Noncommutative ergodic theory of higher rank lattices

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Operator algebras, dynamics and groups

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Introduction

Simple algebraic groups

Let k be any local field i.e. k is a nondiscrete locally compact field.

Definition

Let $\mathbb{G}$ be any simply connected semisimple connected linear algebraic k-group. We assume that

1. $\mathbb{G}$ is **absolutely almost simple** (resp. **almost k-simple**) i.e. the only proper normal algebraic (resp. k-closed) subgroups are finite.
2. $\mathbb{G}$ is **k-isotropic** i.e. $\mathbb{G}$ has a nontrivial k-split torus.

Denote by $\mathrm{rk}_k(\mathbb{G})$ the dimension of a maximal k-split torus in $\mathbb{G}$.

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Example

For every $n \geq 2$, SL_n is a k-isotropic absolutely almost simple algebraic k-group. Moreover, we have $\mathrm{rk}_k(\mathrm{SL}_n) = n - 1$.

Higher rank lattices

Let $I = \{1, \dots, d\}$ be any finite index set with $d \geq 1$.

For every $i \in I$, let k_i be any local field, $\mathbb{G}_i$ any k_i-isotropic almost k_i-simple algebraic k_i-group and set $G_i = \mathbb{G}_i(k_i)$.

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Definition (Higher rank lattice)

Set $G = \prod_{i=1}^{d} G_i$. We say that $\Gamma < G$ is a **higher rank lattice** if

1. $\Gamma < G$ is a discrete subgroup with finite covolume;
2. If $d \geq 2$, then $\Gamma < G$ is with dense projections i.e. for every $j \in \{1, \dots, d\}$, the projection of Γ in $\prod_{i \neq j} G_i$ is dense;
3. $\sum_{i=1}^{d} \mathrm{rk}_{k_i}(\mathbb{G}_i) \geq 2$.

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3. $\sum_{i=1}^{d} \mathrm{rk}_{k_i}(\mathbb{G}_i) \geq 2$.

The higher rank assumption $\sum_{i=1}^{d} \mathrm{rk}_{k_i}(\mathbb{G}_i) \geq 2$ implies that:

- Either $d = 1$, $k = k_1$, $\mathbb{G} = \mathbb{G}_1$ and $\mathrm{rk}_k(\mathbb{G}) \geq 2$ (**simple** case).
- Or $d \geq 2$ (**semisimple** or **product** case).

Examples of higher rank lattices

Denote by $\mathcal{P}$ the set of all prime numbers.

Examples (Borel–Harish-Chandra, Behr, Harder)

Higher rank lattices in **simple** algebraic groups (case $d = 1$):

- $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$
- $\mathrm{SL}_n\left(\mathbb{F}_q[t^{-1}]\right) < \mathrm{SL}_n\left(\mathbb{F}_q((t))\right)$, $n \geq 3$ and $q = p^r$ with $p \in \mathcal{P}$

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Higher rank lattices in **semisimple** algebraic groups (case $d \geq 2$):

- $\mathrm{SL}_n(\mathbb{Z}[\sqrt{a}]) < \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$, $n \geq 2$, $a \geq 2$ square free
- $\mathrm{SL}_n(\mathbb{Z}[S^{-1}]) < \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbb{Q}_{p_k})$, $n \geq 2$, $p_1, \ldots, p_k \in \mathcal{P}$ and $S = p_1 \cdots p_k$
- $\mathrm{SL}_n\left(\mathbb{F}_q[t, t^{-1}]\right) < \mathrm{SL}_n\left(\mathbb{F}_q((t))\right) \times \mathrm{SL}_n\left(\mathbb{F}_q((t^{-1}))\right)$, $n \geq 2$

Motivation

Let $\Gamma < G$ be any **higher rank lattice**.

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1. Γ/N has **property (T)** (Functional Analysis).
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Margulis' Factor Theorem (1978)

For every $i \in \{1, \dots, d\}$, let $\mathbb{P}_i < \mathbb{G}_i$ be any minimal parabolic k_i-subgroup and set $P_i = \mathbb{P}_i(k_i)$. Set $P = \prod_{i=1}^d P_i < G$.

Then any measurable Γ-factor of G/P is Γ-isomorphic to G/Q for some unique intermediate closed subgroup $P < Q < G$.

Goals

We use operator algebras (C^*-algebras and von Neumann algebras) to explore new rigidity phenomena of **higher rank lattices**.

Main Problem

Given a **higher rank lattice** $\Gamma < G$, we investigate:

1. Dynamical properties of the affine action $\Gamma \curvearrowright \mathscr{P}(\Gamma)$
2. Structure of group C^*-algebras $\mathrm{C}^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$
3. Rigidity aspects of the group von Neumann algebra $\mathrm{L}(\Gamma)$

Bibliography

This ICM survey talk is based on the following joint works:

[BH19] R. Boutonnet, C. Houdayer: *Stationary characters on lattices of semisimple Lie groups.* Publications mathématiques de l'IHÉS **133** (2021), 1-46. arXiv:1908.07812

[BBHP20] U. Bader, R. Boutonnet, C. Houdayer, J. Peterson: *Charmenability of arithmetic groups of product type.* To appear in Invent. Math. arXiv:2009.09952

[BBH21] U. Bader, R. Boutonnet, C. Houdayer: *Charmenability of higher rank arithmetic groups.* arXiv:2112.01337

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Dynamics of positive definite functions and character rigidity

Dynamics of $\Lambda \curvearrowright \mathscr{P}(\Lambda)$

For any countable discrete group Λ, set

$$\mathscr{P}(\Lambda) := \{\varphi : \Lambda \to \mathbb{C} \mid \text{normalized positive definite function}\}$$

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To any $\varphi \in \mathscr{P}(\Lambda)$, one associates the GNS triple $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$:

$$\forall \gamma \in \Lambda, \quad \varphi(\gamma) = \langle \pi_\varphi(\gamma)\xi_\varphi, \xi_\varphi \rangle$$

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Definition

A **character** $\varphi \in \mathscr{P}(\Lambda)$ is a fixed point for $\Lambda \curvearrowright \mathscr{P}(\Lambda)$.

Denote by $\mathrm{Char}(\Lambda) \subset \mathscr{P}(\Lambda)$ the convex subset of all characters.

Examples of characters

Denote by $\mathrm{Sub}(\Lambda)$ the compact metrizable space of all subgroups of Λ endowed with the conjugation action $\gamma \cdot H = \gamma H \gamma^{-1}$.

Consider the Λ-equivariant continuous map

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1. If $N \lhd \Lambda$ is a normal subgroup, then $\varphi = 1_N \in \mathrm{Char}(\Lambda)$.
   Its GNS unirep $\pi_\varphi = \lambda_{\Lambda/N}$ is the quasi-regular representation.
   - When $N = \Lambda$, then 1_Λ is the trivial character.
   - When $N = \{e\}$, then $1_{\{e\}}$ is the regular character.

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2. If $\Lambda \curvearrowright (X, \nu)$ is pmp, then $\varphi_\nu : \gamma \mapsto \nu(\mathrm{Fix}(\gamma)) \in \mathrm{Char}(\Lambda)$.
   $\rightsquigarrow$ When $\varphi_\nu = 1_{\{e\}}$, the action $\Lambda \curvearrowright (X, \nu)$ is essentially free.

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   $\rightsquigarrow$ When $\varphi_\nu = 1_{\{e\}}$, the action $\Lambda \curvearrowright (X, \nu)$ is essentially free.
3. If $\pi : \Lambda \to \mathscr{U}(n)$ is a finite dim unirep, then $\mathrm{tr}_n \circ \pi \in \mathrm{Char}(\Lambda)$.

Existence and classification of characters

Theorem (BH19, BBH20, BBH21)

Let $\Gamma < G$ be any **higher rank lattice**. *Then*

1. *Any nonempty Γ-invariant weak-$*$ compact convex subset $\mathscr{C} \subset \mathscr{P}(\Gamma)$ contains a character.*
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Assume moreover that Γ has property (T). Then

3. *Γ is* **character rigid** *i.e. any extremal character $\varphi \in \mathrm{Char}(\Gamma)$ is either supported on $\mathscr{Z}(\Gamma)$ or $\dim(\pi_\varphi) < \infty$.*

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Previous character rigidity results due to Bekka (2006), Peterson–Thom (2013), Creutz–Peterson (2013).

Structure of group C*-algebras $C^*_\pi(\Gamma)$

Corollary (BH19, BBHP20, BBH21)

Let $\Gamma < G$ *be any* **higher rank lattice** *and set* $\Lambda = \Gamma / \mathscr{Z}(\Gamma)$*. Let* $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$ *be any unirep. Then* $C^*_\pi(\Lambda)$ *admits a trace.*

Structure of group C^*-algebras $C^*_\pi(\Gamma)$

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Let $\Gamma < G$ be any **higher rank lattice** *and set $\Lambda = \Gamma/\mathscr{Z}(\Gamma)$. Let $\pi : \Lambda \to \mathscr{U}(\mathscr{H}_\pi)$ be any unirep. Then $C^*_\pi(\Lambda)$ admits a trace.*

Assume moreover that Γ has property (T). If π is weakly mixing, then $\lambda \prec \pi$ i.e. there is a $*$-homomorphism $\Theta : C^*_\pi(\Lambda) \to C^*_\lambda(\Lambda)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Lambda$. Moreover,

1. *$\tau_\Lambda \circ \Theta$ is the unique trace on $C^*_\pi(\Lambda)$.*
2. *$\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Lambda)$.*

This extends results by Bekka–Cowling–de la Harpe (1994) regarding C^*-simplicity and the unique trace property.

Dichotomy for topological dynamics

Theorem (BH19, BBHP20, BBH21)

Let $\Gamma < G$ be any **higher rank lattice** *and set $\Lambda = \Gamma/\mathscr{Z}(\Gamma)$. Let $\Lambda \curvearrowright X$ be any minimal action on a compact metrizable space.*

At least one of the following assertions holds:

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- *There exists a* Λ*-invariant probability measure* $\nu \in \mathrm{Prob}(X)$*.*
- *The action* $\Lambda \curvearrowright X$ *is* **topologically free** *i.e. for every* $\gamma \in \Lambda \setminus \{e\}$, $\mathrm{Fix}(\gamma) := \{x \in X \mid \gamma x = x\}$ *has empty interior.*

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Assume moreover that Γ has property (T). Then

- *Either X is finite or $\Lambda \curvearrowright X$ is topologically free.*

Our result is a topological analogue of Stuck–Zimmer's stabilizer rigidity theorem (1992). It solves a question raised by Glasner–Weiss (2014).

Strategy of proof

Our main results are consequences of a **dynamical dichotomy** for normal ucp Γ-maps $\Theta : M \to \mathrm{L}^\infty(G/P)$ where $\Gamma < G$ is a higher rank lattice and M is an ergodic Γ-von Neumann algebra.

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In [BH19, BBH21], we treat the case $d = 1$ and $\mathrm{rk}_k(\mathbb{G}) \geq 2$ (e.g. $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$). In that case, we prove a much stronger result: the **noncommutative Nevo–Zimmer theorem**.

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In that respect, [BH19], [BBHP20], [BBH21] are complementary.

The noncommutative Nevo–Zimmer theorem

Structure theory of G/P

In this section, we assume that $d = 1$. Then $\mathbb{G}$ is almost k-simple with $\mathrm{rk}_k(\mathbb{G}) \geq 2$ and we set $G = \mathbb{G}(k)$.

Let $\mathbb{P} < \mathbb{G}$ be any minimal parabolic k-subgroup and set $P = \mathbb{P}(k)$. Then $G/P = (\mathbb{G}/\mathbb{P})(k)$.

Example

If $\mathbb{G} = \mathrm{SL}_n$, take $\mathbb{P} < \mathbb{G}$ the subgroup of upper triangular matrices.

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Theorem (Furstenberg 1962, Bader–Shalom 2004)

*For every admissible measure $\mu \in \mathrm{Prob}(G)$, there exists a unique μ-stationary measure $\nu \in \mathrm{Prob}(G/P)$ such that $(G/P, \nu)$ is the (G, μ)-**Furstenberg–Poisson boundary** i.e.*

$$\mathrm{L}^\infty(G/P, \nu) \underset{G\text{-equiv.}}{\cong} \mathrm{Har}^\infty(G, \mu)$$

Recall that ν is μ-**stationary** if $\nu = \mu * \nu = \int_G g_*\nu \, \mathrm{d}\mu(g)$.

Boundary structures on von Neumann algebras

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Definition (BBHP20)

Let $\Theta : M \to \mathrm{L}^\infty(G/P)$ be any normal ucp Γ-map. We simply say that Θ is a **Γ-boundary structure** on M.

We denote by $\mathrm{mult}(\Theta) \subset M$ its multiplicative domain.

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For any Γ-boundary structure $\Theta : M \to \mathrm{L}^\infty(G/P)$, one can define the **induced** G-boundary structure $\widehat{\Theta} : \mathrm{Ind}_\Gamma^G(M) \to \mathrm{L}^\infty(G/P)$.

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Since $\Gamma \curvearrowright G/P$ is amenable, there exists a measurable Γ-map $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$. By duality, we obtain a ucp Γ-map $\mathrm{E} : A \to \mathrm{L}^\infty(G/P) : a \mapsto (b \mapsto \beta_b(a))$.

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Consider the normal extension $\mathrm{E}^{**} : A^{**} \to \mathrm{L}^\infty(G/P)$ and denote by $z \in \mathscr{Z}(A^{**})$ its central support. Letting $M = A^{**}z$, $\Theta = \mathrm{E}^{**}|_M : M \to \mathrm{L}^\infty(G/P)$ is a Γ-boundary structure.

Examples of boundary structures

Note that the action $\Gamma \curvearrowright G/P$ is amenable and ergodic.

Examples (Boundary structures)

Let A be any unital separable C^*-algebra and $\Gamma \curvearrowright A$ any action.

Since $\Gamma \curvearrowright G/P$ is amenable, there exists a measurable Γ-map $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$. By duality, we obtain a ucp Γ-map $\mathrm{E} : A \to \mathrm{L}^\infty(G/P) : a \mapsto (b \mapsto \beta_b(a))$.

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We apply the above construction to the following situations:

1. $A = \mathrm{C}(X)$ where X is a compact metrizable Γ-space.
2. $A = \mathrm{C}^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathscr{U}(\mathscr{H}_\pi)$ is a unitary representation and $\Gamma \curvearrowright \mathrm{C}^*_\pi(\Gamma)$ is the conjugation action.

The noncommutative Nevo–Zimmer theorem

Theorem (BH19, BBH21)

Let $\Gamma < G$ be any lattice. Let M be any von Neumann algebra, $\Gamma \curvearrowright M$ any ergodic action and $\Theta : M \to \mathrm{L}^\infty(G/P)$ any Γ-boundary structure. Then

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- *Either $\Theta(M) = \mathbb{C}1$.*
- *Or there is a unique proper parabolic k-subgroup $\mathbb{P} < \mathbb{Q} < \mathbb{G}$ such that $\mathrm{mult}(\Theta) \underset{\Gamma\text{-equiv.}}{\cong} \mathrm{L}^\infty(G/Q)$ with $Q = \mathbb{Q}(k)$.*

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In case $M = \mathrm{L}^\infty(X)$ and $k = \mathbb{R}$, and considering G-actions instead of Γ-actions, the above theorem is due to Nevo–Zimmer (2000).

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$\rightsquigarrow$ In [BBH21], we further generalized to deal with lattices in simple algebraic groups defined over an arbitrary local field k.

The noncommutative factor theorem and Connes' rigidity conjecture

Connes' rigidity conjecture for higher rank lattices

Connes (1979) showed that whenever Λ is an icc group with property (T), the symmetry groups of $L(\Lambda)$ are at most countable.

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In view of Mostow–Margulis' rigidity results, we state the following version of **Connes' rigidity conjecture** for lattices in higher rank simple real Lie groups.

Connes' rigidity conjecture

For $i \in \{1, 2\}$, let G_i be any real connected simple Lie group with trivial center and $\mathrm{rk}_{\mathbb{R}}(G_i) \geq 2$, and let $\Gamma_i < G_i$ be any lattice.

$$\begin{aligned} \mathrm{L}(\Gamma_1) \cong \mathrm{L}(\Gamma_2) \quad &\Rightarrow \quad G_1 \cong G_2 \\ &\Rightarrow \quad \mathrm{rk}_{\mathbb{R}}(G_1) = \mathrm{rk}_{\mathbb{R}}(G_2) \end{aligned}$$

Let $\Gamma < G$ be any **higher rank lattice**. For every $i \in \{1, \dots, d\}$, let $\mathbb{P}_i < \mathbb{G}_i$ be any minimal parabolic k_i-subgroup and set $P_i = \mathbb{P}_i(k_i)$. Set $P = \prod_{i=1}^{d} P_i < G$.

Set $\Lambda = \Gamma/\mathscr{Z}(\Gamma)$ and consider the ergodic action $\Lambda \curvearrowright G/P$ and its **group measure space** von Neumann algebra $\mathrm{L}(\Lambda \curvearrowright G/P)$.

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Theorem (BBH21, BH22)

For every von Neumann subalgebra $\mathrm{L}(\Lambda) \subset M \subset \mathrm{L}(\Lambda \curvearrowright G/P)$, there exists a unique intermediate closed subgroup $P < Q < G$ such that $M = \mathrm{L}(\Lambda \curvearrowright G/Q)$.

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Corollary (BBH21, BH22)

The rank $\sum_{i=1}^d \mathrm{rk}_{k_i}(\mathbb{G}_i)$ is an invariant of $\mathrm{L}(\Lambda) \subset \mathrm{L}(\Lambda \curvearrowright G/P)$.

Thank you for your attention!