i=1}^{d} P_i < G$. Then any measurable $\Gamma$-factor of $G/P$ is $\Gamma$-isomorphic to $G/Q$ for some unique intermediate closed subgroup $P < Q < G$.*
Goals

We use operator algebras (C*-algebras and von Neumann algebras) to explore new rigidity phenomena of higher rank lattices.

Main Problem

Given a higher rank lattice $\Gamma < G$, we investigate:

1. Dynamical properties of the affine action $\Gamma \curvearrowright \mathcal{P}(\Gamma)$
2. Structure of group C*-algebras $C^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$
3. Rigidity aspects of the group von Neumann algebra $L(\Gamma)$
This ICM survey talk is based on the following joint works:


Dynamics of positive definite functions and character rigidity
For any countable discrete group $\Lambda$, set

$$\mathcal{P}(\Lambda) := \{ \varphi : \Lambda \to \mathbb{C} \mid \text{normalized positive definite function} \}$$

Then $\mathcal{P}(\Lambda) \subset \ell^\infty(\Lambda)$ is a weak-$*$ compact convex set.
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To any $\varphi \in \mathcal{P}(\Lambda)$, one associates the GNS triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$:

$$\forall \gamma \in \Lambda, \quad \varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi, \xi_\varphi \rangle$$
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Consider the affine conjugation action $\Lambda \curvearrowright \mathcal{P}(\Lambda)$ defined by

$$\forall \gamma \in \Lambda, \quad \gamma \varphi := \varphi \circ \text{Ad}(\gamma^{-1})$$
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**Definition**

A **character** $\varphi \in \mathcal{P}(\Lambda)$ is a fixed point for $\Lambda \curvearrowright \mathcal{P}(\Lambda)$.

Denote by $\text{Char}(\Lambda) \subset \mathcal{P}(\Lambda)$ the convex subset of all characters.
Examples of characters

Denote by $\text{Sub}(\Lambda)$ the compact metrizable space of all subgroups of $\Lambda$ endowed with the conjugation action $\gamma \cdot H = \gamma H \gamma^{-1}$.

Consider the $\Lambda$-equivariant continuous map

$$\text{Sub}(\Lambda) \to \mathcal{P}(\Lambda) : H \mapsto 1_H$$

1. If $N \triangleleft \Lambda$ is a normal subgroup, then $\phi_N \in \text{Char}(\Lambda)$.
   Its GNS unirep $\pi_\phi = \lambda_{\Lambda/N}$ is the quasi-regular representation.
   When $N = \Lambda$, then $1_\Lambda$ is the trivial character.
   When $N = \{e\}$, then $1_{\{e\}}$ is the regular character.

2. If $\Lambda \acts (X, \nu)$ is pmp, then $\phi_\nu : \gamma \mapsto \nu(\text{Fix}(\gamma)) \in \text{Char}(\Lambda)$.
   When $\phi_\nu = 1_{\{e\}}$, the action $\Lambda \acts (X, \nu)$ is essentially free.

3. If $\pi : \Lambda \to U(n)$ is a finite dim unirep, then $\text{tr} \circ \pi \in \text{Char}(\Lambda)$.
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3. If \( \pi : \Lambda \to \mathcal{U}(n) \) is a finite dim unirep, then \( \text{tr}_n \circ \pi \in \text{Char}(\Lambda) \).
Existence and classification of characters

Theorem (BH19, BBH20, BBH21)

Let $\Gamma < G$ be any higher rank lattice. Then

1. Any nonempty $\Gamma$-invariant weak-$*$ compact convex subset $\mathcal{C} \subset \mathcal{P}(\Gamma)$ contains a character.

2. Any extremal character $\varphi \in \text{Char}(\Gamma)$ is either supported on $\mathcal{L}(\Gamma)$ or $\pi_{\varphi}$ is amenable.

Assume moreover that $\Gamma$ has property (T). Then

3. $\Gamma$ is character rigid, i.e., any extremal character $\varphi \in \text{Char}(\Gamma)$ is either supported on $\mathcal{L}(\Gamma)$ or $\dim(\pi_{\varphi}) < \infty$.

The classification part (items 2 and 3) is due to Peterson (2014). We obtain a new conceptual proof in [BH19, BBHP20, BBH21].

Theorem (BH19, BBH20, BBH21)

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Corollary (BH19, BBHP20, BBH21)

Let $\Gamma < G$ be any higher rank lattice and set $\Lambda = \Gamma / \mathcal{L}(\Gamma)$. Let $\pi : \Lambda \to \mathcal{U}(\mathcal{H}_\pi)$ be any unirep. Then $C^*_\pi(\Lambda)$ admits a trace.

Assume moreover that $\Gamma$ has property (T). If $\pi$ is weakly mixing, then $\lambda \ll \pi$ i.e. there is a $C^*$-homomorphism $\Theta : C^*_\pi(\Lambda) \to C^*_\lambda(\Lambda)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Lambda$.

Moreover, $\tau_\Lambda \circ \Theta$ is the unique trace on $C^*_\pi(\Lambda)$. $\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Lambda)$.

This extends results by Bekka–Cowling–de la Harpe (1994) regarding $C^*$-simplicity and the unique trace property.
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Theorem (BH19, BBHP20, BBH21)

Let $\Gamma < G$ be any higher rank lattice and set $\Lambda = \Gamma / \mathcal{L}(\Gamma)$. Let $\Lambda \curvearrowright X$ be any minimal action on a compact metrizable space. At least one of the following assertions holds:

1. There exists a $\Lambda$-invariant probability measure $\nu \in \text{Prob}(X)$.
2. The action $\Lambda \curvearrowright X$ is topologically free, i.e., for every $\gamma \in \Lambda \setminus \{e\}$, $\text{Fix}(\gamma) = \{x \in X | \gamma x = x\}$ has empty interior.
3. Assume moreover that $\Gamma$ has property (T). Then either $X$ is finite or $\Lambda \curvearrowright X$ is topologically free.
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Our main results are consequences of a **dynamical dichotomy** for normal ucp $\Gamma$-maps $\Theta : M \to L^\infty(G/P)$ where $\Gamma < G$ is a higher rank lattice and $M$ is an ergodic $\Gamma$-von Neumann algebra.
Strategy of proof

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The proof of the dynamical dichotomy (which is the hard part) uses **von Neumann algebra** theory and depends heavily on whether $d = 1$ (**simple** case) or $d \geq 2$ (**semisimple** case).
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In [BH19, BBH21], we treat the case $d = 1$ and $rk_k(G) \geq 2$ (e.g. $G = SL_n(\mathbb{R})$ for $n \geq 3$). In that case, we prove a much stronger result: the **noncommutative Nevo–Zimmer theorem**.
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In [BBHP20], we treat the case $d \geq 2$ (e.g. $G = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$) exploiting a different method based on the product structure.
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In that respect, [BH19], [BBHP20], [BBH21] are complementary.
The noncommutative Nevo–Zimmer theorem
In this section, we assume that $d = 1$. Then $G$ is almost $k$-simple with $\text{rk}_k(G) \geq 2$ and we set $G = G(k)$.

Let $P < G$ be any minimal parabolic $k$-subgroup and set $P = P(k)$. Then $G/P = (G/P)(k)$.

**Example**

If $G = \text{SL}_n$, take $P < G$ the subgroup of upper triangular matrices.
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**Example**

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**Theorem (Furstenberg 1962, Bader–Shalom 2004)**

For every admissible measure $\mu \in \text{Prob}(G)$, there exists a unique $\mu$-stationary measure $\nu \in \text{Prob}(G/P)$ such that $(G/P, \nu)$ is the $(G, \mu)$-Furstenberg–Poisson boundary i.e.

$$L^\infty(G/P, \nu) \overset{\text{G-equiv.}}{=} \text{Har}^\infty(G, \mu)$$

Recall that $\nu$ is $\mu$-stationary if $\nu = \mu \ast \nu = \int_G g \ast \nu \, d\mu(g)$. 

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Noncommutative ergodic theory of higher rank lattices
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**Definition (BBHP20)**

Let $\Theta : M \to L^\infty(G/P)$ be any normal ucp $\Gamma$-map. We simply say that $\Theta$ is a $\Gamma$-boundary structure on $M$.

We denote by $\text{mult}(\Theta) \subset M$ its multiplicative domain.
Let $\Gamma < G$ be any lattice. Let $M$ be any von Neumann algebra and $\sigma : \Gamma \curvearrowright M$ any action by automorphisms.

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We denote by $\text{mult}(\Theta) \subset M$ its multiplicative domain.

For any $\Gamma$-boundary structure $\Theta : M \rightarrow L^\infty(G/P)$, one can define the *induced* $G$-boundary structure $\hat{\Theta} : \text{Ind}_\Gamma^G(M) \rightarrow L^\infty(G/P)$. 
Examples of boundary structures

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**Examples (Boundary structures)**

Let $A$ be any unital separable $C^*$-algebra and $\Gamma \curvearrowright A$ any action. Since $\Gamma \curvearrowright G/P$ is amenable, there exists a measurable $\Gamma$-map $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$. By duality, we obtain a ucp $\Gamma$-map $E : A \to L^\infty(G/P) : a \mapsto (b \mapsto \beta_b(a))$. 
Examples of boundary structures

Note that the action $\Gamma \curvearrowright G/P$ is amenable and ergodic.

Examples (Boundary structures)

Let $A$ be any unital separable $C^*$-algebra and $\Gamma \curvearrowright A$ any action. Since $\Gamma \curvearrowright G/P$ is amenable, there exists a measurable $\Gamma$-map $\beta : G/P \to \mathcal{S}(A) : b \mapsto \beta_b$. By duality, we obtain a ucp $\Gamma$-map $E : A \to L^\infty(G/P) : a \mapsto (b \mapsto \beta_b(a))$.

Consider the normal extension $E^{**} : A^{**} \to L^\infty(G/P)$ and denote by $z \in \mathcal{Z}(A^{**})$ its central support. Letting $M = A^{**}z$, $\Theta = E^{**} |_M : M \to L^\infty(G/P)$ is a $\Gamma$-boundary structure.
Examples of boundary structures

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We apply the above construction to the following situations:

1. $A = C(X)$ where $X$ is a compact metrizable $\Gamma$-space.
2. $A = C^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation and $\Gamma \curvearrowright C^*_\pi(\Gamma)$ is the conjugation action.
The noncommutative Nevo–Zimmer theorem

<table>
<thead>
<tr>
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Either $\Theta(M) = C_1$.

Or there is a unique proper parabolic $k$-subgroup $P < Q < G$ such that $\text{mult}(\Theta) \sim \Gamma$-equiv. $L^\infty(G/Q)$ with $Q = Q(k)$.

In case $M = L^\infty(X)$ and $k = \mathbb{R}$, and considering $G$-actions instead of $\Gamma$-actions, the above theorem is due to Nevo–Zimmer (2000). In [BH19], we extended NZ theorem to noncommutative von Neumann algebras $M$ and to both $\Gamma$-actions and $G$-actions. In [BBH21], we further generalized to deal with lattices in simple algebraic groups defined over an arbitrary local field $k$. |
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Let $\Gamma < G$ be any lattice. Let $M$ be any von Neumann algebra, $\Gamma \curvearrowright M$ any ergodic action and $\Theta : M \to L^\infty(G/P)$ any $\Gamma$-boundary structure. Then

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The noncommutative factor theorem and Connes’ rigidity conjecture
Connes’ rigidity conjecture for higher rank lattices

Connes (1979) showed that whenever $\Lambda$ is an icc group with property (T), the symmetry groups of $L(\Lambda)$ are at most countable. He conjectured that $L(\Lambda)$ should retain $\Lambda$ for property (T) groups.
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CIOS (2021): First class of $W^*$-superrigid property (T) icc groups.

\( \sim \) Cyril HOUDAYER (Paris-Saclay & IUF)

Noncommutative ergodic theory of higher rank lattices
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In view of Mostow–Margulis’ rigidity results, we state the following version of **Connes’ rigidity conjecture** for lattices in higher rank simple real Lie groups.

**Connes’ rigidity conjecture**

For $i \in \{1, 2\}$, let $G_i$ be any real connected simple Lie group with trivial center and $\text{rk}_R(G_i) \geq 2$, and let $\Gamma_i < G_i$ be any lattice.

\[
L(\Gamma_1) \cong L(\Gamma_2) \Rightarrow G_1 \cong G_2 \\
\Rightarrow \text{rk}_R(G_1) = \text{rk}_R(G_2)
\]
Let $\Gamma < G$ be any **higher rank lattice**. For every $i \in \{1, \ldots, d\}$, let $P_i < G_i$ be any minimal parabolic $k_i$-subgroup and set $P_i = P_i(k_i)$. Set $P = \prod_{i=1}^{d} P_i < G$.

Set $\Lambda = \Gamma / Z(\Gamma)$ and consider the ergodic action $\Lambda \actson G/P$ and its **group measure space** von Neumann algebra $L(\Lambda \actson G/P)$. 
The noncommutative Margulis factor theorem

Let $\Gamma \subset G$ be any higher rank lattice. For every $i \in \{1, \ldots, d\}$, let $P_i \subset G_i$ be any minimal parabolic $k_i$-subgroup and set $\mathcal{P} = \prod_{i=1}^{d} P_i$. Set $\Lambda = \Gamma / \mathcal{L} \Gamma$ and consider the ergodic action $\Lambda \curvearrowright G/P$ and its group measure space von Neumann algebra $L(\Lambda \curvearrowright G/P)$.

**Theorem (BBH21, BH22)**

For every von Neumann subalgebra $L(\Lambda) \subset M \subset L(\Lambda \curvearrowright G/P)$, there exists a unique intermediate closed subgroup $P \subset Q \subset G$ such that $M = L(\Lambda \curvearrowright G/Q)$. 

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**Corollary (BBH21, BH22)**

*The rank $\sum_{i=1}^{d} \text{rk}_{k_i}(G_i)$ is an invariant of $L(\Lambda) \subset L(\Lambda \curvearrowright G/P)$.*
Thank you for your attention!