Structure and classification of C^*-algebras

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Abstract. We give an overview of the development over the last 15 years of the theory of simple C^*-algebras, in particular in regards to their classification and structure. We discuss dimension theory for (simple) C^*-algebras, in particular the so-called stable and real ranks, and we explain how properties of C^*-algebras of low dimension (stable rank one and real rank zero) was used by the author and P. Friis to give a new and simple proof of a theorem of H. Lin that almost commuting self-adjoint matrices are close to exactly commuting self-adjoint matrices. Elliott's classification program is explained and is contrasted with recent examples of C^*-algebras of "high dimension", including an example of a simple C^*-algebra with a finite and an infinite projection.

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1. Introduction

A (represented) C^*-algebra is a norm closed self-adjoint sub-algebra of the bounded operators on a Hilbert space. One can alternatively describe C^*-algebras axiomatically as complex Banach algebras with an involution that satisfies $\|a^*a\| = \|a\|^2$, thanks to a theorem of Gelfand, Naimark and Segal. Each commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, and $C_0(X)$ is isomorphic to $C_0(Y)$ if and only if X and Y are homeomorphic. This justifies the jargon that the study of C^*-algebras is *non-commutative topology*.

One can associate a C^*-algebra to each (locally compact) group, and the representation theory of the group C^*-algebra coincides with with representation theory of the group. Much of the early interest in C^*-algebras lay in their representation theory, and in their connection with other objects (such as groups).

Nearly 50 years ago, Glimm constructed a class of C^*-algebras, now called UHF- or Glimm algebras, that are the C^*-algebra analog of the hyperfinite II_1-factor. Unlike the situation for von Neumann algebras, there is not one UHF-algebra but in fact uncountably many. Glimm classified UHF-algebras by an invariant that we today can identify as the K_0-group of the algebra. Glimm's work was extended by Bratteli and Elliott who classified the larger class of AF-algebras (approximately finite dimensional C^*-algebras) that arise as inductive limits of finite dimensional C^*-algebras (the latter are just direct sums of matrix algebras). All UHF-algebras are *simple*, ie. have no non-trivial closed two-sided ideals. AF-algebras may or may not be simple, and far from all simple AF-algebras are UHF-algebras. Simple AF-algebras need not have unique trace, actually any metrizable Choquet simplex can arise as the trace simplex of a simple unital AF-algebra.

Many other interesting examples of (simple) C^*-algebras saw the light in the 1970s and 1980s. Cuntz invented his algebras $\mathcal{O}_n$. These are, for $2 \leq n < \infty$, generated by n isometries $s_1, s_2, \dots, s_n$ satisfying the relation $1 = s_1s_1^* + s_2s_2^* + \cdots + s_ns_n^*$. (For $n = \infty$ this relation is replaced with the relation that the support projections $s_js_j^*$ are mutually orthogonal.) Cuntz proved that his algebras are simple and purely infinite (see Definition 2.3) and independent on the choice of generators. These were the first explicit examples of simple infinite separable C^*-algebras. Cuntz and Krieger later associated a C^*-algebra to each finite Markov chain. This construction has today been generalized considerably in parts by Pimsner, who associated a Pimsner algebra to each Hilbert bi-module over a C^*-algebra, and with the constructions of C^*-algebras associated with infinite graphs. Many of these C^*-algebras are simple and purely infinite.

One can also obtain simple C^*-algebras from groups. The reduced groups C^*-algebra $C^*_{\mathrm{red}}(G)$ associated with a (discrete) group is simple for many interesting cases of non-amenable groups G, for example when G is a free group (other than $\mathbb{Z}$). These C^*-algebras are often exact, but never nuclear. Non-amenable groups can act amenably on spaces and can in this way give rise to simple, purely infinite, and nuclear C^*-algebra. Dynamical systems in general, also with amenable groups and in particular with $\mathbb{Z}$, give rise to many interesting examples of C^*-algebras, many of which are simple.

The irrational rotation C^*-algebra, A_θ, associated with an irrational number θ, is the universal C^*-algebra generated by two unitaries u and v satisfying the commutation relation $uv = e^{2\pi i\theta}vu$. They were first studied by Rieffel and shown to be simple with a unique trace and being independent of the generators u and v. The irrational rotation C^*-algebra A_θ contains the Harper operator $u + u^* + \lambda(v + v^*)$, where λ is a non-zero real parameter, whose spectrum recently has been shown to be a Cantor set.

These examples, and many more like them, have spurred the interest in understanding, and perhaps classifying, C^*-algebras, in particular the simple ones. This study was first suggested by Dixmier in the 1960s, and later taken up by Cuntz and Blackadar to mention just a few. It was investigated when finite simple C^*-algebras have a trace, and Cuntz studied the purely infinite C^*-algebras (that resemble the type III_1 von Neumann factors). The question, if all simple C^*-algebras are either (stably) finite or purely infinite was left open until a few years ago where the author found a counterexample inspired by ideas of Villadsen.

The most significant progress in our understanding of C^*-algebras comes from the program initiated by Elliott, and known as Elliott's classification program. Elliott predicts that (simple) separable nuclear C^*-algebras can be classified by natural invariants including K-theory as the most prominent ingredient. This conjecture has now been verified for a surprisingly wide class of C^*-algebras, for example for all simple separable nuclear purely infinite K-amenable C^*-algebras (the Kirchberg–Phillips theorem). We also know that we must make modifications to the classification conjecture if we want to turn it into a theorem.

We give here an overview of the theory of simple C^*-algebras including some of the recent examples of exotic "high-dimensional" simple C^*-algebras. We also include a solution to a classical problem, if almost commuting matrices must be close to commuting matrices, as the methods to solve this problem grew out of the methods used to study (simple) C^*-algebras.

2. The structure of simple C^*-algebras

Von Neumann algebra factors were by their inventors, von Neumann and Murray, divided into types: I_n, I_∞, II_1, II_∞, and III. The types are distinguished by the *dimension range* of the projections in the factor, which for the 5 types above are $\{0,1,2,\ldots,n\}$, $\{0,1,2,\ldots,\infty\}$, $[0,1]$, $[0,\infty]$, and $\{0,\infty\}$, respectively. A type I_n-factor, with n finite, is isomorphic to the algebra of $n\times n$ matrices, and, more generally a type I_n factor is the algebra of all bounded operators on an n-dimensional Hilbert space. Type II_1-factors admit a unique tracial state, and type III-factors are traceless. A separable von Neumann algebra is simple (has no non-trivial closed two-sided ideals) if and only if it is a factor of type I_n, with n finite, type II_1, or of type III.

Can one similarly divide the (infinite dimensional) simple C^*-algebras into two types; a *finite type* resembling the type II_1-factors and an *infinite type* resembling the type III-factors? Existence of traces and of finite and infinite projections should be natural dividing criterions:

Definition 2.1. Two projections p and q in a C^*-algebra A are said to be (Murray-von Neumann) *equivalent*, written $p\sim q$, if $p=v^*v$ and $q=vv^*$ for some (partial isometry) v in A; and p is *subequivalent* to q, written $p\precsim q$, if p is equivalent to a subprojection of q.

A projection in a C^*-algebra A is said to be *infinite* if it is equivalent to a proper subprojection of itself; and it is said to be *finite* otherwise.

A simple C^*-algebra A is called *stably infinite* if its stabilization $A\otimes\mathcal{K}$ contains an infinite projection, and it is called *stably finite* otherwise.

The notion of finiteness relate, as we would expect, to the existence of traces. As C^*-algebras need not be unital, we allow our traces to be unbounded and densely (not necessarily everywhere) defined.

The usual construction of a trace on an abstract C^*-algebra goes via a so-called *dimension function* (a "measure" rather than the "integral"), which by "integration" gives rise to a functional, which is slightly short of being a trace: additivity is known to hold only on abelian subalgebras. Such functionals are called *quasitraces.* Uffe Haagerup proved in [16] that quasitraces are in fact traces on *exact* C^*-algebras (Haagerup proved this for unital C^*-algebras, and Kirchberg extended the result to the non-unital case).

We have the following fundamental theorem on the existence of traces on simple C^*-algebra:

Theorem 2.2 (Blackadar-Cuntz-Haagerup). *A simple C^*-algebra A admits a quasitrace (and hence a trace, if A is exact) if and only if A is stably finite.*

Outline of proof: Blackadar and Cuntz proved in [2] that the following three conditions are equivalent for a simple *stable* C^*-algebra A: 1) A contains an infinite projection, 2) A has no dimension function, and 3) A is algebraically simple. Any dimension function lifts to a quasitrace by [3], so the equivalence of 1) and 2) together with Haagerup's result, that quasitraces on exact C^*-algebras are traces, yields the theorem.

How finite are stably finite simple C^*-algebras? and how infinite are the stably infinite ones? The definition below, due to Cuntz, is relevant for the discussion of the latter.

Definition 2.3. A simple C^*-algebra A is said to be *purely infinite* if every non-zero hereditary subalgebra of A contains an infinite projection.

Any subalgebra of the form $\overline{xAx^*}$ is hereditary, and the converse holds in the separable case. In other words, a simple C^*-algebra is purely infinite if one can find infinite projections in all "arbitrarily small corners" of A. A purely infinite C^*-algebra is clearly stably infinite. The opposite does not hold as we shall see in Section 5.

2.1. Dimensions of C^*-algebras.

A commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, and the space X is determined up to homeomorphism by the isomorphism class of the C^*-algebra. In the commutative case we can therefore define the dimension of the C^*-algebra to be the classical dimension of the space X. What about the non-commutative case? It turns out that there are several, and unfortunately mutually disagreeing, ways of extending dimension to the non-commutative setting. The low dimension cases are of most interest in particular in the study of simple C^*-algebras (many nicely behaving simple C^*-algebras are of very low dimension). Two notions of "low dimension" are particularly important:

Definition 2.4. Let A be a C^*-algebra. If the set of invertible elements in A (or in the unitization of A, if A is non-unital) is dense in A, then A is said to be of *stable rank one*, written $\mathrm{sr}(A) = 1$.

If the set of *self-adjoint* invertible elements in A (or in the unitization of A, if A is non-unital) is dense in the set of self-adjoint elements in A, then A is said to be of *real rank zero*, written $\mathrm{RR}(A) = 0$.

Rieffel introduced stable rank in his paper [24], and Brown and Pedersen introduced real rank in [5]. A commutative C^*-algebra $C_0(X)$ is of stable rank one if $\dim(X) \leq 1$, and of real rank zero if $\dim(X) = 0$.

Brown and Pedersen show that a C^*-algebra is of real rank zero if and only if the set of self-adjoint elements with finite spectrum is dense in the set of all self-adjoint elements. All purely infinite simple C^*-algebras are of real rank zero:

Proposition 2.5 (Cuntz [6], Zhang [35]). *The following three conditions are equivalent for any simple C^*-algebra (other than $\mathbb{C}$):*

(i) *A is purely infinite,*

(ii) *for all non-zero positive elements a, b in A there exists $x \in A$ such that $b = x^*ax$,*

(iii) $\mathrm{RR}(A) = 0$ *and all non-zero projections in A are infinite.*

Stably infinite C^*-algebras are never of stable rank one (in fact they have stable rank $+\infty$). It was a surprise when Villadsen in [33] showed that stably finite simple C^*-algebras need not be of stable rank one. Stably finite C^*-algebras can have very few projections and hence have real rank greater than zero.

2.2. Comparison theory for C^*-algebras.

Comparison theory for projections in von Neumann algebras is a crucial ingredient in the classification of von Neumann factors into types and to proving existence of traces on finite von Neumann algebras. Comparison of projections in a von Neumann factor is total: for any two projections p, q one has either $p \precsim q$ or $q \precsim p$ (see Definition 2.1). The comparison theory for C^*-algebras is far more delicate as is in parts reflected by looking at the ordered K_0-group. Any simple dimension group arises as the ordered K_0-group of a simple AF-algebra, and such ordered groups easily fail to be totally ordered. The second best thing after total comparison of projection is weak (or almost) unperforation, described below.

The comparison properties for a C^*-algebra A are contained in the ordered monoids $V(A)$ and $W(A)$ consisting of equivalence classes of projections and positive elements, respectively, in the (non-unital) *-algebra $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$. Equivalence of projections is the usual Murray-von Neumann equivalence (see Definition 2.1). Following Cuntz, comparison of positive elements $a, b \in M_\infty(A)$ is defined as follows: $a \precsim b$ if there is a sequence $\{x_n\}$ in $M_\infty(A)$ such that $x_n^* b x_n \to a$; and by $a \approx b$ iff $a \precsim b$ and $b \precsim a$ one defines an equivalence relation on the positive elements, which by the way does not quite agree with Murray-von Neumann equivalence when restricted to projections.

The sets $V(A)$ and $W(A)$ become ordered abelian semigroups by defining addition to be "orthogonal addition":

$$[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right],$$

and ordering to be induced by $\precsim$. The ordering on $V(A)$ coincides with the algebraic ordering: $x \le y$ iff there is z such that $y = x + z$. The ordering on $W(A)$ is not algebraic. Both semigroups are *positive* in the sense that they have a zero element which at the same time is the smallest element of the semigroup; hence $x \le x + y$ for all x, y. The semigroup $V(A)$ is called the Murray-von Neumann semigroup of A, and $W(A)$ is called the Cuntz semigroup of A.

If A is generated as an ideal by its projections (which is the case for all simple C^*-algebras with a non-trivial projection), then $K_0(A)$ is the Grothendieck group of $V(A)$, and the positive cone, $K_0(A)^+$, is the image of $V(A)$ in $K_0(A)$.

It was shown in [3] that there is a one-to-one correspondence between (lower semi-continuous) states on $W(A)$ and quasitraces on A, and by Haagerup's theorem in [16], quasitraces are traces on exact C^*-algebras. States on $V(A)$ extends (possibly non-uniquely) to (lower semi-continuous) states on $W(A)$ as shown in [4]. It follows in particular that each state on $V(A)$ lifts to a trace on A if A is exact. A new proof of this fact was recently given by Haagerup and Thorbjørnsen, [17], using random matrix methods.

An ordered abelian positive semigroup $(W, +, \leq)$ is said to be *almost unperforated* if

$$\forall n, m \in \mathbb{N}\ \forall x, y \in W : nx \leq my \quad \text{and} \quad n > m \implies x \leq y.$$

(The negation of almost unperforation is *strong perforation.*) One can use a Hahn-Banach type argument (see [15] and [29]) to show that $(W, +, \leq)$ is almost unperforated if and only if the order on W is determined by states on W. It follows in particular, that if A is simple and exact, if $V(A)$ is almost unperforated, and if p, q are two projections in $M_\infty(A)$, then $p \precsim q$ if $\tau(p) < \tau(q)$ for all traces τ on A. A similar statement, with dimension functions in the place of traces, holds for $W(A)$ (see [29]).

A simple C^*-algebra A is purely infinite if and only if $W(A)$ has only one non-zero element; and if $W(A)$ is almost unperforated, then A is either purely infinite or stably finite. It is known that $W(A)$ and $V(A)$ are almost unperforated for many C^*-algebras of interest including, besides all purely infinite C^*-algebras, also all C^*-algebras that tensorially absorb the Jiang-Su algebra $\mathcal{Z}$ (see [29]).

It is quite often the case that the semigroups $V(A)$ and $W(A)$ are almost unperforated, but it is not true in general for simple C^*-algebras as shown in the pioneering work of Villadsen (see Section 5). Almost unperforation can fail spectacularly. For example there is a simple nuclear C^*-algebra A in which one has elements $x, y_1, y_2, y_3, \dots \in V(A)$ satisfying $2x = 2y_1 = 2y_2 = \cdots$ and $x \not\leq y_1 + y_2 + \cdots + y_n$ for all natural numbers n, see [26].

It is not known if such exotic phenomenons can occur for C^*-algebras of real rank zero:

Question 2.6. *Suppose that A is a simple C^*-algebra of real rank zero.*

(i) *Does it follow that A is either stably finite or purely infinite?*

(ii) *Does it follow that $V(A)$ and $W(A)$ are almost unperforated?*

2.3. Tensor products and free products. Takesaki proved that the minimal (= spatial) tensor product of two simple C^*-algebras is again simple. This is at first thought perhaps not surprising, but one should bear in mind that the minimal tensor product of two (non-simple and non-exact) C^*-algebras can have unexpected and exotic ideals.

Following the similar notion from von Neumann factors we say that a simple C^*-algebra is *tensorially prime* if it is not isomorphic to a tensor product $A \otimes B$, where both A and B are (simple and) non-type I (i.e., are not isomorphic to the compact operators on a finite or infinite dimensional Hilbert space). We consider here only the minimal tensor product, which we denote by $\otimes$.

Proposition 2.7 (Kirchberg, see [27]). *Let A and B be simple non-type I C^*-algebras. If A or B is stably infinite, then $A \otimes B$ is purely infinite. If A and B are both stably finite* and exact*, then $A \otimes B$ is stably finite.*

In particular, if D is a simple, exact, and non-tensorially prime C^*-algebra, then D is either stably finite and admits a trace or D is purely infinite.

Note that we do not know if the tensor product of two (non-exact) stably finite simple C^*-algebras is stably finite. This would be the case if we knew that quasitraces on arbitrary (non-exact) C^*-algebras are traces.

Several simple C^*-algebras are non-tensorially prime without obviously being so. Jiang and Su constructed in [18] a simple separable unital stably finite non-type I C^*-algebra $\mathcal{Z}$, called the Jiang–Su algebra, that has the same K-theory (and the same Elliott invariant, see Section 3) as the complex numbers. It has been shown that many C^*-algebras are $\mathcal{Z}$-absorbing, i.e., they satisfy $A \cong A \otimes \mathcal{Z}$; $\mathcal{Z}$-absorbing C^*-algebras are obviously non-tensorially prime.

The most non-commutative product of two C^*-algebras is the free product (= the "largest" C^*-algebra generated by copies of the two C^*-algebras). We also have the unital free product $A *_{\mathbb{C}} B$ of two unital C^*-algebras A and B, which is defined to be the "largest" unital C^*-algebras generated by a unital copy of A and a unital copy of B.

Consider the C^*-algebra $A = M_2 *_{\mathbb{C}} \mathcal{O}_2$, and let $e \in M_2 \subseteq A$ be a 1-dimensional projection in M_2. Then $e \oplus e$ is equivalent to 1_A, which is a (properly) infinite projection in A. The projection e is finite in A, intuitively because it is in free position from $\mathcal{O}_2$ (a rigorous proof of this fact is non-trivial). The free product C^*-algebra A however is very far away from being simple.

Voiculescu introduced the notion of *reduced free products* of C^*-algebras, or rather of non-commutative probability spaces (A, ρ_A) and (B, ρ_B), where A and B are unital C^*-algebras, and ρ_A and ρ_B are states on A and B, respectively. The reduced free product is again a non-commutative probability space, denoted $(A *_{\mathrm{red}} B, \rho_A * \rho_B)$. The associated C^*-algebra $A *_{\mathrm{red}} B$ is very often simple (see [1]), and it is exact if both A and B are exact (see [8] and [11]), but it is almost never nuclear.

At a first glance one might expect that the (simple) reduced free product C^*-algebra $M_2 *_{\mathrm{red}} \mathcal{O}_2$ (with respect to suitable states on M_2 and $\mathcal{O}_2$) would be an example of an infinite C^*-algebra with a finite projection $e \in M_2$ (as above). However, it turns out that most reduced free product C^*-algebras, including $M_2 *_{\mathrm{red}} \mathcal{O}_2$, have excellent comparison theory (eg., their Murray-von Neumann semigroup is almost unperforated), and one can show that the projection e from above is infinite in the reduced free product, and, moreover, that $M_2 *_{\mathrm{red}} \mathcal{O}_2$ (and other C^*-algebras like it) is purely infinite. Results along these lines were obtained by Dykema and the author in [9] and [10].

3. Elliott's classification conjecture

The possibility that C^*-algebras can be classified—up to *-isomorphism—by K-theory was perhaps first suggested by Glimm's classification of UHF-algebras (also called Glimm algebras) by supernatural numbers, or, equivalently, by a subgroup of the rational numbers, their K_0-group. This classification was later extended to AF-algebras by Bratteli and Elliott to yield a one-to-one correspondence between dimension groups and AF-algebras. The former were axiomatically described by Effros, Handelman, and Shen as being the unperforated ordered abelian groups with the Riesz Interpolation Property. In the late 1980's in [12] Elliott extended the classification of AF-algebras to include a class of C^*-algebras, now called A$\mathbb{T}$-algebras, that arise as inductive limits of direct sums of matrix algebras over $C(\mathbb{T})$, with the added assumption that the inductive limit C^*-algebra is of real rank zero. These algebras can have non-trivial K_1-group. Elliott raised in the same paper the possibility that his classification might encompass much more than this apparently rather special class of C^*-algebras: many naturally occurring C^*-algebras might be A$\mathbb{T}$-algebras of real rank zero. Moreover, the same invariant, or possibly an expanded version of it, might classify an even wider class of C^*-algebras. These ideas, expressed in more detail below, are known as the *Elliott classification conjecture.*

Elliott's prediction, that A$\mathbb{T}$-algebras of real rank zero are rather frequently occurring C^*-algebras, was shortly after confirmed by himself and Evans as they discovered that the irrational rotation C^*-algebras mentioned in the introduction are A$\mathbb{T}$-algebras. Putnam showed around the same time that C^*-algebras associated with a minimal action on the Cantor set likewise are A$\mathbb{T}$-algebras.

Turning to the precise formulation of the classification conjecture, we only expect to be able to deal with separable and nuclear C^*-algebras (nuclearity is for C^*-algebras what injectivity, or equivalently, hyperfiniteness, is for von Neumann algebras). The K-theory of a C^*-algebra A consists of two abelian groups $K_0(A)$ and $K_1(A)$. The K_0-group has a distinguished subset, $K_0(A)^+$, (the image of $V(A)$ in $K_0(A)$, cf. Section 2), which gives $K_0(A)$ the structure of an ordered abelian group when A has an approximate unit consisting of projections and when A is stably finite.

To simplify its statement, and to state the conjecture in a situation, where no counterexamples (yet) exist, we state formally the Elliott conjecture only in the real rank zero case:

Conjecture 3.1 (Elliott—the real rank zero case). *Let A and B be simple separable nuclear unital C^*-algebras of real rank zero. Then*

$$A \cong B \iff (K_0(A), K_0(A)^+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B)).$$

The isomorphism on the right-hand side asserts that there exist isomorphisms $\alpha_0\colon K_0(A) \to K_0(B)$ and $\alpha_1\colon K_1(A) \to K_1(B)$ such that $\alpha_0(K_0(A)^+) = K_0(B)^+$ and $\alpha_0([1_A]) = [1_B]$. The invariant can detect whether A is stably finite or stably infinite: $K_0(A)^+ = K_0(A)$ in the latter case, and $K_0(A) \neq 0$ and $K_0(A) \cap -K_0(A)^+ = 0$ in the former case.

The conjecture can—with due care—be extended to non-simple C^*-algebras. We have already mentioned that Elliott's results in [12] confirms his conjecture for A$\mathbb{T}$-algebras of real rank zero. Dadarlat and Gong, [7], later verified the conjecture for the much wider class of so-called AH-algebras (of slow dimension growth) of real rank zero. These classification results also hold in the non-simple case, but the invariant becomes more complicated. It is an open problem if all simple separable nuclear stably finite C^*-algebras of real rank zero are AH-algebras of slow dimension growth and hence classifiable. The range of the invariant has been completely described by Elliott and Gong (see [27, Proposition 3.3]).

K-theory alone will not classify stably finite C^*-algebras not of real rank zero. Intuitively, if a C^*-algebra has very few—or no—projections, then its K_0-group probably say less about the algebra, so we need more information in our invariant. Goodearl produced a class of C^*-algebras (now known as Goodearl algebras) where the trace simplex of the C^*-algebra cannot be detected from its K-theory. This suggests that the trace simplex must be included in the invariant, and—as pointed out by Thomsen—also the pairing between traces and K_0. The resulting invariant (see eg. [27, Chapter 3]) is known as the *Elliott invariant*. The literature contains strong classification results in terms this invariant also for non-real rank zero C^*-algebras, eg. the classification of all simple AH-algebras of bounded dimension by Elliott, Gong and Li, [13], and there is a good description of the range of the invariant for this class due to Villadsen (see [27, Proposition 3.3.7]). A more ultimate result on the range of the invariant within the still not classified class of so-called ASH-algebras due to Elliott and Thomsen can be looked up in [27, Theorem 3.4.4].

The best classification results exist in the stably infinite case. There are no traces on simple stably infinite C^*-algebras, and the order structure on K_0 degenerates: $K_0^+ = K_0$. The Elliott invariant therefore collapses to the two groups $K_0(A)$ and $K_1(A)$ with no other structure except the position of the unit in $K_0(A)$ in the unital case.

The classification result below, that confirms the Elliott conjecture for a sweeping class of stably infinite C^*-algebras, was obtained independently by Kirchberg and Phillips, [19] and [23]:

Theorem 3.2 (Kirchberg–Phillips). *Let A and B be separable, nuclear, simple, purely infinite, K-amenable, unital C^*-algebras. Then*

$$A \cong B \iff (K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B)).$$

A C^*-algebra A is *K-amenable* if it is KK-equivalent to an abelian C^*-algebra; and the class of K-amenable C^*-algebras forms a bootstrap class, see [30]. Two K-amenable C^*-algebras are KK-equivalent if and only if they have isomorphic K-groups. One can remove the condition that A and B are K-amenable by replacing the assumption that the K-groups are isomorphic with the assumption that A and B are KK-equivalent. It is an important open problem if all nuclear C^*-algebras are K-amenable.

The Kirchberg-Philips theorem verifies the Elliott conjecture in the stably infinite, real rank zero case modulo two open problems: Are all separable simple nuclear stably infinite C^*-algebras of real rank zero purely infinite (cf. Question 2.6 (i))? And the problem above if all (separable, simple, purely infinite) nuclear C^*-algebras are K-amenable.

The range of the invariant in the stably infinite case is easy to describe: all pairs of countable abelian groups can arise as K_0 and K_1, and there are no restriction on the position of the unit, see [27, Propositions 4.3.3 and 4.3.4]. The Elliott conjecture would predict that all separable nuclear simple stably infinite C^*-algebras are actually purely infinite. As already mentioned, and as will be shown in Section 5, this is not the case. It may still be that separable nuclear simple stably infinite C^*-algebras of *real rank zero* are purely infinite, cf. Question 2.6 and that the Elliott conjecture holds for these C^*-algebras.

The status for the Elliott conjecture is nonetheless open. It may be that the invariant will be refined, so that it can distinguish also the "high-dimensional" examples that we shall discuss in Section 5, but it may also be that the class of C^*-algebras to be classified must be restricted, for example to the class of $\mathcal{Z}$-absorbing C^*-algebras (that briefly were discussed at the end of Section 2). There are some positive results in this direction, eg. by W. Winter, [34], who verified Elliott's conjecture for $\mathcal{Z}$-stable C^*-algebras of real rank zero and with finite decomposition rank.

4. Almost commuting self-adjoint matrices: an application of real rank zero and stable rank one

The classical problem, if two almost commuting self-adjoint matrices are close to two exactly commuting self-adjoint matrices, was solved in the early 1990's by Huaxin Lin, [20], using techniques from C^*-algebras. His long and technical proof was shortened significantly by Friis and the author, [14], where the analysis was reduced to using known properties of C^*-algebras of real rank zero and stable rank one. We outline the ideas of this argument here, and begin by stating the exact formulation of Lin's theorem:

Theorem 4.1. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for every natural number n and for every pair of self-adjoint $n \times n$ matrices A and B satisfying*

$$\|AB - BA\| < \delta, \qquad \|A\| \leq 1, \quad \|B\| \leq 1,$$

there exists a pair of commuting self-adjoint $n \times n$ matrices A' and B' such that $\|A - A'\| \leq \varepsilon$ and $\|B - B'\| \leq \varepsilon$.

As an instructive example of almost commuting self-adjoint matrices that not obviously are close to commuting self-adjoint matrices, consider the following $n \times n$

matrices:

$$A_n = \begin{pmatrix} 1/n & 0 & 0 & \cdots & 0 \\ 0 & 2/n & 0 & \cdots & 0 \\ 0 & 0 & 3/n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad B_n = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $\|A_nB_n - B_nA_n\| \le 1/n \to 0$ as $n \to \infty$. It follows from Theorem 4.1 that there are commuting $n \times n$ matrices A'_n and B'_n such that $\|A_n - A'_n\| \to 0$ and $\|B_n - B'_n\| \to 0$. Curiously, there are—to the knowledge of the author—no known *explicit* choices for such sequences of matrices $\{A'_n\}$ and $\{B'_n\}$.

The theorem is proved indirectly. If it were wrong, then there would exist a counterexample: $\varepsilon > 0$ and sequences $\{A_n\}$ and $\{B_n\}$ of self-adjoint $k_n \times k_n$ matrices all of norm at most one such that $\|A_nB_n - B_nA_n\| \to 0$ and such that the distance from (A_n, B_n) to a commuting pair of self-adjoint matrices is at least ε for all n. We show that the existence of such a counterexample leads to a contradiction.

Set $T_n = A_n + iB_n$, and note that $\|T_n\| \le 2$ and that $\|T_nT_n^* - T_n^*T_n\| \to 0$. Let $\mathfrak{A} = \prod_{n=1}^\infty M_{k_n}$ be the ℓ^∞-direct product of the matrix algebras and let $\mathfrak{J} = \sum_{n=1}^\infty M_{k_n}$ be the c_0-direct sum of matrix algebras. Then $\mathfrak{J}$ is a closed two-sided ideal in $\mathfrak{A}$, and so we can consider the quotient $\mathfrak{B} = \mathfrak{A}/\mathfrak{J}$ and the quotient mapping $\pi\colon \mathfrak{A} \to \mathfrak{B}$. Put $T = \{T_n\} \in \mathfrak{A}$. Then $TT^* - T^*T$ belongs to $\mathfrak{J}$, and so $\pi(T)$ is a normal operator in the C^*-algebra $\mathfrak{B}$.

If we could lift $\pi(T)$ to a normal operator $S = \{S_n\}$ in $\mathfrak{A}$, then we would have our contradiction: Write $S_n = A'_n + iB'_n$, with A'_n and B'_n self-adjoint—and necessarily commuting, because S_n is normal—and note that $\|A_n - A'_n\| \to 0$ and $\|B_n - B'_n\| \to 0$ because $\{A_n - A'_n\}$ and $\{B_n - B'_n\}$ both belong to $\mathfrak{J}$. However, we do not know if such a normal lift S exits.

To obtain the contradiction we need less: It suffices to find a normal operator T' in $\mathfrak{B}$ within distance $\varepsilon/2$ to $\pi(T)$ such that T' lifts to a normal operator in $\mathfrak{A}$. This is shown in the three propositions below, as we remark that $\mathfrak{B}$ is of real rank zero, stable rank one, and has connected unitary group (these facts are easily seen to hold for matrix algebras, and hence also for $\mathfrak{B}$).

For each $\varepsilon > 0$ let Γ_ε be the one-dimensional grid in the complex plane consisting of those points $x + iy$ where either x or y belongs to $\varepsilon\mathbb{R}$.

Proposition 4.2. *Let T be a normal operator in a unital C^*-algebra $\mathfrak{B}$ of stable rank one. Then for each $\varepsilon > 0$ there is a normal operator $T' \in \mathfrak{B}$ such that $\mathrm{sp}(T') \subseteq \Gamma_\varepsilon$ and $\|T - T'\| < \varepsilon$.*

Outline of proof: By the definition of stable rank one, every element in $\mathfrak{B}$ can be approximated by invertible elements in $\mathfrak{B}$. It was shown in [25] that this implies that any *normal* operator can be approximated by *normal* invertible operators. By translation, one obtains that any normal operator can be approximated by normal operators that do not have a given complex number in its spectrum; and hence—by iteration—by normal operators whose spectrum do not intersect any given finite set. Choosing a suitable finite set of points in the holes of the grid Γ_ε

one obtains a normal operator S close to T for which there is a continuous function $f\colon \mathrm{sp}(S) \to \Gamma_\varepsilon$ (in fact, a retract), such that $|f(t) - t|$ is small for all t; and we can then take T' to be $f(S)$.

The proposition below was first proved by Lin in [21]; a more direct proof is given in [14].

Proposition 4.3. *Let $\mathfrak{B}$ be a unital C^*-algebra of real rank zero and with connected unitary group. Let $\varepsilon > 0$ be given and let T be a normal operator in $\mathfrak{B}$ with $\mathrm{sp}(T) \subseteq \Gamma_\varepsilon$. Then there is a normal operator T' in $\mathfrak{B}$ with $\mathrm{sp}(T')$ finite such that $\|T - T'\| < \varepsilon$.*

By definition, a C^*-algebra is of real rank zero if any normal element with spectrum contained in the real line (a self-adjoint operator) can be approximated by a normal element with finite spectrum. Passing from the spectrum being a subset of the real line (a self-adjoint operator) to a more general one-dimensional spectrum permitting loops (in our case: a closed subset of Γ_ε), introduces extra complications that, besides making the proof harder, also force us to require that the unitary group be connected.

Proposition 4.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $\pi\colon \mathfrak{A} \to \mathfrak{B}$ be a surjective *-homomorphism. Each normal operator in $\mathfrak{B}$ of finite spectrum lifts to a normal operator in $\mathfrak{A}$.*

Proof. Let T be a normal operator in $\mathfrak{B}$ with finite spectrum, and find continuous functions $f\colon \mathrm{sp}(T) \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{C}$ such that $(g \circ f)(t) = t$ for all $t \in \mathrm{sp}(T)$. Lift the self-adjoint operator $f(T)$ to a self-adjoint operator A in $\mathfrak{A}$; then $g(A)$ is a normal operator in $\mathfrak{A}$ that lifts T. (Note that $g(A)$ not necessarily has finite spectrum.) □

5. High dimensional simple C^*-algebras

In Section 2 we discussed properties of a simple C^*-algebra A, including the somewhat technical notions of almost unperforation of the semigroup of equivalence classes of projections, $V(A)$, and of the Cuntz semigroup, $W(A)$. Until the mid 1990's it was believed that all simple C^*-algebras might enjoy these properties; then Jesper Villadsen constructed a counterexample, [32], by taking an inductive limit of algebras of the form $M_{k(n)}(X^{d(n)})$ for a suitable space X (eg. $X = S^2$) and for suitable increasing sequences $k(n)$ and $d(n)$ of natural numbers. It is a crucial point in the construction that the connecting mappings $M_{k(n)}(X^{d(n)}) \to M_{k(n+1)}(X^{d(n+1)})$ be chosen in such a way that the inductive limit C^*-algebra becomes simple and—at the same time—that certain high-dimensional properties of the spaces $X^{d(n)}$ are preserved. These techniques of Villadsen have since then been used by several people, including Villadsen himself, to construct many other simple C^*-algebras with various exotic properties, including the example by the author of a simple

C^*-algebra with a finite and an infinite projection (and hence a simple stably infinite C^*-algebra which is not purely infinite) as well as various counterexamples to Elliott's classification conjecture (as formulated in Section 3).

5.1. The C^*-algebra associated with a multiplier endomorphism. The construction presented here is a special case of Pimsner's construction of a class of C^*-algebras, called the Pimsner algebras, associated with Hilbert bimodules over C^*-algebras. The construction is implicitly contained in our paper [28], where the reader can find more details. Recall that the multiplier algebra, $\mathcal{M}(A)$, of a C^*-algebra A is the largest unital C^*-algebra that contains A as an essential closed two-sided ideal.

To each pair (A, ρ), where A is a (stable) C^*-algebra and $\rho\colon A \to \mathcal{M}(A)$ is a non-degenerate[1] injective *-homomorphism, we associate a C^*-algebra $C^*(A, \rho)$, which in spirit is the crossed product of A by ρ. (We also use the term *multiplier endomorphism* to denote a *-homomorphism from a C^*-algebra into its multiplier algebra.)

The C^*-algebra $C^*(A, \rho)$ is formally constructed as follows. Since ρ is non-degenerate it extends (uniquely) to a strictly continuous unital *-homomorphism $\rho\colon \mathcal{M}(A) \to \mathcal{M}(A)$. Put

$$B = C^*(A, \rho(A), \rho^2(A), \rho^3(A), \dots) \subseteq \mathcal{M}(A),$$

note that ρ restricts to an endomorphism on B; form the inductive limit

$$B \xrightarrow{\rho} B \xrightarrow{\rho} B \xrightarrow{\rho} \cdots \longrightarrow \overline{B},$$

and extend ρ to an automorphism $\overline{\rho}$ on $\overline{B}$. More explicitly, if $\mu_n\colon B \to \overline{B}$ is the inductive limit map from the nth copy of B, $n \geq 0$, then $\overline{\rho}(\mu_n(b)) = \mu_n(\rho(b)) = \mu_{n-1}(b)$, for $b \in B$. The inverse of $\overline{\rho}$ is given by $(\overline{\rho})^{-1}(\mu_n(b)) = \mu_{n+1}(b)$ for $b \in B$. Put $A_{k-\ell} = \mu_\ell(\rho^k(A))$. Then $A_0 = A$, $\overline{\rho}(A_n) = A_{n+1}$ for all $n \in \mathbb{Z}$, and

$$\overline{B} = C^*(\dots, A_{-2}, A_{-1}, A_0, A_1, A_2, \dots).$$

Let $C^*(A, \rho)$ be the crossed product $\overline{B} \rtimes_{\overline{\rho}} \mathbb{Z}$.

Let u be the canonical unitary in the multiplier algebra of $C^*(A, \rho)$ that implements $\overline{\rho}$. Then $C^*(A, \rho)$ is the closure of the span of elements of the form $a_k u^k$, where $k \in \mathbb{Z}$ and $a_k \in C^*(A_n \mid n \in \mathbb{Z})$. $C^*(A_n, A_{n+1}, \dots, A_m)$ is a closed two-sided ideal in $C^*(A_n, A_{n+1}, \dots, A_k)$ whenever $n \leq m \leq k$. In particular, $A_0 = A$ is a closed two-sided ideal in $C^*(A_0, A_1)$. If we view $C^*(A_0, A_1)$ as being a sub-C^*-algebra of $\mathcal{M}(A)$, then $\rho(a) = uau^*$ holds for all a in A.

Proposition 5.1. *If A is nuclear, then so is $C^*(A, \rho)$.*

[1] By non-degenerate we mean that ρ maps an approximate unit for A into a sequence that converges strictly to 1 in $\mathcal{M}(A)$.

Proof. This follows from the construction of $C^*(A,\rho)$ and the fact that the class of nuclear C^*-algebras is closed under extensions, inductive limits, and crossed products by $\mathbb{Z}$. $\square$

Proposition 5.2. *$C^*(A,\rho)$ is simple if ρ is minimal*[2] *and if ρ^n is properly outer*[3] *for every natural number n.*

Proof. If ρ is minimal, then so is the automorphism $\overline{\rho}$ on $\overline{B}$; and if all powers of ρ are properly outer, then the same holds for all powers of $\overline{\rho}$. The proposition therefore follows from [22, Theorem 7.2]. $\square$

If ρ is minimal, and if $\rho(A) \subsetneq A$, as will be the case in the situation we consider in the next subsection, then ρ^n is automatically properly outer for all $n \neq 0$.

We give below conditions that will ensure that a projection p in A is finite, respectively, infinite in $C^*(A,\rho)$.

Proposition 5.3. *Let p be a projection in A.*

(i) *If there exists a projection q in A which is equivalent to p (relatively to A) and is a proper subprojection of $\rho(p)$, then p is infinite in $C^*(A,\rho)$.*

(ii) *If $C^*(A,\rho)$ is simple, and if there is a projection e in A such that $e \not\precsim p \oplus \rho(p) \oplus \rho^2(p) \oplus \cdots \oplus \rho^n(p)$ (relatively to $\mathcal{M}(A)$) for all natural numbers n, then p is finite in $C^*(A,\rho)$.*

Recall that we have a canonical unitary u in the multiplier algebra of $C^*(A,\rho)$ that implements ρ, ie., $\rho(a) = uau^*$ for $a \in A$. In particular, $\rho(p) \sim p$ in $C^*(A,\rho)$, so the assumption in (i) implies that p is equivalent to a proper subprojection of itself, and hence that p is infinite. Part (ii) is a more technical and is verified (in a slightly different setting) in [28, Lemma 6.4].

5.2. A simple C^*-algebra with a finite and an infinite projection.

We apply the crossed product construction from the previous section to the stable C^*-algebra $A = C(Z) \otimes \mathcal{K} = C(Z,\mathcal{K})$ where Z is the infinite Cartesian product of 2-spheres, $Z = \prod_{n=1}^{\infty} S^2$, and where $\mathcal{K}$ denotes the compact operators on a separable Hilbert space. The multiplier algebra $\mathcal{M}(A)$ coincides in this case with the set of all bounded *-strongly continuous functions from Z into $B(H)$, the bounded operators on the Hilbert space on which the compact operators $\mathcal{K}$ acts.

The multiplier endomorphism $\rho\colon A \to \mathcal{M}(A)$ of our construction is of the form $\sum_{j=-\infty}^{\infty} \rho_j$, where each ρ_j is an endomorphism on A, and where the sum $\sum_{j=-\infty}^{\infty} \rho_j(a)$ is strictly convergent for each $a \in A$. (We ensure non-degeneracy of ρ by replacing it with $V^*\rho(\,\cdot\,)V$ for some isometry V in $\mathcal{M}(A)$ if necessary.) Each endomorphism ρ_j is induced by a continuous function $Z \to Z$ of the form

[2] ρ is minimal if there are no non-trivial ρ-invariant closed two-sided ideals in A; and an ideal I in A is said to be ρ-invariant if $A\rho(I)A \subseteq I$.

[3] An endomorphism $\rho\colon A \to \mathcal{M}(A)$ is properly outer if its restriction to each non-zero ρ-invariant ideal has norm distance 2 to a multiplier inner endomorphism.

$(z_1, z_2, \dots) \mapsto (c_1, \dots, c_k, z_{\nu(k+1)}, z_{\nu(k+2)}, \dots)$ (or of the form $(z_1, z_2, \dots) \mapsto (z_{\nu(1)}, z_{\nu(2)}, \dots))$ for suitable $k \in \mathbb{N}$, points $c_i \in S^2$, and for a suitable "shuffle-map" $\nu\colon \mathbb{N} \to \mathbb{N}$ (that all depend on j). The points c_i are chosen such that ρ becomes minimal. As $\rho^n(A) \subsetneq A$ for all n, it follows from Proposition 5.2 that $C^*(A, \rho)$ is simple. The shuffle maps ν (one for each j) are chosen in such a way that certain projections (defined below) have non-trivial Euler class.

If e is a constant 1-dimensional projection in A, then $\rho(e)$ is infinite dimensional and constant, so e is equivalent to a proper subprojection of $\rho(e)$ thus making e infinite in $C^*(A, \rho)$, cf. Proposition 5.3 (i).

It requires more effort to get a finite projection in $C^*(A, \rho)$. For every non-zero projection p in A the projection $\rho(p)$ is a pointwise infinite dimensional in $\mathcal{M}(A)$ (when viewed as a *-strongly continuous function $Z \to B(H)$). We want this projection to be finite in $C^*(A, \rho)$; even more, p must satisfy the condition in Proposition 5.3 (ii) wrt. some projection e.

To this end we take a one-dimensional projection p in $C(S^2, M_2)$ with non-trivial Euler class (p could be the "Bott projection" over S^2). For each $j \in \mathbb{N}$, define $p_j \in C(Z, M_2) \subset A$ by $p_j(z) = p(z_j)$, where $z = (z_1, z_2, \dots) \in Z$; so that p_j is the Bott projection over the jth copy of S^2. For each finite set $I = \{j_1, j_2, \dots, j_k\} \subseteq \mathbb{N}$, let $p_I \in C(Z, M_2 \otimes M_2 \otimes \cdots \otimes M_2) \subseteq A$ be the projection given by

$$p_I(z) = p_{j_1}(z) \otimes p_{j_2}(z) \otimes \cdots \otimes p_{j_k}(z), \qquad z \in Z.$$

It is shown in [28] that p_1, the Bott projection over the first copy of S^2 in Z, is a finite projection in $C^*(A, \rho)$. The proof uses the precise definition of the multiplier endomorphism $\rho\colon A \to \mathcal{M}(A)$, Proposition 5.3 (ii) applied to p_1 and with e being a constant one-dimensional projection, and the proposition below (cf. [28, Proposition 3.2]). (Note that if q is a projection in $C(Z, \mathcal{K})$ with non-trivial Euler class then $e \not\precsim q$ by a fundamental property of the Euler class.)

Proposition 5.4. *Let $I_1, I_2, \dots, I_m$ be non-empty finite subsets of $\mathbb{N}$. Then the following conditions are equivalent:*

(i) *The Euler class of $p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_m}$ is non-trivial.*

(ii) *For all subsets F of $\{1, 2, \dots, m\}$ we have $\left|\bigcup_{j \in F} I_j\right| \geq |F|$.*

(iii) *There is a matching $t_1 \in I_1, t_2 \in I_2, \dots, t_m \in I_m$.*

Putting these results together we obtain the following main result from [28]:

Theorem 5.5. *The C^*-algebra $C^*(A, \rho)$, with $A = C(Z, \mathcal{K})$, with $Z = \prod_{n=1}^{\infty} S^2$, and with $\rho\colon A \to \mathcal{M}(A)$ being the multiplier endomorphism described above, is simple, separable, nuclear, and it contains an infinite projection and a non-zero finite projection.*

Corollary 5.6. *There is a simple, separable, nuclear C^*-algebra that is stably infinite but not purely infinite; and there is a simple, separable, nuclear, unital, finite C^*-algebra that is not stably finite.*

Proof. The C^*-algebra $B = C^*(A, \rho)$ from Theorem 5.5 is stably infinite (containing an infinite projection) and not purely infinite (because it contains a non-zero finite projection). If p is a non-zero finite projection in B, then pBp is finite but not stably finite. □

5.3. Applications and other examples. The example of a simple C^*-algebra with a finite and an infinite projection as well as other examples constructed later by A. Toms give counterexamples to Elliott's conjecture, or at least they show that the Elliott invariant as defined in Section 3 does not suffice to classify separable nuclear simple (unital) C^*-algebras.

Recall from Section 3 that if A is a stably infinite, simple, unital C^*-algebra, then its Elliott invariant reduces to the triple $(K_0(A), [1_A], K_1(A))$.

Theorem 5.7. *There are simple, separable, nuclear, stably infinite unital C^*-algebras A and B such that*

$$(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B)) \quad \textit{and} \quad A \ncong B.$$

Proof. Let A be as in the first part of Corollary 5.6. There is a purely infinite simple nuclear unital C^*-algebra B such that $(K_0(A), [1_A], K_1(A))$ is isomorphic to $(K_0(B), [1_B], K_1(B))$ (see [27, Proposition 4.3.3 and 4.3.4]). As B is purely infinite and A is not, the two C^*-algebras are not isomorphic. □

Note also that it follows from Proposition 2.7 that the C^*-algebra $C^*(A, \rho)$ from Theorem 5.5 is tensorially prime (see Subsection 2.3).

Toms used Villadsen's techniques to construct simple stably finite (AH- and ASH-algebras) with explicit strong perforation in K_0 (eg. with (K_0, K_0^+) isomorphic to $(\mathbb{Z}, S)$ where S can be almost any subsemigroup of $\mathbb{Z}^+$ with $S - S = \mathbb{Z}$). Recently, Toms also constructed ingenious counterexamples to Elliott's conjecture in the stably finite case, ie. pairs of non-isomorphic simple, separable, nuclear, stably finite C^*-algebras with the same Elliott invariant (and for this matter also other invariants, not normally included in the Elliott invariant) (see [31]).

6. References

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