The real rank of certain simple C^*-algebras

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Abstract

In [6] we constructed simple C^*-algebras that contain both finite and infinite projections. We show here, using ideas of Villadsen, that these C^*-algebras are not of real rank zero. However, these C^*-algebras do have the weaker property (SP).

1 Introduction

Simple C^*-algebras with "high-dimensional" behaviour were first constructed by Villadsen in [7], where a finite simple C^*-algebra with *strong perforation* was exhibited. Strong perforation is the converse of *weak unperforation*, and for a simple C^*-algebra A it is equivalent to the existence of two projections p and q in $A \otimes \mathcal{K}$ such that $(n+1)[p] \leq n[q]$ and $[p] \not\leq [q]$ (in the ordered semigroup of Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$). Another high-dimensional phenomenon, which for simple C^*-algebras so far only has been seen in the infinite case, is failure of *(weak) cancellation*, i.e., the existence of (non-zero) projections p, q, r in $A \otimes \mathcal{K}$ such that $[p] + [r] = [q] + [r]$ and $[p] \neq [q]$.

There are two notions of "low dimensionality" for a C^*-algebra A, that the stable rank and the real rank are as small as possible: $\mathrm{sr}(A) = 1$ and $\mathrm{RR}(A) = 0$, respectively. The two properties $\mathrm{sr}(A) = 1$ and $\mathrm{RR}(A) = 0$ are indenpendent: purely infinite simple C^*-algebras are of real rank zero and have infinite stable rank, and there are simple C^*-algebras of stable rank one and real rank one (for example certain simple AI- and A$\mathbb{T}$-algebras, where projections don't separate traces). Villadsen found finite simple AH-algebras of stable rank > 1 and of real rank > 0, [8]. Such AH-algebras cannot have slow dimension growth, and these C^*-algebras are therefor "high-dimensional" in a very explicit way. It is not known if there are finite simple C^*-algebras of real rank zero and stable rank > 1. It is also unknown if all finite simple C^*-algebras must have the weak cancellation property or the cancellation property.

Every infinite simple C^*-algebra is of stable rank ∞, and every purely infinite simple C^*-algebra is of real rank zero. In [6] the author found infinite simple C^*-algebras that are not purely infinite (in fact, they contain both finite and infinite projections). These examples are in some ways similar to those constructed by Villadsen, and they are build from an infinite dimensional space (see Section 3). We show here that these algebras are not of real rank zero. This has relevance for the still open problem if all simple infinite C^*-algebras of real rank zero must be purely infinite.

Acknowledgements: I thank Pere Ara and Francesc Perera for inviting me to CRM, Barcelona, where parts of the work presented in this note was done, and for stimulating conversations on topics related to C^*-algebras of real rank zero. I also thank Andrew Toms for directing my attention to Villadsen's paper [8] in the context of this paper, and Hans Jørgen Munkholm for useful discussions on Proposition 2.2.

2 Preliminaries: Results of Villadsen

We present here two observations that Villadsen in his paper [8] used to decide that his C^*-algebras are not of real rank zero and not of stable rank one.

Lemma 2.1 (Villadsen) *Let A be a unital C^*-algebra, let a be an invertible element in A, and let p be a projection in A such that $pap = 0$. It follows that $p \precsim 1 - p$.*

Proof: Put $x = pa(1-p)$ and $y = (1-p)a^{-1}p$. Then

$$p = paa^{-1}p = papa^{-1}p + pa(1-p)a^{-1}p = xy \sim yx,$$

and yx is a sub-idempotent of $1-p$. $\square$

Proposition 2.2 (Villadsen) *Let M be a compact orientable manifold and let T be a compact manifold which contains a submanifold T_0 with a diffeomorphism $\lambda_0 \colon T_0 \to [-1,1]$. Let $\lambda \colon T \to [-1,1]$ be a continuous extension of λ_0. Let $\pi_1 \colon T \times M \to T$ and $\pi_2 \colon T \times M \to M$ be the coordinate maps.*

There is a dense subset $\mathcal{D}$ of the open ball

$$\mathcal{B} := \{f \in C_{\mathbb{R}}(T \times M) : \|f - \lambda \circ \pi_1\|_\infty < 1\}$$

such that if $f \in \mathcal{D}$, then $N := f^{-1}(\{0\}) \subseteq T \times M$ is a finite CW complex and $\pi^*\colon H^*(M) \to H^*(N)$ is injective, where $\pi = \pi_2|_N\colon N \to M$.

Proof: We show that the proposition follows from Villadsen's [8, Theorem 1] (and its proof). Let $\mathcal{D}$ be the set of smooth functions f in $\mathcal{B}$ such that $0 \in \mathbb{R}$ is a regular value for f and for $f|_{T_0 \times M}$. We show that the set $\mathcal{D}$ is dense in $\mathcal{B}$. Let $h \in \mathcal{B}$ and $\varepsilon > 0$ be given with $\varepsilon < 1 - \|h\|_\infty$. Then there is a Morse function $g\colon T \times M \to \mathbb{R}$ such that $\|h - g\|_\infty < \varepsilon/2$. (That g is a Morse function means that g is smooth and that all but finitely many $c \in \mathbb{R}$ are regular values for g.) By Sard's theorem, the set of regular values for the smooth function $g|_{T_0 \times M}$ is dense in $\mathbb{R}$, and this dense set remains dense after subtracting the finite set of values $c \in \mathbb{R}$ at which $g\colon T \times M \to \mathbb{R}$ is not regular. We can therefore find $c \in \mathbb{R}$ with $|c| < \varepsilon/2$ and such that 0 is a regular value for $g - c$ and for $(g - c)|_{T_0 \times M}$. It follows that $f = g - c$ belongs to $\mathcal{D}$ and $\|f - h\|_\infty < \varepsilon$. The latter implies that f belongs to $\mathcal{B}$.

Take $f \in \mathcal{D}$ and put $N = f^{-1}(\{0\})$. Then N is a submanifold of $T \times M$ (and hence a finite CW complex) by the preimage theorem. Let $\varphi\colon C_{\mathbb{R}}(T \times M) \to C_{\mathbb{R}}([-1,1] \times M)$ be the surjective *-homomorphism given by $\varphi(f) = f \circ (\lambda_0^{-1} \times \mathrm{id}_M)$, and put $N_0 = \varphi(f)^{-1}(\{0\}) \subseteq [-1,1] \times M$. Note that $\varphi(f)$ is smooth and that 0 is a regular value for $\varphi(f)$. It therefore follows from [8, Theorem 1] (and its proof) that N_0 is a submanifold of $[-1,1] \times M$ and that $\sigma^*\colon H^*(M) \to H^*(N_0)$ is injective, when $\sigma_2\colon [-1,1] \times M \to M$ is the coordinate mapping, and $\sigma = \sigma_2|_{N_0}\colon N_0 \to M$.

The commutative diagram

$$\begin{tikzcd} & M & \\ N \arrow[ur, "\pi"] & & N_0 \arrow[ul, "\sigma"'] \arrow[ll, "(\lambda_0^{-1} \times \mathrm{id}_M)|_{N_0}"] \end{tikzcd}$$

at the level of spaces induces the commutative diagram

$$\begin{tikzcd} & H^*(M) \arrow[dl, "\pi^*"'] \arrow[dr, "\sigma^*"] & \\ H^*(N) \arrow[rr, "(\lambda_0^{-1} \times \mathrm{id}_M)|_{N_0}^*"'] & & H^*(N_0) \end{tikzcd}$$

at the level of cohomology groups. As σ^* is injective, so is π^*. $\square$

3 Preliminaries: The construction of two simple C^*-algebras

We remind the reader of the construction in [6, Sections 5 and 6] of a simple, non-exact (and non-separable) C^*-algebra B and of a nuclear, simple, separable C^*-algebra $D \rtimes_\alpha \mathbb{Z}$, both of which contain non-zero finite as well as infinite projections.

Take $Z = \prod_{n=1}^{\infty} S^2$. The C^*-algebra B is the inductive limit

$$\mathcal{M}(C(Z,\mathcal{K})) \xrightarrow{\varphi} \mathcal{M}(C(Z,\mathcal{K})) \xrightarrow{\varphi} \mathcal{M}(C(Z,\mathcal{K})) \xrightarrow{\varphi} \cdots \longrightarrow B,$$

with connecting map φ given as follows. Let $p \in C(S^2, M_2) \subset C(S^2,\mathcal{K})$ be the Bott projection (a 1-dimensional projection with non-zero Euler class[1] $e(p) \in H^2(S^2,\mathbb{Z})$). For each $n \in \mathbb{N}$ define $p_n = p \circ \pi_n \in C(Z,\mathcal{K})$, where $\pi_n \colon Z \to S^2$ is the coordinate map onto the nth copy of S^2. For $I = \{n_1, n_2, \dots, n_k\} \subseteq \mathbb{N}$ define $p_I \in C(Z,\mathcal{K})$ by

$$p_I(x) = p_{n_1}(x) \otimes \cdots \otimes p_{n_k}(x) \in M_2 \otimes \cdots \otimes M_2 \subseteq \mathcal{K}, \qquad x \in Z.$$

Choose an injective map $\nu \colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ and define endomorphisms φ_j on $C(Z,\mathcal{K})$ for each $j \in \mathbb{Z}$ as follows. For $j \leq 0$, for $f \in C(Z,\mathcal{K})$, and for $x = (x_1, x_2, \dots) \in Z$, set

$$\varphi_j(f)(x) = f(x_{\nu(j,1)}, x_{\nu(j,2)}, \dots). \tag{3.1}$$

For $j \geq 1$, set $I_j = \{\nu(j,1), \dots, \nu(j,j)\} \subseteq \mathbb{N}$, and set

$$\varphi_j(f)(x) = \tau\big(f(c_{j,1}, \dots, c_{j,j}, x_{\nu(j,j+1)}, x_{\nu(j,j+2)}, \dots) \otimes p_{I_j}(x)\big), \tag{3.2}$$

where τ is an isomorphism from $\mathcal{K} \otimes \mathcal{K}$ onto $\mathcal{K}$, and where $c_{i,j}$ are suitable points in S^2 (see [6, Section 5] for details). It follows from [6, Lemma 5.1 and (5.6)] that there is a sequence $\{S_j\}_{j \in \mathbb{Z}}$ of isometries in $\mathcal{M}(C(Z,\mathcal{K}))$ such that the sum of the right-hand side of

$$\varphi(f) = \sum_{j \in \mathbb{Z}} S_j \varphi_j(f) S_j^*, \qquad f \in C(Z,\mathcal{K}), \tag{3.3}$$

is strictly convergent and the resulting *-homomorphism $\varphi \colon C(Z,\mathcal{K}) \to \mathcal{M}(C(Z,\mathcal{K}))$ is proper (i.e., maps an approximate unit for $C(Z,\mathcal{K})$ onto a sequence that converges

[1] The Euler class in $H^{2\dim(p)}(X;\mathbb{Z})$ of a projection p in $C(X,\mathcal{K})$ is the Euler class of the associated complex vector bundle over X.

strictly to 1). The *-homomorphism φ can thus be extended to a unital endomorphism of $\mathcal{M}(C(Z,\mathcal{K}))$ that here will be denoted by φ (it is denoted by $\overline{\varphi}$ in [6]). It is proved in [6, Section 5] that B is a simple, infinite C^*-algebra that contains a non-zero finite projection.

Let $\varphi_{\infty,n}$ be the inductive limit map from the nth copy of $\mathcal{M}(C(Z,\mathcal{K}))$ to B. Consider the projection $Q = \varphi_{\infty,0}(p_1) \in B$. Put $Q_n = \varphi^n(p_1) \in \mathcal{M}(C(Z,\mathcal{K}))$.

Lemma 3.1 *For each $n \geq 1$ there is a countable family $\mathbb{I}_n$ of finite subsets of $\mathbb{N}$ such that $Q_n \sim \bigoplus_{I \in \mathbb{I}_n} p_I$.*

None of the natural numbers s_n, $n \geq 0$, defined recursively by $s_0 = 2$ and $s_{n+1} = \nu(0, s_n)$, belong to $\bigcup_{I \in \mathbb{I}_n} I$.

Proof: The first claim follow by induction using [6, Lemma 5.5]. For the second claim, it follows from [6, Lemma 5.4] that the families $\{\mathbb{I}_n\}$ are inductively given by $\mathbb{I}_0 = \{\{1\}\}$ and

$$\mathbb{I}_{n+1} = \{\nu(j,I) \mid j \leq 0,\, I \in \mathbb{I}_n\} \cup \{\nu(j, I \setminus \{1,\dots,j\}) \cup I_j \mid j \geq 1,\, I \in \mathbb{I}_n\}, \qquad n \geq 0.$$

Clearly $s_0 \notin \bigcup_{I \in \mathbb{I}_0} I = \{1\}$.

It now suffices to show that if $s_n \in I$ for some $I \in \mathbb{I}_n$, then $s_{n-1} \in J$ for some $J \in \mathbb{I}_{n-1}$.

Assume therefore that $I \in \mathbb{I}_n$ contains s_n. Since $s_n \in \nu(0,\mathbb{N})$, $I_j \subset \nu(j,\mathbb{N})$, and ν is injective, s_n does not belong to any of the sets I_j. It follows that $s_n \in \nu(j,J)$ for some $j \in \mathbb{Z}$ and some $J \in \mathbb{I}_{n-1}$. Again by injectivity of ν, $j = 0$ and $s_n = \nu(0, s_{n-1}) \in \nu(0,J)$, whence $s_{n-1} \in J$. $\square$

To formulate the next lemma we must set up a little notation.

For each (finite) subset I of $\mathbb{N}$ let $\pi_I \colon Z \to \prod_{j \in I} S^2$ be the coordinate mapping, and let $\pi_{I,J} \colon \prod_{j \in I} S^2 \to \prod_{j \in J} S^2$ and $\pi_{I,k} \colon \prod_{j \in I} S^2 \to S^2$ denote the coordinate mappings whenever $J \subseteq I \subseteq \mathbb{N}$ and $k \in I$.

If X is a closed subset of Z, then the restriction mapping $\psi_X \colon C(Z,\mathcal{K}) \to C(X,\mathcal{K})$ extends to a strictly continuous unital surjective map $\psi_X \colon \mathcal{M}(C(Z,\mathcal{K})) \to \mathcal{M}(C(X,\mathcal{K}))$.

Lemma 3.2 *Let J be a finite subset of $\mathbb{N}$, let s be an element in J, and put $J' = J \setminus \{s\}$. Let $N \subseteq \prod_{j \in J} S^2$ be a closed finite CW-complex, and set $\pi = \pi_{J,J'}|_N \colon N \to \prod_{j \in J'} S^2$. Put $\widetilde{N} = \pi_J^{-1}(N) \subseteq Z$. Let $g \in C(Z,\mathcal{K})$ be a trivial (= constant) 1-dimensional projection, and let $Q_n \in \mathcal{M}(C(Z,\mathcal{K}))$ be as defined above Lemma 3.1.*

Suppose that $\pi^* \colon H^*(\prod_{j \in J'} S^2) \to H^*(N)$ is injective. Then $\psi_{\widetilde{N}}(g) \not\precsim \psi_{\widetilde{N}}(Q_n)$ for each $n \in \mathbb{N}$ provided that $s \notin \bigcup_{I \in \mathbb{I}_n} I$ (cf. Lemma 3.1). If, in addition, $\nu \colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ is chosen as in [6, Section 6], then $\psi_{\widetilde{N}}(g) \not\precsim \bigoplus_{n=0}^m \psi_{\widetilde{N}}(Q_n)$ (relatively to $\mathcal{M}(C(Z,\mathcal{K}))$) for each $m \in \mathbb{N}$ provided that $s \notin \bigcup_{n=0}^m \bigcup_{I \in \mathbb{I}_n} I$.

Proof: It follows from Lemma 3.1 that

$$Q_n \sim \bigoplus_{I \in \mathbb{I}_n} p_I, \qquad \bigoplus_{n=0}^m Q_n \sim \bigoplus_{I \in \bigcup_{n=0}^m \mathbb{I}_n} p_I.$$

With the purpose of combining the proofs of the two statements in the lemma, we show that $\psi_{\widetilde{N}}(g) \not\precsim \psi_{\widetilde{N}}(R)$, when R is a projection in $\mathcal{M}(C(Z,\mathcal{K}))$ with $R \sim \bigoplus_{I \in \mathbb{J}} p_I$ for a countable collection $\mathbb{J}$ of finite subsets of $\mathbb{N}$ for which $|\bigcup_{I \in \mathbb{J}_0} I| \geq |\mathbb{J}_0|$ for every finite subset $\mathbb{J}_0$ of $\mathbb{J}$, and $s \notin \bigcup_{I \in \mathbb{J}} I$. (It follows from the displayed equations above and from inductive use of [6, Lemmas 5.5 and 6.1] that Q_n and $\bigoplus_{n=0}^m Q_n$ are of this form.)

Suppose, to reach a contradiction, that $\psi_{\widetilde{N}}(g) \precsim \psi_{\widetilde{N}}(R)$ in $\mathcal{M}(C(Z,\mathcal{K}))$. Arguing as in [6, Lemma 4.4] we then get

$$g|_{\widetilde{N}} \;=\; \psi_{\widetilde{N}}(g) \;\precsim\; \bigoplus_{I \in \mathbb{J}_0} \psi_{\widetilde{N}}(p_I) \;=\; \big(\bigoplus_{I \in \mathbb{J}_0} p_I\big)|_{\widetilde{N}} \quad \text{in } C(\widetilde{N},\mathcal{K}) \tag{3.4}$$

for some *finite* subset $\mathbb{J}_0$ of $\mathbb{J}$. We observe that $C(\widetilde{N},\mathcal{K})$ is the inductive limit of the sequence

$$C(\pi_{L_1,J}^{-1}(N),\mathcal{K}) \longrightarrow C(\pi_{L_2,J}^{-1}(N),\mathcal{K}) \longrightarrow C(\pi_{L_3,J}^{-1}(N),\mathcal{K}) \longrightarrow \cdots \longrightarrow C(\widetilde{N},\mathcal{K}),$$

when $\{L_n\}$ is an increasing sequence of finite subsets of $\mathbb{N}$, each of which contains J, and with $\mathbb{N} = \bigcup_{n=1}^\infty L_n$. The projection $\big(\bigoplus_{I \in \mathbb{J}_0} p_I\big)|_{\widetilde{N}}$ belongs to (the image in $C(\widetilde{N},\mathcal{K})$ of) $C(\pi_{L_n,J}^{-1}(N),\mathcal{K})$ for some large enough n; and for a possibly larger n, the relation (3.4) holds relatively to $C(\pi_{L_n,J}^{-1}(N),\mathcal{K})$, i.e., $g|_{\overline{N}} \precsim \big(\bigoplus_{I \in \mathbb{J}_0} p_I\big)|_{\overline{N}}$ in $C(\overline{N},\mathcal{K})$, when $\overline{N} = \pi_{L_n,J}^{-1}(N)$. It follows in particular that the Euler class of $\big(\bigoplus_{I \in \mathbb{J}_0} p_I\big)|_{\overline{N}}$ in $H^{2|\mathbb{J}_0|}(\overline{N})$ is zero.

Put $L = L_n$ and put $L' = L \setminus \{s\}$. As $s \notin \bigcup_{I \in \mathbb{J}} I$, the projection $\bigoplus_{I \in \mathbb{J}_0} p_I$ belongs to $C(\prod_{j \in L'} S^2,\mathcal{K})$ and its Euler class in $H^{2|\mathbb{J}_0|}(\prod_{j \in L'} S^2)$ is non-zero by [6, Proposition 3.2 and Lemma 4.1]. Put $\rho = \pi_{L,L'}|_{\overline{N}} \colon \overline{N} \to \prod_{j \in L'} S^2$. By functoriality of the Euler class, ρ^* maps the Euler class of $\bigoplus_{I \in \mathbb{J}_0} p_I$ in $H^{2|\mathbb{J}_0|}(\prod_{j \in L'} S^2)$ to the Euler class of $\big(\bigoplus_{I \in \mathbb{J}_0} p_I\big)|_{\overline{N}}$ in $H^{2|\mathbb{J}_0|}(\overline{N})$. This contradicts the fact, established below, that $\rho^* \colon H^{2|\mathbb{J}_0|}(\prod_{j \in L'} S^2) \to H^{2|\mathbb{J}_0|}(\overline{N})$ is injective.

Note that $\overline{N}$ is the Cartesian product of the spaces N and $\prod_{j \in L \setminus J} S^2$ and hence that

$$\rho = \pi \times \mathrm{id} \colon \overline{N} = N \times \prod_{j \in L \setminus J} S^2 \to \prod_{j \in J'} S^2 \times \prod_{j \in J \setminus L} S^2 = \prod_{j \in L'} S^2.$$

We can therefore use the Künneth formula, which applies because $H^*(\prod_{j \in L \setminus J} S^2)$ is torsion free and N is a finite CW complex (see [5, Theorem A6]), and the fact that π^* is injective to conclude that $\rho^* = (\pi \times \mathrm{id})^*$ is injective. $\square$

We now outline the construction from [6, Section 6] of the separable, *nuclear*, simple C^*-algebra, $D \rtimes_\alpha \mathbb{Z}$. As above, let $\varphi_{\infty,n}$ be the inductive limit map from the nth copy of $\mathcal{M}(C(Z,\mathcal{K}))$ to B. There is an automorphism α on B such that $\alpha \circ \varphi_{\infty,n} = \varphi_{\infty,n} \circ \varphi$ for all $n \geq 0$. Put $D_0 = \varphi_{\infty,0}(C(Z,\mathcal{K}))$ and put $D_n = \alpha^n(D_0)$ for $n \in \mathbb{Z}$. Let D be the sub-C^*-algebra of B generated by $\bigcup_{n\in\mathbb{Z}} D_n$. Then α is an automorphism on D and the crossed product $D \rtimes_\alpha \mathbb{Z}$ is a separable, nuclear, simple C^*-algebra with an infinite and a non-zero finite projection (see [6, Section 6]).

The C^*-algebra D is the inductive limit

$$E \xrightarrow{\varphi'} E \xrightarrow{\varphi'} E \xrightarrow{\varphi'} \cdots \longrightarrow D \tag{3.5}$$

where

$$E = C^*\big(C(Z,\mathcal{K}), \varphi(C(Z,\mathcal{K})), \varphi^2(C(Z,\mathcal{K})), \dots\big) \subseteq \mathcal{M}(C(Z),\mathcal{K}), \tag{3.6}$$

and where $\varphi' = \varphi|_E$. The C^*-algebra E is of type I.

4 The real rank of two simple C^*-algebras

In this section, B and $D \rtimes_\alpha \mathbb{Z}$ are the simple C^*-algebras from Section 3. Take a submanifold T_0 of S^2 such that T_0 is diffeomorphic to $[-1,1]$, let $\lambda\colon S^2 \to [-1,1]$ be a continuous extension of a diffeomorphism from T_0 onto $[-1,1]$, and put $\lambda_n = \lambda \circ \pi_n \colon Z \to [-1,1]$. Let $f \in C(Z,\mathcal{K})$ be given by $f(x) = \lambda_2(x)p_1(x)$ and put

$$a = \varphi_{\infty,0}(f) \in QDQ \subseteq QBQ \subseteq B, \tag{4.1}$$

where $Q = \varphi_{\infty,0}(p_1) \in D$.

Proposition 4.1 *The distance from the element a defined in* (4.1) *to the self-adjoint invertible elements of QBQ is 1.*

Proof: Let $f \in C(Z,\mathcal{K})$ be as above (4.1), and retain the notation from Section 3 (including the endomorphism φ, the projections Q_n, and the isometries S_j). It suffices to show that the distance from $\varphi^n(f)$ to the self-adjoint invertible elements of $Q_n\mathcal{M}(C(Z,\mathcal{K}))Q_n$ is 1 for all $n \in \mathbb{N}$.

Let s_n be as in Lemma 3.1 and define natural numbers r_n by the recursive formula: $r_0 = 1$ and $r_{n+1} = \nu(0, r_n)$. Note that $\varphi^n(f) = f_n + g_n$, where $f_n \in C(Z,\mathcal{K})$ is given by $f_n(x) = \lambda_{s_n}(x)q_n(x)$ for some projection q_n which is equivalent to p_{r_n}, and where

$g_n \in (Q_n - q_n)\mathcal{M}(C(Z,\mathcal{K}))(Q_n - q_n)$. Indeed, this is satisfied for $n = 0$ with $f_0 = f$, $q_0 = Q_0 = p_1$, and $g_0 = 0$. Assuming that the claim is proved for some $n \geq 0$, set $f_{n+1} = S_0\varphi_0(f_n)S_0^*$, set $q_{n+1} = S_0\varphi_0(q_n)S_0^*$, and set $g_{n+1} = \sum_{j\neq 0} S_j\varphi_j(f_n)S_j^* + \varphi(g_n)$. Then $\varphi^{n+1}(f) = f_{n+1} + g_{n+1}$ and

$$\varphi_0(f_n)(x) = f_n(x_{\nu(0,1)}, x_{\nu(0,2)}, \dots) = \lambda_{\nu(0,s_n)}(x)\varphi_0(q_n)(x) = \lambda_{s_{n+1}}(x)\varphi_0(q_n)(x),$$

so $f_{n+1}(x) = \lambda_{s_{n+1}}(x)q_{n+1}(x)$ and q_{n+1} is equivalent to $\varphi_0(p_{r_n}) = p_{\nu(0,r_n)} = p_{r_{n+1}}$.

To arrive at a contradiction, suppose that there is $n \in \mathbb{N}$ and a self-adjoint invertible element $b' \in Q_n\mathcal{M}(C(Z,\mathcal{K}))Q_n$ with $\|\varphi^n(f) - b'\| < 1$. Then $q_nb'q_n = \mu'q_n$ for some continuous function $\mu'\colon Z \to \mathbb{R}$, and $\|\mu' - \lambda_{s_n}\|_\infty \leq \|\varphi^n(f) - b'\| < 1$. Find $\delta > 0$ such that $\delta < 1 - \|\varphi^n(f) - b'\|$ and such that every (self-adjoint) element $c \in Q_n\mathcal{M}(C(Z,\mathcal{K}))Q_n$ within distance less than δ from b' is invertible. There is a finite subset J of $\mathbb{N}$, with $s_n \in J$, and a continuous function $\mu''\colon \prod_{j\in J} S^2 \to \mathbb{R}$ such that $\|\mu' - \mu''\circ\pi_J\|_\infty < \delta/2$.

Now, $\lambda_{s_n} = (\lambda\circ\pi_{J,s_n})\circ\pi_J$ and $\|\mu'' - \lambda\circ\pi_{J,s_n}\|_\infty < 1 - \delta/2$. Apply Proposition 2.2 to find a continuous function $\mu\colon \prod_{j\in J} S^2 \to \mathbb{R}$ with $\|\mu - \mu''\|_\infty < \delta/2$ such that $N := \mu^{-1}(\{0\}) \subseteq \prod_{j\in J} S^2$ is a finite CW complex and

$$\pi^*\colon H^*(\textstyle\prod_{j\in J'} S^2) \to H^*(N)$$

is injective, when $J' = J \setminus \{s_n\}$ and $\pi = \pi_{J,J'}|_N\colon N \to \prod_{j\in J'} S^2$. Set

$$b = b' + (\mu\circ\pi_J - \mu')q_n \in Q_n\mathcal{M}(C(Z,\mathcal{K}))Q_n.$$

Then $\|b - b'\| < \delta$, b is self-adjoint and invertible, and $q_nbq_n = (\mu\circ\pi_J)q_n$.

Put $\widetilde{N} = \pi_J^{-1}(N) \subseteq Z$, and let $\psi_{\widetilde{N}}\colon \mathcal{M}(C(Z,\mathcal{K})) \to \mathcal{M}(C(Z,\mathcal{K}))$ be the strictly continuous extension of the restriction mapping $C(Z,\mathcal{K}) \to C(\widetilde{N},\mathcal{K})$. By construction, $\psi_{\widetilde{N}}(q_nbq_n) = \big((\mu\circ\pi_J)q_n\big)|_{\widetilde{N}} = 0$, and so $\psi_{\widetilde{N}}(q_n) \precsim \psi_{\widetilde{N}}(Q_n - q_n)$ by Lemma 2.1. Hence

$$\psi_{\widetilde{N}}(p_{r_n}) \oplus \psi_{\widetilde{N}}(p_{r_n}) \sim \psi_{\widetilde{N}}(q_n) \oplus \psi_{\widetilde{N}}(q_n) \precsim \psi_{\widetilde{N}}(Q_n),$$

and as the trivial (constant) 1-dimensional projection $g \in C(Z,\mathcal{K})$ is equivalent to a subprojection of $p_r \oplus p_r$ (for all r), cf. [6, Proposition 4.5], we get $\psi_{\widetilde{N}}(g) \precsim \psi_{\widetilde{N}}(Q_n)$. This, however, is in contradiction with Lemma 3.2. $\square$

Corollary 4.2 *The real rank of B is not zero.*

Proof: If $\mathrm{RR}(B) = 0$, then $\mathrm{RR}(QBQ) = 0$, cf. [1]; and it follows immediately from

Proposition 4.1 that $\mathrm{RR}(QBQ) \neq 0$. $\square$

We proceed to consider the real rank of the algebra $D \rtimes_\alpha \mathbb{Z}$. To this end we need some lemmas.

Lemma 4.3 *Let A be a C^*-algebra, let p, q be projections in A, and let $a, b \in A$ be such $\|p - ab\| < 1$, $aq = a$, and $qb = b$. Then $p \precsim q$.*

Proof: The assumption implies that $pabp$ is invertible in pAp, so there is an element $c \in pAp$ with $p = cpabp = cabp$. Put $x = ca$ and $y = bp$. Then $p = xy$ and $yx \leq q$, whence $p \precsim q$. $\square$

Let α be an automorphism on a C^*-algebra A and let $A \rtimes_\alpha \mathbb{Z}$ be the crossed product C^*-algebra. Let u denote the unitary element in the multiplier algebra of $A \rtimes_\alpha \mathbb{Z}$ that implements the automorphism α, and let $\mathcal{E} \colon A \rtimes_\alpha \mathbb{Z} \to A$ denote the canonical conditional expectation.

Lemma 4.4 *Let A be a C^*-algebra, let α be an automorphism on A, and let A_1 be a sub-C^*-algebra of A. Let $p \in A_1$ be a projection, let $m \in \mathbb{N}$, and let b, c be self-adjoint elements in the subspace $\sum_{j=-m}^m A_1 u^j$ of $A \rtimes_\alpha \mathbb{Z}$ such that $pb = b$, $cp = c$, and $\|\mathcal{E}(bc) - p\| < 1$. Let A_2 be a sub-C^*-algebra of A which contains $\bigcup_{j=-m}^m \alpha^j(A_1)$. Let $q \in A_2$ be a sub-projection of p. Let I be a closed two-sided in A_2 which contains $q\mathcal{E}(b)q$, and let $\pi \colon A_2 \to A_2/I$ be the quotient mapping. It follows that*

$$\pi(q \oplus q) \precsim \textstyle\bigoplus_{j=-m}^m \pi(\alpha^j(p)) \quad \text{in } A_2/I.$$

Proof: Write $b = \sum_{j=-m}^m b_j u^j$ and $c = \sum_{j=-m}^m c_j u^j$, where b_j, c_j belong to A_1. Then $\mathcal{E}(bc) = \sum_{j=-m}^m b_j \alpha^j(c_{-j})$. Let $X \in M_{1,2m+1}(A_2)$ and $Y \in M_{2m+1,1}(A_2)$ be the row and the column matrix, respectively, with entries $X_{1j} = b_j$ and $Y_{j1} = \alpha^j(c_{-j})$, $j = -m, \dots, m$, so that $XY = \mathcal{E}(bc)$. Let Q and $\widetilde{q}$ in $M_{2m+1}(A_2)$ be the diagonal matrices with entries $Q_{jj} = \alpha^j(p)$ and $\widetilde{q}_{jj} = \delta_{0,j} q$, $j = -m, \dots, m$. Then $X = XQ$, $QY = Y$, and $\pi(qX\widetilde{q}) = 0$ (where π is extended to all rectangular matrices over A_2 entrywise).

Put $X' = qX(Q - \widetilde{q})$ and $Y' = (Q - \widetilde{q})Yq$. Then

$$\pi(X'Y') = \pi(qX(Q - \widetilde{q})Yq) = \pi(qXYq) - \pi(qX\widetilde{q}Yq) = \pi(q\mathcal{E}(bc)q),$$

which implies that $\|\pi(X'Y') - \pi(q)\| < 1$. It now follows from Lemma 4.3 that $\pi(q) \precsim \pi(Q - \widetilde{q})$. Hence $\pi(q) \oplus \pi(q) \precsim \pi(Q) \sim \bigoplus_{j=-m}^m \pi(\alpha^j(p))$ relatively to A_2/I. $\square$

Proposition 4.5 *The distance from the element a defined in* (4.1) *to the self-adjoint invertible elements of* $Q(D \rtimes_\alpha \mathbb{Z})Q$ *is 1.*

Proof: Suppose, to reach a contradiction, that there is a self-adjoint invertible element $b^{(1)}$ in $Q(D \rtimes_\alpha \mathbb{Z})Q$ with $\|b^{(1)} - a\| < 1$. Take $\delta > 0$, with $\delta + \|b^{(1)} - a\| < 1$, such that the open ball in $Q(D \rtimes_\alpha \mathbb{Z})Q$ with center $b^{(1)}$ and radius δ is contained in the open set of invertible elements of $Q(D \rtimes_\alpha \mathbb{Z})Q$.

Let s_n, r_n be as in the proof of Proposition 4.1. Then, as in the proof of Proposition 4.1, $\varphi^n(f) = f_n + g_n$, where $f_n \in C(Z, \mathcal{K})$ is given by $f_n(x) = \lambda_{s_n}(x)q_n(x)$ for some projection q_n which is equivalent to p_{r_n}, and where $g_n \in (Q_n - q_n)E(Q_n - q_n)$ (where $E \subseteq \mathcal{M}(C(Z, \mathcal{K}))$ is as defined in (3.6)).

For large enough integers $n, m \geq 0$ there are self-adjoint elements $b^{(2)}$ and c in the subspace $\sum_{j=-m}^{m} Q(\varphi_{\infty,n}(E)u^j)Q$ such that $\|b^{(2)} - b^{(1)}\| \leq \delta/2$ and $\|b^{(2)}c - Q\| < 1$. Let $\eta > 0$ be such that $\eta \leq \delta/2$ and $\|bc - Q\| < 1$ for every $b \in Q(D \rtimes_\alpha \mathbb{Z})Q$ with $\|b^{(2)} - b\| < \eta$. As in the proof of Proposition 4.1 there is a finite subset J of $\mathbb{N}$, with $s_n \in J$, and a self-adjoint element

$$b = \sum_{j=-m}^{m} \varphi_{\infty,n}(b_j)u^j, \qquad b_j \in E,$$

such that $\|b^{(2)} - b\| < \eta$, $b = QbQ$, and $q_n b_0 q_n = (\mu \circ \pi_J)q_n$, where $\mu \in C(\prod_{j \in J} S^2, \mathcal{K}) \subseteq C(Z, \mathcal{K})$ has the following property: The set $N = \mu^{-1}(\{0\}) \subseteq \prod_{j \in J} S^2$ is a finite CW complex and $\pi^* \colon H^*(\prod_{j \in J'} S^2) \to H^*(N)$ is injective, when $J' = J \setminus \{s_n\}$ and $\pi = \pi_{J,J'}|_N \colon N \to \prod_{j \in J'} S^2$. Note that $\|bc - Q\| < 1$, whence $\|\mathcal{E}(bc) - Q\| < 1$.

As in the proof of Proposition 4.1, put $\widetilde{N} = \pi_J^{-1}(N) \subseteq Z$, and let $\psi_{\widetilde{N}} \colon \mathcal{M}(C(Z, \mathcal{K})) \to \mathcal{M}(C(Z, \mathcal{K}))$ is the strictly continuous extension of the restriction mapping $C(Z, \mathcal{K}) \to C(\widetilde{N}, \mathcal{K})$. By construction, $\psi_{\widetilde{N}}(q_n b q_n) = \big((\mu \circ \pi_J)q_n\big)|_{\widetilde{N}} = 0$. We shall now apply Lemma 3.2 (with $A_1 = \varphi_{\infty,n}(E)$, $A_2 = \varphi_{\infty,n+m}(E)$, $p = Q$, $q = \varphi_{\infty,n}(q_n)$, and $\pi(\varphi_{\infty,n+m}(x)) = \psi_{\widetilde{N}}(x)$ for $x \in E$). Note that $\alpha^j(\varphi_{\infty,n}(E)) \subseteq \varphi_{\infty,n+m}(E)$ for $|j| \leq m$, because $\alpha^j(\varphi_{\infty,n}(x)) = \varphi_{\infty,n}(\varphi^j(x)) = \varphi_{\infty,n+m}(\varphi^{j+m}(x))$ when $j \geq 0$, and $\alpha^j(\varphi_{\infty,n}(x)) = \varphi_{\infty,n-j}(x)$ when $-n \leq j < 0$. Note also

$$\begin{aligned} \alpha^j(Q) &= \alpha^j(\varphi_{\infty,n+m}(Q_{n+m})) = \alpha^j(\varphi_{\infty,n+m}(\varphi^m(Q_n))) \\ &= \varphi_{\infty,n+m}(\varphi^{m+j}(Q_n)) = \varphi_{\infty,n+m}(Q_{n+m+j}) \end{aligned}$$

for $|j| \leq m$. Let $g \in C(Z, \mathcal{K})$ be a trivial (constant) 1-dimensional projection. We conclude from [6, Proposition 4.5 (ii)], from the expression for $\alpha^j(Q)$ above, and from Lemma 3.2

that

$$\psi_{\tilde{N}}(g) \precsim \psi_{\tilde{N}}(p_{r_n} \oplus p_{r_n}) \sim \psi_{\tilde{N}}(q_n \oplus q_n) \precsim \bigoplus_{j=-m}^{m} \psi_{\tilde{N}}(Q_{n+m+j}) \leq \bigoplus_{j=0}^{n+m} \psi_{\tilde{N}}(Q_j)$$

However, this is in contradiction with Lemma 3.2. □

As in the proof of Corollary 4.2 the corollary below follows immediately from Proposition 4.5

Corollary 4.6 *The real rank of $D \rtimes_\alpha \mathbb{Z}$ is not zero.*

5 Property (SP)

Recall that C^*-algebra A is said to have property (SP) if every non-zero hereditary sub-C^*-algebra of A contains a non-zero projection. A simple C^*-algebra with property (SP) is purely infinite (and hence of real rank zero) if and only if all non-zero projections are infinite. The simple C^*-algebras B and $D \rtimes_\alpha \mathbb{Z}$ (defined in Section 3) are here shown to have property (SP), and were shown to have real rank different from zero in the previous section.

Lemma 5.1 *Let A be a C^*-algebra, let a and b be positive elements in A, and suppose that $\|t^*at - b\| < \lambda < \|b\|$ for some $t \in A$ and some real number λ. Then $\overline{aAa}$ contains a non-zero projection if $\overline{(b-\lambda)_+A(b-\lambda)_+}$ does.*

Proof: The assumption $\|t^*at - b\| < \lambda$ implies that there exists $s \in A$ with $(b-\lambda)_+ = s^*t^*ats$ (see eg. [3]). Put $x = a^{1/2}ts$. Then

$$\overline{(b-\lambda)_+A(b-\lambda)_+} = \overline{x^*xAx^*x} \cong \overline{xx^*Axx^*} \subseteq \overline{aAa},$$

(see eg. [3] for the isomorphism), and this proves the lemma. □

Recall in the next lemma that $\varphi_{\infty,n}$ is the inductive limit map from the nth copy of $\mathcal{M}(C(Z,\mathcal{K}))$ to B.

Lemma 5.2 *Let A be a sub-C^*-algebra of B such that $\bigcup_{n=0}^{\infty} \mathrm{Im}(\varphi_{\infty,n}) \cap A$ is dense in A and $\varphi_{\infty,n}(C(Z,\mathcal{K}))$ is contained in A for every $n \geq 0$. Then A has property (SP).*

Proof: We must show that $\overline{aAa}$ contains a non-zero projection for every non-zero positive element $a \in A$. Put $A_n = \varphi_{\infty,n}^{-1}(A) \subseteq \mathcal{M}(C(Z,\mathcal{K}))$. Then A is the inductive limit of the sequence $A_1 \to A_2 \to \cdots$ with connecting maps $\varphi|_{A_n}$. Each A_n is assumed to contain $C(Z,\mathcal{K})$.

Find an integer $n \geq 0$ and a positive element $b \in A_n$ such that $\|\varphi_{\infty,n}(b) - a\| < \|a\|/2$. Then $\|b\| > \|a\|/2$. If λ is such that $\|\varphi_{\infty,n}(b) - a\| < \lambda < \|b\|$ and if the hereditary sub-C^*-algebra of A generated by $\varphi_{\infty,n}((b-\lambda)_+)$ contains a projection, then so does $\overline{aAa}$. The former holds if $\overline{\varphi_{\infty,n}(c)A\varphi_{\infty,n}(c)}$ contains a non-zero projection for some (non-zero) positive element c in $\overline{(b-\lambda)_+C(Z,\mathcal{K})(b-\lambda)_+} \subseteq C(Z,\mathcal{K}) \subseteq A_n$. Take any such element c.

Identify $C(\prod_{j=1}^m S^2, \mathcal{K})$ with a sub-C^*-algebra of $C(Z,\mathcal{K})$. Then $\bigcup_{m=1}^\infty C(\prod_{j=1}^m S^2, \mathcal{K})$ is dense in $C(Z,\mathcal{K})$. Find a natural number m and a positive element $d \in C(\prod_{j=1}^m S^2, \mathcal{K})$ such that $\|c - d\| < \|c\|/2$. Take a real number μ with $\|c-d\| < \mu < \|c\|/2 \leq \|d\|$, and put $e = (d-\mu)_+ \in C(\prod_{j=1}^m S^2, \mathcal{K})$. Then e is a non-zero positive element in $C(Z,\mathcal{K})$, and the problem is reduced to finding a non-zero projection in $\overline{\varphi_{\infty,n}(e)A\varphi_{\infty,n}(e)}$. This is done below by showing that $\overline{\varphi(e)A_{n+1}\varphi(e)}$ contains a non-zero projection.

Following the definition of φ (see Equations (3.1), (3.2), and (3.3)), for each $j \in \mathbb{Z}$ the hereditary sub-C^*-algebra of A_{n+1} generated by $\varphi(e)$ contains the hereditary sub-C^*-algebra of $C(Z,\mathcal{K})$ generated by $S_j\varphi_j(e)S_j^*$. The hereditary sub-C^*-algebras of $C(Z,\mathcal{K})$ generated by $S_j\varphi_j(e)S_j^*$ and $\varphi_j(e)$, respectively, are isomorphic, and so it suffices to show that $\overline{\varphi_j(e)C(Z,\mathcal{K})\varphi_j(e)}$ contains a non-zero projection for some j. But $\varphi_j(e)$ is a constant function whenever $j > m$. By the density property of the points $c_{j,i} \in S^2$, used in the construction of φ_j for $j \geq 1$, we have $\varphi_j(e) \neq 0$ for some $j > m$. Take such a j. Then $\varphi_j(e)$ is a non-zero, positive, constant function in $C(Z,\mathcal{K})$, and the hereditary sub-C^*-algebra of $C(Z,\mathcal{K})$ generated by $\varphi_j(e)$ clearly contains a non-zero projection. $\square$

If we apply Lemma 5.2 to $A = B$, then we get:

Proposition 5.3 *The C^*-algebra B has property (SP).*

Lemma 5.4 *Let A be a C^*-algebra and let α be an automorphism on A such that all non-zero powers of α are properly outer (cf. [2]). If A has property (SP), then so does $A \rtimes_\alpha \mathbb{Z}$.*

Proof: Take a non-zero positive element a in $A \rtimes_\alpha \mathbb{Z}$. Let $\mathcal{E}\colon A \rtimes_\alpha \mathbb{Z} \to A$ be the (faithful) conditional expectation associated with the crossed product. It is proved in [4, Lemma 3.2] that for each $\varepsilon > 0$ there is a (positive) element h in A such that $\|hah - h\mathcal{E}(a)h\| < \varepsilon$ and $\|h\mathcal{E}(a)h\| \geq \|\mathcal{E}(a)\| - \varepsilon$. It follows in particular that there is $h \in A$ and $\lambda \in \mathbb{R}$

such that $\|hah - h\mathcal{E}(a)h\| < \lambda < \|h\mathcal{E}(a)h\|$. Put $b = (h\mathcal{E}(a)h - \lambda)_+$. The (non-zero) hereditary sub-algebra $\overline{bAb}$ contains a non-zero projection (by assumption on A) and is contained in $\overline{b(A \rtimes_\alpha \mathbb{Z})b}$. It thus follows from Lemma 5.1 that $\overline{a(A \rtimes_\alpha \mathbb{Z})a}$ contains a non-zero projection. □

Proposition 5.5 *The C^*-algebra $D \rtimes_\alpha \mathbb{Z}$ has property (SP).*

Proof: It is shown in [6, Lemma 6.9] that all non-zero powers of α are properly outer (relatively to D). By Lemma 5.4 it thus suffices to prove that D has property (SP). This, however, follows from Lemma 5.2 and (3.6). □

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