Elementary amenable groups are quasidiagonal

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Abstract

We show that the group C*-algebra of any elementary amenable group is quasidiagonal. This is an offspring of recent progress in the classification theory of nuclear C*-algebras.

1 Introduction

Rosenberg proved in 1987, [9], that if the reduced C*-algebra of a group is *quasidiagonal*, then the group is amenable. He suggested that the converse is also true. We confirm this conjecture for *elementary amenable groups.*

Quasidiagonal C*-algebras have been studied since the 1970's. Loosely speaking, a C*-algebra is quasidiagonal if it has a faithful approximately block-diagonal representation on a Hilbert space. Every quasidiagonal C*-algebra is stably finite. There is no known example of a stably finite nuclear C*-algebra which is not quasidiagonal. Rosenberg's conjecture asserts that there is no such example among group C*-algebras. Quasidiagonality plays a central role in the classification program for simple nuclear C*-algebras and for our understanding of stably finite C*-algebras in general, as for example illustrated in the recent paper, [19]. We refer the reader to the nice exposition, [2], by N. Brown for more information about quasidiagonal C*-algebras.

Rosenberg's conjecture was studied in the recent paper [4], where it is shown that the group C*-algebra of an amenable group is quasidiagonal if and only if the group is MF. This, in turn, allowed the authors to conclude for example that amenable LEF groups have quasidiagonal C*-algebras. (LEF stands for locally embeddable into finite groups.) They also proved that the group C*-algebra of an amenable group is not necessarily *strongly quasidiagonal*. For example the lamplighter group, and more generally, a class of wreath products, fail to have this property.

The class of elementary amenable groups is a bootstrap class which is built up from finite and abelian groups by successive elementary operations. All elementary amenable groups are amenable (as the name suggests), but the converse does not hold: groups of intermediate growth discovered by R. Grigorchuck, [7], are amenable but not elementary amenable. Grigorchuck's groups are residually finite (and amenable) and therefore have quasidiagonal C*-algebras. The commutator group of the topological full group of a Cantor minimal system was shown by H. Matui, [16], to be simple and (sometimes) finitely generated, and by K. Juschenko and N. Monod, [10], to be amenable. No infinite, finitely generated simple group can be elementary amenable. It was observed in [4, 11] that the topological full groups, being LEF, [8], have quasidiagonal C*-algebras. The list of amenable groups which are not elementary amenable is long and does not stop here.

Elementary amenable groups can be described more explicitly by transfinite induction, cf. Chou, [5]. We use here a related description of elementary amenable groups due to Osin (see Proposition 3.3). To show that all elementary amenable groups have quasidiagonal C*-algebras, one is faced with the following problem (see also Remark 3.10): if G is the semi-direct product of a group H by the integers $\mathbb{Z}$, and if $C^*_\lambda(H)$ is quasidiagonal, does it follow that $C^*_\lambda(G)$ is quasidiagonal? We reformulate this question by considering the formally stronger property, that we call PQ, of a group G: that the crossed product of G by the Bernoulli shift on the CAR algebra $\bigotimes_G M_2$ is quasidiagonal, cf. Definition 3.1. Using classification theory, specifically Theorem 3.4 from [19], as well as a result of H. Matui on AF-embeddabilty of crossed products of a simple A$\mathbb{T}$-algebra of real rank zero by integers, we can show that the class PQ is closed under extensions by the integers $\mathbb{Z}$ (as well as the other operations as required by Osin's theorem). In this way we obtain our main result, Theorem 3.8. We show in Proposition 3.9 that countable amenable LEF groups, and in fact all countable amenable groups that are locally embeddable into a PQ group, belong to the class PQ. This implies that LEF groups and any extension of such a group by an elementary amenable group have quasidiagonal C*-algebras thus extending the result from [4] mentioned above.

2 Bernoulli action of UHF-algebras

In order to use classification theory of C*-algebras to obtain results about group C*-algebras, we associate to each countable discrete group G a simple and monotracial C*-algebra $B(G)$ by using the Bernoulli shift. Here a C*-algebra is said to be *monotracial* if it has a unique tracial state.

We recall the definition of the Bernoulli shift. Let A be a unital C*-algebra. For a finite set X, we consider the X-fold tensor product $\bigotimes_X A$. Throughout the paper the symbol $\otimes$ means the *minimal tensor product* of C*-algebras. For an inclusion $Y \subseteq X$ of finite sets, the C*-algebra $\bigotimes_Y A$ is naturally identified with the C*-subalgebra $(\bigotimes_Y A) \otimes (\bigotimes_{X \setminus Y} \mathbb{C}1)$ of $\bigotimes_X A$. For an infinite set X, we define $\bigotimes_X A$ to be the inductive limit of $\{\bigotimes_Y A : Y \subseteq X \text{ finite}\}$. Thus it is the closed linear span of the elementary tensors $\bigotimes_{x \in X} a_x$, where $a_x \in A$ for all $x \in X$ and $a_x = 1$ for all but finitely many $x \in X$. It follows that for any inclusion $Y \subseteq X$ of sets, there is a natural embedding $\bigotimes_Y A \subseteq \bigotimes_X A$.

When a group G acts on the index set X by permutations, it gives rise to an action of G on $\bigotimes_X A$ by $*$-automorphisms. This action is called a (noncommutative) *Bernoulli shift* and will be denoted by σ. In other words, σ is defined by $\sigma_g(\bigotimes_{x \in X} a_x) = \bigotimes_{x \in X} a_{g^{-1} \cdot x}$ for an elementary tensor $a = \bigotimes_{x \in X} a_x \in \bigotimes_X A$. We denote by $(\bigotimes_X A) \rtimes_\sigma G$ the corresponding crossed product. Throughout the paper the symbol $\rtimes$ means the *reduced crossed product*. We omit writing σ when there is no fear of confusion.

Let M_n denote the C*-algebra of $n \times n$ matrices over $\mathbb{C}$, and let B denote the CAR algebra $\bigotimes_{\mathbb{N}} M_2$. For each countable discrete group G, we associate the *Bernoulli shift crossed product* C*-algebra $B(G) = (\bigotimes_G B) \rtimes G$, where G acts on itself by the left translation.

Proposition 2.1. *For every countable discrete group G, the C*-algebra $B(G)$ is simple and monotracial. If G is amenable, then $B(G)$ is nuclear and belongs to the UCT class.*

Moreover, the functor $G \mapsto B(G)$ satisfies the following functorial properties.

(i) *If $G_1 \subseteq G_2$, then $B(G_1) \subseteq B(G_2)$ naturally.*

(ii) *If $G_1 \subseteq G_2 \subseteq \cdots$ is an increasing sequence, then $B(\bigcup G_n) = \overline{\bigcup B(G_n)}$.*

(iii) *If H acts on G by automorphisms, then the action extends to an action α of H on $B(G)$ such that $B(G) \rtimes_\alpha H \cong B(G \rtimes H)$.*

(iv) *If $G_1 \subseteq G_2$ is a finite-index inclusion, then there is a faithful embedding of $B(G_2)$ into $B(G_1) \otimes M_{|G_2/G_1|}$.*

Proof. It was shown in [13, Theorem 3.1] that $A \rtimes G$ is simple if A is simple and the action of G on A is outer. Thus it suffices to show that the Bernoulli shift is an outer action. Take a nontrivial central sequence p_n of projections in B and view them as elements in $\bigotimes_{\{1_G\}} B \subseteq \bigotimes_G B$. Then, it is also a central sequence in $\bigotimes_G B$, and $\|\sigma_g(p_n) - p_n\| = 1$ for all $g \in G \setminus \{1_G\}$ and $n \in \mathbb{N}$ (because $\sigma_g(p_n)$ and p_n are distinct but commuting projections). It follows that σ_g is not inner for any $g \in G \setminus \{1_G\}$.

To prove that $B(G)$ is monotracial, it suffices to show that $\tau(x) = \tau(\Phi(x))$ for every tracial state τ on $B(G)$ and $x \in B(G)$, where Φ is the canonical conditional expectation from $B(G)$ onto $\bigotimes_G B$. For this, it further suffices to show that $\tau(au_g) = 0$ for every $g \in G \setminus \{1_G\}$ and $a \in \bigotimes_F \mathcal{B}$ with $F \subseteq G$ finite. Here $\mathcal{B}$ is the dense subalgebra $\bigcup_n (\bigotimes_{i=1}^n M_2)$ of the CAR algebra B, and u_g denotes the canonical unitary element in $B(G)$ that implements the automorphism σ_g. Thus, for any $n \in \mathbb{N}$, one can find a copy C of M_{2^n} in $\bigotimes_{\{1_G\}} B$ that commutes with a. Let $\{p_j\}_{j=1}^{2^n}$ be pairwise orthogonal minimal projections in C such that $\sum_j p_j = 1$. Since $\tau(p_j \sigma_g(p_j)) = 2^{-2n}$ for all j (observe that $\tau|_{\bigotimes_G B}$ is unique and satisfies $\tau(ee') = \tau(e)\tau(e')$ for every $e \in \bigotimes_{\{1_G\}} B$ and $e' \in \bigotimes_{\{g\}} B$), one has

$$|\tau(au_g)| \leq \sum_{j=1}^{2^n} |\tau(p_j au_g p_j)| = \sum_{j=1}^{2^n} |\tau(p_j \sigma_g(p_j) u_g a)| \leq \sum_{j=1}^{2^n} \|a\| \tau(p_j \sigma_g(p_j)) \leq 2^{-n} \|a\|.$$

As $n \geq 1$ was arbitrary, we conclude that $\tau(au_g) = 0$ as desired.

It is well-known that the crossed product of a nuclear C*-algebra by an amenable group is again nuclear, see for example [3, Theorem 4.2.6]. J.-L. Tu proved in [23] that the UCT holds for the C*-algebra of any locally compact amenable groupoid. (It is also known that the C*-algebra of an étale groupoid is nuclear if and only if the groupoid is amenable—hence nuclearity implies that the UCT holds for each C*-algebra in this class.) We show that $B(G)$ is the C*-algebra of an amenable étale groupoid. Let $X = \prod_{G \times \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ be the compact group on which the subgroup $H = \bigoplus_{G \times \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ (viewed as a discrete group) acts from the left. By using the isomorphism $\ell_\infty(\mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong M_2$, one sees the canonical isomorphism $C(X) \rtimes H \cong \bigotimes_{G \times \mathbb{N}} M_2 \cong \bigotimes_G B$. The group G also acts on $G \times \mathbb{N}$ on the first coordinate and hence on X and H, too. These actions are compatible with the H action on X and give rise to an action of the semi-direct product group $H \rtimes G$ on X. It is routine to check

$$C(X) \rtimes (H \rtimes G) \cong (C(X) \rtimes H) \rtimes G \cong (\bigotimes_G B) \rtimes G = B(G).$$

Therefore, $B(G)$ is isomorphic to the C*-algebra of the amenable étale groupoid $X \rtimes (H \rtimes G)$ (see for example [3, Example 5.6.3]).

Now, we proceed to the proof of functoriality. The assertions (i) and (ii) are rather obvious. The C*-subalgebra of $B(G_2)$, generated by $\bigotimes_{G_1} B$ and $\{u_g : g \in G_1\}$, is canonically isomorphic to $B(G_1)$; and the C*-algebra $B(\bigcup G_n)$ is generated by $\bigcup_n (\bigotimes_{G_n} B)$ and $\{u_g : g \in \bigcup G_n\}$.

For (iii), let us fix the notation. We write a typical element in the semi-direct product group $G \rtimes H$ as gs with $g \in G$ and $s \in H$. The H-action on G is denoted by α (which is notationally abusive), i.e., $sgs^{-1} = \alpha_s(g)$. We observe that $B \cong \bigotimes_H B$ and so $B(G) \cong (\bigotimes_{G \rtimes H} B) \rtimes G$ where G acts on $G \rtimes H$ from the left. To avoid a possible confusion, we denote the latter C*-algebra by $\tilde{B}(G)$. The group H also acts on $G \rtimes H$ from the left: $s(ht) = \alpha_s(h)st$, and we denote by α again the corresponding Bernoulli shift action on $\bigotimes_{G \rtimes H} B$. Since, for each $s \in H$, the pair $\bigotimes_{G \rtimes H} B \ni b \mapsto \alpha_s(b)$ and $G \ni g \mapsto u_{\alpha_s(g)}$ is a covariant representation in $\tilde{B}(G)$, the map α extends to an action of H on $\tilde{B}(G)$. It is not difficult to see

$$\tilde{B}(G) \rtimes_\alpha H = ((\bigotimes_{G \rtimes H} B) \rtimes G) \rtimes H \cong (\bigotimes_{G \rtimes H} B) \rtimes (G \rtimes H) = B(G \rtimes H).$$

For (iv), we use the isomorphisms $B \cong \bigotimes_{G_2/G_1} B$ and $B(G_1) \cong (\bigotimes_{G_2} B) \rtimes G_1$. Hence it suffices to show when G_2 acts on a unital C*-algebra C, and G_1 is a finite-index subgroup of G_2,

then $C \rtimes G_2 \subseteq (C \rtimes G_1) \otimes M_{|G_2/G_1|}$. Although this is well-known, we sketch a proof for the convenience of the reader.

Put $n = |G_2/G_1|$. Let π be a faithful representation of C on a Hilbert space H. Then we get a faithful representation $\pi \times \lambda_{G_2}$ of $C \rtimes G_2$ on the Hilbert space $K = H \otimes \ell_2(G_2)$; and a faithful representation $\pi \times \lambda_{G_1}$ of $C \rtimes G_1$ on the Hilbert space $K_0 = H \otimes \ell_2(G_1)$. Write G_2 as a disjoint union $\bigcup_{i=1}^{n} x_i G_1$, for suitable $x_i \in G_2$, and consider the subspaces $K_i = H \otimes \ell_2(x_i G_1)$, $i = 1, \ldots, n$, of K. Let P_i denote the orthogonal projection from K onto K_i and let $U_i = (\pi \times \lambda_{G_2})(u_{x_i}) = I_H \otimes \lambda_{G_2}(x_i)$, where $u_g \in C \rtimes G_2$, $g \in G_2$, are the canonical unitaries that generate the action of G_2 on C. Then $U_i(K_0) = K_i$, and we have a $*$-isomorphism

$$\Phi \colon B(K) \to B(K_0) \otimes M_n, \qquad T \mapsto \left(U_i^* P_i T P_j U_j \right)_{i,j=1}^{n}, \quad T \in B(K).$$

For each $c \in C$ and $g \in G_2$, one checks that

$$U_i^* P_i (\pi \times \lambda_{G_2})(c\, u_g) P_j U_j = \begin{cases} (\pi \times \lambda_{G_1})\left(\alpha_{x_i^{-1}}(c)\, u_{x_i^{-1} g x_j} \right), & \text{if } x_i^{-1} g x_j \in G_1 \\ 0, & \text{else} \end{cases} .$$

It follows that Φ maps $(\pi \times \lambda_{G_2})(C \rtimes G_2)$ into $(\pi \times \lambda_{G_1})(C \rtimes G_1) \otimes M_n$, and we thus obtain the desired embedding of $C \rtimes G_2$ into $(C \rtimes G_1) \otimes M_n$. $\qquad \square$

By the proof of simplicity of $B(G)$ one can obtain that σ is *strongly outer*, cf. [18, Definition 2.5]. Thus the Bernoulli shift action of G on $\bigotimes_G B$ has the *weak Rohlin property*, cf. [18, Definition 2.5], whenever G is countable and elementary amenable by [18, Theorem 3.6].

3   Crossed products by integers

We shall use a bootstrap argument to see that the C*-algebra $B(G)$ is quasidiagonal for every elementary amenable group G. To do so, we introduce the following property:

Definition 3.1. Let PQ be the class of all countable groups G for which the reduced crossed product C*-algebra $\left(\bigotimes_G M_2 \right) \rtimes G$ is quasidiagonal.

Observe that $\mathrm{PQ} \subseteq \mathrm{QD} \subseteq \mathrm{AG}$, where QD is the class of groups G for which $C_\lambda^*(G)$ is quasidiagonal and AG is the class of amenable groups. The former inclusion follows from the fact that $C_\lambda^*(G)$ embeds into $\left(\bigotimes_G M_2 \right) \rtimes G$, and the latter inclusion is Rosenberg's theorem. We show in this section that PQ contains the class of countable elementary amenable groups.

Remark 3.2. We observe that a countable group G belongs to PQ if and only if the Bernoulli shift crossed product $B(G)$ is quasidiagonal. Indeed, any unital embedding of M_2 into the CAR-algebra B induces an embedding of $\left(\bigotimes_G M_2 \right) \rtimes G$ into $B(G)$, so if the latter is quasidiagonal, then so is the former. Assume that $\left(\bigotimes_G M_2 \right) \rtimes G$ is quasidiagonal and consider the diagonal action δ of G on $G \times G$. Then

$$\left(\bigotimes_G \left(\bigotimes_G M_2 \right) \right) \rtimes G \cong \left(\bigotimes_{G^2} M_2 \right) \rtimes_\delta G \hookrightarrow \left(\bigotimes_{G^2} M_2 \right) \rtimes G^2 \cong \left(\left(\bigotimes_G M_2 \right) \rtimes G \right) \otimes \left(\left(\bigotimes_G M_2 \right) \rtimes G \right)$$

where the first isomorphism is induced by a suitable set bijection $G \times G \to G^2$, and where the inclusion of the second C*-algebra into the third arises from the diagonal embedding $G \hookrightarrow G^2$. The C*-algebra on the right-hand side is quasidiagonal (being the minimal tensor product of quasidiagonal C*-algebras), and the C*-algebra on the left-hand side is isomorphic to $B(G)$ if G is infinite. If G is finite, then $B(G)$ embeds into $B \otimes M_{|G|}$, cf. Proposition 2.1 (iv). In either case we see that $B(G)$ is quasidiagonal.

Before proceeding, let us recall some facts about the class EG of elementary amenable groups. By definition, EG is the smallest class of groups that contains all finite and all abelian groups, and which is closed under the following four operations: taking subgroups, quotients, direct limits,

and extensions. Let B_0 denote the union of all finite groups and the infinite cyclic group $\mathbb{Z}$. Note that B_0 is closed under taking subgroups and quotients. It was shown by C. Chou, [5], and refined by D. V. Osin, [20, Theorem 2.1], that all groups in EG can be built up from the basis B_0 by transfinite induction just using direct limits and extensions. We let $\mathrm{EG_c}$ denote the class of countable elementary amenable groups.

Proposition 3.3 (Chou, [5] and Osin, [20]). *EG is the smallest class of groups which contains the trivial group $\{1\}$ and which is closed under taking direct limits and extensions by groups from B_0. In particular, $\mathrm{EG_c}$ is the smallest class of groups which contains the trivial group $\{1\}$ and which is closed under taking countable direct limits and extensions by groups from B_0.*

A class C of groups is closed under extensions by groups from B_0 if for all short exact sequences $1 \to N \to G \to G/N \to 1$, that $N \in C$ and $G/N \in B_0$ implies $G \in C$. Thus, by Proposition 2.1, to prove $\mathrm{EG_c} \subseteq \mathrm{PQ}$, it remains to show $B(G) \rtimes_\alpha \mathbb{Z}$ is quasidiagonal for every $G \in \mathrm{PQ}$. This follows from the following two results from the classification theory of C*-algebras.

The first is proved in [19, Corollary 6.2] using classification theorems by H. Lin–Z. Niu, [14], and W. Winter, [24]. We denote by $\mathcal{U}$ the universal UHF-algebra.

Theorem 3.4 ([14], [19]). *Let A be a unital separable simple nuclear monotracial quasidiagonal C*-algebra in the UCT class. Then $A \otimes \mathcal{U}$ is an $A\mathbb{T}$-algebra of real rank zero. In particular, A is embeddable into an AF-algebra.*

The classification theorems mentioned above imply that if A and B are C*-algebras satisfying the assumptions of Theorem 3.4 as well as the *strict comparison*, then $A \cong B$ if and only if $K_0(A) \cong K_0(B)$ (as ordered abelian groups with distinguished order units) and $K_1(A) \cong K_1(B)$ (as groups). Theorem 3.4 follows from this because $A \otimes \mathcal{U}$ has strict comparison, [21], and one can check that the K-theory of $A \otimes \mathcal{U}$ agrees with the K-theory of a simple $A\mathbb{T}$-algebra of real rank zero. AF-embeddability of such an algebra is a consequence of Elliott's classification result, [6], see also [22, Proposition 4.1] for a simpler proof of this fact.

The second is Matui's theorem, [15, Theorem 2], about AF-embeddability. We recall that every AF-algebra is quasidiagonal, and hence every AF-embeddable C*-algebra is quasidiagonal. It is not known whether there is a nuclear quasidiagonal C*-algebra which is not AF-embeddable.

Theorem 3.5 (Matui, [15]). *Let A be a unital separable simple $A\mathbb{T}$-algebra of real rank zero. Then for any $*$-automorphism α, the crossed product $A \rtimes_\alpha \mathbb{Z}$ is embeddable into an AF-algebra. In particular, $A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal.*

Corollary 3.6. *Let A be a unital separable simple nuclear monotracial quasidiagonal C*-algebra in the UCT class. Then for any $*$-automorphism α, the crossed product $A \rtimes_\alpha \mathbb{Z}$ is embeddable into an AF-algebra. In particular, $A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal.*

Proof. Apply Theorems 3.4 and 3.5 to $(A \otimes \mathcal{U}) \rtimes_{\alpha \otimes \mathrm{id}} \mathbb{Z}$, which contains $A \rtimes_\alpha \mathbb{Z}$. $\qquad\square$

Remark 3.7. We note that $A \rtimes_\alpha G$ is again monotracial if A is a unital monotracial C*-algebra, G is a countable amenable group, and the action α of G has the weak Rohlin property. This follows easily from the definition of the weak Rohlin property, cf. [18, Definition 2.5], arguing for example as in [17, Remark 2.8].

Here is the main result of this paper.

Theorem 3.8.

 (i) *The class PQ is closed under the following operations: direct limits, subgroups, and extensions by countable elementary amenable groups.*

 (ii) $\mathrm{EG_c} \subset \mathrm{PQ}$.

(iii) $C^*_\lambda(G)$ *is AF-embeddable for every group G in* PQ.

(iv) $C^*_\lambda(G)$ *quasidiagonal for all elementary amenable groups G.*

Proof. (i). The first two claims follow from Proposition 2.1.

Let QQ denote the class of those countable groups H which satisfy the following property: whenever G is a group having a normal subgroup N from PQ such that $G/N \cong H$, the group G belongs to PQ. Since the trivial group $\{1\}$ belongs to PQ, one has QQ $\subseteq$ PQ. We shall prove that $EG_c \subseteq$ QQ. By Proposition 3.3, it suffices to show that QQ is closed under taking countable direct limits and extensions by groups from B_0. Let a surjective homomorphism $q\colon G \to H$ such that $\ker q \in$ PQ be given. First, suppose that $H = \bigcup_n H_n$ is a directed union of $H_n \in$ QQ. Then, one has $G_n := q^{-1}(H_n) \in$ PQ and $G = \bigcup_n G_n$. Hence G belongs to PQ. Next, suppose that H has finite-index subgroup H_0 which belongs to QQ. Then, $G_0 := q^{-1}(H_0)$ is a finite-index subgroup of G which belongs to PQ. Hence $G \in$ PQ. Lastly, suppose that H has a normal subgroup H_0 such that $H_0 \in$ QQ and $H/H_0 \cong \mathbb{Z}$. Then, since $\mathbb{Z}$ is a free group, $G \cong G_0 \rtimes \mathbb{Z}$ for $G_0 := q^{-1}(H_0) \in$ PQ. That $G \in$ PQ follows from Proposition 2.1.(iii) and Corollary 3.6. This finishes the proof of (i).

(ii) follows from (i) and the fact that the trivial group belongs to PQ.

(iii). If G belongs to PQ, then $B(G)$ satisfies the hypothesis of Theorem 3.4, cf. Proposition 2.1 and Remark 3.2. Hence $B(G)$ is AF-embeddable by Theorem 3.4, whence $C^*_\lambda(G) \subseteq B(G)$ is also AF-embeddable.

(iv). It follows from (ii) and (iii) that $C^*_\lambda(G)$ is quasidiagonal for all countable elementary amenable groups. When G is uncountable and elementary amenable, it is a direct limit of countable subgroups, all of which are elementary amenable. Since quasidiagonality is separably determined, the reduced group C*-algebra $C^*_\lambda(G)$ is quasidiagonal. □

The class PQ is strictly larger than the class EG_c of countable elementary amenable groups, as the proposition below shows. Recall that a group G is said to be LEF if for every finite subset $F \subset G$ there are a finite group H and an injective map $\pi\colon F \to H$ such that $\pi(gh) = \pi(g)\pi(h)$ whenever $g, h, gh \in F$. In [4, 11], it is shown that amenable LEF groups have quasidiagonal reduced group C*-algebras. We adapt this and prove the following.

Proposition 3.9. *Let G be a countable amenable group which satisfies the following property: for every finite subset $F \subset G$ there are a group H from* PQ *and an injective map $\pi\colon F \to H$ such that $\pi(gh) = \pi(g)\pi(h)$ whenever $g, h, gh \in F$. Then, G belongs to* PQ.

In particular, any countable amenable LEF group belongs to PQ, *and more particular, any countable amenable residually finite group, the Grigorchuk group, and topological full groups of Cantor minimal systems belong to* PQ.

Proof. Let $G = \bigcup_n F_n$ be a directed union of finite subsets. For every n, there are a group H_n from PQ and an injective map $\pi_n\colon F_n \to H_n$ that satisfy the above-stated condition. We construct an embedding of $(\bigotimes_G M_2) \rtimes G$ into $\prod_n A_n / \bigoplus_n A_n$ where $A_n = (\bigotimes_{H_n} M_2) \rtimes H_n$ and $\prod_n$ (resp. $\bigoplus_n$) denotes the ℓ_∞ (resp. c_0) direct sum of C*-algebras. Each A_n is quasidiagonal and therefore an MF algebra (in the sense of [1]). It therefore follows from [1, Corollary 3.4.3] that the separable C*-algebra $(\bigotimes_G M_2) \rtimes G$ is an MF algebra, hence also an NF algebra (in the sense of [1]) because it is nuclear, and hence quasidiagonal by [1, Theorem 5.2.2].

We first look at $\bigotimes_G M_2$. Let $a \in \bigcup_n(\bigotimes_{F_n} M_2)$ be given. Then $a \in \bigotimes_E M_2$ for some finite subset $E \subset G$. Take $N \in \mathbb{N}$ such that $E \subset F_N$. Then, for every $n \geq N$, the injective map π_n induces a canonical *-homomorphism ρ_n from $\bigotimes_E M_2$ into $\bigotimes_{H_n} M_2 \subset A_n$. Set $\rho_n(a) = 0$ when $n < N$ and define $\rho(a) \in \prod A_n / \bigoplus A_n$ by $\rho(a) = [(\rho_n(a))_{n=1}^\infty]$. Then ρ extends to a *-homomorphism from $\bigotimes_G M_2$ into $\prod A_n / \bigoplus A_n$. Next, for each $g \in G$, we define $u_g \in \prod A_n / \bigoplus A_n$ by $u_g = [(u_{\pi_n(g)})_{n=1}^\infty]$. The values $u_{\pi_n(g)}$ where they are not defined do not matter, and u is a unitary representation of G into $\prod A_n / \bigoplus A_n$. These representations are covariant and since G is amenable, they give rise to a *-homomorphism from the reduced crossed product $(\bigotimes_G M_2) \rtimes G$ into

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$\prod_n A_n / \bigoplus_n A_n$. Since the former is simple (or since the canonical tracial states are compatible), it is a faithful embedding.

The Grigorchuk group and the topological full groups of Cantor minimal systems are LEF and so satisfy the assumption of this proposition. $\qquad \square$

Remark 3.10. Observe that our proof that the class PQ is closed under extensions by $\mathbb{Z}$ relies on the classification theory for C*-algebras. We do not know if the class QD is closed under extensions by $\mathbb{Z}$. If it is, then one can obtain part (iv) of Theorem 3.8 directly from Osin's theorem.

Remark 3.11. Kerr and Nowak proved in [12, Theorem 3.5] that the reduced crossed product $A \rtimes_\alpha G$ is quasidiagonal whenever G is a countable group in QD, A is a separable nuclear C*-algebra A, and α is a *quasidiagonal* action of G on A (in the sense of [12, Definition 3.2]). It follows that a countable group G in QD belongs to the class PQ if the Bernoulli action of G on $\bigotimes_G M_2$ is quasidiagonal. The latter holds for example when G is residually finite.

We conclude our paper remarking that our proofs imply the following corollary that may be of independent interest:

Corollary 3.12. *For any group G in PQ there is an action α of G on the universal UHF-algebra $\mathcal{U}$ such that $\mathcal{U} \rtimes_\alpha G$ is a simple A$\mathbb{T}$-algebra of real rank zero.*

Proof. It follows from Proposition 2.1, Theorem 3.8, and Theorem 3.4 that $((\bigotimes_G B) \otimes \mathcal{U}) \rtimes_{\sigma \otimes \mathrm{id}} G$ is a simple A$\mathbb{T}$-algebra of real rank zero. We note that $(\bigotimes_G B) \otimes \mathcal{U} \cong \mathcal{U}$. $\qquad \square$

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