Analysis with simple Lie groups and lattices

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Section 7 : Lie Theory and Generalizations
Section 8 : Analysis

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Plan

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2. Some analysis questions on $\mathrm{SL}_d(\mathbf{Z})$

   Fourier analysis

   Approximation properties

   Actions on low-dimensional manifolds : Zimmer's program

   Group actions on Banach spaces and their geometry

3. A tool : rank 0 reduction

An example of arithmetic group : $\mathrm{SL}_d(\mathbf{Z})$

$\mathrm{SL}_d(\mathbf{Z})$ = the group of all $d \times d$ matrices with determinant 1 and integer coefficients.

$d = 2$: $\mathrm{PSL}_2(\mathbf{Z}) \simeq (\mathbf{Z}/3\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z})$. Consequence : $\mathrm{SL}_2(\mathbf{Z})$ has **many** actions.

$d \geq 3$: (General expectation) $\mathrm{SL}_d(\mathbf{Z})$ has very few actions.

Theorem (Kazhdan 67)

$\mathrm{SL}_{d\geq 3}(\mathbf{Z})$ has Kazhdan's property (T) : its trivial representation is isolated in its space of unitary representations.

The classical method to study arithmetic groups

1. Passing from $\mathrm{SL}_d(\mathbf{Z})$ to $\mathrm{SL}_d(\mathbf{R})$.
   Minkowski/Borel-Harish-Chandra : $\mathrm{SL}_d(\mathbf{Z})$ is a **lattice** in $\mathrm{SL}_d(\mathbf{R})$ (discrete subgroup with $\mathrm{Haar}(\mathrm{SL}_d(\mathbf{R})/\mathrm{SL}_d(\mathbf{Z})) < \infty$).
2. SL_2-reduction : study $\mathrm{SL}_d(\mathbf{R})$ through the copies of $\mathrm{SL}_2(\mathbf{R})$ contained in $\mathrm{SL}_d(\mathbf{R})$.

Example (Jacobson-Morozov) every unipotent $u \in G$ (spectrum(u)$= \{1\}$) is the image of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by a homomorphism $\mathrm{SL}_2 \to G$.

Some analysis questions on $\mathrm{SL}_d(\mathbf{Z})$

Group von Neumann algebras

If Γ is a countable group, define $\mathcal{L}\Gamma \subset B(\ell_2(\Gamma))$ the algebra of bounded left-convolution operators on $\ell_2(\Gamma)$:

$$\lambda(a)\xi = a * \xi : s \mapsto \sum_{t\in\Gamma} a(t)\xi(t^{-1}s).$$

It is stable by adjoint, and w-* closed (**von Neumann algebra of** Γ).

Example : if $\Gamma = \mathbf{Z}$, by Fourier transform, $\mathcal{L}\mathbf{Z} \simeq L_\infty(\mathbf{R}/\mathbf{Z})$ $(f \in L_\infty(\mathbf{R}/\mathbf{Z}) \mapsto \lambda(\hat{f}))$.

So $\mathcal{L}Z \simeq \mathcal{L}\Gamma$ for every countable infinite abelian group Γ.

Conjecture (Connes 80)

If $d \neq n$, $\mathcal{L}\mathrm{PSL}_d(\mathbf{Z})$ and $\mathcal{L}\mathrm{PSL}_n(\mathbf{Z})$ are not isomorphic.

Fourier series : L^p convergence

Very classical theorem (Riesz 24)

If $1 < p < \infty$ and $f \in L_p(\mathbf{R}/\mathbf{Z})$. Define $S_N f(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2i\pi nt}$. Then

$$\lim_N \|f - S_N f\|_p = 0.$$

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False for $p = 1, \infty$, but there are more clever summation methods :

Theorem (Fejér 1900)

If $1 \leq p \leq \infty$ and $f \in L_p(\mathbf{R}/\mathbf{Z})$ (with f continuous if $p = \infty$), then

$$\lim_N \|f - W_N f\|_p = 0, \text{ where } W_N f(t) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{2i\pi nt}.$$

Such clever summation methods exist for $\mathbf{Z}$ replaced by Γ countable abelian group, and $\mathbf{R}/\mathbf{Z}$ by $\hat{\Gamma}$, its Pontryagin dual.

Fourier synthesis for other (non-abelian) groups?

Define $L_p(\mathcal{L}\Gamma)$ the completion of $\mathcal{L}\Gamma$ for the norm

$$\|f\|_p = (\tau(|f|^p))^{\frac{1}{p}}, \text{ where } \tau(\lambda(a)) = a(1_\gamma).$$

Every $f \in L_p(\mathcal{L}\Gamma)$ has Fourier coefficients f " $=$" $\sum_\gamma \hat{f}(\gamma)\lambda(\gamma)$.

We say that Γ has an L_p-**Fourier summation method** if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbf{C}$ such that $\forall f \in L_p(\mathcal{L}\Gamma)$,

$$\lim_N \|f - T_{\varphi_N}(f)\|_p = 0, \text{ where } T_{\varphi_N}(f) = \sum_\gamma \varphi_N(\gamma)\hat{f}(\gamma)\lambda(\gamma).$$

Open : let $p \neq 2$. Does there exist a group without L_p-Fourier summation method?

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Open : let $p \neq 2$. Does there exist a group without L_p-Fourier summation method?

Conjecture

$\mathrm{SL}_{d\geq 3}(\mathbf{Z})$ has no L_p-Fourier summation method for any $p > 4$.

Equivalently for $p < \frac{4}{3}$. Perhaps even $p \neq 2$?

Operator space variant

Let S_p=Schatten p-class $= \{T \in B(\ell_2) \mid Tr(|T|^p) < \infty\}$.

Define $L_p(\mathcal{L}\Gamma; S_p)$ the completion of $S_p \otimes \mathcal{L}\Gamma \subset B(\ell_2(\mathbf{N} \times \Gamma))$ for the norm

$$\|f\|_p = (\mathrm{Tr} \otimes \tau(|f|^p))^{\frac{1}{p}}.$$

Again, every $f \in L_p(\mathcal{L}\Gamma; S_p)$ has a Fourier series $f = \sum_\gamma \hat{f}(\gamma) \otimes \lambda(\gamma)$ with $\hat{f}(\gamma) \in S_p$.

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We say that Γ has a **completely bounded L_p-Fourier summation method** if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbf{C}$ such that $\forall f \in L_p(\mathcal{L}\Gamma; S_p)$,

$$\lim_N \|f - T_{\varphi_N}(f)\|_p = 0, \text{ where } T_{\varphi_N}(f) = \sum_\gamma \varphi_N(\gamma)\hat{f}(\gamma) \otimes \lambda(\gamma).$$

No completely bounded Fourier synthesis for SL_3

Theorem (Lafforgue–dlS 11, de Laat–dlS 16)

Let $\Gamma = \mathrm{SL}_3(\mathbf{Z})$. For every $4 < p < \infty$ or $1 \leq p < \frac{4}{3}$, there is $f \in L_p(\mathcal{L}\Gamma; S_p)$ such that, for every finitely supported $\varphi : \Gamma \to \mathbf{C}$,

$$\|f - T_\varphi(f)\|_p \geq 1, \text{ where } T_\varphi(f) = \sum_\gamma \varphi(\gamma)\hat{f}(\gamma) \otimes \lambda(\gamma).$$

If $\Gamma = \mathrm{SL}_{d\geq 3}(\mathbf{Z})$, the same holds for $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$.

Banach space approximation property

Definition

A Banach space X has the approximation property if $\mathrm{id} : X \to X$ belongs to the closure of finite rank operator for the topology of uniform convergence on compact subsets of X.

- Equivalently : X has AP if $\forall Y, F(Y,X)$ is dense in $K(Y,X)$.
- (Grothendieck's thesis 55) Conjecture : every X has AP.
- (Grothendieck's *résumé* 53) Conjecture : $\exists X$ without AP.
- Only one natural example without AP : $B(\ell_2)$ (Szankowski 81).

Conjecture

For $\Gamma = \mathrm{SL}_3(\mathbf{Z})$, $C^*_\lambda(\Gamma)$ does not have the AP;

$L_p(\mathcal{L}\Gamma)$ does not have the AP for $p > 4$.

Theorem (Lafforgue–dlS 11, de Laat–dlS 16)

Assume either

- $\Gamma = \mathrm{SL}_3(\mathbf{Z})$ and $4 < p < \infty$ or $1 \leq p < \frac{4}{3}$,
- (more general) $\Gamma = \mathrm{SL}_{d\geq 3}(\mathbf{Z})$ and $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$.

Then $L_p(\mathcal{L}\Gamma)$ ($C^*_\lambda\Gamma$ if $p = \infty$) does not have the operator space approximation property.

Remark : if the condition $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$ was also necessary (open), this would settle Connes' conjecture.

Actions on low-dimensional manifolds : Zimmer's program

Theorem (Brown-Fisher-Hurtado 2020)

If $\alpha : \mathrm{SL}_d(\mathbf{Z}) \to \mathrm{Diff}(M)$ is an action by C^∞-diffeomorphisms on a compact manifold M of dimension $< d - 1$, then α has finite image.

The proof has several parts. One of them is :

Theorem (combine BFH 2020+dlS 2019)

If $\alpha : \mathrm{SL}_d(\mathbf{Z}) \to \mathrm{Diff}(M)$ has subexponential growth of derivatives :

$$\lim_{|\gamma| \to \infty} \frac{1}{|\gamma|} \sum_{x \in X} \log \|D_x \alpha(\gamma)\| = 0,$$

then there is a Riemanian metric on M for which α acts by isometries.

Group actions on Banach spaces and their geometry

Conjecture (Bader-Furman-Gelander-Monod 08)

Every action by isometries of $\mathrm{SL}_{d\geq 3}(\mathbf{Z})$ on a uniformly convex Banach space has a fixed point.

True for :

- Hilbert spaces (Kazhdan 67, Delorme 77),
- L_p spaces (BFGM 08),
- $\mathrm{SL}_d(\mathbf{Z})$ replaced by $\mathrm{SL}_d(\mathbf{F}_q[T])$ (Lafforgue 09),
- all d large enough, if X is a Banach space and $\exists C > 0, \beta < \frac{1}{2}$ such that every finite-dimensional subspace of X is $\leq C\dim(E)^\beta$-isomorphic to a Euclidean space (de Laat-Mimura-dlS 16).

A tool : rank $\circ$ reduction

A tool : rank 0 reduction

For all the previous questions, we start by translating the question to $\mathrm{SL}_d(\mathbf{R})$ (induction).

Then the usual techniques of SL_2-reduction are not effective. All the results mentionned are proved with a same new method, which originates from Vincent Lafforgue's work on strong property (T).

This method proceeds in two steps :

- do analysis on compact subgroups.
- Exploit how compact subgroups are distorted.

Illustration : Lafforgue's proof of (T) for $G = \mathrm{SL}_3(\mathbf{R})$

Denote $K = \mathrm{SO}(3)$ and $U \simeq \mathrm{SO}(2) \subset K$: $U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K$.

Step 1 : Hölder $\frac{1}{2}$-continuity of U-biinvariant matrix coefficients of unitary representations of K.

Step 2, from $U \subset K$ to $K \subset G$: promote this Hölder $\frac{1}{2}$-continuity to K-biinvariant matrix coefficients of unitary representations of G, with **exponentially decaying** Hölder constants.

Conclusion : K-biinvariant matrix coefficients converge exponentially fast, property (T) follows.

Matrix coefficients of $K = \mathrm{SO}(3)$

Proposition

Let $(\pi, \mathcal{H})$ be unitary representation of K, and $\xi, \eta \in \mathcal{H}$ be $\pi(U)$-invariant unit vectors.

For every $k, k' \in K$ with $k'_{1,1} = 0$,

$$|\langle \pi(k)\xi, \eta\rangle - \langle \pi(k')\xi, \eta\rangle| \leq 2|k_{1,1}|^{\frac{1}{2}}$$

Equivalent formulation, in terms of Harmonic analysis on the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$.

For $\delta \in [-1, 1]$, define an operator T_δ on $L_2(\mathbf{S}^2)$

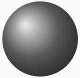
$$T_\delta f(x) = \text{average of } f \text{ on } \{y \in \mathbf{S}^2 \mid \langle x, y\rangle = \delta\}.$$

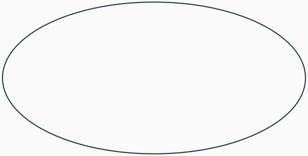
The Proposition is equivalent to $\|T_\delta - T_0\|_{L_2(\mathbf{S}^2)\to L_2(\mathbf{S}^2)} \leq 2|\delta|^{\frac{1}{2}}$.

Weyl chamber exploration for $\mathrm{SL}_3(\mathbf{R})$ (Lafforgue)

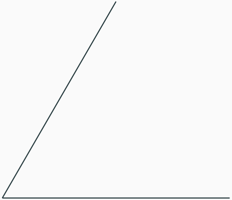
$G = \mathrm{SL}_3(\mathbf{R})$,
$K = \mathrm{SO}_3$ maximal compact.

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K \simeq \mathrm{SO}(2).$$

K/U 

 G/K

$U\backslash K/U$ 

 $K\backslash G/K$

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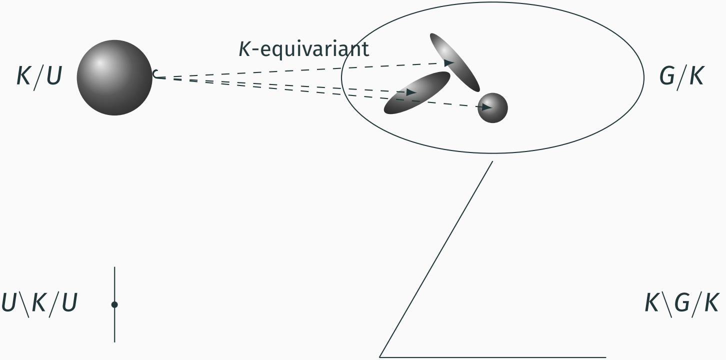

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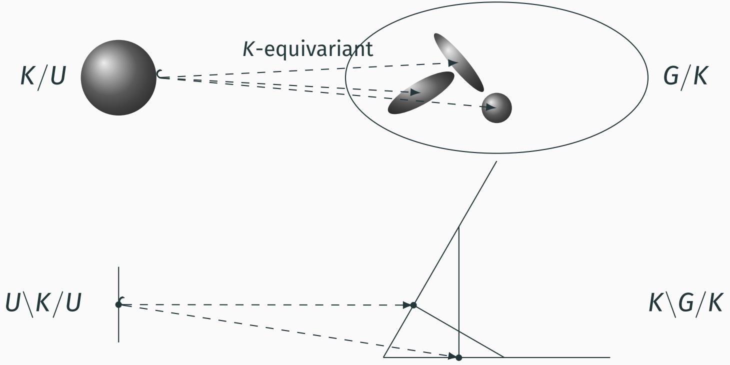

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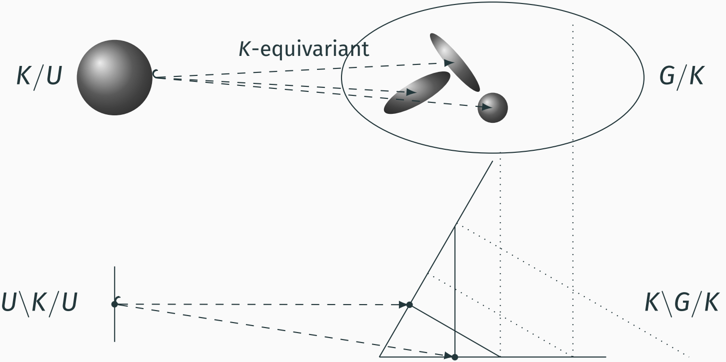

Beyond property (T)

To summarize :

Step 1 : analysis on compact groups.

Step 2 : combinatorics/geometry of the Weyl chambers.

When we change the setting, the challenge comes from the first part : understanding analysis with compact groups. Often very easy.

Beyond : a few examples

- No L_∞-Fourier summation method for $\mathrm{SL}_3(\mathbf{R})$.
  Ingredient : L_∞-completely bounded Fourier multipliers of a compact group coincides with matrix coefficients of unitary representations.
- No L_p-Fourier summation method if $p > 4$.
  Ingredient : $\delta \mapsto T_\delta \in S_p(L_2\mathbf{S}^2)$ is Hölder-continuous if $p > 4$.
- Strong (T) : Form of property (T) for non-unitary representations on Hilbert spaces.
  Ingredient : every representation of a compact group on a Hilbert space is similar to a unitary representation.
- Banach-space representations.
  Ingredient/challenge : understand regularity properties of $\delta \mapsto T_\delta \in B(L_2(\mathbf{S}^2; X))$ in terms of the geometry of X.