Wreath-like product groups and rigidity of their von Neumann algebras

joint work with Ionut Chifan, Denis Osin and Bin Sun

Adrian Ioana

UCSD

Operator algebras, dynamics and groups - an ICM satellite conference
University of Copenhagen, July 3

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Definition

The group von Neumann algebra $\mathrm{L}(G) \subset \mathbb{B}(\ell^2 G)$ is defined as

$$\mathrm{L}(G) := \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{WOT}}$$

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Definition

The group von Neumann algebra $\mathrm{L}(G) \subset \mathbb{B}(\ell^2 G)$ is defined as

$$\mathrm{L}(G) := \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{WOT}}$$

Central problem

Classify $\mathrm{L}(G)$ in terms of the group G.

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Definition

The group von Neumann algebra $\mathrm{L}(G) \subset \mathbb{B}(\ell^2 G)$ is defined as

$$\mathrm{L}(G) := \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{WOT}}$$

Central problem

Classify $\mathrm{L}(G)$ in terms of the group G.

Facts

1. If G is infinite abelian, then $\mathrm{L}(G) \cong \mathrm{L}^\infty(\hat{G}, Haar) \cong \mathrm{L}^\infty([0,1], Leb)$.

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Definition

The group von Neumann algebra $\mathrm{L}(G) \subset \mathbb{B}(\ell^2 G)$ is defined as

$$\mathrm{L}(G) := \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{WOT}}$$

Central problem

Classify $\mathrm{L}(G)$ in terms of the group G.

Facts

1. If G is infinite abelian, then $\mathrm{L}(G) \cong \mathrm{L}^\infty(\hat{G}, Haar) \cong \mathrm{L}^\infty([0,1], Leb)$.
2. $\mathrm{L}(G)$ is a II_1 factor $\Leftrightarrow$ G is icc: $|\{hgh^{-1} \mid h \in G\}| = \infty, \forall g \neq e$.

Group von Neumann algebras (Murray-von Neumann, '43)

Let G be a countable discrete group.
Let $u : G \to \mathcal{U}(\ell^2 G)$ be the left regular representation: $u_g(\delta_h) = \delta_{gh}$.

Definition

The group von Neumann algebra $\mathrm{L}(G) \subset \mathbb{B}(\ell^2 G)$ is defined as

$$\mathrm{L}(G) := \overline{\mathrm{span}\{u_g \mid g \in G\}}^{\mathrm{WOT}}$$

Central problem

Classify $\mathrm{L}(G)$ in terms of the group G.

Facts

1. If G is infinite abelian, then $\mathrm{L}(G) \cong \mathrm{L}^\infty(\hat{G}, Haar) \cong \mathrm{L}^\infty([0,1], Leb)$.
2. $\mathrm{L}(G)$ is a II_1 factor $\Leftrightarrow$ G is icc: $|\{hgh^{-1} \mid h \in G\}| = \infty, \forall g \neq e$.

Convention. In the rest of the talk, all groups are icc countable discrete.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}} \mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.
3. $\mathrm{L}(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.
3. $\mathrm{L}(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

Definition. A group G is amenable if its regular rep. has almost invariant vectors: $\exists$ unit vectors $(\xi_n) \subset \ell^2 G$ such that $\|u_g\xi_n - \xi_n\|_2 \to 0, \forall g \in G$.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.
3. $\mathrm{L}(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

Definition. A group G is amenable if its regular rep. has almost invariant vectors: $\exists$ unit vectors $(\xi_n) \subset \ell^2 G$ such that $\|u_g\xi_n - \xi_n\|_2 \to 0, \forall g \in G$.

Connes, 1975

$\mathrm{L}(G) \cong R$, $\forall$ G icc amenable. Any amenable II_1 factor is isomorphic to R.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.
3. $\mathrm{L}(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

Definition. A group G is amenable if its regular rep. has almost invariant vectors: $\exists$ unit vectors $(\xi_n) \subset \ell^2 G$ such that $\|u_g\xi_n - \xi_n\|_2 \to 0, \forall g \in G$.

Connes, 1975

$\mathrm{L}(G) \cong R$, $\forall$ G icc amenable. Any amenable II_1 factor is isomorphic to R.

Absence of rigidity: vN algebras forget any properties of amenable groups.

Classification of group von Neumann algebras

Murray-von Neumann, 1936-43

1. $\exists!$ approximately finite dimensional II_1 factor, $R = \overline{\otimes_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})}^{\mathrm{WOT}}$.
2. $\mathrm{L}(G) \cong R$, $\forall$ G icc locally finite, for instance $G = S_\infty = \cup_{n\in\mathbb{N}} S_n$.
3. $\mathrm{L}(\mathbb{F}_2) \ncong R$, where $\mathbb{F}_2$ is the free group on two generators.

Definition. A group G is amenable if its regular rep. has almost invariant vectors: $\exists$ unit vectors $(\xi_n) \subset \ell^2 G$ such that $\|u_g\xi_n - \xi_n\|_2 \to 0, \forall g \in G$.

Connes, 1975

$\mathrm{L}(G) \cong R$, $\forall$ G icc amenable. Any amenable II_1 factor is isomorphic to R.

Absence of rigidity: vN algebras forget any properties of amenable groups.

The rest of the talk: rigidity for vN algebras of nonamenable groups.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.

2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.

2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).

3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).
3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.
Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).
3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Connes' rigidity conjecture, 1980

If G, H are icc property (T) groups such that $\mathrm{L}(G) \cong \mathrm{L}(H)$, then $G \cong H$.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.
Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).
3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Connes' rigidity conjecture, 1980

If G, H are icc property (T) groups such that $\mathrm{L}(G) \cong \mathrm{L}(H)$, then $G \cong H$.

- **Connes-Jones (1983)** Property (T) is an invariant of $\mathrm{L}(G)$.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.

2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).

3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Connes' rigidity conjecture, 1980

If G, H are icc property (T) groups such that $\mathrm{L}(G) \cong \mathrm{L}(H)$, then $G \cong H$.

- **Connes-Jones (1983)** Property (T) is an invariant of $\mathrm{L}(G)$.
- **Cowling-Haagerup (1988)** If $G < \mathrm{Sp}(1, n)$ and $H < \mathrm{Sp}(1, m)$ are lattices and $n \neq m$, then $\mathrm{L}(G) \ncong \mathrm{L}(H)$.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.

Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).
3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Connes' rigidity conjecture, 1980

If G, H are icc property (T) groups such that $\mathrm{L}(G) \cong \mathrm{L}(H)$, then $G \cong H$.

- **Connes-Jones (1983)** Property (T) is an invariant of $\mathrm{L}(G)$.
- **Cowling-Haagerup (1988)** If $G < \mathrm{Sp}(1, n)$ and $H < \mathrm{Sp}(1, m)$ are lattices and $n \neq m$, then $\mathrm{L}(G) \ncong \mathrm{L}(H)$.
- **Ozawa (2002)** $\exists$ uncountably many noniso. prop. (T) group factors.

Property (T) group von Neumann algebras, I

Definition. A group G has Kazhdan's property (T) if any unitary rep. of G with almost invariant vectors has nonzero invariant vectors.
Examples. 1) Higher rank lattices, e.g., $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.
2) Random groups: Gromov **d**ensity model $\frac{1}{3} < \mathbf{d} < \frac{1}{2}$ (also hyperbolic).
3) $\mathrm{Aut}(\mathbb{F}_k)$ (Novak-Kaluba-Ozawa $k = 5$; Novak-Kaluba-Kielak $k > 5$).

Connes (1980) If G is icc property (T), then the outer automorphism group $\mathrm{Out}(\mathrm{L}(G))$ and fundamental group $\mathcal{F}(\mathrm{L}(G))$ are countable.

Connes' rigidity conjecture, 1980

If G, H are icc property (T) groups such that $\mathrm{L}(G) \cong \mathrm{L}(H)$, then $G \cong H$.

- **Connes-Jones (1983)** Property (T) is an invariant of $\mathrm{L}(G)$.
- **Cowling-Haagerup (1988)** If $G < \mathrm{Sp}(1, n)$ and $H < \mathrm{Sp}(1, m)$ are lattices and $n \neq m$, then $\mathrm{L}(G) \ncong \mathrm{L}(H)$.
- **Ozawa (2002)** $\exists$ uncountably many noniso. prop. (T) group factors.
- **Popa (2006)** $G \mapsto \mathrm{L}(G)$ is countable-to-1 for icc prop. (T) groups.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.
Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for some $t > 0$.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.
Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for some $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g)u_{\delta(g)}, \forall g \in G$.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.
Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for some $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any H, then $G \cong H$,

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.
Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for some $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g)u_{\delta(g)}, \forall g \in G$.

In particular, G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any H, then $G \cong H$, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Property (T) group von Neumann algebras, II

Problem (Connes, 1994) Compute $\mathcal{F}(\mathrm{L}(G))$, for icc property (T) groups G.

Conjecture (Jones, 2000) Show that $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, for icc property (T) groups G.

Popa's strengthening of Connes' rigidity conjecture, 2006

Let G be an icc property (T) group and H be any group.
Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for some $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g)u_{\delta(g)}, \forall g \in G$.

In particular, G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any H, then $G \cong H$, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathsf{L}(\mathbb{Z}^2 \rtimes \mathsf{SL}_2(\mathbb{Z})) = \{1\}$.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathrm{Out}(M) = \{e\}$.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathrm{Out}(M) = \{e\}$.
- **Popa-Vaes (2006)** icc G with $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathrm{Out}(M) = \{e\}$.
- **Popa-Vaes (2006)** icc G with $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$.
- **I-Popa-Vaes (2010)** examples of W^*-superrigid icc groups G.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathsf{L}(\mathbb{Z}^2 \rtimes \mathsf{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathsf{L}(\mathbb{Z} \wr G) \cong \mathsf{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathsf{Out}(M) = \{e\}$.
- **Popa-Vaes (2006)** icc G with $\mathsf{Out}(\mathsf{L}(G)) \cong \mathsf{Char}(G) \rtimes \mathsf{Out}(G)$.
- **I-Popa-Vaes (2010)** examples of W*-superrigid icc groups G.

Remark. These results do not apply to property (T) group factors.

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathsf{L}(\mathbb{Z}^2 \rtimes \mathsf{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathsf{L}(\mathbb{Z} \wr G) \cong \mathsf{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathrm{Out}(M) = \{e\}$.
- **Popa-Vaes (2006)** icc G with $\mathrm{Out}(\mathsf{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$.
- **I-Popa-Vaes (2010)** examples of W^*-superrigid icc groups G.

Remark. These results do not apply to property (T) group factors. This is because deformation/rigidity applies to II_1 factors which admit deformations, whose presence is typically incompatible with property (T).

Deformation/rigidity theory

Popa's deformation/rigidity theory has led to remarkable advances in the classification of group factors and calculation of invariants.

- **Popa (2001)** $\mathcal{F}(\mathrm{L}(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) = \{1\}$.
- **Popa (2004)** If G, H icc property (T), $\mathrm{L}(\mathbb{Z} \wr G) \cong \mathrm{L}(\mathbb{Z} \wr H) \Rightarrow G \cong H$. Here, $A \wr B = (\bigoplus_{b \in B} A) \rtimes B$ is the wreath-product of A and B.
- **I-Peterson-Popa (2005)** $\exists$ II_1 factors M with $\mathrm{Out}(M) = \{e\}$.
- **Popa-Vaes (2006)** icc G with $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$.
- **I-Popa-Vaes (2010)** examples of W^*-superrigid icc groups G.

Remark. These results do not apply to property (T) group factors. This is because deformation/rigidity applies to II_1 factors which admit deformations, whose presence is typically incompatible with property (T).

Chifan-Das-Houdayer-Khan (2020) examples of icc property (T) groups G such that $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B,
in symbols $G \in \mathcal{WR}(A, B)$,

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_bg^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_b g^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.

Example. $A \wr B = (\bigoplus_{b \in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b\in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_b g^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b\in B} A$.

Example. $A \wr B = (\bigoplus_{b\in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\langle H\rangle\rangle = *_{t\in T} tHt^{-1}$, with T a left transversal for the normal subgrp $\langle\langle H\rangle\rangle \lhd G$ generated by H.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_b g^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.

Example. $A \wr B = (\bigoplus_{b \in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\langle H \rangle\rangle = *_{t \in T} tHt^{-1}$, with T a left transversal for the normal subgrp $\langle\langle H \rangle\rangle \lhd G$ generated by H.

Examples (1) $H < H * K$ (Proof. $\langle\langle H \rangle\rangle = *_{k \in K} kHk^{-1}$.)

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_b g^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.

Example. $A \wr B = (\bigoplus_{b \in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\langle H \rangle\rangle = *_{t \in T} tHt^{-1}$, with T a left transversal for the normal subgrp $\langle\langle H \rangle\rangle \lhd G$ generated by H.

Examples (1) $H < H * K$ (Proof. $\langle\langle H \rangle\rangle = *_{k \in K} kHk^{-1}$.)

(2) **(Cohen-Lyndon, 1963)** any maximal cyclic subgrp $H = \mathbb{Z} < G = \mathbb{F}_n$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_b g^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.

Example. $A \wr B = (\bigoplus_{b \in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\langle H \rangle\rangle = *_{t \in T}\, tHt^{-1}$, with T a left transversal for the normal subgrp $\langle\langle H \rangle\rangle \lhd G$ generated by H.

Examples (1) $H < H * K$ (Proof. $\langle\langle H \rangle\rangle = *_{k \in K}\, kHk^{-1}$.)

(2) **(Cohen-Lyndon, 1963)** any maximal cyclic subgrp $H = \mathbb{Z} < G = \mathbb{F}_n$.

Proposition Let $S = \langle [tHt^{-1}, t'Ht'^{-1}] \mid t, t' \in T, t \neq t' \rangle$, for a C-L subgroup $H < G$. Then $S < \langle\langle H \rangle\rangle$, $S \lhd G$ and $G/S \in \mathcal{WR}(H, G/\langle\langle H \rangle\rangle)$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B, in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b\in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_bg^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b\in B} A$.

Example. $A \wr B = (\bigoplus_{b\in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\langle H\rangle\rangle = *_{t\in T}\, tHt^{-1}$, with T a left transversal for the normal subgrp $\langle\langle H\rangle\rangle \lhd G$ generated by H.

Examples (1) $H < H * K$ (Proof. $\langle\langle H\rangle\rangle = *_{k\in K}\, kHk^{-1}$.)

(2) **(Cohen-Lyndon, 1963)** any maximal cyclic subgrp $H = \mathbb{Z} < G = \mathbb{F}_n$.

Proposition Let $S = \langle [tHt^{-1}, t'Ht'^{-1}] \mid t, t' \in T, t \neq t'\rangle$, for a C-L subgroup $H < G$. Then $S < \langle\langle H\rangle\rangle$, $S \lhd G$ and $G/S \in \mathcal{WR}(H, G/\langle\langle H\rangle\rangle)$.

Proof. We have $\langle\langle H\rangle\rangle/S \cong \bigoplus_{t\in T} tHt^{-1} \cong \bigoplus_{G/\langle\langle H\rangle\rangle} H$ and a short exact sequence $\{e\} \to \langle\langle H\rangle\rangle/S \to G/S \to G/\langle\langle H\rangle\rangle \to \{e\}$.

Wreath-like product groups, I

Definition

A group G is a wreath-like product of two groups A and B,
in symbols $G \in \mathcal{WR}(A, B)$, if there is a short exact sequence

$$\{e\} \to \bigoplus_{b \in B} A \to G \xrightarrow{\varepsilon} B \to \{e\}$$

such that $gA_bg^{-1} = A_{\varepsilon(g)b}$, with A_b the b-labelled copy of A in $\bigoplus_{b \in B} A$.
Example. $A \wr B = (\bigoplus_{b \in B} A) \rtimes B \in \mathcal{WR}(A, B)$.

Definition. $H < G$ is a Cohen-Lyndon subgroup if $\langle\!\langle H \rangle\!\rangle = *_{t \in T} tHt^{-1}$,
with T a left transversal for the normal subgrp $\langle\!\langle H \rangle\!\rangle \lhd G$ generated by H.
Examples (1) $H < H * K$ (Proof. $\langle\!\langle H \rangle\!\rangle = *_{k \in K} kHk^{-1}$.)
(2) **(Cohen-Lyndon, 1963)** any maximal cyclic subgrp $H = \mathbb{Z} < G = \mathbb{F}_n$.

Proposition Let $S = \langle [tHt^{-1}, t'Ht'^{-1}] \mid t, t' \in T, t \neq t' \rangle$, for a C-L subgroup $H < G$. Then $S < \langle\!\langle H \rangle\!\rangle$, $S \lhd G$ and $G/S \in \mathcal{WR}(H, G/\langle\!\langle H \rangle\!\rangle)$.
Proof. We have $\langle\!\langle H \rangle\!\rangle / S \cong \bigoplus_{t \in T} tHt^{-1} \cong \bigoplus_{G/\langle\!\langle H \rangle\!\rangle} H$ and
a short exact sequence $\{e\} \to \langle\!\langle H \rangle\!\rangle / S \to G/S \to G/\langle\!\langle H \rangle\!\rangle \to \{e\}$.
Remark. When $G = H * K$, we have $G/\langle\!\langle H \rangle\!\rangle = K$ and $G/S = H \wr K$.

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic.

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic. Moreover, if K has property (T), then so does $G \in \mathcal{WR}(A, B)$.

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic. Moreover, if K has property (T), then so does $G \in \mathcal{WR}(A, B)$.

Remarks:

1. This is surprising because wreath products $A \wr B$ never have (T).

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic. Moreover, if K has property (T), then so does $G \in \mathcal{WR}(A, B)$.

Remarks:

1. This is surprising because wreath products $A \wr B$ never have (T).
2. If A is abelian, $\mathrm{L}(A^{(B)}) \cong \mathrm{L}^\infty(\widehat{A}^B)$ and $\mathrm{L}(G) \cong \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, with $B \curvearrowright^\sigma \widehat{A}^B$ Bernoulli & $c \in \mathrm{Z}^2(\sigma, \mathbb{T})$.

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic. Moreover, if K has property (T), then so does $G \in \mathcal{WR}(A, B)$.

Remarks:

1. This is surprising because wreath products $A \wr B$ never have (T).
2. If A is abelian, $\mathrm{L}(A^{(B)}) \cong \mathrm{L}^\infty(\widehat{A}^B)$ and $\mathrm{L}(G) \cong \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, with $B \curvearrowright^\sigma \widehat{A}^B$ Bernoulli & $c \in \mathrm{Z}^2(\sigma, \mathbb{T})$. It follows $\mathrm{H}^2(\sigma, \mathbb{T}) \neq \mathrm{H}^2(\Gamma, \mathbb{T})$, answering a question of Popa (recovering a result of Jiang, 2015).

Wreath-like product groups, II

Theorem (Dahmani-Guirardel-Osin 2011, Sun 2020) If G is hyperbolic relative to H, then $\exists F \subset H$ finite s.t. $\forall N \lhd H$ with $N \cap F = \emptyset$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} tNt^{-1}$, where T is a left transversal for $H\langle\langle N \rangle\rangle < G$, and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

Using this theorem, we prove:

Theorem A (Chifan-I-Osin-Sun, 2021)

Let K be an icc hyperbolic group. Then for any finitely generated group A, $\exists$ a quotient G of K such that $G \in \mathcal{WR}(A, B)$, for B icc hyperbolic. Moreover, if K has property (T), then so does $G \in \mathcal{WR}(A, B)$.

Remarks:

1. This is surprising because wreath products $A \wr B$ never have (T).
2. If A is abelian, $\mathrm{L}(A^{(B)}) \cong \mathrm{L}^\infty(\widehat{A}^B)$ and $\mathrm{L}(G) \cong \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, with $B \curvearrowright^\sigma \widehat{A}^B$ Bernoulli & $c \in \mathrm{Z}^2(\sigma, \mathbb{T})$. It follows $\mathrm{H}^2(\sigma, \mathbb{T}) \neq \mathrm{H}^2(\Gamma, \mathbb{T})$, answering a question of Popa (recovering a result of Jiang, 2015).
3. The case $A = \mathbb{F}_2$ allows us to prove that every separable II_1 factor embeds into one with property (T). (Chifan-Drimbe-I, 2022).

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups,

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g)u_{\delta(g)}, \forall g \in G$.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Additionally, we can take G with no characters and prescribed Out:

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g)u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Additionally, we can take G with no characters and prescribed Out:

Corollary C (CIOS, 2021)

$\forall$ f.p. group Q, $\exists$ a continuum of icc property (T) groups $\{G_i\}_{i \in I}$ s.t.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Additionally, we can take G with no characters and prescribed Out:

Corollary C (CIOS, 2021)

$\forall$ f.p. group Q, $\exists$ a continuum of icc property (T) groups $\{G_i\}_{i \in I}$ s.t.

1. $\mathrm{L}(G_i) \ncong \mathrm{L}(G_j), \forall i \neq j$.
2. $\mathrm{Out}(\mathrm{L}(G_i)) \cong Q$ and $\mathcal{F}(\mathrm{L}(G_i)) = \{1\}, \forall i \in I$.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Additionally, we can take G with no characters and prescribed Out:

Corollary C (CIOS, 2021)

$\forall$ f.p. group Q, $\exists$ a continuum of icc property (T) groups $\{G_i\}_{i \in I}$ s.t.

1. $\mathrm{L}(G_i) \not\cong \mathrm{L}(G_j), \forall i \neq j$.
2. $\mathrm{Out}(\mathrm{L}(G_i)) \cong Q$ and $\mathcal{F}(\mathrm{L}(G_i)) = \{1\}, \forall i \in I$.

Connes' rigidity conjecture for wreath-like products

Theorem B (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ and $H \in \mathcal{WR}(C, D)$ be property (T) groups, where A, C are nontrivial abelian or icc and B, D are icc hyperbolic.

Let $\theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ be $*$-isomorphism, for any $t > 0$.

Then $t = 1$ and $\exists$ a group isomorphism $\delta : G \to H$ and character $\eta : G \to \mathbb{T}$ such that (up to unitary conjugacy) $\theta(u_g) = \eta(g) u_{\delta(g)}, \forall g \in G$.

In particular, $\mathrm{Out}(\mathrm{L}(G)) \cong \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ and $\mathcal{F}(\mathrm{L}(G)) = \{1\}$.

Additionally, we can take G with no characters and prescribed Out:

Corollary C (CIOS, 2021)

$\forall$ f.p. group Q, $\exists$ a continuum of icc property (T) groups $\{G_i\}_{i \in I}$ s.t.

1. $\mathrm{L}(G_i) \ncong \mathrm{L}(G_j), \forall i \neq j$.
2. $\mathrm{Out}(\mathrm{L}(G_i)) \cong Q$ and $\mathcal{F}(\mathrm{L}(G_i)) = \{1\}, \forall i \in I$.

These are the first calculations of $\mathrm{Out}(\mathrm{L}(G))$, for icc property (T) G.

W^*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W^*-superrigid: if $L(G) \cong L(H)$, for any group H, then $G \cong H$.

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.

I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.
I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic.

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.

I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic. Then G is W*-superrigid.

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.
I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic. Then G is W*-superrigid.

Corollary E (CIOS, 2021)

Let G be an icc hyperbolic property (T) group and $g \in G$ be an element of infinite order.

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $L(G) \cong L(H)$, for any group H, then $G \cong H$.
I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic. Then G is W*-superrigid.

Corollary E (CIOS, 2021)

Let G be an icc hyperbolic property (T) group and $g \in G$ be an element of infinite order. Then $\exists d \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$ divisible by d, the quotient group $G/[\langle\langle g^k \rangle\rangle, \langle\langle g^k \rangle\rangle]$ is a W*-superrigid icc group with property (T).

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.

I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic. Then G is W*-superrigid.

Corollary E (CIOS, 2021)

Let G be an icc hyperbolic property (T) group and $g \in G$ be an element of infinite order. Then $\exists d \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$ divisible by d, the quotient group $G/[\langle\langle g^k \rangle\rangle, \langle\langle g^k \rangle\rangle]$ is a W*-superrigid icc group with property (T).

These are the first examples of W*-superrigid groups with property (T).

W*-superrigid groups with property (T)

Connes' rigidity conjecture (1980) Any icc property (T) group G is W*-superrigid: if $\mathrm{L}(G) \cong \mathrm{L}(H)$, for any group H, then $G \cong H$.
I-Popa-Vaes 2010, Berbec-Vaes 2012, Chifan-I 2017, and **Chifan-DiazArias-Drimbe 2021**: W*-superrigid grps *without* prop. (T) (generalized wreath products and amalgamated free products).

Theorem D (Chifan-I-Osin-Sun, 2021)

Let $G \in \mathcal{WR}(A, B)$ be a property (T) group, where A is nontrivial abelian and B is icc hyperbolic. Then G is W*-superrigid.

Corollary E (CIOS, 2021)

Let G be an icc hyperbolic property (T) group and $g \in G$ be an element of infinite order. Then $\exists d \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$ divisible by d, the quotient group $G/[\langle\langle g^k \rangle\rangle, \langle\langle g^k \rangle\rangle]$ is a W*-superrigid icc group with property (T).

These are the first examples of W*-superrigid groups with property (T).
CIOS, 2022: uncountably many W*-superrigid property (T) groups.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$. Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$. Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

- As B is hyperbolic, $\mathrm{L}(A^{(B)})$ is the unique Cartan subalgebra of $\mathrm{L}(G)$ (**Popa-Vaes, 2012**). We may assume that $\theta(\mathrm{L}(A^{(B)}) = \mathrm{L}(A^{(B)})$. Therefore, θ induces an automorphism of $\mathcal{R}(B \curvearrowright \widehat{A}^B)$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

- As B is hyperbolic, $\mathrm{L}(A^{(B)})$ is the unique Cartan subalgebra of $\mathrm{L}(G)$ (**Popa-Vaes, 2012**). We may assume that $\theta(\mathrm{L}(A^{(B)})) = \mathrm{L}(A^{(B)})$. Therefore, θ induces an automorphism of $\mathcal{R}(B \curvearrowright \widehat{A}^B)$.
- Since B is icc property (T), $B \curvearrowright \widehat{A}^B$ is OE superrigid (**Popa, 2005**). Since $\mathrm{Out}(B) = \{e\}$, we get that $\theta(u_g) \in \mathrm{L}(A^{(B)})u_g, \forall g \in G$.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

- As B is hyperbolic, $\mathrm{L}(A^{(B)})$ is the unique Cartan subalgebra of $\mathrm{L}(G)$ (**Popa-Vaes, 2012**). We may assume that $\theta(\mathrm{L}(A^{(B)}) = \mathrm{L}(A^{(B)})$. Therefore, θ induces an automorphism of $\mathcal{R}(B \curvearrowright \widehat{A}^B)$.
- Since B is icc property (T), $B \curvearrowright \widehat{A}^B$ is OE superrigid (**Popa, 2005**). Since $\mathrm{Out}(B) = \{e\}$, we get that $\theta(u_g) \in \mathrm{L}(A^{(B)})u_g, \forall g \in G$.
- Let $\theta(u_g) = w_g u_g$. Then $(w_g)_{g \in G} \subset \mathcal{U}(\mathrm{L}(A^{(B)})$ is a 1-cocycle for α.

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

- As B is hyperbolic, $\mathrm{L}(A^{(B)})$ is the unique Cartan subalgebra of $\mathrm{L}(G)$ (**Popa-Vaes, 2012**). We may assume that $\theta(\mathrm{L}(A^{(B)}) = \mathrm{L}(A^{(B)})$. Therefore, θ induces an automorphism of $\mathcal{R}(B \curvearrowright \widehat{A}^B)$.
- Since B is icc property (T), $B \curvearrowright \widehat{A}^B$ is OE superrigid (**Popa, 2005**). Since $\mathrm{Out}(B) = \{e\}$, we get that $\theta(u_g) \in \mathrm{L}(A^{(B)})u_g, \forall g \in G$.
- Let $\theta(u_g) = w_g u_g$. Then $(w_g)_{g\in G} \subset \mathcal{U}(\mathrm{L}(A^{(B)})$ is a 1-cocycle for α. Since G has property (T), α is cocycle superrigid (**Popa, 2005**).

Property (T) group factors with trivial Out

Let $G \in \mathcal{WR}(A, B)$ with property (T) and $\mathrm{Char}(G) = \{1\}$, where $A \neq \{1\}$ is abelian, B is icc hyperbolic and $\mathrm{Out}(B) = \{e\}$.Then $\mathrm{Out}(\mathrm{L}(G)) = \{e\}$.

Sketch of proof. Let θ be an automorphism of $\mathrm{L}(G)$.
Identifying $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ with $\mathrm{L}^\infty(\widehat{A}^B) \subset \mathrm{L}^\infty(\widehat{A}^B) \rtimes_{\sigma,c} B$, we have
(i) $\mathrm{L}(A^{(B)}) \subset \mathrm{L}(G)$ is a Cartan subalgebra and its equivalence relation is the orbit equivalence relation of the Bernoulli action $B \curvearrowright^\sigma \widehat{A}^B$.
(ii) The conjugation action $G \curvearrowright^\alpha \mathrm{L}(A^{(B)}) = \mathrm{L}^\infty(\widehat{A}^B)$ given by $\alpha_g = \mathrm{Ad}(u_g)$ is a generalized Bernoulli action.

- As B is hyperbolic, $\mathrm{L}(A^{(B)})$ is the unique Cartan subalgebra of $\mathrm{L}(G)$ (**Popa-Vaes, 2012**). We may assume that $\theta(\mathrm{L}(A^{(B)})) = \mathrm{L}(A^{(B)})$. Therefore, θ induces an automorphism of $\mathcal{R}(B \curvearrowright \widehat{A}^B)$.
- Since B is icc property (T), $B \curvearrowright \widehat{A}^B$ is OE superrigid (**Popa, 2005**). Since $\mathrm{Out}(B) = \{e\}$, we get that $\theta(u_g) \in \mathrm{L}(A^{(B)})u_g, \forall g \in G$.
- Let $\theta(u_g) = w_g u_g$. Then $(w_g)_{g \in G} \subset \mathcal{U}(\mathrm{L}(A^{(B)}))$ is a 1-cocycle for α. Since G has property (T), α is cocycle superrigid (**Popa, 2005**). As $\mathrm{Char}(G) = \{1\}$, it follows that $u_g = u\sigma_g(u)^*, \forall g \in G$. Hence $\theta = \mathrm{Ad}(u)$ is an inner automorphism.