Simplicity and the ideal intersection property for essential groupoid C*-algebras

joint with Matthew Kennedy, Se-Jin Kim, Xin Li and Dan Ursu

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Overview

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Groupoids and their C*-algebras

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Relative Powers averaging

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Key insight from Kalantar-Kennedy: C^*-simplicity of G is related to the dynamics on the Furstenberg boundary $\partial_{\mathrm{F}} G$.

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Key: universality/injectivity + rigidity/essentiality combined.

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Key: classification of equivariant ucp maps $C^*_{\mathrm{red}}(G) \to C(\partial_F G)$.

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$$\tau : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathbb{C} : \tau(u_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e\,. \end{cases}$$

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- τ is faithful (i.e. $\tau(x^*x) = 0$ implies $x = 0$) and a *-homomorphism is faithful if and only if it is injective.

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- $\exists!\ \mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}(\partial_{\mathrm{F}} G)$ equivariant ucp map, namely τ.

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Definition

A groupoid is a small category in which all morphisms are invertible.

Source and range maps $\mathrm{s}, \mathrm{r} : \mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ such that

$$g \cdot h \text{ is defined if and only if } \mathrm{s}(g) = \mathrm{r}(h)\,.$$

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For the purpose of a simple presentation:
Ample, i.e. $\mathcal{G}$ has a basis of compact open subsets.

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- $G \ltimes X$ for a discrete group G, acting on a totally disconnected, locally compact Hausdorff space X.
- The restrictions $(G \ltimes X)|_U$ to open subsets and quotients with open kernel.

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$\mathcal{G}$ étale groupoid: replace $\mathbb{C}[\Gamma_{\mathrm{comp}}(\mathcal{G})]$ by suitable function algebra.

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Theorem (Kawabe 2017)

Let X be a compact G-space. Then $\mathrm{C}(X) \subseteq \mathrm{C}(X) \rtimes G$ has the intersection property if and only if every G-invariant closed subset of $\{(H, x) \in \mathcal{C}(G) \times X \mid H \leqslant G_x\}$ contains $\{e\} \times X$.

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Theorem (Borys 2020)

Let $\mathcal{G}$ be a Hausdorff étale groupoid with compact unit space that does not have confined amenable sections of isotropy groups , and in which the orbit of any unit in the groupoid contains at least two points. Then $\mathrm{C}(\mathcal{G}^{(0)}) \subseteq \mathrm{C}^*_{\mathrm{red}}(\mathcal{G})$ *has the intersection property.*

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Characterising C*-simplicity: Hausdorff groupoids

Theorem (Kennedy-Kim-Li-R-Ursu 2021)

Let $\mathcal{G}$ be an étale Hausdorff groupoid with locally compact space of units. Then the following are equivalent:

- $C_0(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{red}}(\mathcal{G})$ *has the ideal intersection property.*
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Key new ideas

- Introduction of a *new notion of groupoid* C^**-algebras*, in order to obtain a Hamana boundary with sufficiently strong universal property.
- *Compactification of unit space of a groupoid* in order apply C^*-simplicity techniques also in the locally compact case.

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We turn $C^*_{max}(\mathcal{G})$ and its quotients into $\mathcal{G}$-C*-algebras, generalising inner actions of groups $a \mapsto u_g a u_g^*$.

Compactification of groupoids

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Formally

We apply the Alexandrov compactification to the unit space $\mathcal{G}^+ = \mathcal{G} \cup \{\infty\} \supseteq \mathcal{G}^{(0)} \cup \{\infty\} = (\mathcal{G}^{(0)})^+$.

Non-Hausdorff groupoids: a motivating example

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Example

Let G be a discrete group and $G \curvearrowright X$ an action. The *groupoid of germs* $\mathcal{G} = (G \ltimes X)/\mathrm{Iso}(G \ltimes X)^\circ$ is Hausdorff if and only if points stabilisers equal neighbourhood stabilisers.

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Example

The action of Thompson's group T on a totally disconnected cover of S^1 gives rise to a non-Hausdorff groupoid of germs.

Kwaśniewski-Meyer: essential groupoid C*-algebras

Example (Khoshkam-Skandalis 2002)

Consider $\mathcal{G} = [0,1] \times \mathbb{F}_2/\sim\ =_{\text{set}} \{0\} \times \mathbb{F}_2 \cup (0,1]$ where $(t,g) \sim (t,h)$ if $t \neq 0$. Then $\mathrm{C}^*_{\text{red}}(\mathcal{G}) = \mathrm{C}^*_{\text{red}}(\mathbb{F}_2) \oplus \mathrm{C}([0,1])$.

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Proposition / Definition (Kwaśniewski-Meyer 2021)

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Restriction of functions from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ extends to a generalised conditional expectation $\mathrm{E}_{\text{ess}} : \mathrm{C}^*_{\max}(\mathcal{G}) \to \frac{\mathcal{B}^\infty}{\mathcal{M}}(\mathcal{G}^{(0)}) = \mathrm{I}(\mathrm{C}_0(\mathcal{G}^{(0)}))$.

Kwaśniewski-Meyer: essential groupoid C^*-algebras

Example (Khoshkam-Skandalis 2002)

Consider $\mathcal{G} = [0,1] \times \mathbb{F}_2/\sim =_{\text{set}} \{0\} \times \mathbb{F}_2 \cup (0,1]$ where $(t,g) \sim (t,h)$ if $t \neq 0$. Then $C^*_{\text{red}}(\mathcal{G}) = C^*_{\text{red}}(\mathbb{F}_2) \oplus C([0,1])$.

- Ideal intersection properties can fail for topologically free non-Hausdorff groupoids.
- Replacement for reduced groupoid C^*-algebra needed.

Proposition / Definition (Kwaśniewski-Meyer 2021)

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff space of units. Restriction of functions from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ extends to a generalised conditional expectation $E_{\text{ess}} : C^*_{\max}(\mathcal{G}) \to \frac{\mathcal{B}^\infty}{\mathcal{M}}(\mathcal{G}^{(0)}) = I(C_0(\mathcal{G}^{(0)}))$*.*
The essential groupoid C^*-algebra of $\mathcal{G}$ is

$$C^*_{\text{ess}}(\mathcal{G}) = C^*_{\max}(\mathcal{G})/\{a \mid E_{\text{ess}}(a^*a) = 0\}.$$

Characterising C^*-simplicity: non-Hausdorff groupoids

Theorem (KKLRU 2021)

Let $\mathcal{G}$ be an étale Hausdorff groupoid with locally compact space of units.
Then the following are equivalent:

- $C_0(\mathcal{G}^{(0)}) \subseteq C^*_{\mathrm{red}}(\mathcal{G})$ *has the ideal intersection property.*
- $\mathcal{G}$ *has no confined, amenable sections of isotropy groups.*

Characterising C*-simplicity: non-Hausdorff groupoids

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Let $\mathcal{G}$ be an étale ~~Hausdorff~~ groupoid with locally compact Hausdorff space of units. Assume that $\mathcal{G}$ is Hausdorff or σ-compact.
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Key new ideas / ingredients

- *Inclusion* $C^*_{\mathrm{ess}}(\mathcal{G}) \subseteq C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$. Needs extra assumptions.
- *Hausdorffification* of groupoids.
- $C^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_F \mathcal{G})$ is isomorphic to a reduced groupoid C*-algebra.

Overview

C*-simplicity for groups

Groupoids and their C*-algebras

C*-simplicity for groupoids

Relative Powers averaging

Examples of relative Powers averaging

Exploiting the full strength of our characterisation, strong structural results can be derived from simplicity.

Theorem (KKLRU 2021)

Let G be a countable discrete group and $G \curvearrowright X$ a boundary action. Let $\mathcal{G} = (G \ltimes X)/\mathrm{Iso}(G \ltimes X)^\circ$ be its groupoid of germs and denote by $\pi : G \to \mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ the associated unitary representation of G.

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Then for every $a \in \mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ with $\mathrm{E}(a) = 0$, there is a sequence of elements $g_1, g_2, \ldots$ from G such that

$$\frac{1}{n}\sum_{i=1}^{n} \pi(g_i) a \pi(g_i)^* \xrightarrow{n\to\infty} 0 \quad \textit{in norm.}$$

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Example

The action of Thompson's group T on a totally disconnected cover of S^1, gives rise to a unitary representation into the Cuntz algebra $\pi : \mathrm{T} \to \mathcal{O}_2$ enjoying relative Powers averaging property in the above sense.

Proving relative Powers averaging

Theorem (KKLRU 2021)

Let $\mathcal{G}$ be a minimal étale groupoid with compact Hausdorff space of units and S a contractive and covering convex semigroup of generalised probability measures on $\mathcal{G}$. Then the following statements are equivalent.

- $\mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ *is simple.*
- $A \subseteq \mathrm{C}^*_{\mathrm{ess}}(\mathcal{G})$ *is* C^**-irreducible for every A supporting S.*
- *Given* $a \in \mathrm{C}^*_{\mathrm{ess}}(\mathcal{G} \ltimes \partial_{\mathrm{F}}\mathcal{G})$ *with* $\mathrm{E}_{\mathrm{ess}}(a) = 0$*, we have* $0 \in \overline{\{\mu * a \mid \mu \in S\}}$.

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Key new ideas

- Notion of strongly proximal groupoid actions.
- Furstenberg and Hamana boundary constructed and identified.

More details can be found in:

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Thanks for your attention!