Hecke C*-algebras of right-angled Coxeter groups

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Main notions - an overview

- *Graph product* $*_{v\in V}(A_v, \varphi_v)$ of a family of unital C^*-algebras A_v endowed with GNS faithful states, with V set of vertices in a simplicial graph. Interpolates between free and direct products.
- *Coxeter system* (W, S), a group W with generating set S and relations including $s^2 = e$ for every s in S.
- *Multi-parameter* $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with q_s in $\mathbb{R}_{>0}$ having value independent of conjugacy class in W.
- *Hecke C^*-algebra* $C^*_{r,q}(W)$ of (W, S) and q. A kind of deformation of $C^*_r(W)$ with a canonical tracial state τ_q.
- Special class of (W, S), the *right-angled* ones, with relations given by commutation among tuples in subsets of S.

More formal definitions to be presented afterwards.

A Khintchine type inequality for graph products

Theorem (M. Caspers, M. Klisse and NSL (2021))

Given a graph product $*_{v,\Gamma}(A_v, \varphi_v)$ over a finite simplicial graph Γ, let $\chi_n : A \to A$ denote the projection onto the space of operators of length n in A, with $n \geq 1$. Then for every $n \geq 1$ there are an operator space X_n and maps

$$j_n : \chi_n(A) \to X_n,\ \pi_n : \mathrm{dom}(\pi_n) \subset X_n \to \chi_n(A),$$

so that $\pi_n \circ j_n$ is the identity on $\chi_n(A)$ and π_n is completely bounded with

$$\|\pi_n\|_{cb} \leq Cn,$$

where C is a constant depending on Γ.

Here length is in relation to "words" $v_1 v_2 \cdots v_n$ formed by concatenation, where $v_i \in V\Gamma$. Furthermore, the space X_n is a Haagerup tensor product of column and row spaces.

A Haagerup type inequality for R-A (W, S)

Theorem (M. Caspers, M. Klisse and NSL (2021))

Suppose that (W, S) is a right-angled Coxeter system with finite S. Let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter, let $C^*_{r,q}(W)$ be the associated reduced Hecke C^*-algebra acting on $\ell^2(W)$ with o.n.b. $(\delta_w)_{w \in W}$. Then for every $n \geq 1$ and $x \in \chi_n(C^*_{r,q}(W))$, we have that

$$\|\chi_n(x)\| \leq C' n \|x\delta_e\|_2,$$

where the constant C' depends on the multi-parameter q and on the graph Γ with vertex set S and edges prescribed by the group relations.

Here length is in relation to expressions of the form $w = s_1 s_2 \cdots s_n$ for $w \in W$, $s_i \in S$.

Simplicity of reduced Hecke C^*-algebras

Theorem (M. Caspers, M. Klisse and NSL (2021))

Suppose that (W, S) is an irreducible right-angled Coxeter system with $3 \leq |S| < \infty$. Let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter and let $C^*_{r,q}(W)$ be the associated reduced Hecke C^*-algebra acting on $\ell^2(W)$.
Then there exists an open neighbourhood U of $1 \in \mathbb{R}_{>0}^{(W,S)}$ such that for all $q \in U$ we have that $C^*_{r,q}(W)$ is simple. Furthermore, $C^*_{r,q}(W)$ has unique tracial state.

The single-parameter case $q_s = 1$ for all s in S leads to $C^*_r(W)$, and the conclusion is known, *P. de la Harpe* (2007).

For the spherical and affine type (W, S), there is always a character on $C^*_{r,q}(W)$.

Motivational earlier work

Theorem (L. Garncarek (2016))

Let (W, S) be an irreducible right-angled Coxeter system with $|S| \geq 3$ and let $q_s = q$ in $\mathbb{R}_{>0}$ for all s. Then the single parameter Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is a factor if and only if $q \in [\rho, \rho^{-1}]$ where ρ is the radius of convergence of the growth series $\sum_{w \in W} z^{|w|}$ of W.

Motivational earlier work

For a Coxeter group W with generating set S, let W_s be the subgroup determined by $s^2 = e$, so $W_s \cong \mathbb{Z}_2$.

Theorem (M. Caspers (2020))

Given a right-angled Coxeter system (W, S) with graph Γ, i.e. $V\Gamma = S$ and edges encode possible commutation of pairs of generators, let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter, let $C^*_{r,q_s}(W_s)$ be the reduced Hecke C^*-algebra with its canonical faithful trace τ_{q_s} for all $s \in S$. Then

$$\big(C^*_{r,q}(W), \tau_q\big) \cong *_{s,\Gamma}\big(C^*_{r,q_s}(W_s), \tau_{q_s}\big).$$

More precisely, if $w = s_1 \cdots s_n$ is reduced, $w \in W$, the isomorphism is implemented on generators $T_w^{(q)}$ and $T_s^{(q_s)}$

$$T_w^{(q)} \mapsto T_{s_1}^{(q_{s_1})} \ldots T_{s_n}^{(q_{s_n})}.$$

Ensuing refinement and strengthening (I)

Theorem (S. Raum and A. Skalski (2021))

Let (W, S) be an irreducible right-angled Coxeter system with $3 \leq |S| < \infty$ and $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$ a multi-parameter with $q_s \leq 1$ for all $s \in S$. Then the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ satisfies

$$\mathcal{N}_q(W) \cong \mathcal{M} \oplus \bigoplus_P \mathbb{C},$$

where $\mathcal{M}$ is a factor and the summands are indexed over one-dim. central projections.

This extends Garncarek's and Caspers-Klisse-L factoriality to all multi-parameters.

Ensuing refinement and strengthening (II)

A complete characterisation of simplicity in the right-angled case:

Theorem (M. Klisse (2021))

Given an irreducible right-angled Coxeter system (W, S) and any multi-parameter q, the Hecke C^*-algebra $C^*_{r,q}(W)$ is simple precisely when q lies outside the closure of a set $R(W, S)$ of parameters associated to the region of convergence of the multivariate growth series $\sum_{w \in W} z_w$ of W.

Note: The methods are inspired by *M. Kalantar and M. Kennedy's* boundary theory and differ from [CKL].

Preliminaries on graphs

Simplicial graph Γ:

- vertex set $V\Gamma$,
- edge set $E\Gamma$,
- undirected, no double edges, no loops at a vertex.

A *word* $\mathbf{v}$ on Γ is a concatenation

$$\mathbf{v} = v_1 v_2 \cdots v_n, v_i \in V\Gamma,$$

with $v \in V\Gamma$ regarded as *letters*.

Simplicial graphs: words

Given $\Gamma = (V\Gamma, E\Gamma)$ simplicial graph, two words $\mathbf{w}$ and $\mathbf{w}'$ are *equivalent* if they are in the same equivalence class w.r.t

- *shuffle of letters*

  $$v_1 \cdots v_{i-1} v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n$$

  whenever $(v_i, v_{i+1}) \in E\Gamma$,
- *absorbtion of letters*

  $$v_1 \cdots v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_i v_{i+2} \cdots v_n \text{ if } v_i = v_{i+1}.$$

Say $\mathbf{v} = v_1 \cdots v_n$ is *reduced* if $v_i = v_j$ for $i < j$ implies

$$\exists i < k < j : (v_i, v_k), (v_j, v_k) \notin E\Gamma.$$

The *length* of a reduced word is the smallest number of letters in a representative, up to equivalence.

Graph products of groups

E. Green (1990). Setup consists of

- Simplicial graph: Γ, with vertex set $V\Gamma$ and edge set $E\Gamma$, undirected, no double edges, no loops at a vertex.
- Group G_v for every $v \in V\Gamma$.

The *graph product* of $\{G_v\}_{v\in V\Gamma}$ is a group $*_{v\in V\Gamma} G_v$ built from the free product of the G_v's by adding new relations

$$g_v g_w = g_w g_v, \quad \text{if } v \text{ and } w \text{ share an edge.}$$

Graph products of C^*-algebras

M. Caspers and P. Fima (2017). Setup consists of

- Simplicial graph $\Gamma = (V\Gamma, E\Gamma)$.
- Unital C^*-algebra $\{A_v, \varphi_v\}$ endowed with a GNS-faithful state φ_v for each $v \in V\Gamma$.

The *graph product* $(A, \varphi) = *_{v,\Gamma}(A_v, \varphi_v)$ is given by a C^*-algebra A, unique up to isomorphism, with a GNS faithful state φ s.t.

1. There are unital inclusions $A_v \subseteq A$ for all v so that $\bigcup_v A_v$ generates A and elements of A_v commute with those of A_w whenever v, w share an edge;
2. $\varphi|_{A_v} = \varphi_v$, for all $v \in V\Gamma$ and
3. if $\mathbf{v} = v_1 \cdots v_n$ is reduced and $\varphi_{v_j}(a_j) =$ for $j = 1, \ldots, n$, then $\varphi(a_{v_1} \cdots a_{v_n}) = 0$.

Generalises free products of C^*-algebras, e.g. in the foundation of free probability of *D.-V. Voiculescu*.

Graph products - alternative picture

Idea: use Hilbert space representations accounting for shuffle equivalence, namely

$$v_1 \cdots v_{i-1} v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n$$

whenever $(v_i, v_{i+1}) \in E\Gamma$.
Useful observation: for reduced words, equivalence is determined by shuffle equivalence.

Graph products - alternative picture

Given Γ finite simplicial graph and $(A_v, \varphi)_{v\in V\Gamma}$ unital C^*-algebras with corresponding faithful GNS states, let $\Omega_v \in L^2(A_v, \varphi_v)$ (or $L^2(A_v)$) be the unit of A_v, let

$$A_v^\circ := \{a \in A_v \mid \varphi_v(a) = 0\},$$

let $L^2(A_v^\circ, \varphi_v)$ (or $L^2(A_v^\circ)$) be the closure of A_v° in the GNS-space $L^2(A_v)$, for each $v \in V\Gamma$.

For $\mathbf{v} = v_1 \cdots v_n$ reduced, set

$$\mathcal{H}_{\mathbf{v}} = L^2(A_{v_1}^\circ) \otimes \cdots \otimes L^2(A_{v_n}^\circ).$$

It follows that $\mathcal{H}_{\mathbf{v}} \simeq \mathcal{H}_{\mathbf{w}}$ whenever reduced words $\mathbf{v}, \mathbf{w}$ are (shuffle) equivalent. Then the graph product is (up to isom.) determined by representing $x \in A_v$ on $\mathcal{H} = \oplus_{[\mathbf{v}]}\mathcal{H}_{\mathbf{v}}$.

Graph products: alternative picture

Let $\mathbf{w} = w_1 \cdots w_d$ be reduced of length d. If $v \in V\Gamma$ is not an initial letter in $\mathbf{w}$, the action of $x \in A_v$ on $\xi_1 \otimes \cdots \otimes \xi_d \in \mathcal{H}_{\mathbf{w}}$ is (faithfully) represented by

$$x^\circ \Omega_v \otimes \xi_1 \otimes \cdots \otimes \xi_d + \varphi_v(x) \xi_1 \otimes \cdots \otimes \xi_d,$$

with $x^\circ = x - \varphi_v(x)$. If v is an initial letter of $\mathbf{w}$, may assume $w_1 = v$ and set $x \cdot (\xi_1 \otimes \cdots \otimes \xi_d)$ to be

$$(x\xi_1 - \langle x\xi_1, \Omega_v \rangle \Omega_v) \otimes \xi_2 \otimes \cdots \otimes \xi_d + \langle x\xi_1, \Omega_v \rangle \xi_2 \otimes \cdots \otimes \xi_d.$$

The *reduced graph product* $(A, \varphi) := *_{v,\Gamma}(A_v, \varphi_v)$ is the C^*-algebra generated by $A_v, v \in V\Gamma$ acting on $\mathcal{H}$, with (faithful) state $\varphi(x) := \langle x\Omega, \Omega \rangle$. Here Ω is a vacuum state so that $\mathcal{H}_\emptyset := \mathbb{C}\Omega$. Next, define operators of finite length.

Operators of fixed finite length

Let $(A, \varphi) := *_{v,\Gamma}(A_v, \varphi_v)$ be a graph product C^*-algebra represented faithfully on $\mathcal{H} = \oplus_{[\mathbf{v}]}\mathcal{H}_{\mathbf{v}}$. An operator of length n in A is given by $a_1 \cdots a_n$ with $a_i \in A^\circ_{v_i}$ and $\mathbf{v} = v_1 \cdots v_n$ reduced.

For $v \in V\Gamma$, let P_v be the orthogonal projection of $\mathcal{H}$ onto $\oplus_{[\mathbf{v}]\in I_v}\mathcal{H}_{\mathbf{v}}$, where I_v are representatives of all words that start with v up to shuffle equivalence. For $n \in \mathbb{N}_{\geq 0}$, let

$$\chi_n : A \to A,\ a_1 \cdots a_r \mapsto \delta_{r,n} a_1 \cdots a_r, \tag{1}$$

where $a_1 \cdots a_r$ is reduced. Thus, χ_n is the word length projection of length n.

About the proof of the Khintchine inequality

- Identify reduced words admitting a permutation
$$(v_{\sigma(1)}\cdots v_{\sigma(k)})(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})(v_{\sigma(k+l+1)}\cdots v_{\sigma(n)})$$
with first bracket ending in a clique $V\Gamma_1$, then a clique $V\Gamma_0$ of length l in the middle, and lastly starting at a clique $V\Gamma_2$.
- Mirror this for an operator $a_1\cdots a_n$ in reduced form as a direct sum of creation-diagonal-annihilation. Set $C_i = P_{v_{\sigma(i)}} a_{\sigma(i)} P^{\perp}_{v_{\sigma(i)}}$, let $D_i = P_{v_{\sigma(i)}} a_{\sigma(i)} P_{v_{\sigma(i)}}$ and $A_i = P^{\perp}_{v_{\sigma(i)}} a_{\sigma(i)} P_{v_{\sigma(i)}}$.
- So, identify $a_1\cdots a_n$ in reduced form for which the corresp. $v_1\cdots v_n$ decomposes with
$$\bigoplus_{l=0}^{n}\bigoplus_{k=0}^{n-l}\bigoplus_{\Gamma_0}\bigoplus_{(\Gamma_1,\Gamma_2)}(C_1\cdots C_k)(D_{k+1}\cdots D_{k+l})(A_{k+l+1}\cdots A_n)$$

This generalises Ricard-Xu's method to the graph product.

About the proof of the Khintchine inequality

- Use the free product $*_v(A_v, \varphi_v)$ to define X_n as

$$\bigoplus_{l=0}^{n} \bigoplus_{k=0}^{n-l} \bigoplus_{\Gamma_0 \in Cliq(\Gamma, l)} \bigoplus_{(\Gamma_1, \Gamma_2) \in \mathrm{Comm}(\Gamma_0)} L_k \otimes_h A_{\Gamma_0} \otimes_h K_{n-k-l}.$$

  where L_* is a column space, K_* a row space, and A_{Γ_0} diagonal.
- Construct π_n by interpolating between $*_{v,\Gamma} A_v$ and $*_v A_v$.

Coxeter groups

A *Coxeter system* (W, S) consists of a group W freely generated by a set S with relations

$$(st)^{m_{s,t}} = e$$

for exponents $m_{s,t} \in \{1, 2, \ldots, \infty\}$ so that:

- $m_{s,s} = 1$ for all $s \in S$;
- $m_{s,t} \geq 2$ for all $s \neq t$, and
- $m_{s,t} = m_{t,s}$ for all $s, t \in S$.

If $m_{s,t} = \infty$, no relation is imposed on s and t. The system (W, S) is

1. *right-angled* if $m_{s,t} \in \{2, \infty\}$ for all distinct $s, t \in S$;
2. *irreducible* if W is not isomorphic to a direct product of special subgroups W_T for $T \subset S$.

Words and multi-parameter expressions

Fix a Coxeter system (W, S). For $\mathbf{w}$ in W, an expression $\mathbf{w} = s_1 s_2 \cdots s_n$ with $s_i \in S$ is *reduced* if $n \leq m$ for any expression of $\mathbf{w}$ as a product of m generators. Set $|\mathbf{w}| := n$ as the word-length of $\mathbf{w}$ and write $\mathbf{w} =_r s_1 s_2 \cdots s_n$.
A concatenation $s\mathbf{w}$ can lead to $|s\mathbf{w}| = |\mathbf{w}| - 1$, if s is *exchanged* for some s_i appearing in $\mathbf{w}$ (in general, exchange, deletion and folding are equivalent "operations" on expressions for elements in W). Else $|s\mathbf{w}| = |\mathbf{w}| + 1$.
Write $s \sim t$ if s, t are conjugate in W. Employ the notation:

$$q := (q_s)_{s \in S} \in A^{(W,S)} \text{ if } q_s \in A \text{ and } q_s = q_t \text{ for } s \sim t.$$

Relevant choices for A are $\mathbb{R}$, $\mathbb{C}$ or $\{-1, 1\}$. Given $\mathbf{w} =_r s_1 s_2 \cdots s_n$, put

$$q_{\mathbf{w}} := q_{s_1} \cdots q_{s_n}.$$

Aim: associate a $*$-algebra $\mathbb{C}_q[W]$ and C^*-completions.

Operator algebras associated with (W, S)

A (possibly incomplete) list:

- Hecke-von Neumann algebras and L^2-cohomology, *J. Dymara* (2006), *M. Davis, J. Dymara, T. Januskiewicz, B. Okun* (2008)
- Hecke-von Neumann algebras for a single parameter ($q = q_s$ for $s \in S$) and factoriality in the right-angled, irreducible case, *L. Garncarek*.
- Recovers in special cases results of *K. Dykema* (1993) on factoriality of interpolated free products.
- The Hecke C^*-algebra completion of $\mathbb{C}_q[W]$ inside $B(\ell^2(W))$ and absence of Cartan subalgebras in the right-angled, irreducible case, *M. Caspers* (2020).

The Hecke $*$-algebra of (W, S) with a parameter

Definition (Cf. e.g. Davis, Garncarek)

Given $q := (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}$, *let*

$$p_s(q) := q_s^{-\frac{1}{2}}\left(q_s - 1\right), s \in S,$$

and define $\mathbb{C}_q[W]$ *as the* $*$*-algebra spanned by* $\{T_{\mathbf{w}} \mid \mathbf{w} \in W\}$ *with relations*

$$T_s^{(q)} T_{\mathbf{w}}^{(q)} = \begin{cases} T_{s\mathbf{w}}^{(q)} & \text{if } |s\mathbf{w}| > |\mathbf{w}|, \\ T_{s\mathbf{w}}^{(q)} + p_s(q) T_{\mathbf{w}}^{(q)} & \text{if } |s\mathbf{w}| < |\mathbf{w}|, \end{cases}$$

and adjoint

$$\left(T_{\mathbf{w}}^{(q)}\right)^* = T_{\mathbf{w}^{-1}}^{(q)}.$$

Representation on Hilbert space

Represent $\mathbb{C}_q[W]$ on $\ell^2(W)$ (with canonical o.n.b. $(\delta_{\mathbf{w}})_{\mathbf{w}\in W}$) by setting, for $s \in S$,

$$T_s^{(q)}\delta_{\mathbf{w}} := \begin{cases} \delta_{s\mathbf{w}} & \text{if } |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(q)\delta_{\mathbf{w}} & \text{if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

This action extends to a faithful $*$-representation $\mathbb{C}_q[W] \to B(\ell^2(W))$.

Definition

The reduced Hecke C^*-algebra $C^*_{r,q}(W)$ associated with (W, S) and $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}$ is the norm closure of $\mathbb{C}_q[W]$ in $B\left(\ell^2(W)\right)$.

Hecke C^*-algebras as graph products

For a Coxeter system (W, S), define a graph Γ by $V\Gamma := S$ and declaring (s, t) an edge, for $s, t \in S$, when $m_{s,t} = 2$. Let W_s denote the special subgroup $W_{\{s\}}$ with relation $s^2 = 1$. Then

$$W \cong *_{s,\Gamma} W_s = *_{s,\Gamma} \mathbb{Z}_2.$$

Recall (M. Caspers) that

$$\big(C^*_{r,q}(W), \tau_q\big) \cong *_{s,\Gamma} \big(C^*_{r,q_s}(W_s), \tau_{q_s}\big).$$

About the proof of the Haagerup inequality

In the right-angled case for (W, S), we have that A_{Γ_0} is one-dimensional and

$$X_n = \bigoplus_{l=0}^{n} \bigoplus_{k=0}^{n-l} \bigoplus_{\Gamma_0 \in Cliq(\Gamma, l)} \bigoplus_{(\Gamma_1, \Gamma_2) \in \mathrm{Comm}(\Gamma_0)} M_{k, n-k-l}(\mathbb{C}).$$

Apply the Khintchine inequality to $*_{s,\Gamma}\big(C^*_{r,q_s}(W_s), \tau_{q_s}\big)$, and show that j_n extends to a bounded map

$$L^2(\chi_n(C^*_{r,q}(W)), \tau_q) \to L^2(X_n, \mathrm{Tr})$$

and estimate using $\mathrm{Tr} = \mathrm{Tr}_{n-k-l}$.

Simplicity and unique trace property

Employ the Haagerup inequality in the context of $C^*_{r,q}(W)$, in connection with an adaptation of Powers's technique by using a deformed averaging operator depending on q, which builds upon Ching's variant of Pukanszky's 14ε-inequality.

THANK YOU!