a noncommutative nonlinear condenser capacity

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quasicentral modulus $k_{\mathcal{J}}(\tau)$

$\mathcal{H}$ separable ∞-dim Hilbert space

$\mathcal{B}(\mathcal{H})$ bdd. ops. $\mathcal{K}(\mathcal{H})$ compact ops.

$(\mathcal{J}, |\ |_{\mathcal{J}})$ normed ideal, $\mathcal{J} \subset \mathcal{K}$ ideal

$$|AXB|_{\mathcal{J}} \leq \|A\| \, |X|_{\mathcal{J}} \, \|B\|, \quad A, B \in \mathcal{B}(\mathcal{H}), X \in \mathcal{J}$$

$\mathcal{R} \subset \mathcal{K}$ finite rank ops.

$\mathcal{R}_1^+ = \{A \in \mathcal{R} \mid 0 \leq A \leq I\}$

$\tau = (T_j)_{1 \le j \le n}$ bdd. ops., $(\mathcal{J}, |\ |_{\mathcal{J}})$ normed ideal

$k_{\mathcal{J}}(\tau) =$ smallest $C \in [0, \infty]$ for which

$$\exists\ A_m \in \mathcal{R}_1^+,\ A_m \uparrow I,\ \underbrace{|[A_m, \tau]|_{\mathcal{J}}}_{\max\limits_{1 \le j \le n} |[A_m, T_j]|_{\mathcal{J}}} \xrightarrow[m \to \infty]{} C$$

$\mathcal{J} = \mathcal{C}_p$ $\quad k_p(\tau)$ Schatten-v. Neumann p-class

$\mathcal{J} = \mathcal{C}_p^-$ $\quad k_p^-(\tau)$, $(\mathcal{C}_p^-, |\ |_p^-)$, $(p,1)$ Lorentz ideal

$|\ |_p^-$ roughly largest norm $\quad |P|_p^- \sim |P|_p$

projection

invariance mod $\mathcal{J}$

$$\tau = (T_j)_{1 \le j \le n}, \quad \sigma = (S_j)_{1 \le j \le n}$$

$$T_j - S_j \in \mathcal{J}, \quad 1 \le j \le n$$

$\mathcal{R}$ dense in $\mathcal{J}$

$$\Longrightarrow \quad k_{\mathcal{J}}(\tau) = k_{\mathcal{J}}(\sigma)$$

invariance of absolutely continuous parts

$\alpha = (A_j)_{1\le j\le n}$, $\beta = (B_j)_{1\le j\le n}$

n-tuples of commuting selfadjoint ops.

$$A_j - B_j \in \mathcal{C}_n^- \quad 1 \le j \le n$$

$\Longrightarrow$ α_{ac} unitarily equivalent β_{ac}

(absolutely continuous parts w.r.t. n-dim. Lebesgue meas.)

$n=1$ corollary of Kato–Rosenblum Thm.

$n \geq 2$ $k_n^-(\cdot)$ plays key role in proof (V)

$\mathcal{C}_n^-$ optimal

commutant mod : $\mathcal{E}(\tau; \mathcal{J})$

$\tau = (T_j)_{1 \le j \le n}$, $T_j = T_j^*$, $1 \le j \le n$

$\mathcal{E}(\tau; \mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J}, 1 \le j \le n\}$

$|||X||| = \|X\| + |[\tau, X]|_{\mathcal{J}}$

Banach *-algebra w. isometric involution

$\mathcal{K}(\tau; \mathcal{J}) = \mathcal{E}(\tau; \mathcal{J}) \cap \mathcal{K}$

closed ideal

Facts: $k_{\mathcal{J}}(\tau) = 0$ iff $\mathcal{K}(\tau; \mathcal{J})$ has norm 1 approximate unit

$k_{\mathcal{J}}(\tau) < \infty$ iff $\mathcal{K}(\tau; \mathcal{J})$ has bounded approximate unit

Banach space dualities

1°. $\mathcal{R}$ dense in $\mathcal{J}$ and $\mathcal{J}^{\text{dual}}$, $k_{\mathcal{J}}(\tau)=0$

$\Longrightarrow$ $\mathcal{E}(\tau;\mathcal{J}) \sim$ bidual of $\mathcal{K}(\tau;\mathcal{J})$

2°. $\mathcal{J}$ reflexive and $k_{\mathcal{J}}(\tau)=0$

$\Longrightarrow$ $\mathcal{E}(\tau;\mathcal{J})$ has unique predual

$\mathcal{E}\mathcal{E}(\tau;\mathcal{J})$ the bicommutant mod

$$\mathcal{E}\mathcal{E}(\tau;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, \mathcal{E}(\tau;\mathcal{J})] \subset \mathcal{J}\}$$

$$\mathcal{K}\mathcal{E}(\tau;\mathcal{J}) = \mathcal{E}\mathcal{E}(\tau;\mathcal{J}) \cap \mathcal{K}$$

$$\mathcal{E}/\mathcal{K}\mathcal{E}(\tau;\mathcal{J}) = \mathcal{E}\mathcal{E}(\tau;\mathcal{J}) / \mathcal{K}\mathcal{E}(\tau;\mathcal{J})$$

$$|||X||| = \|X\| + \sup\{\|[X,Y]\|_{\mathcal{J}} \mid Y \in \mathcal{E}(\tau;\mathcal{J}), |||Y||| < 1\}$$

$$X \in \mathcal{E}\mathcal{E}(\tau;\mathcal{J})$$

<u>Fact</u> $(\mathcal{E}\mathcal{E}(\tau;\mathcal{J}), |||\cdot|||)$ Banach algebra
w. isometric involution

$$\mathcal{E}\mathcal{E}(\tau;\mathcal{J}) = \mathcal{E}\mathcal{E}(\tau;\mathcal{J}) \cap (\tau)'' + \mathcal{J}$$

(uses Ber–Huang–Levitina–Sukochev)

$\mathcal{D}(\tau;\mathcal{J}) = \mathcal{E}\mathcal{E}(\tau;\mathcal{J}) \cap (\tau)''$ smooth alg. of τ w. r. t. $\mathcal{J}$

$$\tau \equiv \tau' (\mathrm{mod}\,\mathcal{J}) \Longrightarrow \mathcal{D}(\tau;\mathcal{J})/\mathcal{J} = \mathcal{D}(\tau';\mathcal{J})/\mathcal{J}.$$

ubiquity of quasicentral modulus

- multivariable generalizations of Thms. of Kato–Rosenblum and Weyl–v. Neumann–Kuroda
- Banach space and Banach algebra properties of $\mathcal{E}(\tau;\mathcal{J})$, $\mathcal{K}(\tau;\mathcal{J})$, $\mathcal{E}(\tau;\mathcal{J})/\mathcal{K}(\tau;\mathcal{J})$
- connection with dynamical entropy

possible explanation: doing some kind of noncommutative nonlinear potential theory — sans le savoir

Monsieur Jourdain

Par ma foi! Il y a plus de quarante ans que je fais du potentiel nonlinéaire noncommutatif sans que j'en susse rien.

condenser capacity in nonlinear potential theory

$\Omega \subset \mathbb{R}^n$ open

$K, L \Subset \Omega$, $K \cap L = \emptyset$ condenser

$\mathrm{cap}_p(K, L; \Omega) =$

$= \inf \{ \|\nabla u\|_p^p \,/\, u \in C_0^\infty(\Omega), 0 \le u \le 1, u|_K \equiv 1, u|_L \equiv 0 \}$

$L = \emptyset$

$$\mathrm{cap}_p(K;\Omega) = \inf\{\|\nabla u\|_p^p \mid u \in C_0^\infty(\Omega),\ 0 \le u \le 1,\ u|_K \equiv 1\}$$

$$\mathrm{cap}_p(\Omega) = \sup_{K \Subset \Omega} \mathrm{cap}_p(K;\Omega)$$

$p = 2$ linear potential theory

generalization: Lorentz p,q-norms

$\|\ \|_p \rightsquigarrow \|\ \|_{p,q}$

power scaling

$$\mathrm{cap}_p(K, L; \Omega) \rightsquigarrow (\mathrm{cap}_p(K, L; \Omega))^{1/p}$$

$$\| \ \|_p^p \rightsquigarrow \| \ \|_p$$

more general:

$\| \ \|_p$ replaced by general rearrangement invariant norm $\| \ \|_\Phi$

$$\inf\{ \|\nabla u\|_\Phi \mid u \in C_0^\infty(\Omega), 0 \le u \le 1, u|_K \equiv 1, u|_L \equiv 0 \}$$

noncommutative analogy dictionary

$K, L \Subset \Omega$ $\rightsquigarrow$ P, Q finite rank projections

$K \cap L = \emptyset$ $\qquad$ $PQ = 0$

∇ $\rightsquigarrow$ $[\cdot, \tau]$, $\tau = (T_j)_{1 \le j \le n}$

$C_0^\infty(\Omega)$ $\rightsquigarrow$ $\mathcal{R}$

$\| \ \|_\Phi$ $\rightsquigarrow$ $| \ |_{\mathcal{J}}$ normed ideal norm

rearrangement invariant norm

condenser quasicentral modulus

P, Q finite rank projections $PQ = 0$

$$k_{\mathcal{J}}(\tau; P, Q) =$$
$$= \inf\{|[A,\tau]|_{\mathcal{J}} \mid A \in \mathcal{R}_1^+, AP = P, AQ = 0\}$$

$$k_{\mathcal{J}}(\tau; P) = \inf\{|[A,\tau]|_{\mathcal{J}} \mid A \in \mathcal{R}_1^+, AP = P\}$$

$$k_{\mathcal{J}}(\tau) = \sup\{k_{\mathcal{J}}(\tau; P) \mid P \text{ finite rank projection}\}$$

fact: same as $k_{\mathcal{J}}(\tau)$ discussed before.

discrete groups

G group with generator $\{g_1, \cdots, g_n\}$

λ regular representation of G on $\ell^2(G)$.

$\ell_{\mathcal{J}}(G)$ functions $f : G \longrightarrow \mathbb{C}$ with $\mathcal{J}$-norm

i.e. diagonal operators in $\mathcal{J}(\ell^2(G))$.

$X, X_1, X_2 \subset G$ finite, $X_1 \cap X_2 = \emptyset$

$$\mathrm{cap}_{\mathcal{J}}(X_1, X_2) = $$
$$= \inf \{ \max_{1 \le j \le n} |u(g_j \cdot) - u(\cdot)|_{\mathcal{J}} \mid 0 \le u \le 1,\ u/_{X_1} \equiv 1,\ u/_{X_2} \equiv 0 \ \text{supp}\, u \text{ finite} \}$$

$X \subset G$ finite

$$\operatorname{cap}_\gamma(X) = \operatorname{cap}_\gamma(X, \emptyset)$$

$$\operatorname{cap}_\gamma(G) = \sup_{X \subset G} \operatorname{cap}_\gamma(X)$$

<u>Fact</u>: $\gamma = \{g_1, \cdots, g_n\}$

$$\operatorname{cap}_\gamma(X_1, X_2) = k_\gamma(\lambda(\gamma); P_{\ell^2(X_1)}, P_{\ell^2(X_2)})$$

$$\operatorname{cap}_\gamma(X) = k_\gamma(\lambda(\gamma); P_{\ell^2(X)})$$

$$\operatorname{cap}_\gamma(G) = k_\gamma(\lambda(\gamma))$$

$\operatorname{cap}_p(G) > 0$ Yamasaki hyperbolicity

$\operatorname{cap}_p(G) = 0$ Yamasaki parabolicity example

$G = \mathbb{Z}$, $\gamma = \{1\}$, $J = \mathcal{C}_p$, $U = \lambda(1)$

$M, N \subset \mathbb{Z}$ finite, $a = \inf(M \cup N)$, $b = \sup(M \cup N)$

$a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le b$

(a_j, b_j) maximal open intervals so that

$(a_j, b_j) \cap (M \cup N) = \emptyset$, $\#(\{a_j, b_j\} \cap M) = \#(\{a_j, b_j\} \cap N) = 1$

then:

$$k_p(U; P_M, P_N) = \Big(\sum_j (b_j - a_j)^{1-p} \Big)^{1/p} \text{ if } 1 < p < \infty$$

$$k_1(U; P_M, P_N) = m + \#(\{a, b\} \cap M).$$

<u>symmetry</u>

if $k_y(\tau) = 0$ then

$$k_y(\tau; P, Q) = k_y(\tau; Q, P)$$

Lower bound

$\tau = (T_j)_{1 \le j \le n}$, $T_j = T_j^*$ $1 \le j \le n$

dual normed ideal $(\mathcal{J}^d, |\ |_{\mathcal{J}^d})$ of $(\mathcal{J}, |\ |_{\mathcal{J}})$

$\Omega = \{ X_j = X_j^*, 1 \le j \le n \mid i \sum_j [T_j, X_j] \in \mathcal{B}(\mathcal{H})_+ + \mathcal{C}_1, \sum_j |X_j|_{\mathcal{J}^d} = 1 \}$

Then:

$$k_{\mathcal{J}}(\tau; P, Q) \geq \sup \{ \operatorname{Tr} PYP - \operatorname{Tr}((I-P-Q)Y(I-P-Q))_- \mid Y = i \sum_j [T_j, X_j], (X_j)_{1 \le j \le n} \in \Omega \}$$

if $k_{\mathcal{J}}(\tau; P, Q) > 0$ equality holds.

condenser variational problem

$\mathcal{J} = \mathcal{C}_p \,, \quad 2 \leq p < \infty$

modification

$$\max_{1 \leq j \leq n} |[T_j, A]|_p \rightsquigarrow I(A) = \left| \begin{pmatrix} [T_1, A] \\ \vdots \\ [T_n, A] \end{pmatrix} \right|_p$$

P, Q finite rank projections) condenser
$PQ = 0$

$$\mathcal{T}^0_{PQ} = \{ A \in \mathcal{R}^+_1 \mid AP = P, AQ = 0 \}$$

$$\mathcal{C}_{PQ} = \{ 0 \leq B \leq I \mid BP = P, BQ = 0 \}$$

minimization of $I(\cdot)$

$k_p(\tau) = 0$

$\Rightarrow \inf\{I(A) \mid A \in \tau^{\circ}_{PQ}\} = \inf\{I(B) \mid B \in \tau_{PQ}\}$

$\exists X \in \tau_{PQ}$

$\bar{I}(X) = \inf\{I(A) \mid A \in \tau^{\circ}_{PQ}\}$

X unique mod $N = (\tau)'$

$X \in \mathcal{G}_{PQ}$ minimizer

$P \leq P_1$, $Q \leq Q_1$, $P_1 Q_1 = 0$ projections

$X P_1 = P_1$, $X Q_1 = 0$

$$(I - P - Q_1)\, H\, (I - P - Q_1) \geq 0$$

$$(I - P_1 - Q)\, H\, (I - P_1 - Q) \leq 0$$

$$H = \sum_k \Big[T_k, [X, T_k]\Big(-\sum_j [X,T_j]^2\Big)^{\frac{p}{2}-1} + \Big(-\sum_j [X,T_j]^2\Big)^{\frac{p}{2}-1}[X, T_k]\Big]$$

[analogue of p-Laplace equn.

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$$

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