The Stable rank of $C^*_{\text{red}}(F_n)$ is one — A Survey

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1 Introduction

In a recent paper, [1], by Ken Dykema, Uffe Haagerup and the author of this note it was proved that any reduced free product of $C^*$-algebras with respect to tracial states has stable rank one, provided that the ingoing $C^*$-algebras satisfy a certain (mild) condition (called the Avitzour condition). Recall that a unital $C^*$-algebra $A$ has stable rank one if its group $\text{GL}(A)$ of invertible elements is a norm dense subset of $A$. This definition is due to Marc Rieffel, [4], who associates the stable rank, a number in $\{1, 2, 3, \ldots \} \cup \{\infty\}$, to each $C^*$-algebra, as an analogue of dimension for topological spaces.

Rieffel posed in his paper the problem of calculating the stable rank of some concrete (simple, finite) $C^*$-algebras of interest, namely the irrational rotation $C^*$-algebras and the $C^*$-algebra $C^*_{\text{red}}(F_n)$ arising from the free group $F_n$ of $n$ generators, where $2 \leq n \leq \infty$. Ian Putnam, [3], settled the first question by proving that all the irrational rotation $C^*$-algebras have stable rank one. It is proved in [1], as a corollary to its main theorem, that $C^*_{\text{red}}(F_n)$ has stable rank one for all $2 \leq n \leq \infty$.

The purpose of this note is to give a direct and self contained proof of this corollary. The proof given here does not contain any new ideas, not already contained in [1], and the papers it is based upon ([2] and [5]), but it is shorter, and perhaps also less technical, having the privilege of dealing only with a special case of main theorem of [1].

It is a commonly asked question if every finite, simple $C^*$-algebra has stable rank one. (A unital $C^*$-algebra is said to be finite if it contains no non-unitary isometries.) In Section 5 we give an example that shows that the approach taken in [1] cannot be generalized (in any obvious way) to settle this conjecture for all finite, simple $C^*$-algebras. This example, I believe, has not been published before.
After a first draft of this note was written, Jesper Villadsen has constructed an example of a finite, simple, unital $C^*$-algebra which is not of stable rank one ([6]).

2 The distance to the invertible elements

We shall in this section give a direct proof of a theorem from [5] that states that if the set of invertible elements in a unital $C^*$-algebra is not dense, then the $C^*$-algebra contains an element with the largest possible distance to the invertibles.

Let $A$ be a unital $C^*$-algebra, and denote, as above, the group of invertible elements in $A$ by $\text{GL}(A)$. Upon representing $A$ faithfully on a Hilbert space $H$, we may assume that $A \subseteq B(H)$. Each element $a \in A$ has a polar decomposition $a = v|a|$, where $v$ is a partial isometry in $B(H)$ and $|a| = (a^*a)^{1/2} \in A$. For each $\alpha > 0$, define projections

$$p_\alpha = 1_{(\alpha, \infty)}(|a|), \quad q_\alpha = 1_{(\alpha, \infty)}(|a^*|),$$

on $H$. Observe that $vp_\alpha v^* = q_\alpha$.

**Lemma 2.1** If $vp_\alpha = yp_\alpha$ for some $\alpha > 0$ and for some $y \in \text{GL}(A)$, then $\text{dist}(a, \text{GL}(A)) \leq \alpha$.

**Proof:** Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(t) = \max\{0, t - \alpha\}$. Then $(b =) v f(|a|) = y f(|a|)$. Since $y(f(|a|) + \varepsilon \cdot 1) \in \text{GL}(A)$ for all $\varepsilon > 0$, we see that $b$ belongs to the closure of $\text{GL}(A)$. Hence

$$\text{dist}(a, \text{GL}(A)) \leq \|a - b\| = \|v(|a| - f(|a|))\| = \sup_{t \in (|a|)} |t - f(t)| \leq \alpha.$$ 

□

For each $\alpha > 0$, let $g_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by $g_\alpha(t) = \min\{1, \alpha^{-1}t\}$. Set $b_\alpha = v g_\alpha(|a|) \in A$.

**Lemma 2.2** If $0 < \alpha < \beta$ and if $\|b_\alpha - y\| < 1$ for some $y \in \text{GL}(A)$, then there exists $y' \in \text{GL}(A)$ such that $vp_\beta = y'p_\beta$. 


Proof: Observe first that

\[ \| (1 - y^*v)p_\alpha \| = \| (v^* - y^*)vp_\alpha \| = \| (v^* - y^*)q_\alpha v \| \]

\[ = \| (v^*g_\alpha (|a^*|) - y^*)q_\alpha \| \leq \| v^*g_\alpha (|a^*|) - y^* \| \]

\[ = \| b_\alpha^* - y^* \| = \| b_\alpha - y \| < 1. \]

Let \( h: \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous function satisfying \( 0 \leq h \leq 1 \), \( h \) is zero on the interval \([0, \alpha]\), and \( h \) is equal to 1 on the interval \([\beta, \infty)\). Put

\[ z = (1 - y^*v)h(|a|) = (1 - y^*v)p_\alpha h(|a|). \]

Then \( z \in A \) because \( h(0) = 0 \). Moreover, \( \| z \| \leq \| (1 - y^*v)p_\alpha \| < 1 \) and \( zp_\beta = (1 - y^*v)p_\beta \). Hence \( 1 - z \in \text{GL}(A) \) and \( (1 - z)p_\beta = y^*vp_\beta \). It follows that \( vp_\beta = y'p_\beta \), when \( y' = (y^*)^{-1}(1 - z) \in \text{GL}(A) \).

\[ \square \]

**Theorem 2.3** ([5, Theorem 2.6]) If \( A \) has stable rank not equal to one, then there exists an element \( x \) in \( A \) with

\[ \text{dist}(x, \text{GL}(A)) = \| x \| = 1. \]

**Proof:** Assume that \( A \) is a unital \( C^* \)-algebra of stable rank different from one. Then there is an element \( a \in A \) not in the closure of \( \text{GL}(A) \). Choose \( \alpha, \beta \) such that

\[ 0 < \alpha < \beta < \text{dist}(a, \text{GL}(A)). \]

If \( \| b_\alpha - y \| < 1 \) for some \( y \in \text{GL}(A) \), then \( vp_\beta = y'p_\beta \) for some \( y' \in \text{GL}(A) \) by Lemma 2.2. However, from Lemma 2.1, this would entail that \( \text{dist}(a, \text{GL}(A)) \leq \beta \), in contradiction with the choice of \( \beta \).

It follows that \( \text{dist}(b_\alpha, \text{GL}(A)) \geq 1 \). In combination with the obvious fact that \( \| b_\alpha \| \leq 1 \), this yields that \( \| x \| = \text{dist}(x, \text{GL}(A)) = 1 \), when \( x = b_\alpha \).

\[ \square \]
3 A norm estimate

The two first lemmas of this section are from Uffe Haagerup’s paper [2]. Proposition 3.3 is an easy consequence of these two lemmas, and it will, together with Theorem 2.3, go into the proof of Theorem 4.2.

Let $e_i$ denote the generators of $F_n$. Each element of the free group $F_n$ is a finite word in $e_i$ and their inverses. Such a word in $F_n$ is called reduced if no occurrence of $e_i$ follows or is followed by $e_i^{-1}$. The length of a $g \in F_n$, which is written $l(g)$, is the number of factors $e_i$ or $e_i^{-1}$ appearing in the reduced representation of $g$.

Let $\lambda : F_n \to B(\ell^2(F_n))$ be the left regular representation. Set

$$\mathfrak{A}_n = \text{span}\{\lambda(g) \mid g \in F_n\}, \quad \mathfrak{A}_n^{(j)} = \text{span}\{\lambda(g) \mid g \in F_n, l(g) = j\}.$$ 

Then $C^*_\text{red}(F_n)$ is the norm closure of $\mathfrak{A}_n$.

$C^*_\text{red}(F_n)$ has a unique trace $\tau$, which gives rise to an inner product on $C^*_\text{red}(F_n)$ defined by $\langle a, b \rangle = \tau(b^*a)$ and to the norm $\|a\|_2 = \langle a, a \rangle^{1/2}$. Denote by $E_j$ the orthogonal projection from $\mathfrak{A}_n$ onto $\mathfrak{A}_n^{(j)}$.

Lemma 3.1 ([2, Lemma 1.3]) Let $a \in \mathfrak{A}_n^{(k)}$ and $b \in \mathfrak{A}_n^{(l)}$ be given, and let $j \in \mathbb{N}$. Then

$$\|E_j(ab)\|_2 \leq \|a\|_2\|b\|_2.$$ 

Moreover, if $j < |k - l|$, if $j > k + l$, or if $k + l - j$ is odd, then $E_j(ab) = 0$.

Proof: The last claims (about $E_j(ab)$ being zero) follow from the fact that if $g, h \in F_n$, then $l(gh) = l(g) + l(h) - 2m$ for some $0 \leq m \leq \min\{l(g), l(h)\}$.

Suppose now that $|k - l| \leq j \leq k + l$ and that $k + l - j = 2m$ for some $m \in \mathbb{N}$. Let $g, h \in F_n$ with $l(g) = k$ and $l(h) = l$. Write $g = g_1g_2$ and $h = h_2h_1$ as reduced words with $l(g_2) = l(h_2) = m$ (and consequently, $l(g_1) = k - m$ and $l(h_1) = l - m$). Then $l(gh) = j$ if and only if $g_2 = h_2^{-1}$ and $g_1h_1$ is reduced. Hence

$$E_j(\lambda(gh)) = \langle \lambda(g_2), \lambda(h_2^{-1}) \rangle E_j(\lambda(g_1h_1))$$

$$= \begin{cases} 
\langle \lambda(g_2), \lambda(h_2^{-1}) \rangle \lambda(g_1h_1), & \text{if } g_1h_1 \text{ is reduced,} \\
0, & \text{otherwise.}
\end{cases}$$
Write

\[ a = \sum \alpha_{g_1 g_2} \lambda(g_1 g_2), \quad b = \sum \beta_{h_2 h_1} \lambda(h_2 h_1), \]

summing over all \( g_1, g_2 \), respectively \( h_1, h_2 \), such that \( l(g_1) = k - m \), \( l(h_1) = l - m \), \( l(g_2) = l(h_2) = m \) and such that \( g_1 g_2 \) and \( h_2 h_1 \) are reduced. Then,

\[
\| E_j(ab) \|_2^2 = \| \sum_{g_1, h_1} \left( \sum_{g_2, h_2} \alpha_{g_1 g_2} \beta_{h_2 h_1} \langle \lambda(g_2), \lambda(h_2^{-1}) \rangle \right) E_j(\lambda(g_1 h_1)) \|_2^2 \\
\leq \sum_{g_1, h_1} \left| \sum_{g_2, h_2} \alpha_{g_1 g_2} \beta_{h_2 h_1} \langle \lambda(g_2), \lambda(h_2^{-1}) \rangle \right|^2 \\
= \sum_{g_1, h_1} \left| \sum_{g_2} \alpha_{g_1 g_2} \lambda(g_2) \right|^2 \left| \sum_{h_2} \beta_{h_2 h_1} \lambda(h_2) \right|^2 \\
\leq \sum_{g_1, h_1} \left\| \sum_{g_2} \alpha_{g_1 g_2} \lambda(g_2) \right\|_2^2 \left\| \sum_{h_2} \beta_{h_2 h_1} \lambda(h_2) \right\|_2^2 \\
= \sum_{g_1, h_1} \left\| \sum_{g_2} \alpha_{g_1 g_2} \right\|^2 \left\| \sum_{h_2} \beta_{h_2 h_1} \right\|^2 \\
= \|a\|_2^2 \|b\|_2^2.
\]

\[ \square \]

**Lemma 3.2** ([2, Lemma 1.4]) For each \( k \in \mathbb{N} \) and for each \( a \in \text{span}\mathfrak{A}_n^{(k)} \),

\[ \|a\| \leq (2k + 1)\|a\|_2. \]

**Proof:** It suffices to show that

\[ \|ab\|_2 \leq (2k + 1)\|a\|_2\|b\|_2 \]
for all $b \in \mathcal{A}_n$. Put $b_l = E_l(b)$. It follows from Lemma 3.1 that

$$
\|E_j(ab)\|_2 = \left\| \sum_{l=|j-k|}^{j+k} E_j(ab_l) \right\|_2 \leq \sum_{l=|j-k|}^{j+k} \|E_j(ab_l)\|_2 \\
\leq \sum_{l=|j-k|}^{j+k} \|a\|_2 \|b_l\|_2 \leq (2k + 1)^{1/2} \|a\|_2 \left( \sum_{l=|j-k|}^{j+k} \|b_l\|_2^2 \right)^{1/2}.
$$

Hence

$$
\|ab\|_2^2 = \sum_{j=0}^{\infty} \|E_j(ab)\|_2^2 \leq (2k + 1) \|a\|_2^2 \sum_{j=0}^{\infty} \sum_{l=|j-k|}^{j+k} \|b_l\|_2^2 \\
\leq (2k + 1)^2 \|a\|_2^2 \sum_{j=0}^{\infty} \|b_j\|_2^2 = (2k + 1)^2 \|a\|_2^2 \|b\|_2^2.
$$

Proposition 3.3 (c.f. [1, Lemma 3.5]) For each $k \in \mathbb{N}$ and each $a \in \text{span}_{j \leq k} \mathcal{A}_n^{(j)}$,

$$
\|a\| \leq (2k + 1)^{3/2} \|a\|_2.
$$

Proof: Put $a_j = E_j(a)$. It follows from Lemma 3.2 that

$$
\|a\| = \left\| \sum_{j=0}^{k} a_j \right\| \leq \sum_{j=0}^{k} \|a_j\| \\
\leq \sum_{j=0}^{k} (2j + 1) \|a_j\|_2 \leq (2k + 1) \sum_{j=0}^{k} \|a_j\|_2 \\
\leq (2k + 1)(k + 1)^{1/2} \sum_{j=0}^{k} \|a_j\|_2^2 = (2k + 1)(k + 1)^{1/2} \|a\|_2 \\
\leq (2k + 1)^{3/2} \|a\|_2.
$$

One can replace the constant $2k + 1$ in Lemma 3.2 and in Proposition 3.3 with $k + 1$ by using that $E_j(ab) = 0$ whenever $k + l - j$ is odd (c.f. Lemma 3.1).
4 The stable rank of $C^*_\text{red}(F_n)$

As in Section 3 let $\mathfrak{A}_n$ denote the dense subalgebra of $C^*_\text{red}(F_n)$ spanned by $\lambda(g), g \in F_n$, where $2 \leq n \leq \infty$.

**Lemma 4.1** (c.f. [1, Lemma 3.7]) For each $a \in \mathfrak{A}_n$ there exist unitaries $u, v \in \mathfrak{A}_n$ such that $\|(uv)^m\|_2 = \|a\|_2^m$ for all $m \in \mathbb{N}$.

**Proof:** As before we let $e_i$ denote the generators of $F_n$. It follows by the property of the free groups that if $g_1, g_2, \ldots, g_m$ and $h_1, h_2, \ldots, h_m$ are elements in $F_n$ all of length $\leq k$, and if

$$e_1^{2k+1}g_1e_2^{2k+1}g_2e_2^{2k+1}\cdots e_1^{2k+1}g_me_2^{2k+1} = e_1^{2k+1}h_1e_2^{2k+1}h_2e_2^{2k+1}\cdots e_1^{2k+1}h_me_2^{2k+1},$$

then $g_1 = h_1, g_2 = h_2, \ldots, g_m = h_m$.

We can find $k \in \mathbb{N}$ such that $a \in \text{span}_{j \leq k} \mathfrak{A}^{(j)}_n$. Put $u = \lambda(e_1^{2k+1})$ and put $v = \lambda(e_2^{2k+1})$. Write $a = \sum_{l(g) \leq k} \alpha_g \lambda(g)$.

Then

$$uv = \sum_{l(g) \leq k} \alpha_g \lambda(e_1^{2k+1}g_1e_2^{2k+1}),$$

and consequently

$$(uv)^m = \sum_{g_1} \sum_{g_2} \cdots \sum_{g_m} \alpha_{g_1}\alpha_{g_2} \cdots \alpha_{g_m} \lambda(e_1^{2k+1}g_1e_2^{2k+1}g_2e_2^{2k+1}e_1^{2k+1}g_me_2^{2k+1}).$$

Since all $g_j$ have length $\leq k$ the argument in the first paragraph shows that the group elements appearing in the expression above for $(uv)^m$ are mutually distinct. It therefore follows that

$$\|(uv)^m\|_2^2 = \sum_{g_1} \sum_{g_2} \cdots \sum_{g_m} \left| \alpha_{g_1}\alpha_{g_2} \cdots \alpha_{g_m} \right|^2$$

$$= \left( \sum_{g_1} \left| \alpha_{g_1} \right|^2 \right) \cdot \left( \sum_{g_2} \left| \alpha_{g_2} \right|^2 \right) \cdots \left( \sum_{g_m} \left| \alpha_{g_m} \right|^2 \right) = \|a\|_2^{2m}.$$
The spectral radius of an element $x$ in a $C^*$-algebra will be denoted by $r(x)$. If $t > r(wx)$ for some unitary $w$ in $A$, then $x - tw^* = w^*(wx - t \cdot 1)$ is invertible, and so

$$\text{dist}(x, GL(A)) \leq \|x - (x - tw^*)\| = t.$$ 

This proves that

$$\text{dist}(x, GL(A)) \leq \inf_{w \in U(A)} r(wx).$$

**Theorem 4.2** ([1, Corollary 3.9]) The $C^*$-algebras $C^*_\text{red}(F_n)$ have stable rank one for all $2 \leq n \leq \infty$.

**Proof:** We begin by proving that

$$\text{dist}(a, GL(C^*_\text{red}(F_n))) \leq \|a\|_2$$

for all $a \in C^*_\text{red}(F_n)$. By continuity it suffices to prove this for $a \in \mathfrak{A}_n$. Let $u, v \in \mathfrak{A}_n$ be as in Lemma 4.1. Then $uav \in \text{span}_{j \leq k} \mathfrak{A}_n^{(j)}$ for some $k$, and $(uav)^m \in \text{span}_{j \leq mk} \mathfrak{A}_n^{(j)}$. By Proposition 3.3 and Lemma 4.1 we get

$$d(a, GL(C^*_\text{red}(F_n))) \leq r(uva) = r(uav)$$

$$= \liminf_{m \to \infty} \|(uav)^m\|^{1/m}$$

$$\leq \liminf_{m \to \infty} (2mk + 1)^{3/2m}\|(uav)^m\|_2^{1/m} = \|a\|_2.$$ 

Now, if $C^*_\text{red}(F_n)$ had stable rank different from one, then by Theorem 2.3 there would exist an element $x$ in $C^*_\text{red}(F_n)$ of norm one and distance one from the invertibles. That would imply

$$1 = \|x\| = \text{dist}(x, GL(C^*_\text{red}(F_n))) \leq \|x\|_2 \leq \|x\|,$$

and hence $\|x\| = \|x\|_2 = 1$. Consequently, $\tau(1 - xx^*) = \tau(1 - x^*x) = 1 - \|x\|_2^2 = 0$, and also $1 - xx^* \geq 0, 1 - x^*x \geq 0$. Since $\tau$ is faithful, this shows that $x$ is unitary. But unitary elements are invertible and do not have distance one to the invertibles. □
5 An example

In the proof of Theorem 4.2 it was shown that

$$\inf_{u \in U(C^*_{\text{red}}(F_n))} r(u x) \leq \|x\|_2$$

for all $x$ in a dense sub-*-algebra of $C^*_{\text{red}}(F_n)$, where $r(\cdot)$ is the spectral radius. Once this is established, density of the invertibles in $C^*_{\text{red}}(F_n)$ follows easily from Theorem 2.3. One might proceed to establish this estimate on the spectral radius for general $C^*$-algebras (with a unique trace), or appropriate generalizations thereof. For example, it is plausible that if $A$ is any simple, unital $C^*$-algebra, and if $x \in A$ is a non-zero element such that $ax = 0 = xa$ for some non-zero positive $a \in A$, then

$$\inf_{u \in U(A)} r(u x) < \|x\|.$$ 

One could moreover hope that this holds for general (non-simple) unital $C^*$-algebras provided that the element $a$ above is assumed to be full. This is not the case, however, as shown in Theorem 5.1 below.

Let $A$ be any unital $C^*$-algebra, let $n, k, l \in \mathbb{N}$ be such that $k + l \leq n$, and let $x$ be any element in $M_n(A)$ such that at most $k$ rows and $l$ columns in the $n \times n$ matrix of $x$ are non-zero. Then for some (unitary) permutation matrices $u$ and $v$, $uxv$ is strictly upper triangular. It follows that $uxv$ and $vux$ are nilpotent and hence that $r((vu)x) = 0$.

Let $k, l \in \mathbb{N}$, and let $X_{k, l}$ be the space of all complex $k \times l$-matrices of (operator) norm $\leq 1$. For each $n \geq \max\{k, l\}$ let $z_{k, l}^{(n)} \in M_n(C(X_{k, l})) = C(X_{k, l}, M_n(\mathbb{C}))$ be given by

$$z_{k, l}^{(n)}(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in X_{k, l}.$$

Clearly $\|z_{k, l}^{(n)}\| = 1$, and the argument above shows that if $k + l \leq n$, then $uz_{k, l}^{(n)}$ is nilpotent for some unitary $u \in M_n(C(X_{k, l}))$, and $z_{k, l}^{(n)}$ belongs to the closure of the invertible elements in $M_n(C(X_{k, l}))$.

**Theorem 5.1** If $k + l > n$, then

$$\text{dist}(z_{k, l}^{(n)}, \text{GL}(M_n(C(X_{k, l})))) = \|z_{k, l}^{(n)}\| = 1.$$
In particular, \( r(u z_{k,l}^{(n)}) = 1 \) for all unitaries \( u \in M_n(C(X_{k,l})) \).

Proof: We need only prove that \( \| z_{k,l}^{(n)} - a \| \geq 1 \) for all invertible \( a \in M_n(C(X_{k,l})) \). (The formula for the spectral radius will then follow from the inequality above Theorem 4.2.)

Suppose that \( a \in M_n(C(X_{k,l})) \) and that \( \| z_{k,l}^{(n)} - a \| < 1 \). We show that \( a \) is not invertible. Let \( a_0 \in M_{k,l}(C(X_{k,l})) = C(X_{k,l}, M_{k,l}(\mathbb{C})) \) be the upper left \( k \times l \) block of the matrix of \( a \). We begin by proving that \( a_0(x_0) = 0 \) for some \( x_0 \in X_{k,l} \).

Assume, to reach a contradiction, that \( a_0(x) \) is non-zero for all \( x \in X_{k,l} \). Let \( Y_{k,l} \subseteq X_{k,l} \) be the set of all \( k \times l \) matrices of norm equal to 1. Observe that \( Y_{k,l} \) is homeomorphic to the sphere \( S^{2k-1} \), and that \( Y_{k,l} \) therefore is not contractible. Observe also, that

\[
\| x - a_0(x) \| \leq \| z_{k,l}^{(n)}(x) - a(x) \| < 1, \quad x \in X_{k,l}.
\]

Define a function \( f: Y_{k,l} \times [0, 2] \to M_{k,l}(\mathbb{C}) \) by

\[
f(y, t) = \begin{cases} 
a_0(ty), & 0 \leq t \leq 1 \\
(t-1)y + (2-t)a_0(y), & 1 \leq t \leq 2
\end{cases}
\]

Then \( f \) is continuous, the function \( y \mapsto f(y, 0) \) is constant, \( f(y, 2) = y \), and \( f(y, t) \neq 0 \) for all \((y, t)\). The function \( h: Y_{k,l} \times [0, 2] \to Y_{k,l} \) given by \( h(y, t) = f(y, t)/\| f(y, t) \| \) is therefore continuous, \( y \mapsto h(y, 0) \) is constant, and \( h(y, 2) = y \). However, no such function \( h \) exists because \( Y_{k,l} \) is not contractible.

To prove that \( a \) is non-invertible, it suffices to show that \( a(x_0) \) is non-invertible. Let \( v_1, v_2, \ldots, v_n \in \mathbb{C}^n \) be the column vectors of \( a(x_0) \). Because \( a_0(x_0) = 0 \), it follows that \( v_1, v_2, \ldots, v_l \) all lie in an \((n - k)\)-dimensional subspace of \( \mathbb{C}^n \). Since \( l > n - k \), the set \( (v_1, v_2, \ldots, v_n) \) cannot be linearly independent, and therefore \( a(x_0) \) is not invertible.

\[ \square \]

References


