Simplicity and the ideal intersection property for essential groupoid C*-algebras
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Operator algebras, dynamics and groups
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Overview

C*-simplicity for groups

Groupoids and their C*-algebras

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Relative Powers averaging
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Simple group $C^*$-algebras

Theorem (Kennedy 2015, using previous work of Kalantar-Kennedy and Breuillard-KK-Ozawa)

A discrete group $G$ is $C^*$-simple, i.e. $C^*_{\text{red}}(G)$ is simple, if and only if it does not have any confined, amenable subgroups.
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Confined subgroups

- Group theoretic generalisation of non-trivial normal subgroups.
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Group theoretic generalisation of *non-trivial* normal subgroups.

Formally: $H \leq G$ such that its orbit closure in the Chabauty space $C(G)$ does not contain the trivial subgroup.
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Formally: \( H \leq G \) such that its orbit closure in the Chabauty space \( \mathcal{C}(G) \) does not contain the trivial subgroup.

More combinatorially: \( H \leq G \) for which there is a finite set \( F \subseteq G \setminus \{e\} \) such that for all \( g \in G \) we have \( gHg^{-1} \cap F \neq \emptyset \).
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Key insight from Kalantar-Kennedy: C*-simplicity of $G$ is related to the dynamics on the Furstenberg boundary $\partial_F G$. 
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Hamana 1980’s: Studied injective hull of $G$-operator systems.
Observes equality of $I_G(\mathbb{C})$ and $C(\partial_F G)$. 
Short history of the Furstenberg boundary

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For simple Lie groups $\partial_F G = G/B$ for $B \leq G$ Borel. For discrete groups $\partial_F G$ is extremally disconnected.


Key: universality/injectivity + rigidity/essentiality combined.
Key: classification of equivariant ucp maps $C^*_\text{red}(G) \to C(\partial_F G)$. 
Proving $C^*$-simplicity à la Kalantar-Kennedy

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Key example given by the natural trace

$$\tau : C^*_{\text{red}}(G) \to \mathbb{C} : \tau(u_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e. \end{cases}$$
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Strategy

- $\tau$ is faithful (i.e. $\tau(x^*x) = 0$ implies $x = 0$) and a *-homomorphism is faithful if and only if it is injective.
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- $\forall C^*_\text{red}(G) \to A$ there is an equivariant ucp map $A \to C(\partial_F G)$.
- $\exists! C^*_\text{red}(G) \to C(\partial_F G)$ equivariant ucp map, namely $\tau$. 

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If $G \rhd X$ is an action of a group on a topological space, then its transformation groupoid $G \ltimes X$ replaces the quotient $X/G$. 
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Definition
A groupoid is a small category in which all morphisms are invertible.
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Definition
A groupoid is a small category in which all morphisms are invertible.

Source and range maps \( s, r : \mathcal{G} \Rightarrow \mathcal{G}^{(0)} \) such that

\[
g \cdot h \text{ is defined if and only if } s(g) = r(h).
\]
Topological groupoids

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- $G \ltimes X$ for a discrete group $G$, acting on a locally compact Hausdorff space $X$. 
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- $G \rtimes X$ for a discrete group $G$, acting on a locally compact Hausdorff space $X$.
- The restrictions $(G \rtimes X)|_U$ to open subsets and quotients with open kernel.
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Assumptions for the definition of groupoid C*-algebras

Necessary for C*-algebras:
locally compact Hausdorff space of units.

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Necessary for $\mathbb{C}^*$-algebras:
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For the purpose of a simple presentation:
Ample, i.e. $G$ has a basis of compact open subsets.

Example

- $G \ltimes X$ for a discrete group $G$, acting on a totally disconnected, locally compact Hausdorff space $X$.
- The restrictions $(G \ltimes X)|_U$ to open subsets and quotients with open kernel.
Groupoid C*-algebras

Definition

A topological groupoid $\mathcal{G}$ is a bisection if $s|_B$ and $r|_B$ are injective.

Closure in a suitable regular representation.

Gétale groupoid: replace $C^*\Gamma$ by suitable function algebra.
Definition

$\mathcal{G}$ topological groupoid:

$B \subseteq \mathcal{G}$ is a bisection if $s|_B$ und $r|_B$ are injective.

$\mathcal{G}$ ample:

$\Gamma_{\text{comp}}(\mathcal{G})$ *-semigroup of compact, open bisections.
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$$B \cdot C = \{ g \cdot h \mid g \in B, \ h \in C, \ s(g) = r(h) \},$$

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\( \mathcal{G} \) étale groupoid: replace \( C[\Gamma_{\text{comp}}(\mathcal{G})] \) by suitable function algebra.
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Simple groupoid $C^*$-algebras: a chronology

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Studies $G \curvearrowright X$ for countable $G$ and compact $X$. 
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$C(X) \rtimes G$ is simple if and only if $G \curvearrowright X$ is minimal and $C(X) \subseteq C(X) \rtimes G$ has the ideal intersection property.
Studies $G \act X$ for countable $G$ and compact $X$. $C(X) \rtimes G$ is simple if and only if $G \act X$ is minimal and $C(X) \subseteq C(X) \rtimes G$ has the ideal intersection property.

Hamana-type boundary replaces Furstenberg boundary:
$\partial_F(G \act X) = \text{spec} I_G(C(X))$. 
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**Theorem (Kawabe 2017)**

Let $X$ be a compact $G$-space. Then $C(X) \subseteq C(X) \rtimes G$ has the intersection property if and only if every $G$-invariant closed subset of $$\{(H, x) \in C(G) \times X \mid H \leq G_x\}$$ contains $\{e\} \times X$. 
Borys: a Hamana boundary for groupoids

Studies étale Hausdorff groupoids with compact space of units.
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$$\partial_F(G) = \text{spec I}_{\text{fib. } G\text{-op. sys.}}(C(X)).$$

Introduces sections of isotropy groups, living in $\bigcup_{x \in G^{(0)}} C(G^x_x)$. 

Theorem (Borys 2020)

Let $G$ be a Hausdorff étale groupoid with compact unit space that does not have confined amenable sections of isotropy groups, and in which the orbit of any unit in the groupoid contains at least two points. Then $C_p G^{p_0 q}$ has the intersection property.
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$$C(\mathcal{G}^{(0)}) \subseteq C^*_\text{red}(\mathcal{G})$$ has the intersection property.
Theorem (Kennedy-Kim-Li-R-Ursu 2021)

Let $\mathcal{G}$ be an étale Hausdorff groupoid with locally compact space of units. Then the following are equivalent:

- $C_0(\mathcal{G}^{(0)}) \subseteq C^*_\text{red}(\mathcal{G})$ has the ideal intersection property.
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Characterising $\mathbb{C}^*$-simplicity: Hausdorff groupoids

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Key new ideas

- Introduction of a new notion of groupoid $\mathbb{C}^*$-algebras, in order to obtain a Hamana boundary with sufficiently strong universal property.
- Compactification of unit space of a groupoid in order apply $\mathbb{C}^*$-simplicity techniques also in the locally compact case.
A new notion of groupoid $C^*$-algebras

Before KKLRU
Groupoids acting on bundles of $C^*$-algebras (or operator spaces) over the space of units $V \to G^{(0)}$. 
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This usually excludes the groupoid $C^*$-algebra $C^*_{\text{red}}(G)$.
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Solution
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We turn $C^*_{\text{max}}(\mathcal{G})$ and its quotients into $\mathcal{G}$-C*-algebras, generalising inner actions of groups $a \mapsto u_g a u_g^*$.
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**Idea**
Introduce an isolated unit to compactify the unit space.
Compactification of groupoids

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Important
Well-developed notion of confinedness to pass between compactification and original space.
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**Important**
Well-developed notion of confinedness to pass between compactification and original space.

**Formally**
We apply the Alexandrov compactification to the unit space $G^+ = G \cup \{\infty\} \supseteq G^{(0)} \cup \{\infty\} = (G^{(0)})^+$. 

**Compactification of groupoids**
Non-Hausdorff groupoids: a motivating example

Non-Hausdorff groupoids pose additional challenges, but arise naturally.
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Example

Let $G$ be a discrete group and $G \curvearrowright X$ an action. The groupoid of germs $\mathcal{G} = (G \ltimes X)/\text{Iso}(G \ltimes X)^\circ$ is Hausdorff if and only if points stabilisers equal neighbourhood stabilisers.
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Example
The action of Thompson’s group $T$ on a totally disconnected cover of $S^1$ gives rise to a non-Hausdorff groupoid of germs.
Example (Khoshkam-Skandalis 2002)
Consider $G = [0, 1] \times \mathbb{F}_2/\sim = \text{set} \left\{0\right\} \times \mathbb{F}_2 \cup (0, 1] \text{ where } (t, g) \sim (t, h) \text{ if } t \neq 0$. Then $C^*_\text{red}(G) = C^*_\text{red}(\mathbb{F}_2) \oplus C([0, 1])$. 
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Proposition / Definition (Kwaśniewski-Meyer 2021)

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Restriction of functions from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ extends to a generalised conditional expectation $E_{\text{ess}} : C^*_\text{max}(\mathcal{G}) \to \mathcal{B}^\infty_M(\mathcal{G}^{(0)}) = I(\mathcal{C}_0(\mathcal{G}^{(0)}))$. 

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$$C^*_\text{ess}(\mathcal{G}) = C^*_\text{max}(\mathcal{G})/\{a \mid E_{\text{ess}}(a^*a) = 0\}. $$
Characterising $\mathbb{C}^*$-simplicity: non-Hausdorff groupoids

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Characterising $\mathbb{C}^*$-simplicity: non-Hausdorff groupoids

Theorem (KKLRU 2021)

Let $\mathcal{G}$ be an étale Hausdorff groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff or $\sigma$-compact. Then the following are equivalent:

- $C_0(\mathcal{G}^{(0)}) \subseteq C^*_{\text{ess}}(\mathcal{G})$ has the ideal intersection property.
- $\mathcal{G}$ has no essentially confined, amenable sections of isotropy groups.

Surprise that non-Hausdorff case can be made to fit by carefully adapting notions.

Key new ideas / ingredients

- Inclusion $C^*_{\text{ess}}(\mathcal{G}) \subseteq C^*_{\text{ess}}(\mathcal{G}^H)$. Needs extra assumptions.
- Hausdorffification of groupoids.
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- *Inclusion* $C^{*}_{\text{ess}}(\mathcal{G}) \subseteq C^{*}(\mathcal{G} \times \partial_F \mathcal{G})$. Needs extra assumptions.
- *Hausdorffification* of groupoids.
- $C^{*}_{\text{ess}}(\mathcal{G} \times \partial_F \mathcal{G})$ is isomorphic to a reduced groupoid $C^{*}$-algebra.
Overview

C*-simplicity for groups

Groupoids and their C*-algebras

C*-simplicity for groupoids

Relative Powers averaging
Examples of relative Powers averaging

Exploiting the full strength of our characterisation, strong structural results can be derived from simplicity.

**Theorem (KKLRU 2021)**

Let $G$ be a countable discrete group and $G \curvearrowright X$ a boundary action. Let $\mathcal{G} = (G \times X)/\text{Iso}(G \times X)^\circ$ be its groupoid of germs and denote by $\pi : G \rightarrow \mathcal{C}_{\text{ess}}^*(\mathcal{G})$ the associated unitary representation of $G$. Then for every $a \in \mathcal{C}_{\text{ess}}^*(G)$ with $E_p a = 0$, there is a sequence of elements $g_1, g_2, \ldots$ from $G$ such that $\lim_{n \to \infty} \sum_{i=1}^n \pi_{g_i} a \pi_{g_i} \overset{\text{in norm}}{\to} 0$.

Example: The action of Thompson's group $T$ on a totally disconnected cover of $S^1$, gives rise to a unitary representation into the Cuntz algebra $\pi : T \rightarrow O_2$ enjoying relative Powers averaging property in the above sense.
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Then for every $a \in C^*_{\text{ess}}(\mathcal{G})$ with $E(a) = 0$, there is a sequence of elements $g_1, g_2, \ldots$ from $G$ such that

$$\frac{1}{n} \sum_{i=1}^{n} \pi(g_i) a \pi(g_i)^* \xrightarrow{n \to \infty} 0 \quad \text{in norm.}$$
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Example

The action of Thompson’s group $T$ on a totally disconnected cover of $S^1$, gives rise to a unitary representation into the Cuntz algebra $\pi : T \to \mathcal{O}_2$ enjoying relative Powers averaging property in the above sense.
Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff space of units and $S$ a contractive and covering convex semigroup of generalised probability measures on $\mathcal{G}$. Then the following statements are equivalent.

- $\mathcal{C}_\text{ess}^*(\mathcal{G})$ is simple.
- $A \subseteq \mathcal{C}_\text{ess}^*(\mathcal{G})$ is $\mathcal{C}^*$-irreducible for every $A$ supporting $S$.
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Kennedy, Haagerup, Amrutam-Ursu: use boundary techniques to understand dynamics on state spaces of group(oid) $\mathcal{C}^*$-algebras.

Key new ideas
- Notion of strongly proximal groupoid actions.
- Furstenberg and Hamana boundary constructed and identified.
Proving relative Powers averaging

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*The ideal intersection property for essential groupoid C*-algebras*
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Thanks for your attention!