On Example 5.11 from my paper "Irreducible inclusions of simple C^*-algebras

We provide here details of the claim made in [5, Example 5.11]. One needs the following version of [5, Lemma 5.10]:

Lemma 1. *Let $n \geq 2$ be an integer and consider the tensor powers $\mathcal{A} = \bigotimes_{j \in \mathbb{Z}} M_n(\mathbb{C})$ and $\mathcal{B} = \bigotimes_{j \in \mathbb{N}} M_n(\mathbb{C})$, and view $\mathcal{B}$ as a subalgebra of $\mathcal{A}$ in the canonical way. Let σ denote the Bernoulli shift on $\mathcal{A}$ and denote also by σ its restriction to $\mathcal{B}$ (which is the one-sided shift).*

Then, for each $N \geq 1$ and each $\varepsilon > 0$, there exist $m \geq 1$ and unitaries $v_1, \dots, v_m$ in $\mathcal{B}$ (and hence also in $\mathcal{A}$) such that

$$\Big\| \frac{1}{m} \sum_{j=1}^{m} v_j \sigma^k(v_j)^* \Big\| \leq \varepsilon, \qquad \Big\| \frac{1}{m} \sum_{j=1}^{m} \sigma^k(v_j) v_j^* \Big\| \leq \varepsilon$$

for all $1 \leq k \leq N$.

Proof. The existence of the unitaries $v_1, \dots, v_m$ in $\mathcal{A}$ with the stated properties follows from [5, Lemma 5.10] (with $b_j = 1$) since all non-zero powers of the shift σ are outer. Since $\lim_{p \to \infty} \operatorname{dist}(\sigma^p(x), \mathcal{B}) = 0$, for each $x \in \mathcal{A}$, upon replacing the unitaries v_j with $\sigma^p(v_j)$ for a suitably large $p \geq 1$, and then perturbing them sligthly, we obtain unitaries in $\mathcal{B}$ with the desired properties. $\square$

Example 2. For $2 \leq n < \infty$ consider the Cuntz algebra $\mathcal{O}_n$ and its sub-C^*-algebra $\mathcal{B}_n$ isomorphic to the UHF-algebra of type n^∞, which arises as the fixed point algebra under the canonical circle action of $\mathcal{O}_n$. Let $E \colon \mathcal{O}_n \to \mathcal{B}_n$ be the conditional expectation (obtained by integrating with respect to the circle action). Then E is faithful and has the pinching property, cf. [2] (use the projection Q constructed in the proof of Proposition 1.7). Hence $\mathcal{B}_n \subseteq \mathcal{O}_n$ is C^*-irreducible.

It is well-known that $\mathcal{O}_n = C^*(\mathcal{B}_n, s_1)$, and that the isometry s_1 induces an endomorphism ρ on $\mathcal{B}_n$ by $\rho(b) = s_1 b s_1^*$. It this sense we can write $\mathcal{O}_n$ as a crossed product $\mathcal{B}_n \rtimes_\rho \mathbb{N}$ over the semigroup $\mathbb{N}$. Similar to the situation of [5, Theorem 5.9], each proper intermediate C^*-algebra of the inclusion $\mathcal{B}_n \subseteq \mathcal{O}_n$ is equal to $\mathcal{B}_n \rtimes_\rho d\mathbb{N}$, for some $d \geq 2$. The crossed product $\mathcal{B}_n \rtimes_\rho d\mathbb{N}$ is equal to $C^*(\mathcal{B}_n, s_1^d)$ and also to the fixed-point algebra $\mathcal{O}_n^{\mathbb{Z}/d}$ with respect to the order d automorphism on $\mathcal{O}_n$ given by $s_j \mapsto \omega s_j$, where ω is a primitive dth root of the unit.

We proceed to justify the claim made above. The proof follows the same lines as the proof of [5, Theorem 5.9], but one must adjust for dealing with endomorphisms instead of automorphisms. Each $x \in \mathcal{O}_n$ can be written as a formal sum

$$x = \sum_{k=1}^{\infty} (s_1^*)^k a_{-k} + a_0 + \sum_{k=1}^{\infty} a_k s_1^k,$$

for some $a_k \in \mathcal{B}_n$, with $a_k = a_k s_1^k (s_1^*)^k$ and $a_{-k} = s_1^k (s_1^*)^k a_{-k}$, for all $k \geq 1$. As in the proof of [5, Theorem 5.9], it suffices to show that if $\ell \in \operatorname{supp}(x)$, i.e., $a_\ell \neq 0$, then s_1^ℓ, respectively, $(s_1^*)^{-\ell}$, belong to $C^*(\mathcal{B}_n, x)$ (the former when $\ell \geq 0$ and the latter when $\ell < 0$). Consider the case $\ell \geq 1$ (the case where $\ell \leq -1$ is treated in a similar manner). By (algebraic) simplicty of $\mathcal{B}_n$ we can find $c_i, d_i \in \mathcal{B}_n$ such that $\sum_{i=1}^r c_i a_\ell d_i = 1$. Put $e_i = (s_1^*)^\ell d_i s_1^\ell \in \mathcal{B}_n$. Then $y := \sum_{i=1}^r c_i x e_i$ belongs to $C^*(\mathcal{B}_n, x)$, and $y = \sum_{k=1}^\infty (s_1^*)^k b_{-k} + b_0 + \sum_{k=1}^\infty b_k s_1^k$, with

$$b_\ell s_1^\ell = \sum_{i=1}^r c_i a_\ell s_1^\ell e_i = \sum_{i=1}^r c_i a_\ell s_1^\ell (s_1^*)^\ell d_i s_1^\ell = \sum_{i=1}^r c_i a_\ell d_i s_1^\ell = s_1^\ell.$$

Let $\varepsilon > 0$ and take $y_0 = \sum_{k=1}^N (s_1^*)^k b'_{-k} + b'_0 + \sum_{k=1}^N b'_k s_1^k$ in $\mathcal{O}_n$ with $b'_k \in \mathcal{B}_n$ and $\|x - y\| \leq \varepsilon/3$. Use Lemma 1 to find unitaries $v_1, \dots, v_m \in \mathcal{B}_n$ such that

$$\|\frac{1}{m}\sum_{j=1}^m v_j \sigma^k(v_j)^*\| \leq \frac{\varepsilon}{12N}, \qquad \|\frac{1}{m}\sum_{j=1}^m \sigma^k(v_j) v_j^*\| \leq \frac{\varepsilon}{12N}, \quad 1 \leq k \leq N.$$

Identifying $\mathcal{B}_n$ with $\bigotimes_{j\in\mathbb{N}} M_n(\mathbb{C})$, the (one-sided) shift σ from Lemma 1 is given by $\sigma(b) = \sum_{j=1}^n s_j b s_j^*$. Hence $s_1^k b (s_1^*)^k = \sigma^k(b) s_1^k (s_1^*)^k = s_1^k (s_1^*)^k \sigma^k(b)$, for all $b \in \mathcal{B}$ and all $k \geq 1$. One now checks that

$$\sigma^\ell(v_j) b'_k s_1^k v_j^* = \sigma^\ell(v_j) b'_k \sigma^k(v_j)^* s_1^k, \qquad \sigma^\ell(v_j)(s_1^*)^k b'_{-k} v_j^* = (s_1^*)^k \sigma^{\ell+k}(v_j) b'_{-k} v_j^*.$$

Since $[\sigma^p(b), b'] \to 0$ as $p \to \infty$, for all $b, b' \in \mathcal{B}_n$, upon replacing each v_j with $\sigma^p(v_j)$ for a sufficiently large p, both expressions above can be made less than $\varepsilon/(6N)$, when $k \neq \ell$. Hence we obtain

$$\Big\|\frac{1}{m}\sum_{j=1}^m \sigma^\ell(v_j)(y_0 - b'_\ell s_1^\ell) v_j^*\Big\| \leq \varepsilon/3,$$

which in turn yields

$$\Big\|\frac{1}{m}\sum_{j=1}^m \sigma^\ell(v_j) y v_j^* - s_1^\ell\Big\| = \Big\|\frac{1}{m}\sum_{j=1}^m \sigma^\ell(v_j)(y - s_1^\ell) v_j^*\Big\| \leq \varepsilon.$$

Since $m^{-1}\sum_{j=1}^m \sigma^\ell(v_j) y v_j^* \in C^*(\mathcal{B}_n, x)$, this proves that $s_1^\ell \in C^*(\mathcal{B}_n, x)$.

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