TRACIALLY COMPLETE C^*-ALGEBRAS

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Joint work with: Carrión, Castillejos, Evington, Gabe, Schafhauser, Tikuisis.

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Proof steps in abstract classification of maps $A \to B$.

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PROOF STEPS IN ABSTRACT CLASSIFICATION OF MAPS $A \to B$.

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$$\begin{tikzcd} & ((\pi_\tau)(B)'')^\omega \\ A \arrow[ur, "\theta"'] & \end{tikzcd}$$

Proof steps in abstract classification of maps $A \to B$.

1. Classify weakly nuclear θ by traces.

Recall

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Proof steps in abstract classification of maps $A \to B$.

1. Classify weakly nuclear θ by traces.
2. Classify lifts to ϕ by KK.

These KK computations need J to be reasonably nice

- J is "separably stable": given a separable $J_0 \subset J$, there is a stable separable J_1 with $J_0 \subset J_1 \subset J$
- Get this from regularity hypotheses on B.

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Proof steps in abstract classification of maps $A \to B$.

1. Classify weakly nuclear θ by traces.
2. Classify lifts to ϕ by KK.
3. Compute the relevant KK using the UCT.
4. Intertwine to classify maps $A \to B$.

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These *KK* computations need *J* to be reasonably nice

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- Get this from regularity hypotheses on B.

When *B* has infinitely many extremal traces

- Tempting to use $(B^{**}_{\mathrm{fin}})^\omega$ in place of $(\pi_\tau(B)'')^\omega$ as one has classification by traces
- But the corresponding J is not separably stable - no chance of pulling off the KK-computations.

CONSTRUCTING $(\mathcal{R}_X, X)$

Given a metrisable Choquet simplex X

1. Write as an inverse limit of finite dimensional simplices

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2. $$(\mathcal{R}^{\oplus \partial_e X_1}, X_1) \xrightarrow{\theta_1} (\mathcal{R}^{\oplus \partial_e X_2}, X_2) \xrightarrow{\theta_2} \dots$$

   with θ_n inducing α_n.

Constructing $(\mathcal{R}_X, X)$

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3. Inductive limit in the category of tracially complete C^*-algebras:
   - Take C^*-algebra inductive limit
   - and tracially complete with respect to X

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$(\mathcal{R}_X, X)$ IS INDEPENDENT OF ALL CHOICES

- analogous to Murray and von Neumann's existence and uniqueness of $\mathcal{R}$.

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Classification of factorial separable amenable tracially complete C^*-algebras with property Γ by traces	(Murray & von Neumann): $\exists$! injective II_1 factor $\mathcal{R}$
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For a factorial separable amenable tracially complete $(\mathcal{M}, X)$, TFAE: • $(\mathcal{M}, X)$ has property Γ • $(\mathcal{M}, X)$ is McDuff • $(\mathcal{M}, X)$ has hyperfinite	(Connes): A separable injective II_1 factor $\mathcal{M}$: • has property Γ • is McDuff: $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}$ • is hyperfinite.

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