Wreath-like product groups and rigidity of their von Neumann algebras

joint work with Ionut Chifan, Denis Osin and Bin Sun

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Let $G$ be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$. 

Definition
The group von Neumann algebra $L(G) \subset B(\ell^2 G)$ is defined as $L(G) := \text{span}\{u_g | g \in G\}$ WOT.

Central problem
Classify $L(G)$ in terms of the group $G$.

Facts
1. If $G$ is infinite abelian, then $L(G) \sim L(\hat{G}, \text{Haar}) \sim L([0,1], \text{Leb})$.
2. $L(G)$ is a II$_1$ factor $\iff$ $G$ is icc: $|\{hgh^{-1} | h \in G\}| = \infty$, $\forall g \neq e$.

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In the rest of the talk, all groups are icc countable discrete.
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Definition.

A group \( G \) is amenable if its regular rep. has almost invariant vectors:

\[ \exists \text{ unit vectors } (\xi_n) \subset \ell^2(G) \text{ such that } \|u_g \xi_n - \xi_n\|_2 \to 0, \forall g \in G. \]

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Definition. A group $G$ has Kazhdan’s property (T) if any unitary rep. of $G$ with almost invariant vectors has nonzero invariant vectors.

Examples.
1) Higher rank lattices, e.g., $\text{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov density model $1 < d < 1/2$ (also hyperbolic).
3) $\text{Aut}(F_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If $G$ is icc property (T), then the outer automorphism group $\text{Out}(L(G))$ and fundamental group $\pi_1(L(G))$ are countable.

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Conjecture (Jones, 2000) Show that $\text{Out}(L(G)) \cong \text{Char}(G) \rtimes \text{Out}(G)$, for icc property (T) groups $G$. 

Popa’s strengthening of Connes’ rigidity conjecture, 2006

Let $G$ be an icc property (T) group and $H$ be any group. Let $\theta: L(G) \to L(H)$ be $^\ast$-isomorphism, for some $t > 0$. Then $t = 1$ and there exists a group isomorphism $\delta: G \to H$ and character $\eta: G \to T$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}$, $\forall g \in G$. In particular, $G$ is $W^\ast$-superrigid: if $L(G) \cong L(H)$, for any $H$, then $G \cong H$. 

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Remark. These results do not apply to property (T) group factors. This is because deformation/rigidity applies to II$_1$ factors which admit deformations, whose presence is typically incompatible with property (T).
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- **I-Peterson-Popa (2005)** \( \exists \text{ II}_1 \) factors \( M \) with \( \text{Out}(M) = \{e\} \).
- **Popa-Vaes (2006)** icc \( G \) with \( \text{Out}(L(G)) \cong \text{Char}(G) \rtimes \text{Out}(G) \).
- **I-Popa-Vaes (2010)** examples of \( \text{W}^* \)-superrigid icc groups \( G \).

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**Remark.** These results do not apply to property (T) group factors. This is because deformation/rigidity applies to II$_1$ factors which admit deformations, whose presence is typically incompatible with property (T).

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Wreath-like product groups, I

Definition

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**Examples** (1) $H < H \ast K$ (Proof. $\langle\langle H \rangle\rangle = \ast_{k \in K} k H k^{-1}$.)

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Proposition Let $S = \langle [tHt^{-1}, t'Ht'^{-1}] \mid t, t' \in T, t \neq t' \rangle$, for a C-L subgroup $H < G$. Then $S < \langle \langle H \rangle \rangle$, $S \triangleleft G$ and $G/S \in \mathcal{WR}(H, G/\langle \langle H \rangle \rangle)$. 
Wreath-like product groups, I

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Remark. When $G = H \ast K$, we have $G/\langle \langle H \rangle \rangle = K$ and $G/S = H \wr K$. 

52 / 90
Wreath-like product groups, II

**Theorem** (Dahmani-Guirardel-Osin 2011, Sun 2020) If $G$ is hyperbolic relative to $H$, then $\exists F \subset H$ finite s.t. $\forall N \triangleleft H$ with $N \cap F = \emptyset$ we have:

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1. This is surprising because wreath products $A \wr B$ never have (T).
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Using this theorem, we prove:

**Theorem A** (Chifan-I-Osin-Sun, 2021) Let $K$ be an icc hyperbolic group. Then for any finitely generated group $A$, $\exists$ a quotient $G$ of $K$ such that $G \in \text{WR}(A, B)$, for $B$ icc hyperbolic.

Moreover, if $K$ has property (T), then so does $G \in \text{WR}(A, B)$.

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Wreath-like product groups, II

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Connes’ rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)
Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where $A, C$ are nontrivial abelian or icc and $B, D$ are icc hyperbolic. Let $\theta: L(G) \to L(H)$ be $^\ast$-isomorphism, for any $t > 0$. Then $t = 1$ and $\exists$ a group isomorphism $\delta: G \to H$ and character $\eta: G \to T$ such that (up to unitary conjugacy) $\theta(\nu_g) = \eta(g) \nu_\delta(g)$, $\forall g \in G$.

In particular, $\text{Out}(L(G)) \cong \text{Char}(G) \rtimes \text{Out}(G)$ and $\text{F}(L(G)) = \{1\}$.

Additionally, we can take $G$ with no characters and prescribed $\text{Out}$:

Corollary C (CIOS, 2021)
For all f.p. group $Q$, $\exists$ a continuum of icc property (T) groups $\{G_i\}_{i \in I}$ s.t. $1 \neq L(G_i) \cong L(G_j), \forall i \neq j$.

$\text{Out}(L(G_i)) \cong Q$ and $\text{F}(L(G_i)) = \{1\}, \forall i \in I$.

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Out($L(G_i)$) $\sim = Q$ and $F(L(G_i)) = \{1\}, \forall i \in I$. 

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These are the first calculations of $\text{Out}(\text{L}(G))$, for icc property (T) $G$. 
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Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where $A$ is nontrivial abelian and $B$ is icc hyperbolic.
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CIOS, 2022: uncountably many $W^*$-superrigid property (T) groups.
Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\text{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, $B$ is icc hyperbolic and $\text{Out}(B) = \{e\}$. Then $\text{Out}(L(G)) = \{e\}$.

Sketch of proof. Let $\theta$ be an automorphism of $L(G)$. Identifying $L(A(B)) \subset L(G)$ with $L_{\infty}(\hat{A}B) \subset L_{\infty}(\hat{A}B) \rtimes \sigma, cB$, we have (i) $L(A(B)) \subset L(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \rtimes \hat{A}B$.

(ii) The conjugation action $G \rtimes \alpha L(A(B)) = L_{\infty}(\hat{A}B)$ given by $\alpha g = \text{Ad}(u g)$ is a generalized Bernoulli action. As $B$ is hyperbolic, $L(A(B))$ is the unique Cartan subalgebra of $L(G)$ (Popa-Vaes, 2012). We may assume that $\theta(L(A(B)) = L(A(B))$.

Therefore, $\theta$ induces an automorphism of $R(B \rtimes \hat{A}B)$. Since $B$ is icc property (T), $B \rtimes \hat{A}B$ is OE superrigid (Popa, 2005). Since $\text{Out}(B) = \{e\}$, we get that $\theta(u g) \in L(A(B)) u g$, $\forall g \in G$.

Let $\theta(u g) = w g u g$. Then $(w g) g \in G \subset U(L(A(B)))$ is a 1-cocycle for $\alpha$.

Since $G$ has property (T), $\alpha$ is cocycle superrigid (Popa, 2005). As $\text{Char}(G) = \{1\}$, it follows that $u g = u \sigma g (u^*)$, $\forall g \in G$.

Hence $\theta = \text{Ad}(u)$ is an inner automorphism.
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Property (T) group factors with trivial Out

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Let \( G \in \mathcal{W}R(A, B) \) with property (T) and \( \text{Char}(G) = \{1\} \), where \( A \neq \{1\} \) is abelian, \( B \) is icc hyperbolic and \( \text{Out}(B) = \{e\} \). Then \( \text{Out}(L(G)) = \{e\} \).

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86 / 90
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\(87 / 90\)
Let \( G \in \mathcal{WR}(A, B) \) with property (T) and \( \text{Char}(G) = \{ 1 \} \), where \( A \neq \{ 1 \} \) is abelian, \( B \) is icc hyperbolic and \( \text{Out}(B) = \{ e \} \). Then \( \text{Out}(L(G)) = \{ e \} \).

**Sketch of proof.** Let \( \theta \) be an automorphism of \( L(G) \).

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- Since \( B \) is icc property (T), \( B \curvearrowright \hat{A}B \) is OE superrigid ([Popa, 2005](#)). Since \( \text{Out}(B) = \{ e \} \), we get that \( \theta(u_g) \in L(A(B))u_g, \forall g \in G \).
- Let \( \theta(u_g) = w_g u_g \). Then \( (w_g)_{g \in G} \subset \mathcal{U}(L(A(B))) \) is a 1-cocycle for \( \alpha \). Since \( G \) has property (T), \( \alpha \) is cocycle superrigid ([Popa, 2005](#)).
Let $G \in \mathcal{W}(A, B)$ with property (T) and $\text{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, $B$ is icc hyperbolic and $\text{Out}(B) = \{e\}$. Then $\text{Out}(L(G)) = \{e\}$.

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