Properly infinite $C(X)$-algebras and K_1-injectivity

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Abstract

We investigate if a unital $C(X)$-algebra is properly infinite when all its fibres are properly infinite. We show that this question can be rephrased in several different ways, including the question if every unital properly infinite C^*-algebra is K_1-injective. We provide partial answers to these questions, and we show that the general question on proper infiniteness of $C(X)$-algebras can be reduced to establishing proper infiniteness of a specific $C([0,1])$-algebra with properly infinite fibres.

1 Introduction

The problem that we mainly are concerned with in this paper is if any unital $C(X)$-algebra with properly infinite fibres is itself properly infinite (see Section 2 for a brief introduction to $C(X)$-algebras). An analogous study was carried out in the recent paper [8] where it was decided when $C(X)$-algebras, whose fibres are either stable or absorb tensorially a given strongly self-absorbing C^*-algebra, itself has the same property. This was answered in the affirmative in [8] under the crucial assumption that the dimension of the space X is finite, and counterexamples were given in the infinite dimensional case.

Along similar lines, Dadarlat, [5], recently proved that $C(X)$-algebras, whose fibres are Cuntz algebras, are trivial under some K-theoretical conditions provided that the space X is finite dimensional.

The property of being properly infinite turns out to behave very differently than the property of being stable or of absorbing a strongly self-absorbing C^*-algebra. It is relative easy to see (Lemma 2.10) that if a fibre A_x of a $C(X)$-algebra A is properly infinite, then A_F is properly infinite for some closed neighborhood F of x. The (possible) obstruction to proper infiniteness of the $C(X)$-algebra is hence not local. Such an obstruction is also not related to the possible complicated structure of the space X, as we can show that a counterexample, if it exists, can be taken to be a (specific) $C([0,1])$-algebra (Example 4.1 and Theorem 5.5). The problem appears to be related with some rather subtle internal structure properties of properly infinite C^*-algebras.

Cuntz studied purely infinite—and in the process also properly infinite—C^*-algebras, [4], where he among many other things (he was primarily interested in calculating the

K-theory of his algebras $\mathcal{O}_n$) showed that any unital properly infinite C^*-algebra A is K_1-surjective, i.e., the mapping $\mathcal{U}(A) \to K_1(A)$ is onto; and that any purely infinite simple C^*-algebra A is K_1-injective, i.e., the mapping $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ is injective (and hence an isomorphism). He did not address the question if any properly infinite C^*-algebra is K_1-injective. That question has not been raised formally to our knowledge—we do so here—but it does appear implicitly, eg. in [10] and in [14], where K_1-injectivity of properly infinite C^*-algebras has to be assumed.

Proper infiniteness of C^*-algebras has relevance for existence (or rather non-existence) of traces and quasitraces. Indeed, a unital C^*-algebra admits a 2-quasitrace if and only if no matrix algebra over the C^*-algebra is properly infinite, and a unital *exact* C^*-algebra admits tracial state again if and only if no matrix algebra over the C^*-algebra is properly infinite.

In this paper we show that every properly infinite C^*-algebra is K_1-injective if and only if every $C(X)$-algebra with properly infinite fibres itself is properly infinite. We also show that a *matrix algebra* over any such $C(X)$-algebra is properly infinite. Examples of unital C^*-algebras A, where $M_n(A)$ is properly infinite for some natural number $n \geq 2$ but where $M_{n-1}(A)$ is not properly infinite, are known, see [12] and [11], but still quite exotic.

We relate the question if a given properly infinite C^*-algebra is K_1-injective to questions regarding homotopy of projections (Proposition 5.1). In particular we show that our main questions are equivalent to the following question: is any non-trivial projection in the first copy of $\mathcal{O}_\infty$ in the full unital universal free product $\mathcal{O}_\infty * \mathcal{O}_\infty$ homotopic to any (non-trivial) projection in the second copy of $\mathcal{O}_\infty$? The specific $C([0,1])$-algebra, mentioned above, is perhaps not surprisingly a sub-algebra of $C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$.

Using ideas implicit in Rieffel's paper, [9], we construct in Section 4 a $C(\mathbb{T})$-algebra $\mathcal{B}$ for each C^*-algebra A and for each unitary $u \in A$ for which $\operatorname{diag}(u,1)$ is homotopic to $1_{M_2(A)}$; and $\mathcal{B}$ is non-trivial if u is not homotopic to 1_A. In this way we relate our question about proper infiniteness of $C(X)$-algebras to a question about K_1-injectivity.

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2 $C(X)$-algebras with properly infinite fibres

A powerful tool in the classification of C^*-algebras is the study of their projections. A projection in a C^*-algebra is said to be *infinite* if it is equivalent to a proper subprojection of itself, and it is said to be *properly infinite* if it is equivalent to two mutually orthogonal subprojections of itself.

A projection which is not infinite is said to be *finite*. A unital C^*-algebra is said to be finite, infinite, or properly infinite if its unit is finite, infinite, or properly infinite, respectively. If A is a C^*-algebra for which $M_n(A)$ is finite for all positive integers n, then A is *stably finite*.

In this section we will study stability properties of proper infiniteness under (upper-semi-)continuous deformations using the *Cuntz-Toeplitz algebra* which is defined as follows.

For all integers $n \geq 2$ the Cuntz-Toeplitz algebra $\mathcal{T}_n$ is the universal C^*-algebra generated by n isometries $s_1, \dots, s_n$ satisfying the relation

$$s_1 s_1^* + \cdots + s_n s_n^* \leq 1.$$

Remark 2.1 A unital C^*-algebra A is properly infinite if and only if $\mathcal{T}_n$ embeds unitally into A for some $n \geq 2$, in which case $\mathcal{T}_n$ embeds unitally into A for all $n \geq 2$.

In order to study deformations of such algebras, let us recall a few notions from the theory of $C(X)$-algebras.

Let X be a compact Hausdorff space and $C(X)$ be the C^*-algebra of continuous functions on X with values in the complex field $\mathbb{C}$.

Definition 2.2 A $C(X)$-algebra is a C^*-algebra A endowed with a unital *-homomorphism from $C(X)$ to the center of the multiplier C^*-algebra $\mathcal{M}(A)$ of A.

If A is as above and $Y \subseteq X$ is a closed subset, then we put $I_Y = C_0(X \setminus Y)A$, which is a closed two-sided ideal in A. We set $A_Y = A/I_Y$ and denote the quotient map by π_Y.

For an element $a \in A$ we put $a_Y = \pi_Y(a)$, and if Y consists of a single point x, we will write A_x, I_x, π_x and a_x in the place of $A_{\{x\}}, I_{\{x\}}, \pi_{\{x\}}$ and $a_{\{x\}}$, respectively. We say that A_x is the *fibre* of A at x.

The function

$$x \mapsto \|a_x\| = \inf\{\| [1 - f + f(x)]a\| : f \in C(X)\}$$

is upper semi-continuous for all $a \in A$ (as one can see using the right-hand side identity above). A $C(X)$-algebra A is said to be *continuous* (or to be a *continuous C^*-bundle over* X) if the function $x \mapsto \|a_x\|$ is actually continuous for all element a in A.

For any unital C^*-algebra A we let $\mathcal{U}(A)$ denote the group of unitary elements in A, $\mathcal{U}^0(A)$ denotes its connected component containing the unit of A, and $\mathcal{U}_n(A)$ and $\mathcal{U}_n^0(A)$ are equal to $\mathcal{U}(M_n(A))$ and $\mathcal{U}^0(M_n(A))$, respectively.

An element in a C^*-algebra A is said to be *full* if it is not contained in any proper closed two-sided ideal in A.

It is well-known (see for example [13, Exercise 4.9]) that if p is a properly infinite, full projection in a C^*-algebra A, then $e \precsim p$, i.e., e is equivalent to a subprojection of p, for every projection $e \in A$.

We state below more formally three more or less well-known results that will be used frequently throughout this paper, the first of which is due to Cuntz, [4].

Proposition 2.3 (Cuntz) *Let A be a C^*-algebra which contains at least one properly infinite, full projection.*

(i) *Let p and q be properly infinite, full projections in A. Then $[p] = [q]$ in $K_0(A)$ if and only if $p \sim q$.*

(ii) *For each element $g \in K_0(A)$ there is a properly infinite, full projection $p \in A$ such that $g = [p]$.*

The second statement is a variation of the Whitehead lemma.

Lemma 2.4 *Let A be a unital C^*-algebra.*

(i) *Let v be a partial isometry in A such that $1 - vv^*$ and $1 - v^*v$ are properly infinite and full projections. Then there is a unitary element u in A such that $[u] = 0$ in $K_1(A)$ and $v = uv^*v$, i.e., u extends v.*

(ii) *Let u be a unitary element A such that $[u] = 0$ in $K_1(A)$. Suppose there exists a projection $p \in A$ such that $\|up - pu\| < 1$ and p and $1 - p$ are properly infinite and full. Then u belongs to $\mathcal{U}^0(A)$.*

Proof: (i). It follows from Proposition 2.3 (i) that $1 - v^*v \sim 1 - vv^*$, so there is a partial isometry w such that $1 - v^*v = w^*w$ and $1 - vv^* = ww^*$. Now, $z = v + w$ is a unitary element in A with $zv^*v = v$. The projection $1 - v^*v$ is properly infinite and full, so $1 \precsim 1 - v^*v$, which implies that there is an isometry s in A with $ss^* \leq 1 - v^*v$. As $-[z] = [z^*] = [sz^*s^* + (1 - ss^*)]$ in $K_1(A)$ (see eg. [13, Exercise 8.9 (i)]), we see that $u = z(sz^*s^* + (1 - ss^*))$ is as desired.

(ii). Put $x = pup + (1 - p)u(1 - p)$ and note that $\|u - x\| < 1$. It follows that x is invertible in A and that $u \sim_h x$ in $\mathrm{GL}(A)$. Let $x = v|x|$ be the polar decomposition of x, where $|x| = (x^*x)^{1/2}$ and $v = x|x|^{-1}$ is unitary. Then $u \sim_h v$ in $\mathcal{U}(A)$ (see eg. [13, Proposition 2.1.8]), and $pv = vp$. We proceed to show that v belongs to $\mathcal{U}^0(A)$ (which will entail that u belongs to $\mathcal{U}^0(A)$).

Write $v = v_1v_2$, where

$$v_1 = pvp + (1 - p), \qquad v_2 = p + (1 - p)v(1 - p).$$

As $1 - p \precsim p$ we can find a symmetry t in A such that $t(1 - p)t \leq p$. As t belongs to $\mathcal{U}^0(A)$ (being a symmetry), we conclude that $v_2 \sim_h tv_2t$, and one checks that tv_2t is of the form $w + (1 - p)$ for some unitary w in pAp. It follows that v is homotopic to a unitary of the form $v_0 + (1 - p)$, where v_0 is a unitary in pAp. We can now apply eg. [13, Exercise 8.11] to conclude that $v \sim_h 1$ in $\mathcal{U}(A)$. □

We remind the reader that if p, q are projections in a unital C^*-algebra A, then p and q are homotopic, in symbols $p \sim_h q$, (meaning that they can be connected by a continuous path of projections in A) if and only if $q = upu^*$ for some $u \in \mathcal{U}^0(A)$, eg. cf. [13, Proposition 2.2.6].

Proposition 2.5 *Let A be a unital C^*-algebra. Let p and q be two properly infinite, full projections in A such that $p \sim q$. Suppose that there exists a properly infinite, full projection $r \in A$ such that $p \perp r$ and $q \perp r$. Then $p \sim_h q$.*

Proof: Take a partial isometry $v_0 \in A$ such that $v_0^*v_0 = p$ and $v_0v_0^* = q$. Take a subprojection r_0 of r such that r_0 and $r - r_0$ both are properly infinite and full. Put $v = v_0 + r_0$.

Then $vpv^* = q$ and $vr_0 = r_0 = r_0v$. Note that $1 - v^*v$ and $1 - vv^*$ are properly infinite and full (because they dominate the properly infinite, full projection $r - r_0$). Use Lemma 2.4 (i) to extend v to a unitary $u \in A$ with $[u] = 0$ in $K_1(A)$. Now, $upu^* = q$ and $ur_0 = vr_0 = r_0 = r_0v = r_0u$. Hence $u \in \mathcal{U}^0(A)$ by Lemma 2.4 (ii), and so $p \sim_h q$ as desired. $\square$

Definition 2.6 A unital C^*-algebra A is said to be K_1-*injective* if the natural mapping

$$\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$$

is injective. In other words, if A is K_1-injective, and if u is a unitary element in A, then $u \sim_h 1$ in $\mathcal{U}(A)$ if (and only if) $[u] = 0$ in $K_1(A)$.

One could argue that K_1-injectivity should entail that the natural mappings $\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A)$ be injective for every natural number n. However there seem to be an agreement for defining K_1-injectivity as above. As we shall see later, in Proposition 5.2, if A is properly infinite, then the two definitions agree.

Proposition 2.7 *Let A be a unital C^*-algebra that is the pull-back of two unital, properly infinite C^*-algebras A_1 and A_2 along the *-epimorphisms $\pi_1\colon A_1 \to B$ and $\pi_2\colon A_2 \to B$:*

$$\begin{tikzcd}
 & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\
 & B &
\end{tikzcd}$$

Then $M_2(A)$ is properly infinite. Moreover, if B is K_1-injective, then A itself is properly infinite.

Proof: Take unital embeddings $\sigma_i\colon \mathcal{T}_3 \to A_i$ for $i = 1, 2$, where $\mathcal{T}_3$ is the Cuntz-Toeplitz algebra (defined earlier), and put

$$v = \sum_{j=1}^{2} (\pi_1 \circ \sigma_1)(t_j)(\pi_2 \circ \sigma_2)(t_j^*),$$

where t_1, t_2, t_3 are the canonical generators of $\mathcal{T}_3$. Note that v is a partial isometry with $(\pi_1 \circ \sigma_1)(t_j) = v(\pi_2 \circ \sigma_2)(t_j)$ for $j = 1, 2$. As $(\pi_1 \circ \sigma_1)(t_3t_3^*) \leq 1 - vv^*$ and $(\pi_2 \circ \sigma_2)(t_3t_3^*) \leq 1 - v^*v$, Lemma 2.4 (i) yields a unitary $u \in B$ with $[u] = 0$ in $K_1(B)$ and with $(\pi_1 \circ \sigma_1)(t_j) = u(\pi_2 \circ \sigma_2)(t_j)$ for $j = 1, 2$.

If B is K_1-injective, then u belongs to $\mathcal{U}^0(B)$, whence u lifts to a unitary $v \in A_2$. Define $\widetilde{\sigma}_2\colon \mathcal{T}_2 \to A_2$ by $\widetilde{\sigma}_2(t_j) = v\sigma_2(t_j)$ for $j = 1, 2$ (observing that t_1, t_2 generate $\mathcal{T}_2$). Then $\pi_1 \circ \sigma_1 = \pi_2 \circ \widetilde{\sigma}_2$, which by the universal property of the pull-back implies that σ_1

and $\widetilde{\sigma}_2$ lift to a (necessarily unital) embedding $\sigma\colon \mathcal{T}_2 \to A$, thus forcing A to be properly infinite.

In the general case (where B is not necessarily K_1-injective) u may not lift to a unitary element in A_2, but $\mathrm{diag}(u,u)$ does lift to a unitary element v in $M_2(A_2)$ by Lemma 2.4 (ii) (applied with $p = \mathrm{diag}(1,0)$). Define unital embeddings $\widetilde{\sigma}_i\colon \mathcal{T}_2 \to M_2(A_i)$, $i = 1,2$, by

$$\widetilde{\sigma}_1(t_j) = \begin{pmatrix} \sigma_1(t_j) & 0 \\ 0 & \sigma_1(t_j) \end{pmatrix}, \qquad \widetilde{\sigma}_2(t_j) = v \begin{pmatrix} \sigma_2(t_j) & 0 \\ 0 & \sigma_2(t_j) \end{pmatrix},$$

for $j = 1,2$. As $(\pi_1 \otimes \mathrm{id}_{M_2}) \circ \widetilde{\sigma}_1 = (\pi_2 \otimes \mathrm{id}_{M_2}) \circ \widetilde{\sigma}_2$, the unital embeddings $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ lift to a (necessarily unital) embedding of $\mathcal{T}_2$ into $M_2(A)$, thus completing the proof. □

Question 2.8 Is the pull-back of any two properly infinite unital C^*-algebras again properly infinite?

As mentioned in the introduction, one cannot in general conclude that A is properly infinite if one knows that $M_n(A)$ is properly infinite for some $n \geq 2$.

One obvious way of obtaining an answer to Question 2.8, in the light of the last statement in Proposition 2.7, is to answer the question below in the affirmative:

Question 2.9 Is every properly infinite unital C^*-algebra K_1-injective?

We shall see later, in Section 5, that the two questions above in fact are equivalent.

The lemma below, which shall be used several times in this paper, shows that one can lift proper infiniteness from a fibre of a $C(X)$-algebra to a whole neighborhood of that fibre.

Lemma 2.10 *Let X be a compact Hausdorff space, let A be a unital $C(X)$-algebra, let $x \in X$, and suppose that the fibre A_x is properly infinite. Then A_F is properly infinite for some closed neighborhood F of x.*

Proof: Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a decreasing net of closed neighborhoods of $x \in X$, fulfilling that $\bigcap_{\lambda\in\Lambda} F_\lambda = \{x\}$, and set $I_\lambda = C_0(X \setminus F_\lambda)A$. Then $\{I_\lambda\}_{\lambda\in\Lambda}$ is an increasing net of ideals in A, $A_{F_\lambda} = A/I_\lambda$, $I := \overline{\bigcup_{\lambda\in\Lambda} I_\lambda} = C_0(X \setminus \{x\})$, and $A_x = A/I$.

By the assumption that A_x is properly infinite there is a unital *-homomorphism $\psi\colon \mathcal{T}_2 \to A_x$, and since $\mathcal{T}_2$ is semi-projective there is a $\lambda_0 \in \Lambda$ and a unital *-homomorphism $\varphi\colon \mathcal{T}_2 \to A_{F_{\lambda_0}}$ making the diagram

$$\begin{tikzcd} & A_{F_{\lambda_0}} \arrow[d, "\pi_x"] \\ \mathcal{T}_2 \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & A_x \end{tikzcd}$$

commutative. We can thus take F to be F_{λ_0}. □

Theorem 2.11 *Let A be a unital $C(X)$-algebra where X is a compact Hausdorff space. If all fibres A_x, $x \in X$, are properly infinite, then some matrix algebra over A is properly infinite.*

Proof: By Lemma 2.10, X can be covered by finitely many closed sets $F_1, F_2, \dots, F_n$ such that A_{F_j} is properly infinite for each j. Put $G_j = F_1 \cup F_2 \cup \dots \cup F_j$. For each $j = 1, 2, \dots, n-1$ we have a pull-back diagram

$$\begin{tikzcd} & A_{G_{j+1}} \arrow[dl] \arrow[dr] & \\ A_{G_j} \arrow[dr] & & A_{F_{j+1}} \arrow[dl] \\ & A_{G_j \cap F_{j+1}} & \end{tikzcd}$$

We know that $M_{2^{j-1}}(A_{G_j})$ is properly infinite when $j = 1$. Proposition 2.7 (applied to the diagram above tensored with $M_{2^{j-1}}(\mathbb{C})$) tells us that $M_{2^j}(A_{G_{j+1}})$ is properly infinite if $M_{2^{j-1}}(A_{G_j})$ is properly infinite. Hence $M_{2^{n-1}}(A)$ is properly infinite. $\square$

Remark 2.12 Uffe Haagerup has suggested another way to prove Theorem 2.11: If no matrix-algebra over A is properly infinite, then there exists a bounded non-zero lower semi-continuous 2-quasi-trace on A, see [7] and [1, page 327], and hence also an extremal 2-quasi-trace. Now, if A is also a $C(X)$-algebra for some compact Hausdorff space X, this implies that there is a bounded non-zero lower semi-continuous 2-quasitrace on A_x for (at least) one point $x \in X$ (see eg. [8, Proposition 3.7]). But then the fibre A_x cannot be properly infinite.

Question 2.13 Is any unital $C(X)$-algebra A properly infinite if all its fibres A_x, $x \in X$, are properly infinite?

We shall show in Section 5 that the question above is equivalent to Question 2.8 which again is equivalent to Question 2.9.

3 Lower semi-continuous fields of properly infinite C^*-algebras

Let us briefly discuss whether the results from Section 2 can be extended to lower semi-continuous C^*-bundles $(A, \{\sigma_x\})$ over a compact Hausdorff space X. Recall that any such *separable* lower semi-continuous C^*-bundle admits a faithful $C(X)$-linear representation on a Hilbert $C(X)$-module E such that, for all $x \in X$, the fibre $\sigma_x(A)$ is isomorphic to the induced image of A in $\mathcal{L}(E_x)$, [2]. Thus, the problem boils down to the following: Given a separable Hilbert $C(X)$-module E with infinite dimensional fibres E_x, such that the unit p of the C^*-algebra $\mathcal{L}_{C(X)}(E)$ of bounded adjointable $C(X)$-linear operators acting on E

has a properly infinite image in $\mathcal{L}(E_x)$ for all $x \in X$. Is the projection p itself properly infinite in $\mathcal{L}_{C(X)}(E)$?

Dixmier and Douady proved that this is always the case if the space X has finite topological dimension, [6]. But it does not hold anymore in the infinite dimensional case, see [6, §16, Corollaire 1] and [11], not even if X is contractible, [3, Corollary 3.7].

4 Two examples

We describe here two examples of continuous fields; the first is over the interval and the second (which really is a class of examples) is over the circle.

Example 4.1 Let $(\mathcal{O}_\infty * \mathcal{O}_\infty, (\iota_1, \iota_2))$ be the universal unital free product of two copies of $\mathcal{O}_\infty$, and let $\mathcal{A}$ be the unital sub-C^*-algebra of $C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$ given by

$$\mathcal{A} = \{f \in C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty) : f(0) \in \iota_1(\mathcal{O}_\infty),\ f(1) \in \iota_2(\mathcal{O}_\infty)\}.$$

Observe that $\mathcal{A}$ (in a canonical way) is a $C([0,1])$-algebra with fibres

$$\mathcal{A}_t = \begin{cases} \iota_1(\mathcal{O}_\infty), & t = 0, \\ \mathcal{O}_\infty * \mathcal{O}_\infty, & 0 < t < 1, \\ \iota_2(\mathcal{O}_\infty), & t = 1 \end{cases} \cong \begin{cases} \mathcal{O}_\infty, & t = 0, \\ \mathcal{O}_\infty * \mathcal{O}_\infty, & 0 < t < 1, \\ \mathcal{O}_\infty, & t = 1. \end{cases}$$

In particular, all fibres of $\mathcal{A}$ are properly infinite.

One claim to fame of the example above is that the question below is equivalent to Question 2.13 above. Hence, to answer Question 2.13 in the affirmative (or in the negative) we need only consider the case where $X = [0,1]$, and we need only worry about this one particular $C([0,1])$-algebra (which of course is bad enough!).

Question 4.2 Is the $C([0,1])$-algebra $\mathcal{A}$ from Example 4.1 above properly infinite?

The three equivalent statements in the proposition below will in Section 5 be shown to be equivalent to Question 4.2.

Proposition 4.3 *The following three statements concerning the $C([0,1])$-algebra $\mathcal{A}$ and the C^*-algebra $(\mathcal{O}_\infty * \mathcal{O}_\infty, (\iota_1, \iota_2))$ defined above are equivalent:*

(i) *$\mathcal{A}$ contains a non-trivial projection (i.e., a projection other than 0 and 1).*

(ii) *There are non-zero projections $p, q \in \mathcal{O}_\infty$ such that $p \neq 1$, $q \neq 1$, and $\iota_1(p) \sim_h \iota_2(q)$.*

(iii) *Let s be any isometry in $\mathcal{O}_\infty$. Then $\iota_1(ss^*) \sim_h \iota_2(ss^*)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$.*

We warn the reader that all three statements above could be false.

Proof: (i) $\Rightarrow$ (ii). Let e be a non-trivial projection in $\mathcal{A}$. Let $\pi_t\colon \mathcal{A} \to \mathcal{A}_t$, $t \in [0,1]$, denote the fibre map. As $\mathcal{A} \subseteq C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$, the mapping $t \mapsto \pi_t(e) \in \mathcal{O}_\infty * \mathcal{O}_\infty$ is continuous, so in particular, $\pi_0(e) \sim_h \pi_1(e)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$. The mappings ι_1 and ι_2 are injective, so there are projections $p, q \in \mathcal{O}_\infty$ such that $\pi_0(e) = \iota_1(p)$ and $\pi_1(e) = \iota_2(q)$. The projections p and q are non-zero because the mapping $t \mapsto \|\pi_t(e)\|$ is continuous and not constant equal to 0. Similarly, and $1-p$ and $1-q$ are non-zero because $1-e$ is non-zero.

(ii) $\Rightarrow$ (iii). Take non-trivial projections $p, q \in \mathcal{O}_\infty$ such that $\iota_1(p) \sim_h \iota_2(q)$. Take a unitary v in $\mathcal{U}^0(\mathcal{O}_\infty * \mathcal{O}_\infty)$ with $\iota_2(q) = v\iota_1(p)v^*$. Let $s \in \mathcal{O}_\infty$ be an isometry. If s is unitary, then $\iota_1(ss^*) = 1 = \iota_2(ss^*)$ and there is nothing to prove. Suppose that s is non-unitary. Then ss^* is homotopic to a subprojection p_0 of p and to a subprojection q_0 of q (use that p and q are properly infinite and full, then Lemma 2.4 (i), and last the fact that the unitary group of $\mathcal{O}_\infty$ is connected). Hence $\iota_1(ss^*) \sim_h \iota_1(p_0) \sim_h v\iota_1(p_0)v^*$ and $\iota_2(ss^*) \sim_h \iota_2(q_0)$, so we need only show that $v\iota_1(p_0)v^* \sim_h \iota_2(q_0)$. But this follows from Proposition 2.5 with $r = 1 - \iota_2(q) = \iota_2(1-q)$, as we note that $p_0 \sim 1 \sim q_0$ in $\mathcal{O}_\infty$, whence

$$\iota_2(q_0) \sim \iota_2(1) = 1 = \iota_1(1) \sim \iota_1(p_0) \sim v\iota_1(p_0)v^*.$$

(iii) $\Rightarrow$ (i). Take a non-unitary isometry $s \in \mathcal{O}_\infty$. Then $\iota_1(ss^*) \sim_h \iota_2(ss^*)$, and so there is a continuous function $e\colon [0,1] \to \mathcal{O}_\infty * \mathcal{O}_\infty$ such that $e(t)$ is a projection for all $t \in [0,1]$, $e(0) = \iota_1(ss^*)$ and $e(1) = \iota_2(ss^*)$. But then e is a non-trivial projection in $\mathcal{A}$. $\square$

It follows from Theorem 2.11 that some matrix algebra over $\mathcal{A}$ (from Example 4.1) is properly infinite. We can sharpen that statement as follows:

Proposition 4.4 *$M_2(\mathcal{A})$ is properly infinite; and if $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective, then $\mathcal{A}$ itself is properly infinite.*

It follows from Theorem 5.5 below that $\mathcal{A}$ is properly infinite if *and only if* $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective.

Proof: We have a pull-back diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl] \arrow[dr] & \\ \mathcal{A}_{[0,\frac{1}{2}]} \arrow[dr, "\pi_{1/2}"'] & & \mathcal{A}_{[\frac{1}{2},1]} \arrow[dl, "\pi_{1/2}"] \\ & \mathcal{O}_\infty * \mathcal{O}_\infty & \end{tikzcd}$$

One can unitally embed $\mathcal{O}_\infty$ into $\mathcal{A}_{[0,\frac{1}{2}]}$ via ι_1, so $\mathcal{A}_{[0,\frac{1}{2}]}$ is properly infinite, and a similar argument shows that $\mathcal{A}_{[\frac{1}{2},1]}$ is properly infinite. The two statements now follow from Proposition 2.7. $\square$

The example below, which will be the focus of the rest of this section, and in parts also of Section 5, is inspired by arguments from Rieffel's paper [9].

Example 4.5 Let A be a unital C^*-algebra, and let v be a unitary element in A such that

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in} \quad \mathcal{U}_2(A).$$

Let $t \mapsto u_t$ be a continuous path of unitaries in $\mathcal{U}_2(A)$ such that $u_0 = 1$ and $u_1 = \operatorname{diag}(v, 1)$. Put

$$p(t) = u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \in M_2(A),$$

and note that $p(0) = p(1)$. Identifying, for each C^*-algebra D, $C(\mathbb{T}, D)$ with the algebra of all continuous functions $f\colon [0,1] \to D$ such that $f(1) = f(0)$, we see that p belongs to $C(\mathbb{T}, M_2(A))$. Put

$$\mathcal{B} = pC(\mathbb{T}, M_2(A))p,$$

and note that $\mathcal{B}$ is a unital (sub-trivial) $C(\mathbb{T})$-algebra, being a corner of the trivial $C(\mathbb{T})$-algebra $C(\mathbb{T}, M_2(A))$. The fibres of $\mathcal{B}$ are

$$\mathcal{B}_t = p(t)M_2(A)p(t) \cong A$$

for all $t \in \mathbb{T}$.

Summing up, for each unital C^*-algebra A, for each unitary v in A for which $\operatorname{diag}(v, 1) \sim_h 1$ in $\mathcal{U}_2(A)$, and for each path $t \mapsto u_t \in \mathcal{U}_2(A)$ implementing this homotopy we get a $C(\mathbb{T})$-algebra $\mathcal{B}$ with fibres $\mathcal{B}_t \cong A$. We shall investigate this class of $C(\mathbb{T})$-algebras below.

Lemma 4.6 *In the notation of Example 4.5,*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - p \;\sim\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{in} \quad C(\mathbb{T}, M_2(A)).$$

In particular, p is stably equivalent to $\operatorname{diag}(1, 0)$.

Proof: Put

$$v_t = u_t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,1].$$

Then

$$v_0 = u_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_1 = u_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so v belongs to $C(\mathbb{T}, M_2(A))$. It is easy to see that $v_t^* v_t = \operatorname{diag}(0, 1)$ and $v_t v_t^* = 1 - p(t)$, and so the lemma is proved. □

Proposition 4.7 *Let A, $v \in \mathcal{U}(A)$, and $\mathcal{B}$ be as in Example 4.5. Conditions* (i) *and* (ii) *below are equivalent for any unital C^*-algebra A, and all three conditions are equivalent if A in addition is assumed to be properly infinite.*

(i) $v \sim_h 1$ *in* $\mathcal{U}(A)$.

(ii) $p \sim \operatorname{diag}(1_A, 0)$ *in* $C(\mathbb{T}, M_2(A))$.

(iii) *The* $C(\mathbb{T})$*-algebra* $\mathcal{B}$ *is properly infinite.*

Proof: (ii) $\Rightarrow$ (i). Suppose that $p \sim \operatorname{diag}(1, 0)$ in $C(\mathbb{T}, M_2(A))$. Then there is a $w \in C(\mathbb{T}, M_2(A))$ such that

$$w_t w_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w_t^* w_t = p_t$$

for all $t \in [0, 1]$ and $w_1 = w_0$ (as we identify $C(\mathbb{T}, M_2(A))$ with the set of continuous functions $f\colon [0, 1] \to M_2(A)$ with $f(1) = f(0)$). Upon replacing w_t with $w_0^* w_t$ we can assume that $w_1 = w_0 = \operatorname{diag}(1, 0)$. Now, with $t \mapsto u_t$ as in Example 4.5,

$$w_t u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_t & 0 \\ 0 & 0 \end{pmatrix},$$

where $t \mapsto a_t$ is a continuous path of unitaries in A. Because $u_0 = \operatorname{diag}(1, 1)$ and $u_1 = \operatorname{diag}(v, 1)$ we see that $a_0 = 1$ and $a_1 = v$, whence $v \sim_h 1$ in $\mathcal{U}(A)$.

(i) $\Rightarrow$ (ii). Suppose conversely that $v \sim_h 1$ in $\mathcal{U}(A)$. Then we can find a continuous path $t \mapsto v_t \in \mathcal{U}(A)$, $t \in [1-\varepsilon, 1]$, such that $v_{1-\varepsilon} = v$ and $v_1 = 1$ for an $\varepsilon > 0$ (to be determined below). Again with $t \mapsto u_t$ as in Example 4.5, define

$$\widetilde{u}_t = \begin{cases} u_{(1-\varepsilon)^{-1}t}, & 0 \le t \le 1 - \varepsilon, \\ \operatorname{diag}(v_t, 1), & 1 - \varepsilon \le t \le 1. \end{cases}$$

Then $t \mapsto \widetilde{u}_t$ is a continuous path of unitaries in $\mathcal{U}_2(A)$ such that $\widetilde{u}_{1-\varepsilon} = u_1 = \operatorname{diag}(v, 1)$ and $\widetilde{u}_0 = \widetilde{u}_1 = 1$. It follows that $\widetilde{u}$ belongs to $C(\mathbb{T}, M_2(A))$. Provided that $\varepsilon > 0$ is chosen small enough we obtain the following inequality:

$$\left\| \widetilde{u}_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{u}_t^* - p(t) \right\| = \left\| \widetilde{u}_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{u}_t^* - u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \right\| < 1$$

for all $t \in [0, 1]$, whence $p \sim \widetilde{u} \operatorname{diag}(1, 0) \widetilde{u}^* \sim \operatorname{diag}(1, 0)$ as desired.

(iii) $\Rightarrow$ (ii). Suppose that $\mathcal{B}$ is properly infinite. From Lemma 4.6 we know that $[p] = [\operatorname{diag}(1_A, 0)]$ in $K_0(C(\mathbb{T}, A))$. Because $\mathcal{B}$ and A are properly infinite, it follows that p and $\operatorname{diag}(1_A, 0)$ are properly infinite (and full) projections, and hence they are equivalent by Proposition 2.3 (i).

(ii) $\Rightarrow$ (iii). Since A is properly infinite, $\operatorname{diag}(1_A, 0)$ and hence p (being equivalent to $\operatorname{diag}(1_A, 0)$) are properly infinite (and full) projections, whence $\mathcal{B}$ is properly infinite. $\square$

We will now use (the ideas behind) Lemma 4.6 and Proposition 4.7 to prove the following general statement about C^*-algebras.

Corollary 4.8 *Let A be a unital C^*-algebra such that $C(\mathbb{T}, A)$ has the cancellation property. Then A is K_1-injective.*

Proof: It suffices to show that the natural maps $\mathcal{U}_{n-1}(A)/\mathcal{U}^0_{n-1}(A) \to \mathcal{U}_n(A)/\mathcal{U}^0_n(A)$ are injective for all $n \geq 2$. Let $v \in \mathcal{U}_{n-1}(A)$ be such that $\mathrm{diag}(v, 1_A) \in \mathcal{U}^0_n(A)$ and find a continuous path of unitaries $t \mapsto u_t$ in $\mathcal{U}_n(A)$ such that

$$u_0 = 1_{M_n(A)} = \begin{pmatrix} 1_{M_{n-1}(A)} & 0 \\ 0 & 1_A \end{pmatrix} \quad \text{and} \quad u_1 = \begin{pmatrix} v & 0 \\ 0 & 1_A \end{pmatrix}.$$

Put

$$p_t = u_t \begin{pmatrix} 1_{M_{n-1}(A)} & 0 \\ 0 & 0 \end{pmatrix} u_t^*, \quad t \in [0,1],$$

and note that $p_0 = p_1$ so that p defines a projection in $C(\mathbb{T}, M_n(A))$. Repeating the proof of Lemma 4.6 we find that $1_{M_n(A)} - p \sim \mathrm{diag}(0, 1_A)$ in $C(\mathbb{T}, M_n(A))$, whence $p \sim \mathrm{diag}(1_{M_{n-1}(A)}, 0)$ by the cancellation property of $C(\mathbb{T}, A)$, where we identify projections in $M_n(A)$ with constant projections in $C(\mathbb{T}, M_n(A))$. The arguments going into the proof of Proposition 4.7 show that $v \sim_h 1_{M_{n-1}(A)}$ in $\mathcal{U}_{n-1}(A)$ if (and only if) $p \sim \mathrm{diag}(1_{M_{n-1}(A)}, 0)$. Hence v belongs to $\mathcal{U}^0_{n-1}(A)$ as desired. $\square$

5 K_1-injectivity of properly infinite C^*-algebras

In this section we prove our main result that relate K_1-injectivity of arbitrary unital properly infinite C^*-algebras to proper infiniteness of $C(X)$-algebras and pull-back C^*-algebras. More specifically we shall show that Question 2.9, Question 2.13, Question 2.8, and Question 4.2 are equivalent.

First we reformulate in two different ways the question if a given properly infinite unital C^*-algebra is K_1-injective.

Proposition 5.1 *The following conditions are equivalent for any unital properly infinite C^*-algebra A:*

(i) *A is K_1-injective.*

(ii) *Let p, q be projections in A such that $p \sim q$ and p, q, $1-p$, $1-q$ are properly infinite and full. Then $p \sim_h q$.*

(iii) *Let p and q be properly infinite, full projections in A. There exist properly infinite, full projections $p_0, q_0 \in A$ such that $p_0 \leq p$, $q_0 \leq q$, and $p_0 \sim_h q_0$.*

Proof: (i) $\Rightarrow$ (ii). Let p, q be properly infinite, full projections in A with $p \sim q$ such that $1-p, 1-q$ are properly infinite and full. Then by Lemma 2.4 (i) there is a unitary $v \in A$ such that $vpv^* = q$ and $[v] = 0$ in $K_1(A)$. By the assumption in (i), $v \in \mathcal{U}^0(A)$, whence $p \sim_h q$.

(ii) ⇒ (i). Let $u \in \mathcal{U}(A)$ be such that $[u] = 0$ in $K_1(A)$. Take, as we can, a projection p in A such that p and $1-p$ are properly infinite and full. Set $q = upu^*$. Then $p \sim_h q$ by (ii), and so there exists a unitary $v \in \mathcal{U}^0(A)$ with $p = vqv^*$. It follows that

$$pvu = vqv^*vu = v(upu^*)v^*vu = vup.$$

Therefore $vu \in \mathcal{U}^0(A)$ by Lemma 2.4 (ii), which in turn implies that $u \in \mathcal{U}^0(A)$.

(ii) ⇒ (iii). Let p, q be properly infinite and full projections in A. There exist mutually orthogonal projections e_1, f_1 such that $e_1 \le p$, $f_1 \le p$ and $e_1 \sim p \sim f_1$, and mutually orthogonal projections e_2, f_2 such that $e_2 \le q$, $f_2 \le q$ and $e_2 \sim q \sim f_2$. Being equivalent to either p or q, the projections e_1, e_2, f_1 and f_2 are properly infinite and full. There are properly infinite, full projections $p_0 \le e_1$ and $q_0 \le e_2$ such that $[p_0] = [q_0] = 0$ in $K_0(A)$ and $p_0 \sim q_0$ (cf. Proposition 2.3). As $f_1 \le 1 - p_0$ and $f_2 \le 1 - q_0$, we see that $1 - p_0$ and $1 - q_0$ are properly infinite and full, and so we get $p_0 \sim_h q_0$ by (ii).

(iii) ⇒ (ii). Let p, q be equivalent properly infinite, full projections in A such that $1-p, 1-q$ are properly infinite and full. From (iii) we get properly infinite and full projections $p_0 \le p$, $q_0 \le q$ which satisfy $p_0 \sim_h q_0$. Thus there is a unitary $v \in \mathcal{U}_0(A)$ such that $vp_0v^* = q_0$. Upon replacing p by vpv^* (as we may do because $p \sim_h vpv^*$) we can assume that $q_0 \le p$ and $q_0 \le q$. Now, q_0 is orthogonal to $1-p$ and to $1-q$, and so $1 - p \sim_h 1 - q$ by Proposition 2.5, whence $p \sim_h q$. □

Proposition 5.2 *Let A be a unital properly infinite C^*-algebra. The following conditions are equivalent:*

(i) *A is K_1-injective, ie., the natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ is injective.*

(ii) *The natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective.*

(iii) *The natural maps $\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A)$ are injective for each natural number n.*

Proof: (i) ⇒ (ii) holds because the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ factors through the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$.

(ii) ⇒ (i). Take $u \in \mathcal{U}(A)$ and suppose that $[u] = 0$ in $K_1(A)$. Then $\operatorname{diag}(u, 1_A) \in \mathcal{U}_2^0(A)$ by Lemma 2.4 (ii) (with $p = \operatorname{diag}(1_A, 0)$). Hence $u \in \mathcal{U}_0(A)$ by injectivity of the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$.

(i) ⇒ (iii). Let $n \ge 1$ be given and consider the natural maps

$$\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A).$$

The first map is onto, as proved by Cuntz in [4], see also [13, Exercise 8.9], and the composition of the two maps is injective by assumption, hence the second map is injective.

(iii) ⇒ (i) is trivial. □

We give below another application of K_1-injectivity for properly infinite C^*-algebras. First we need a lemma:

Lemma 5.3 *Let A be a unital, properly infinite C^*-algebra, and let $\varphi, \psi \colon \mathcal{O}_\infty \to A$ be unital embeddings. Then ψ is homotopic to a unital embedding $\psi' \colon \mathcal{O}_\infty \to A$ for which there is a unitary $u \in A$ with $[u] = 0$ in $K_1(A)$ and for which $\psi'(s_j) = u\varphi(s_j)$ for all j (where $s_1, s_2, \dots$ are the canonical generators of $\mathcal{O}_\infty$).*

Proof: For each n set

$$v_n = \sum_{j=1}^{n} \psi(s_j)\varphi(s_j)^* \in A, \qquad e_n = \sum_{j=1}^{n} s_j s_j^* \in \mathcal{O}_\infty.$$

Then v_n is a partial isometry in A with $v_n v_n^* = \psi(e_n)$, $v_n^* v_n = \varphi(e_n)$, and $\psi(s_j) = v_n\varphi(s_j)$ for $j = 1, 2, \dots, n$. Since $1 - e_n$ is full and properly infinite it follows from Lemma 2.4 that each v_n extends to a unitary $u_n \in A$ with $[u_n] = 0$ in $K_1(A)$. In particular, $\psi(s_j) = u_n\varphi(s_j)$ for $j = 1, 2, \dots, n$.

We proceed to show that $n \mapsto u_n$ extends to a continuous path of unitaries $t \mapsto u_t$, for $t \in [2, \infty)$, such that $u_t\varphi(e_n) = u_n\varphi(e_n)$ for $t \geq n + 1$. Fix $n \geq 2$. To this end it suffices to show that we can find a continuous path $t \mapsto z_t$, $t \in [0, 1]$, of unitaries in A such that $z_0 = 1$, $z_1 = u_n^* u_{n+1}$, and $z_t\varphi(e_{n-1}) = \varphi(e_{n-1})$ (as we then can set u_t to be $u_n z_{t-n}$ for $t \in [n, n + 1]$).

Observe that

$$u_{n+1}\varphi(e_n) = v_{n+1}\varphi(e_n) = v_n = u_n\varphi(e_n).$$

Set $A_0 = (1 - \varphi(e_{n-1}))A(1 - \varphi(e_{n-1}))$, and set $y = u_n^* u_{n+1}(1 - \varphi(e_{n-1}))$. Then y is a unitary element in A_0 and $[y] = 0$ in $K_1(A_0)$. Moreover, y commutes with the properly infinite full projection $\varphi(e_n) - \varphi(e_{n-1}) \in A_0$. We can therefore use Lemma 2.4 to find a continuous path $t \mapsto y_t$ of unitaries in A_0 such that $y_0 = 1_{A_0} = 1 - \varphi(e_{n-1})$ and $y_1 = y$. The continuous path $t \mapsto z_t = y_t + \varphi(e_{n-1})$ is then as desired.

For each $t \geq 2$ let $\psi_t \colon \mathcal{O}_\infty \to A$ be the *-homomorphism given by $\psi_t(s_j) = u_t\varphi(s_j)$. Then $\psi_t(s_j) = \psi(s_j)$ for all $t \geq j + 1$, and so it follows that

$$\lim_{t \to \infty} \psi_t(x) = \psi(x)$$

for all $x \in \mathcal{O}_\infty$. Hence ψ_2 is homotopic to ψ, and so we can take ψ' to be ψ_2. □

Proposition 5.4 *Any two unital *-homomorphisms from $\mathcal{O}_\infty$ into a unital K_1-injective (properly infinite) C^*-algebra are homotopic.*

Proof: In the light of Lemma 5.3 it suffices to show that if $\varphi, \psi \colon \mathcal{O}_\infty \to A$ are unital *-homomorphisms such that, for some unitary $u \in A$ with $[u] = 0$ in $K_1(A)$, $\psi(s_j) = u\varphi(s_j)$ for all j, then $\psi \sim_h \varphi$. By assumption, $u \sim_h 1$, so there is a continuous path $t \mapsto u_t$ of unitaries in A such that $u_0 = 1$ and $u_1 = u$. Letting $\varphi_t \colon \mathcal{O}_\infty \to A$ be the *-homomorphism given by $\varphi_t(s_j) = u_t\varphi(s_j)$ for all j, we get $t \mapsto \varphi_t$ is a continuous path of *-homomorphisms connecting $\varphi_0 = \varphi$ to $\varphi_1 = \psi$. □

Our main theorem below, which in particular implies that Question 2.9, Question 2.13, Question 2.8 and Question 4.2 all are equivalent, also give a special converse to Proposition 5.4: Indeed, with $\iota_1, \iota_2 \colon \mathcal{O}_\infty \to \mathcal{O}_\infty * \mathcal{O}_\infty$ the two canonical inclusions, if $\iota_1 \sim_h \iota_2$, then condition (iv) below holds, whence $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective, which again implies that all unital properly infinite C^*-algebras are K_1-injective. Below we retain the convention that $\mathcal{O}_\infty * \mathcal{O}_\infty$ is the universal *unital* free product of two copies of $\mathcal{O}_\infty$ and that ι_1 and ι_2 are the two natural inclusions of $\mathcal{O}_\infty$ into $\mathcal{O}_\infty * \mathcal{O}_\infty$.

Theorem 5.5 *The following statements are equivalent:*

(i) *Every unital, properly infinite C^*-algebra is K_1-injective.*

(ii) *For every compact Hausdorff space X, every unital $C(X)$-algebra A, for which A_x is properly infinite for all $x \in X$, is properly infinite.*

(iii) *Every unital C^*-algebra A, that is the pull-back of two unital, properly infinite C^*-algebras A_1 and A_2 along *-epimorphisms $\pi_1 \colon A_1 \to B$, $\pi_2 \colon A_2 \to B$:*

$$\begin{tikzcd} & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

is properly infinite.

(iv) *There exist non-zero projections $p, q \in \mathcal{O}_\infty$ such that $p \neq 1$, $q \neq 1$, and $\iota_1(p) \sim_h \iota_2(p)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$.*

(v) *The specific $C([0,1])$-algebra $\mathcal{A}$ considered in Example 4.1 (and whose fibres are properly infinite) is properly infinite.*

(vi) *$\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective.*

Note that statement (i) is reformulated in Propositions 5.1, 5.2, and 5.4; and that statement (iv) is reformulated in Proposition 4.3. We warn the reader that all these statements may turn out to be false (in which case, of course, there will be counterexamples to all of them).

Proof: (i) $\Rightarrow$ (iii) follows from Proposition 2.7.

(iii) $\Rightarrow$ (ii). This follows from Lemma 2.10 as in the proof of Theorem 2.11, except that one does not need to pass to matrix algebras.

(ii) $\Rightarrow$ (i). Suppose that A is unital and properly infinite. Take a unitary $v \in \mathcal{U}(A)$ such that $\mathrm{diag}(v, 1) \in \mathcal{U}_2^0(A)$. Let $\mathcal{B}$ be the $C(\mathbb{T})$-algebra constructed in Example 4.5 from A, v, and a path of unitaries $t \mapsto u_t$ connecting $1_{M_2(A)}$ to $\mathrm{diag}(v, 1)$. Then $\mathcal{B}_t \cong A$ for all

$t \in \mathbb{T}$, so all fibres of $\mathcal{B}$ are properly infinite. Assuming (ii), we can conclude that $\mathcal{B}$ is properly infinite. Proposition 4.7 then yields that $v \in \mathcal{U}^0(A)$. It follows that the natural map $\mathcal{U}(A)/\mathcal{U}_0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective, whence A is K_1-injective by Proposition 5.2.

(ii) $\Rightarrow$ (v) is trivial (because $\mathcal{A}$ is a $C([0,1])$-algebra with properly infinite fibres).

(v) $\Rightarrow$ (iv) follows from Proposition 4.3.

(iv) $\Rightarrow$ (i). We show that Condition (iii) of Proposition 4.3 implies Condition (iii) of Proposition 5.1.

Let A be a properly infinite C^*-algebra and let p, q be properly infinite, full projections in A. Then there exist (properly infinite, full) projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim 1 \sim q_0$ and such that $1-p_0$ and $1-q_0$ are properly infinite and full, cf. Propositions 2.3. Take isometries $t_1, r_1 \in A$ with $t_1t_1^* = p_0$ and $r_1r_1^* = q_0$; use the fact that $1 \precsim 1 - p_0$ and $1 \precsim 1 - q_0$ to find sequences of isometries $t_2, t_3, t_4, \dots$ and $r_2, r_3, r_4, \dots$ in A such that each of the two sequences $\{t_jt_j^*\}_{j=1}^\infty$ and $\{r_jr_j^*\}_{j=1}^\infty$ consist of pairwise orthogonal projections.

By the universal property of $\mathcal{O}_\infty$ there are unital *-homomorphisms $\varphi_j \colon \mathcal{O}_\infty \to A$, $j = 1, 2$, such that $\varphi_1(s_j) = t_j$ and $\varphi_2(s_j) = r_j$, where $s_1, s_2, s_3, \dots$ are the canonical generators of $\mathcal{O}_\infty$. In particular,

$$\varphi_1(s_1s_1^*) = p_0 \quad \text{and} \quad \varphi_2(s_1s_1^*) = q_0.$$

By the property of the universal unital free products of C^*-algebras, there is a unique unital *-homomorphism $\varphi \colon \mathcal{O}_\infty * \mathcal{O}_\infty \to A$ making the diagram

$$\begin{tikzcd} & \mathcal{O}_\infty * \mathcal{O}_\infty \arrow[dd, "\varphi"] & \\ \mathcal{O}_\infty \arrow[ur, "\iota_1"] \arrow[dr, "\varphi_1"'] & & \mathcal{O}_\infty \arrow[ul, "\iota_2"'] \arrow[dl, "\varphi_2"] \\ & A & \end{tikzcd}$$

commutative. It follows that $p_0 = \varphi(\iota_1(s_1s_1^*))$ and $q_0 = \varphi(\iota_2(s_1s_1^*))$. By Condition (iii) of Proposition 4.3, $\iota_1(s_1s_1^*) \sim_h \iota_2(s_1s_1^*)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$, whence $p_0 \sim_h q_0$ as desired.

(i) $\Rightarrow$ (vi) is trivial.

(vi) $\Rightarrow$ (v) follows from Proposition 4.4. $\square$

6 Concluding remarks

We do not know if all unital properly infinite C^*-algebras are K_1-injective, but we observe that K_1-injectivity is assured in the presence of certain central sequences:

Proposition 6.1 *Let A be a unital properly infinite C^*-algebras that contains an asymptotically central sequence $\{p_n\}_{n=1}^\infty$, where p_n and $1-p_n$ are properly infinite, full projections for all n. Then A is K_1-injective*

Proof: This follows immediately from Lemma 2.4 (ii). $\square$

It remains open if arbitrary $C(X)$-algebras with properly infinite fibres must be properly infinite. If this fails, then we already have a counterexample of the form $\mathcal{B} = pC(\mathbb{T}, A)p$, cf. Example 4.5, for some unital properly infinite C^*-algebra A and for some projection $p \in C(\mathbb{T}, A)$. (The C^*-algebra $\mathcal{B}$ is a $C(\mathbb{T})$-algebra with fibres $\mathcal{B}_t \cong A$.)

On the other hand, any trivial $C(X)$-algebra $C(X, D)$ with constant fibre D is clearly properly infinite if its fibre(s) D is unital and properly infinite (because $C(X, D) \cong C(X) \otimes D$). We extend this observation in the following easy:

Proposition 6.2 *Let X be a compact Hausdorff space, let $p \in C(X, D)$ be a projection, and consider the sub-trivial $C(X)$-algebra $pC(X, D)p$ whose fibre at x is equal to $p(x)Dp(x)$.*

If p is Murray-von Neumann equivalent to a constant projection q, then $pC(X, D)p$ is $C(X)$-isomorphic to the trivial $C(X)$-algebra $C(X, D_0)$, where $D_0 = qDq$. In this case, $pC(X, D)p$ is properly infinite if and only if D_0 is properly infinite.

In particular, if X is contractible, then $pC(X, D)p$ is $C(X)$-isomorphic to a trivial $C(X)$-algebra for any projection $p \in C(X, D)$ and for any C^*-algebra D.

Proof: Suppose that $p = v^*v$ and $q = vv^*$ for some partial isometry $v \in C(X, D)$. The map $f \mapsto vfv^*$ defines a $C(X)$-isomorphism from $pC(X, D)p$ onto $qC(X, D)q$, and $qC(X, D)q = C(X, D_0)$.

If X is contractible, then any projection $p \in C(X, D)$ is homotopic, and hence equivalent, to the constant projection $x \mapsto p(x_0)$ for any fixed $x_0 \in X$. $\square$

Remark 6.3 One can elaborate a little more on the construction considered above. Take a unital C^*-algebra D such that for some natural number $n \geq 2$, $M_n(D)$ is properly infinite, but $M_{n-1}(D)$ is not properly infinite (see [12] or [11] for such examples). Take any space X, preferably one with highly non-trivial topology, eg. $X = S^n$, and take, for some $k \geq n$, a sufficiently non-trivial n-dimensional projection p in $C(X, M_k(D))$ such that $p(x)$ is equivalent to the trivial n dimensional projection $1_{M_n(D)}$ for all x (if X is connected we need only assume that this holds for one $x \in X$). The $C(X)$-algebra

$$\mathcal{A} = p\, C(X, M_k(D))\, p,$$

then has properly infinite fibres $\mathcal{A}_x = p(x)Dp(x) \cong M_n(D)$. Is A always properly infinite? We guess that a possible counterexample to the questions posed in this paper could be of this form (for suitable D, X, and p).

Let us end this paper by remarking that the answer to Question 2.13, which asks if any $C(X)$-algebra with properly infinite fibres is itself properly infinite, does not depend (very much) on X. If it fails, then it fails already for $X = [0, 1]$ (cf. Theorem 5.5), and $[0, 1]$ is a contractible space of low dimension. However, if we make the dimension of X even lower than the dimension of $[0, 1]$, then we do get a positive anwer to our question:

Proposition 6.4 *Let X be a totally disconnected space, and let A be a $C(X)$-algebra such that all fibres A_x, $x \in X$, of A are properly infinite. Then A is properly infinite.*

Proof: Using Lemma 2.10 and the fact that X is totally disconnected we can write X as the disjoint union of clopen sets $F_1, F_2, \dots, F_n$ such that A_{F_j} is properly infinite for all j. As

$$A = A_{F_1} \oplus A_{F_2} \oplus \cdots \oplus A_{F_n},$$

the claim is proved. $\square$

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