From local to global lifting ?

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Let me start by setting the stage
by defining the **lifting problems**
we wish to discuss:

Let C a C^*-algebra. Let $C/\mathcal{I}$ be a quotient C^*-algebra.
Let A be another C^*-algebra.

LIFTING

Global Lifting Problem :

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "\widehat{u}"] \arrow[r, "u"'] & C/\mathcal{I} \end{tikzcd}$$

Local Lifting Problem :

$$\begin{tikzcd} & & C \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "u^E"] \arrow[rr, bend right, "u_{|E}"'] & A \arrow[r, "u"'] & C/\mathcal{I} \end{tikzcd}$$

Discussion: contractions, positive contractions, global case open

vNa: $C^{**} = \mathcal{I}^{**} \oplus (C/\mathcal{I})^{**}$

Let A be a unital C^*-algebra.

Definition

A (*separable*) has the lifting property (LP in short) if $\forall C/\mathcal{I}$, $\forall u : A \to C/\mathcal{I}$ u.c.p. $\exists \widehat{u} : A \to C$ u.c.p. lifting u

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[ur, "\widehat{u}"] \arrow[r, "u"] & C/\mathcal{I} \end{tikzcd}$$

(There is also a non-separable variant)

Definition

A has the local lifting property (LLP in short) if $\forall C/\mathcal{I}$ $\forall u : A \to C/\mathcal{I}$ u.c.p. u is **locally liftable** i.e. $\forall E \subset A$ f.d. op. sys. $u_{|E} : E \to C/\mathcal{I}$ admits a u.c.p. lifting $u^E : E \to C$.

$$\begin{tikzcd} & & C \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "u^E"] \arrow[rr, bend right, "u_{|E}"'] & A \arrow[r, "u"] & C/\mathcal{I} \end{tikzcd}$$

In the general case,
we say A has LP (resp. LLP) if its unitization does

CLASSICAL FACT: Any separable unital A can be written as

$$A = C^*(\mathbb{F}_\infty)/\mathcal{I} \text{ for some ideal } \mathcal{I}$$

Therefore it suffices to consider the lifting problem for

$$C = C^*(\mathbb{F}_\infty) \text{ and } u = Id : A \to C/\mathcal{I}$$

Definition (Equivalent definition)

A has the **LP** if any unital $*$-homomorphism $u : A \to C/\mathcal{I}$ is **liftable**, i.e. admits a u.c.p. lifting $\widehat{u} : A \to C$.

Definition (Equivalent definition)

A has the **LLP** if any unital $*$-homomorphism $u : A \to C/\mathcal{I}$ is **locally liftable**, i.e. for any $E \subset A$ f.d. operator system the restriction $u_{|E} : E \to C/\mathcal{I}$ admits a u.c.p. lifting $u^E : E \to C$.

Examples of C^*-algebras with LP

• Nuclear C^*-algebras (Choi-Effros 1977)
(Typically: $C^*(G)$ for G amenable countable discrete group)

• $C^*(\mathbb{F}_N)$ where $\mathbb{F}_N$ is a free group $(2 \leq N \leq \infty)$
(Kirchberg, 1994)

Both have the LP

Remark (digression): If $C_\lambda(G)$ is QWEP (no counterexample known) then

$$C_\lambda(G) \text{ LLP } \Leftrightarrow G \text{ amenable}$$

In particular $C_\lambda(\mathbb{F}_N)$ fails LLP for $N \geq 2$

Groups with the LP

Let us say that a discrete group G has LP if $C^*(G)$ does.
Note:

$$\{amenable\} \cup \{\text{free groups}\} \subset \{LP\}$$

$\rightarrow$ not so easy to find counterexamples !

Open Problem Does $\mathbb{F}_2 \times \mathbb{F}_2$ (or a product of free groups) have the LP or the LLP ?

Counter-examples to LP: Property (T)

Among $C^*(G)$ for G discrete group (reduced case easier)
All counterexamples are **Kazhdan property (T) groups**
Ozawa (PAMS 2004): $\exists G$ with $C^*(G)$ failing LP
Thom (2010) produced an explicit example with $C^*(G)$ failing LLP
Ioana, Spaas and Wiersma (2020) showed

Theorem (ISW 2020)

For $G = SL(n, \mathbb{Z})$ for $n \geq 3$ $C^*(G)$ fails LLP.

Still open for general (T) groups
They also showed:

Theorem (ISW 2020)

If G has (T) and is NOT finitely presented
then $C^*(G)$ fails LP.
Moreover: There are uncountably many such groups

From Local to Global ?

Open Problem (Kirchberg 1993) :

$$\text{LLP} \Rightarrow \text{LP ?}$$

(in the separable case)

Partial Motivation :
If the Connes embedding problem
has a positive [1]
solution
then (Kirchberg)
the LLP implies the LP

[1] **A recent paper entitled MIP* = RE posted on arxiv in Jan. 2020 by Ji, Natarajan, Vidick, Wright, and Yuen contains a negative solution**

Two opposite lines of attack ($\cdots$ stalled !)

I recently constructed an unusual example of LLP C^*-algebra namely one with the WEAK EXPECTATION PROPERTY (WEP) (and not exact)

Project 1: Produce a similar example **failing** LP

$\cdots$ probably quite difficult since it implies a negative solution to the Connes Embedding Problem

Project 2: Produce a similar example **satisfying** LP

$* * *$

Plan of talk:

- A new characterization of the LP involving tensor products
- A reduction of Project 2 to the validity of a simple inequality

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Tensor products of C^*-algebras (background)

Originating in Japan in the late 1950's
Turumaru, then
Takesaki (1958), Guichardet (1965),
Lance (1973)
Choi-Effros (+Connes), Kirchberg (1976-7) Wasserman (1976)
Effros-Lance (1977)
Archbold-Batty (1980) Effros-Haagerup (1985)
Kirchberg (1993) $\cdots$

Minimal and maximal tensor products

$$A \otimes_{\min} B \text{ and } A \otimes_{\max} B$$

$$A \otimes_{\min} B = (A \otimes_{alg} \widehat{B, \|}\ \|_{\min}), \quad A \otimes_{\max} B = (A \otimes_{alg} \widehat{B, \|}\ \|_{\max})$$

When $A \subset B(H)$ and $B \subset B(K)$ then $\forall t \in A \otimes_{alg} B$

$$\|t\|_{\min} = \|t\|_{B(H \otimes_2 K)} \text{ “spatial norm”}$$

$$\|t\|_{\max} = \sup\{\|\pi \cdot \sigma(t)\|_{B(\mathcal{H})} \mid \pi, \sigma \text{ with commuting ranges}\}$$

where sup runs over all $\mathcal{H}$’s and all pairs (π, σ) of $*$-homomorphisms

$$\begin{tikzcd} & B(\mathcal{H}) & \\ A \arrow[ur, "\pi"] & & B \arrow[ul, "\sigma"'] \end{tikzcd}$$

In case A or B (or both) is a von Neumann algebra

$$\begin{tikzcd} & B(\mathcal{H}) & \\ A \arrow[ur, "\pi"] & & B \arrow[ul, "\sigma"'] \end{tikzcd}$$

Effros-Lance 1977:

If A is a vNa : $(\forall t \in A \otimes_{alg} B)$

$$\|t\|_{\mathrm{nor}} = \sup\{\|\pi\cdot\sigma(t)\|_{B(\mathcal{H})} \mid \pi \text{ normal }, \sigma \text{ with commuting ranges}\}$$

If A and B are both vNa :

$$\|t\|_{\mathrm{bin}} = \sup\{\|\pi\cdot\sigma(t)\|_{B(\mathcal{H})} \mid \pi, \sigma \text{ both normal with commuting ranges}\}$$

$$(A^{**} \otimes_{\mathrm{bin}} B^{**}) \subset (A \otimes_{\max} B)^{**} \text{ isometrically}$$

Tensor products of C^*-algebras (basic facts)

Let A, C be C^*-algebras
for any C^*-norm $\| \ \|$ on $A \otimes_{alg} C$

$$\| \ \|_{\min} \leq \| \ \| \leq \| \ \|_{\max}$$

Def: A is nuclear if $A \otimes_{\min} C = A \otimes_{\max} C \quad \forall C$

$$A \otimes_{\max} [C/\mathcal{I}] = [A \otimes_{\max} C]/[A \otimes_{\max} \mathcal{I}] \qquad \text{“projectivity”}$$

$$\forall B \subset C \qquad A \otimes_{\min} B \subset A \otimes_{\min} C \qquad \text{“injectivity”}$$

Tensor products of C^*-algebras (Warning !)

Let A, C be C^*-algebras

$$A \otimes_{\min} [C/\mathcal{I}] \neq [A \otimes_{\min} C]/[A \otimes_{\min} \mathcal{I}]$$

$$\forall B \subset C \qquad A \otimes_{\max} B \not\subset A \otimes_{\max} C$$

Kirchberg's characterization of LLP

Theorem

Let A be a C^*-algebra

$$A \text{ has LLP } \Leftrightarrow A \otimes_{\min} B(\ell_2) = A \otimes_{\max} B(\ell_2)$$

$$A \text{ has LLP } \Leftrightarrow A \otimes_{\min} \mathbb{B} = A \otimes_{\max} \mathbb{B}$$

where

$$\mathbb{B} = (\oplus \sum\nolimits_{n \geq 1} M_n)_\infty.$$

(often denoted $\prod_{n\geq 1} M_n$ in C^*-literature)

Stability properties

LP and LLP are stable under (finite) direct sums (easy)

LP and LLP are stable under extensions

LLP stable under (maximal) free products of arbitrary family (P. 1996)

LP stable under (maximal) free products of any countable family (Boca 1996, easy by Boca 1991)
Indeed, if A_i $(i \in I)$ is such that id_{A_i} is liftable up in $C_i = C(\mathbb{F}_\infty)$

$$\begin{tikzcd} & C_i \arrow[d, "q_i"] \\ A_i \arrow[ur, "u_i"] \arrow[r, "id_{A_i}"'] & A_i \end{tikzcd} \qquad \begin{tikzcd} & *_{i\in I} C_i \arrow[d, "*_{i\in I} q_i"] \\ *_{i\in I} A_i \arrow[ur, "*_{i\in I} u_i"] \arrow[r, "id"'] & *_{i\in I} A_i \end{tikzcd}$$

Boca 1991: u_i u.c.p. $\Rightarrow$ $*_{i\in I} u_i$ u.c.p.

More stability properties

$$id_A : A \xrightarrow{u} C \xrightarrow{v} A$$

If id_A factors through C with decomposable maps u, v then

$$C \; LP \text{ (resp. LLP)} \Rightarrow \; A \; LP \text{ (resp. LLP)}$$

$$* * * * * * * * * * * * * * * *$$

LLP stable under closure of union of arbitrary nested family

$$A = \overline{\cup A_i}$$

$$A_i \text{ LLP } \forall i \in I \Rightarrow A \text{ LLP}$$

(easy using $\mathbb{B} \otimes_{\min} A = \mathbb{B} \otimes_{\max} A$)

$\rightarrow$**analogue unclear for LP**

New Characterization of LP for a separable C^* -algebra A

For any family $(D_i)_{i\in I}$ of C^*-algebras,
consider the following property: We have a natural isometric embedding

$$(*) \qquad \ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\}).$$

Equivalently $\ell_\infty(\{D_i\}) \otimes_{\max} A$ can be identified with the closure of $\ell_\infty(\{D_i\}) \otimes_{alg} A$ in $\ell_\infty(\{D_i \otimes_{\max} A\})$.

Main result: $LP \Leftrightarrow (*)$

Remark: Suffices $I = \mathbb{N}$ and $D_i = C^*(\mathbb{F}_\infty)$ $\forall i \in \mathbb{N}$
(recall A separable)
Remark: (*) is always true for the min-norm, $\Rightarrow$ case A nuclear
Remark: (*) in case $A = C^*(\mathbb{F}_\infty)$ checked by linearization trick

Digression: (*) $\Rightarrow$ LLP

$$(*) \qquad \ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\}).$$

Take $\{D_i\} = \{M_n \mid n \geq 1\}$. Then

$$(*) \Rightarrow \ell_\infty(\{M_n\}) \otimes_{\max} A \subset \ell_\infty(\{M_n \otimes_{\max} A\}) = \ell_\infty(\{M_n \otimes_{\min} A\})$$

and hence

$$\ell_\infty(\{M_n\}) \otimes_{\max} A \subset \ell_\infty(\{M_n\}) \otimes_{\min} A.$$

equivalently

$$\mathbb{B} \otimes_{\max} A = \mathbb{B} \otimes_{\min} A$$

where

$$\mathbb{B} = \ell_\infty(\{M_n\})$$

which is Kirchberg's criterion for LLP

Main tool: Maximally bounded maps

Let $E \subset A$ be an operator space (A a C^*-algebra) Let D be another C^*-algebra. We denote (abusively)

$$D \otimes_{\max} E = \overline{D \otimes_{alg} E}^{\| \ \|_{\max}} \subset D \otimes_{\max} A$$

Definition

$u : E \to C$ is called maximally bounded if for any C^*-algebra D

$$\|u\|_{mb} := \|Id_D \otimes u : D \otimes_{\max} E \to D \otimes_{\max} C\| < \infty$$

We denote by $MB(E, C)$ the normed space of such u's

Similar definition for maximally positive for E operator system

Characterization of MB maps

Theorem

Let $E \subset A$ be an operator subspace, $u : E \to C$

$$\|u\|_{mb} = \inf \|\tilde{u}\|_{dec},$$

where the infimum runs over all maps $\tilde{u} : A \to C^{**}$ such that $\tilde{u}_{|E} = i_C u$ (infimum attained),
where $i_C : C \to C^{**}$ is canonical inclusion

$$\begin{tikzcd}
A \arrow[drr, dashed, "\tilde{u}"] & & \\
E \arrow[u, hook] \arrow[r, "u"'] & C \arrow[r, hook, "i_C"'] & C^{**}
\end{tikzcd}$$

Based on Kirchberg (unpublished): $\|u\|_{mb} = \|i_C u\|_{dec}$ when $E = A$

Main result

Theorem

The following are equivalent:

- (i) *A has the LP*
- (ii) $(*)$ $\ell_\infty(\{D_i\}) \otimes_{\max} A \subset \ell_\infty(\{D_i \otimes_{\max} A\})$ $\forall (D_i)$
- (iii) $\forall E \subset A$ *f.d.* $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ *contractively*
- (iv) $\forall D$ $D^{**} \otimes_{\max} A \subset (D \otimes_{\max} A)^{**}$ *isometrically*
- (v) $\forall M$ *vNa* $M \otimes_{\max} A = M \otimes_{\mathrm{nor}} A$ *isometrically*
- (vi) *For any family $(D_i)_{i \in I}$ of C^*-algebras and any ultrafilter on I we have a natural isometric embedding*

$$[\prod_{i \in I} D_i/\mathcal{U}] \otimes_{\max} A \subset \prod_{i \in I} [D_i \otimes_{\max} A]/\mathcal{U}.$$

Sketch:

Main new point is (ii) $\Rightarrow$ (iii)

(ii) A satisfies (*) (for any (D_i))

(iii) $\forall E \subset A$ f.d. $\forall C$ $MB(E, C^{**}) \subset MB(E, C)^{**}$ contractively

We set
$MB(E, C)^* = C^* \otimes_\alpha E$ (recall $\dim(E) < \infty$)
Then (*) implies a property of α that leads to (iii)

Back to Project 2

$$\cdots E_n \xrightarrow{T_n} E_{n+1} \cdots$$

$$\exists A(\{E_n, T_n\})$$

with WEP, LLP and such that $\mathcal{C} = C^*(\mathbb{F}_\infty)$ locally embeds in $A(\{E_n, T_n\}$ (and conversely).

Our project 2 asked whether the following algebra has the LP

Theorem (P. Inv 2020)

There is a C^*-algebra $A(\{E_n, T_n\})$ which is WEP and LLP such that $\mathcal{C} = C^*(\mathbb{F}_\infty)$ locally embeds in $A(\{E_n, T_n\})$.

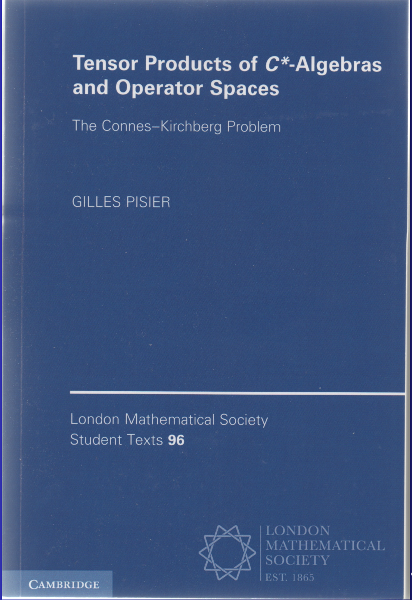
Theorem

The following **(true or false !)** *assertions are equivalent*

- (i) $A(\{E_n, T_n\})$ *has the LP*
- (ii) $LLP \Rightarrow LP \quad \forall C^* - algebra \quad with\ WEP$
- (iii) *Any faithful representation $j : \mathcal{C} \to B(H)$ extends to a contractive morphism*

$$Id_{\ell_\infty(\mathcal{C})} \otimes j : \ell_\infty(\mathcal{C}) \otimes_{\min} \mathcal{C} \to \ell_\infty(\mathcal{C}) \otimes_{\max} B(H)$$

- (iv) *There is a completely isometric embedding $f : \mathcal{C} \to \mathcal{C}$ such that the min and max norms coincide on $f(\mathcal{C}) \otimes \mathcal{C} \subset \mathcal{C} \otimes \mathcal{C}$.*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Thank you !