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# Superrigidity for group actions on the $n$-sphere and their skew products

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# Isometric actions on the hyperbolic plane

Consider the group of isometries $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

**Theorem (Drimbe - V, 2021)**

Let $\mathcal{S}$ be a set of prime numbers and $G = \mathrm{PSL}(2,\mathbb{Z}[\mathcal{S}^{-1}])$. Consider the orbit equivalence (OE) relation $\mathcal{R} = \mathcal{R}(G \curvearrowright \mathbb{H}^2) = \{(z, g\cdot z) \mid z \in \mathbb{H}^2, g \in G\}$.

- (well known) If $|\mathcal{S}| = 0$, then $\mathcal{R}$ admits a fundamental domain.
- If $|\mathcal{S}| = 1$, then $\mathcal{R}$ is ergodic, nonamenable and treeable, so that $\mathcal{R} \cong \mathcal{R}(\Lambda \curvearrowright Z)$ for uncountably many essentially different $\Lambda \curvearrowright Z$.
- If $|\mathcal{S}| \geq 2$, we have OE superrigidity: if $\mathcal{R} \cong \mathcal{R}(\Lambda \curvearrowright Z)$, we essentially have $\Lambda \cong G$ and $\Lambda \curvearrowright Y$ conjugate with $G \curvearrowright \mathbb{H}^2$.


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# Actions on the $(n-1)$-sphere

View $S^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^*_+$. In this way, $S^{n-1}$ is a homogeneous space: $\mathrm{SL}(n,\mathbb{R}) \curvearrowright S^{n-1}$.

**Theorem (Popa-V, 2008)**

For $n \geq 5$, the action $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright S^{n-1}$ is OE superrigid.

Whenever $\mathcal{A} \subset \mathbb{R}$ is a countable subring, consider $\mathrm{SL}(n,\mathcal{A}) \subset \mathrm{SL}(n,\mathbb{R})$.
Note that if $\mathcal{A} \neq \mathbb{Z}$, then $\mathcal{A} \subset \mathbb{R}$ is dense.

**Theorem (Drimbe-V 2021 and V-Verjans 2022)**

Let $\mathcal{A} \subset \mathbb{R}$ be any countable subring containing an algebraic number that is not an integer.
Let $n \geq 3$. Then, $\mathrm{SL}(n,\mathcal{A}) \curvearrowright S^{n-1}$ is OE superrigid.

$\rightsquigarrow$ Actions of type $\mathrm{III}_1$. Nuances of OE superrigidity still to be clarified.

# Skew product actions

Notation: $E(n, \mathcal{A}) \subset \mathrm{SL}(n, \mathcal{A})$ is the subgroup generated by the elementary matrices.

In many cases, $E(n, \mathcal{A}) = \mathrm{SL}(n, \mathcal{A})$, e.g. when $\mathcal{A} = \mathbb{Z}[\mathcal{S}^{-1}]$ or $\mathcal{A} = \mathcal{O}_K$.

**Theorem (V-Verjans, 2022)**

Let $\mathcal{A} \subset \mathbb{R}$ be any countable subring containing an algebraic number that does not belong to $\mathbb{Z}$. Let $n \geq 3$ be an odd integer.

Whenever $\mathbb{R} \curvearrowright (Y, \eta)$ is a properly ergodic flow, consider $E(n, \mathcal{A}) \curvearrowright (\mathbb{R}^n \times Y)/\mathbb{R}$, where the quotient is by $t \cdot (x, y) = (e^{t/n} x, t \cdot y)$.

This action is OE superrigid and of type $\mathrm{III}_0$.

$\rightsquigarrow$ Again: nuances of OE superrigidity still to be clarified.

$\rightsquigarrow$ The associated flow is **not** $\mathbb{R} \curvearrowright (Y, \eta)$, but its **adjoint flow**, to be defined.

# Orbit equivalence and stable orbit equivalence

Consider free, ergodic, nonsingular actions $G \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Z,\zeta)$.

- **Orbit equivalence:** there exists a nonsingular isomorphism $\Delta : (X,\mu) \to (Z,\zeta)$ such that $\Delta(G \cdot x) = \Lambda \cdot \Delta(x)$ for a.e. $x \in X$.
- **Stable orbit equivalence:** there exist nonnegligible $\mathcal{U} \subset X$, $\mathcal{V} \subset Z$ and a nonsingular isomorphism $\Delta : \mathcal{U} \to \mathcal{V}$ such that $\Delta(\mathcal{U} \cap G \cdot x) = \mathcal{V} \cap \Lambda \cdot \Delta(x)$ for a.e. $x \in \mathcal{U}$.

$\leadsto$ The nuance is important in the probability measure preserving (pmp) setting, with $\zeta(\mathcal{V})/\mu(\mathcal{U})$ being the coupling constant.

$\leadsto$ For actions of type III, both notions are the same.

$\leadsto$ (Stable) orbit equivalence implies (stable) isomorphism of $L^\infty(X) \rtimes G$ and $L^\infty(Z) \rtimes \Lambda$.

# Krieger type and associated flow

Consider a free, ergodic, nonsingular action $G \curvearrowright (X, \mu)$, with $\mu$ nonatomic.

- **Type II$_1$ :** there exists a $G$-invariant probability measure $\nu \sim \mu$.
- **Type II$_\infty$ :** there exists a $G$-invariant infinite measure $\nu \sim \mu$.
- **Type III :** the others.

Denote $\omega(g, x) = \log \frac{d(g^{-1} \cdot \mu)}{d\mu}(x)$. Then $\omega : G \times X \to \mathbb{R}$ is a 1-cocycle.

**Maharam extension:** $G \curvearrowright X \times \mathbb{R} : g \cdot (x, s) = (g \cdot x, \omega(g, x) + s)$, which commutes with $\mathbb{R} \curvearrowright X \times \mathbb{R} : t \cdot (x, s) = (x, t + s)$.

$\rightsquigarrow$ Ergodic decomposition $\pi : X \times \mathbb{R} \to Y$ of the Maharam extension, with $\mathbb{R} \curvearrowright Y$.

**Krieger's associated flow**; also **Connes-Takesaki flow of weights** of $L^\infty(X) \rtimes G$.



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# Krieger type and associated flow

Consider a free, ergodic, nonsingular action $G \curvearrowright (X, \mu)$, with $\mu$ nonatomic.

Let $\mathbb{R} \curvearrowright (Y, \eta)$ be the associated flow, which is ergodic.

- Type $\mathrm{II}_1$ or type $\mathrm{II}_\infty$ iff the associated flow is $\mathbb{R} \curvearrowright \mathbb{R}$ by translation.
- Type $\mathrm{III}_1$ iff $Y$ is one point.
- Type $\mathrm{III}_\lambda$ with $0 < \lambda < 1$ iff the associated flow is periodic: $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z} \log \lambda$.
- Type $\mathrm{III}_0$ iff the associated flow is properly ergodic.

$\rightsquigarrow$ The types and the associated flow are stable orbit equivalence invariants, and are invariants of the crossed product factors.

$\rightsquigarrow$ Connes, Connes-Feldman-Weiss, Takesaki, Haagerup, Krieger: complete invariants if $G$ is amenable.

# Versions of OE superrigidity

**Definition: OE superrigidity (v0) of a free ergodic nonsingular $G \curvearrowright (X, \mu)$**

Any free, ergodic, nonsingular action $\Lambda \curvearrowright (Z, \zeta)$ that is orbit equivalent with $G \curvearrowright (X, \mu)$, must be conjugate to $G \curvearrowright (X, \mu)$.

**Conjugacy:** isomorphisms $\delta : G \to \Lambda$ and $\Delta : X \to Z$ with $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.

$\leadsto$ OE superrigidity (v0) can only hold in the pmp setting, because $G \curvearrowright X$ is OE with $G \times \Lambda_1 \curvearrowright X \times \Lambda_1$ in the infinite case.

**Theorem (Popa, 2005)**

Let $G$ be an infinite group without nontrivial finite normal subgroups. Assume that $G$ has property (T), or that $G = G_1 \times G_2$ is the product of an infinite and a nonamenable group.

Then the Bernoulli action $G \curvearrowright [0, 1]^G$ is OE superrigid (v0).

# Versions of OE superrigidity

We say that $\Lambda \curvearrowright (Z,\zeta)$ is **induced** from $\Lambda_0 \curvearrowright Z_0$ if $Z_0$ is $\Lambda_0$-invariant and if $(g \cdot Z_0)_{g\in\Lambda/\Lambda_0}$ is a partition of $Z$, up to measure zero.

$\rightsquigarrow$ By construction, $\Lambda \curvearrowright Z$ is stably orbit equivalent with $\Lambda_0 \curvearrowright Z_0$.

**Definition: OE superrigidity (v1) of a free ergodic nonsingular $G \curvearrowright (X,\mu)$**

Any free, ergodic, nonsingular action $\Lambda \curvearrowright (Z,\zeta)$ that is stably orbit equivalent with $G \curvearrowright (X,\mu)$, must be conjugate to an induction of $G \curvearrowright (X,\mu)$.

**The following type $\mathrm{III}_1$ actions are OE superrigid (v1)**

- $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright S^{n-1}$ for $n \geq 5$ odd (Popa-V, 2008).
- $\mathrm{SL}(n,\mathcal{A}) \curvearrowright S^{n-1}$ for $n \geq 3$ odd and $\mathcal{A} \subset \mathbb{R}$ a countable subring containing an algebraic number that does not belong to $\mathbb{Z}$ (Drimbe-V 2021, V-Verjans 2022).

# Versions of OE superrigidity

⤳ (V-Verjans, 2022) An action of type $\mathrm{III}_0$ will basically **never** be OE superrigid (v1).

We need another **"trivial" stable orbit equivalence:**

if $\Lambda \curvearrowright (Z, \zeta)$ and $\Sigma \lhd \Lambda$ is a normal subgroup whose action on $Z$ admits a fundamental domain, then $\Lambda \curvearrowright Z$ is stably orbit equivalent with $\Lambda/\Sigma \curvearrowright Z/\Sigma$.

**Definition: OE superrigidity (v2) of a free ergodic nonsingular $G \curvearrowright (X, \mu)$**

Any free, ergodic, nonsingular action $\Lambda \curvearrowright (Z, \zeta)$ that is stably orbit equivalent with $G \curvearrowright (X, \mu)$, must be induced from $\Lambda_0 \curvearrowright Z_0$ that admits a quotient $\Lambda_0/\Sigma \curvearrowright Z_0/\Sigma$ that is conjugate to $G \curvearrowright X$, for some $\Sigma \lhd \Lambda_0$ acting with fundamental domain.

**Theorem (V-Verjans, 2022)**

Given an ergodic flow $\mathbb{R} \curvearrowright (Y, \eta)$, for $n \geq 3$ odd and $\mathcal{A} \subset \mathbb{R}$ as before, the action $E(n, \mathcal{A}) \curvearrowright (\mathbb{R}^n \times Y)/\mathbb{R}$ is OE superrigid (v2).



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# Terminology: simple actions

We call a free nonsingular action $G \curvearrowright (X, \mu)$ **simple** if the action is **ergodic**, **not induced** and if **no nontrivial normal subgroup** $\Sigma \lhd G$ acts with fundamental domain.

$\rightsquigarrow$ These actions have "no trivially stably orbit equivalent actions".

Let $G \curvearrowright (X, \mu)$ be a free nonsingular **simple** action.

- **Every** version of OE superrigidity implies the following: if another free nonsingular **simple** action $\Lambda \curvearrowright (Z, \zeta)$ is stably orbit equivalent with $G \curvearrowright (X, \mu)$, then these actions are conjugate.
- **V-Verjans, 2022:**
  a free nonsingular **simple** action of type $\mathrm{III}_0$ is never OE superrigid (v1).



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# OE superrigidity (v1) for actions of type $\mathrm{III}_\lambda$ when $0 < \lambda < 1$

**Theorem (V-Verjans, 2022)**

Let $p$ be a prime number and $0 < \lambda < 1$.

Define $\mathcal{A} = \mathbb{Z}[p^{-1}, \lambda]$ and $G = \{A \in \mathrm{GL}(n, \mathcal{A}) \mid \det A \in \lambda^{\mathbb{Z}}\}$.

For every $n \geq 3$, the affine action $G \ltimes \mathcal{A}^n \curvearrowright \mathbb{R}^n : (A, a) \cdot x = A(a + x)$ is simple, of type $\mathrm{III}_\lambda$ and OE superrigid (v1).



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# How to prove OE superrigidity

If $\Delta : X \to Z$ is an orbit equivalence between $G \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Z, \zeta)$, we have the **Zimmer 1-cocycle** $\omega : G \times X \to \Lambda$ defined by $\Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$.

- Similar for stable orbit equivalence.
- Two 1-cocycles $\omega$ and $\omega'$ are **cohomologous** if there exists $\varphi : X \to \Lambda$ such that $\omega'(g, x) = \varphi(g \cdot x)^{-1} \omega(g, x) \varphi(x)$.
- One says that $G \curvearrowright (X, \mu)$ is **cocycle superrigid with countable targets** if every 1-cocycle with values in a countable group $\Lambda$ is cohomologous to a group homomorphism $\delta : G \to \Lambda$, viewed as the 1-cocycle $(g, x) \mapsto \delta(g)$.

**Proposition** (V-Verjans, 2022): a free nonsingular **simple** action $G \curvearrowright (X, \mu)$ is OE superrigid (v1) **if and only if** $G \curvearrowright (X, \mu)$ is cocycle superrigid with countable targets.



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# How to prove cocycle superrigidity

Popa's **deformation/rigidity theory**.

Illustration for the Bernoulli action $G \curvearrowright X = [0,1]^G$ and a 1-cocycle $\omega : G \times X \to \Lambda$.

- **Deformation, Popa's malleability:** one-parameter group $\alpha_t \in \mathrm{Aut}(X \times X)$, commuting with the diagonal $G$-action $g \cdot (x,y) = (g \cdot x, g \cdot y)$, and $\alpha_0 = \mathrm{id}$ and $\alpha_1 = \mathrm{flip}$.
- Define $\omega_t : G \times (X \times X) \to \Lambda$, by $\omega_0(g,(x,y)) = \omega(g,x)$ and $\omega_t(g,(x,y)) = \omega_0(g, \alpha_t(x,y))$.
- **Rigidity:** property (T) for $G$ or spectral gap. Then, $\omega_t \sim \omega_s$ when $|t-s|$ is small.
- Then $\omega_0 \sim \omega_1$ and we find that $\omega$ is cohomologous to a group homomorphism.

# Cocycle superrigidity for dense subgroups

**Definition (special case of Drimbe-V, 2021)**

A dense subgroup $\Gamma$ of a connected Lie group $G$ is **essentially cocycle superrigid** if, using the universal cover $\pi : \widetilde{G} \to G$, the translation action of $\pi^{-1}(\Gamma)$ on $\widetilde{G}$ is cocycle superrigid with countable targets.

**Proposition.** If $G$ is a connected Lie group and $P \subset G$ is a closed connected subgroup such that $\pi_1(P) \to \pi_1(G)$ is surjective, then essential cocycle superrigidity of $\Gamma \subset G$ implies plain cocycle superrigidity for $\Gamma \curvearrowright G/P$.

**Example:** the actions $\Gamma \curvearrowright \mathbb{R}^n$ and $\Gamma \curvearrowright S^{n-1}$ are cocycle superrigid whenever $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ is essentially cocycle superrigid.



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# A cocycle superrigidity theorem for dense subgroups

**Theorem (Drimbe-V, 2021)**

Let $G$ and $H$ be lcsc groups with $H$ compactly generated. Let $\Gamma \subset G \times H$ be a lattice.

Assume that either $H$ has property (T), or that $H = H_1 \times H_2$ such that $H_1 \curvearrowright (G \times H)/\Gamma$ is **ergodic** and $H_2 \curvearrowright (G \times H)/\Gamma$ is **strongly ergodic**.

Then, the projection $\Gamma_0 \subset G$ of $\Gamma$ is essentially cocycle superrigid.

- A 1-cocycle $\omega : \Gamma_0 \times G \to \Lambda$ can be viewed as a $1$-cocycle for the left-right action $(\{e\} \times H) \times \Gamma \curvearrowright G \times H$.
- It can thus be viewed as a $1$-cocycle $\omega'$ for the action $\{e\} \times H \curvearrowright (G \times H)/\Gamma$.
- For every $g \in G$, we have a 1-cocycle $\omega'_g(h,(x,k)\Gamma) = \omega'(h,(gx,k)\Gamma)$.
- By rigidity, $\omega'_g \sim \omega'$ if $g$ is close to $e$ ...... conclusion.

# Examples

**Drimbe-V, 2021**

The following dense subgroups $\Gamma$ are essentially cocycle superrigid.

- $\mathrm{SL}(n, \mathbb{Z}[\mathcal{S}^{-1}]) \subset \mathrm{SL}(n, \mathbb{R})$ if $(n-1)\,|\mathcal{S}| \geq 2$. Here $\mathcal{S}$ is a set of prime numbers.
- $\mathrm{SL}(n, \mathcal{O}_K) \subset \mathrm{SL}(n, \mathbb{R})$ if $(n-1)(a+b-1) \geq 2$.

  Here $\mathbb{Q} \subset K \subset \mathbb{R}$ is a real algebraic number field, with ring of integers $\mathcal{O}_K$, with $a$ real embeddings and $2b$ complex embeddings.

On the other hand, the translation actions of $\mathrm{SL}(2, \mathbb{Z}[1/p])$ and $\mathrm{SL}(2, \mathbb{Z}[\sqrt{N}])$ on $\mathrm{SL}(2, \mathbb{R})$ are **treeable** and therefore do not satisfy any cocycle superrigidity.

**V-Verjans, 2022:** $\mathrm{SL}(n, \mathcal{A}) \subset \mathrm{SL}(n, \mathbb{R})$ is essentially cocycle superrigid whenever $n \geq 3$ and $\mathcal{A} \subset \mathbb{R}$ contains an algebraic number that does not belong to $\mathbb{Z}$.

$\rightsquigarrow$ Each time: $\Gamma \curvearrowright \mathbb{R}^n$ and $\Gamma \curvearrowright S^{n-1}$ are cocycle superrigid.

# OE superrigidity of type $\text{III}_0$

**Input:** an ergodic action $G \curvearrowright (X, \mu)$ of type $\text{III}_1$ whose Maharam extension is **simple** and **cocycle superrigid** with countable targets, with $G$ finitely generated and trivial center.

Radon-Nikodym cocycle: $\omega : G \times X \to \mathbb{R}$.

**Example:** $E(n, \mathcal{A}) \curvearrowright S^{n-1}$ when $n \geq 3$ is odd and $\mathcal{A} \subset \mathbb{R}$ as before.

**Theorem (V-Verjans, 2022)**

Let $\mathbb{R} \curvearrowright (Y, \eta)$ be a properly ergodic flow. Put $G \curvearrowright X \times Y : g \cdot (x, y) = (g \cdot x, \omega(g, x) \cdot y)$.

Then, $G \curvearrowright X \times Y$ is of type $\text{III}_0$ and OE superrigid (v2).

The associated flow is the **adjoint flow:** for any ergodic flow $\mathbb{R} \curvearrowright^{\alpha} Y$, there is a unique nonsingular action $\mathbb{R}^2 \curvearrowright (Z, \zeta)$ on an infinite measure space such that $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ scale the measure $\zeta$ and $\mathbb{R} \curvearrowright Y$ is given by $\mathbb{R} \curvearrowright Z/(\{0\} \times \mathbb{R})$.

**Adjoint flow** $\widehat{\alpha} : \mathbb{R} \curvearrowright Z/(\mathbb{R} \times \{0\})$.


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