Analysis with simple Lie groups and lattices

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Section 8 : Analysis

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Plan

1. An example of arithmetic group: $\text{SL}_d(\mathbb{Z})$

2. Some analysis questions on $\text{SL}_d(\mathbb{Z})$
   - Fourier analysis
   - Approximation properties
   - Actions on low-dimensional manifolds: Zimmer’s program
   - Group actions on Banach spaces and their geometry

3. A tool: rank 0 reduction
An example of arithmetic group: 

$\text{SL}_d(\mathbb{Z})$
$\text{SL}_d(\mathbb{Z})$ = the group of all $d \times d$ matrices with determinant 1 and integer coefficients.

\[ d = 2 : \text{PSL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/3\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}). \]

Consequence: $\text{SL}_2(\mathbb{Z})$ has many actions.

\[ d \geq 3 : \text{(General expectation)} \text{SL}_d(\mathbb{Z}) \text{ has very few actions.} \]

**Theorem (Kazhdan 67)**

$\text{SL}_{d \geq 3}(\mathbb{Z})$ has Kazhdan’s property (T): its trivial representation is isolated in its space of unitary representations.
1. Passing from $\text{SL}_d(\mathbb{Z})$ to $\text{SL}_d(\mathbb{R})$.
   Minkowski/Borel-Harish-Chandra: $\text{SL}_d(\mathbb{Z})$ is a lattice in $\text{SL}_d(\mathbb{R})$ (discrete subgroup with $\text{Haar}(\text{SL}_d(\mathbb{R})/\text{SL}_d(\mathbb{Z})) < \infty$).

2. $\text{SL}_2$-reduction: study $\text{SL}_d(\mathbb{R})$ through the copies of $\text{SL}_2(\mathbb{R})$ contained in $\text{SL}_d(\mathbb{R})$.

Example (Jacobson-Morozov) every unipotent $u \in G$ (spectrum($u$) = $\{1\}$) is the image of \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\] by a homomorphism $\text{SL}_2 \rightarrow G$. 
Some analysis questions on $SL_d(Z)$
If $\Gamma$ is a countable group, define $L\Gamma \subset B(\ell_2(\Gamma))$ the algebra of bounded left-convolution operators on $\ell_2(\Gamma)$:

$$\lambda(a)\xi = a \ast \xi : s \mapsto \sum_{t \in \Gamma} a(t)\xi(t^{-1}s).$$

It is stable by adjoint, and $w$-* closed (von Neumann algebra of $\Gamma$).

Example: if $\Gamma = \mathbb{Z}$, by Fourier transform, $L\mathbb{Z} \simeq L_\infty(\mathbb{R}/\mathbb{Z})$ ($f \in L_\infty(\mathbb{R}/\mathbb{Z}) \mapsto \lambda(\hat{f})$).

So $L\mathbb{Z} \simeq L\Gamma$ for every countable infinite abelian group $\Gamma$.

**Conjecture (Connes 80)**

If $d \neq n$, $L\text{PSL}_d(\mathbb{Z})$ and $L\text{PSL}_n(\mathbb{Z})$ are not isomorphic.
Fourier series: $L^p$ convergence

Very classical theorem (Riesz 24)

If $1 < p < \infty$ and $f \in L^p(\mathbb{R}/\mathbb{Z})$. Define $S_Nf(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{2i\pi nt}$. Then

$$\lim_{N} \|f - S_Nf\|_p = 0.$$
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$$\lim_{N} \|f - S_N f\|_p = 0.$$  

False for $p = 1, \infty$, but there are more clever summation methods:

**Theorem (Fejér 1900)**

If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}/\mathbb{Z})$ (with $f$ continuous if $p = \infty$), then

$$\lim_{N} \|f - W_N f\|_p = 0, \text{ where } W_N f(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n)e^{2i\pi nt}.$$  

Such clever summation methods exist for $\mathbb{Z}$ replaced by $\Gamma$ countable abelian group, and $\mathbb{R}/\mathbb{Z}$ by $\hat{\Gamma}$, its Pontryagin dual.
Fourier synthesis for other (non-abelian) groups?

Define $L_p(\mathcal{L}\Gamma)$ the completion of $\mathcal{L}\Gamma$ for the norm

$$
\|f\|_p = (\tau(|f|^p))^{\frac{1}{p}}, \text{ where } \tau(\lambda(a)) = a(1_\gamma).
$$

Every $f \in L_p(\mathcal{L}\Gamma)$ has Fourier coefficients $f^{\sim} = \sum_\gamma \hat{f}(\gamma)\lambda(\gamma)$.

We say that $\Gamma$ has an $L_p$-Fourier summation method if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbb{C}$ such that

$$
\forall f \in L_p(\mathcal{L}\Gamma),
\lim_N \|f - T_{\varphi_N}(f)\|_p = 0, \text{ where } T_{\varphi_N}(f) = \sum_\gamma \varphi_N(\gamma)\hat{f}(\gamma)\lambda(\gamma).
$$

Open: let $p \neq 2$. Does there exist a group without $L_p$-Fourier summation method?
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$$\|f\|_p = (\tau(|f|^p))^{\frac{1}{p}}, \text{ where } \tau(\lambda(a)) = a(1_{\gamma}).$$

Every $f \in L_p(\mathcal{L}\Gamma)$ has Fourier coefficients $f \sim \sum_{\gamma} \hat{f}(\gamma) \lambda(\gamma)$.

We say that $\Gamma$ has an $L_p$-Fourier summation method if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbb{C}$ such that

$$\forall f \in L_p(\mathcal{L}\Gamma), \lim_{N} \|f - T_{\varphi_N}(f)\|_p = 0, \text{ where } T_{\varphi_N}(f) = \sum_{\gamma} \varphi_N(\gamma) \hat{f}(\gamma) \lambda(\gamma).$$

Open: let $p \neq 2$. Does there exist a group without $L_p$-Fourier summation method?

**Conjecture**

$SL_{d \geq 3}(\mathbb{Z})$ has no $L_p$-Fourier summation method for any $p > 4$.

Equivalently for $p < \frac{4}{3}$. Perhaps even $p \neq 2$?
Let $S_p =$ Schatten $p$-class $= \{ T \in B(\ell_2) \mid \text{Tr}(|T|^p) < \infty \}$.

Define $L_p(\mathcal{L} \Gamma; S_p)$ the completion of $S_p \otimes \mathcal{L} \Gamma \subset B(\ell_2(\mathbb{N} \times \Gamma))$ for the norm

$$\|f\|_p = (\text{Tr} \otimes \tau(|f|^p))^{\frac{1}{p}}.$$ 

Again, every $f \in L_p(\mathcal{L} \Gamma; S_p)$ has a Fourier series $f = \sum_{\gamma} \hat{f}(\gamma) \otimes \lambda(\gamma)$ with $\hat{f}(\gamma) \in S_p$. 
Let $S_p$-Schatten $p$-class $= \{ T \in B(\ell_2) \mid \text{Tr}(|T|^p) < \infty \}$.

Define $L_p(\mathcal{L}\Gamma; S_p)$ the completion of $S_p \otimes \mathcal{L}\Gamma \subset B(\ell_2(\mathbb{N} \times \Gamma))$ for the norm

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Again, every $f \in L_p(\mathcal{L}\Gamma; S_p)$ has a Fourier series $f = \sum \gamma \hat{f}(\gamma) \otimes \lambda(\gamma)$ with $\hat{f}(\gamma) \in S_p$.

We say that $\Gamma$ has a **completely bounded** $L_p$-Fourier summation method if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbb{C}$ such that $\forall f \in L_p(\mathcal{L}\Gamma; S_p)$,

$$\lim_N \|f - T_{\varphi_N}(f)\|_p = 0,$$

where $T_{\varphi_N}(f) = \sum \gamma \varphi_N(\gamma)\hat{f}(\gamma) \otimes \lambda(\gamma)$. 

No completely bounded Fourier synthesis for $SL_3$

**Theorem (Lafforgue–dlS 11, de Laat–dlS 16)**

Let $\Gamma = SL_3(\mathbb{Z})$. For every $4 < p < \infty$ or $1 \leq p < \frac{4}{3}$, there is $f \in L_p(\mathcal{L}\Gamma; S_p)$ such that, for every finitely supported $\varphi : \Gamma \to \mathbb{C}$,

$$\|f - T\varphi(f)\|_p \geq 1,$$

where $T\varphi(f) = \sum_{\gamma} \varphi(\gamma)\hat{f}(\gamma) \otimes \lambda(\gamma)$.

If $\Gamma = SL_{d \geq 3}(\mathbb{Z})$, the same holds for $\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{c}{d-2}$. 
Banach space approximation property

Definition

A Banach space $X$ has the approximation property if $\text{id} : X \to X$ belongs to the closure of finite rank operator for the topology of uniform convergence on compact subsets of $X$.

- Equivalently: $X$ has AP if $\forall Y, F(Y, X)$ is dense in $K(Y, X)$.
- (Grothendieck’s thesis 55) Conjecture: every $X$ has AP.
- (Grothendieck’s résumé 53) Conjecture: $\exists X$ without AP.
- Only one natural example without AP: $B(\ell_2)$ (Szankowski 81).

Conjecture

For $\Gamma = \text{SL}_3(\mathbb{Z})$, $C^*_\lambda(\Gamma)$ does not have the AP;

$L_p(\mathcal{L}\Gamma)$ does not have the AP for $p > 4.$
### Theorem (Lafforgue–dlS 11, de Laat–dlS 16)

Assume either

- $\Gamma = \text{SL}_3(\mathbb{Z})$ and $4 < p < \infty$ or $1 \leq p < \frac{4}{3}$,
- (more general) $\Gamma = \text{SL}_{d\geq3}(\mathbb{Z})$ and $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$.

Then $L_p(\mathcal{L}\Gamma)$ ($C^*_\lambda\Gamma$ if $p = \infty$) does not have the operator space approximation property.

Remark: if the condition $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$ was also necessary (open), this would settle Connes’ conjecture.
Actions on low-dimensional manifolds: Zimmer’s program

**Theorem (Brown-Fisher-Hurtado 2020)**

If $\alpha : \text{SL}_d(\mathbb{Z}) \to \text{Diff}(M)$ is an action by $C^\infty$-diffeomorphisms on a compact manifold $M$ of dimension $< d - 1$, then $\alpha$ has finite image.

The proof has several parts. One of them is:

**Theorem (combine BFH 2020+dlS 2019)**

If $\alpha : \text{SL}_d(\mathbb{Z}) \to \text{Diff}(M)$ has subexponential growth of derivatives:

$$\lim_{|\gamma| \to \infty} \frac{1}{|\gamma|} \sum_{x \in X} \log \|D_x \alpha(\gamma)\| = 0,$$

then there is a Riemannian metric on $M$ for which $\alpha$ acts by isometries.
Conjecture (Bader-Furman-Gelander-Monod 08)

Every action by isometries of $\text{SL}_{d \geq 3}(\mathbb{Z})$ on a uniformly convex Banach space has a fixed point.

True for:

- Hilbert spaces (Kazhdan 67, Delorme 77),
- $L_p$ spaces (BFGM 08),
- $\text{SL}_d(\mathbb{Z})$ replaced by $\text{SL}_d(\mathbb{F}_q[T])$ (Lafforgue 09),
- all $d$ large enough, if $X$ is a Banach space and $\exists C > 0$, $\beta < \frac{1}{2}$ such that every finite-dimensional subspace of $X$ is $\leq C \text{dim}(E)^\beta$-isomorphic to a Euclidean space (de Laat-Mimura-dlS 16).
A tool: rank o reduction
For all the previous questions, we start by translating the question to $\text{SL}_d(\mathbb{R})$ (induction).

Then the usual techniques of $\text{SL}_2$-reduction are not effective. All the results mentioned are proved with a same new method, which originates from Vincent Lafforgue’s work on strong property (T).

This method proceeds in two steps:

- do analysis on compact subgroups.
- Exploit how compact subgroups are distorted.
Denote $K = \text{SO}(3)$ and $U \cong \text{SO}(2) \subset K$:

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K.$$

Step 1: Hölder $\frac{1}{2}$-continuity of $U$-biinvariant matrix coefficients of unitary representations of $K$.

Step 2, from $U \subset K$ to $K \subset G$: promote this Hölder $\frac{1}{2}$-continuity to $K$-biinvariant matrix coefficients of unitary representations of $G$, with \textbf{exponentially decaying} Hölder constants.

Conclusion: $K$-biinvariant matrix coefficients converge exponentially fast, property (T) follows.
Matrix coefficients of $K = \text{SO}(3)$

**Proposition**

Let $(\pi, \mathcal{H})$ be unitary representation of $K$, and $\xi, \eta \in \mathcal{H}$ be $\pi(U)$-invariant unit vectors.

For every $k, k' \in K$ with $k'_{1,1} = 0$,

$$|\langle \pi(k)\xi, \eta \rangle - \langle \pi(k')\xi, \eta \rangle| \leq 2|k_{1,1}|^{\frac{1}{2}}$$

Equivalent formulation, in terms of Harmonic analysis on the unit sphere $S^2 \subset \mathbb{R}^3$.

For $\delta \in [-1, 1]$, define an operator $T_\delta$ on $L_2(S^2)$

$$T_\delta f(x) = \text{average of } f \text{ on } \{y \in S^2 \mid \langle x, y \rangle = \delta\}.$$ 

The Proposition is equivalent to $\|T_\delta - T_0\|_{L_2(S^2) \rightarrow L_2(S^2)} \leq 2|\delta|^{\frac{1}{2}}$. 

Weyl chamber exploration for $SL_3(\mathbb{R})$ (Lafforgue)

$G = SL_3(\mathbb{R})$, 
$K = SO_3$ maximal compact.

$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K \simeq SO(2)$. 

$K/U \quad G/K \quad U/K/U \quad K/G/K$
Weyl chamber exploration for $\text{SL}_3(\mathbb{R})$ (Lafforgue)

$G = \text{SL}_3(\mathbb{R})$, $K = \text{SO}_3$ maximal compact.

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$K\text{-equivariant}$
Weyl chamber exploration for $\text{SL}_3(\mathbb{R})$ (Lafforgue)

$G = \text{SL}_3(\mathbb{R})$, 
$K = \text{SO}_3$ maximal compact.

$U = \begin{cases} 
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix} 
\end{cases} \cap K \simeq \text{SO}(2)$. 

$K/U \quad \text{K-equivariant} \quad G/K$

$U \setminus K/U \quad K \setminus G/K$
Weyl chamber exploration for $\text{SL}_3(\mathbb{R})$ (Lafforgue)

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$K/U$ is $K$-equivariant.
To summarize:

Step 1: analysis on compact groups.

Step 2: combinatorics/geometry of the Weyl chambers.

When we change the setting, the challenge comes from the first part: understanding analysis with compact groups. Often very easy.
Beyond: a few examples

• No $L_\infty$-Fourier summation method for $SL_3(\mathbb{R})$.
  Ingredient: $L_\infty$-completely bounded Fourier multipliers of a compact group coincides with matrix coefficients of unitary representations.

• No $L_p$-Fourier summation method if $p > 4$.
  Ingredient: $\delta \mapsto T_\delta \in S_p(L_2\mathbb{S}^2)$ is Hölder-continuous if $p > 4$.

• Strong (T): Form of property (T) for non-unitary representations on Hilbert spaces.
  Ingredient: every representation of a compact group on a Hilbert space is similar to a unitary representation.

• Banach-space representations.
  Ingredient/challenge: understand regularity properties of $\delta \mapsto T_\delta \in B(L_2(\mathbb{S}^2; X))$ in terms of the geometry of $X$. 