UNIVERSAL PROPERTIES OF GROUP ACTIONS ON LOCALLY COMPACT SPACES

HIROKI MATUI AND MIKAEL RØRDAM

Abstract. We study universal properties of locally compact G-spaces for countable infinite groups G. In particular we consider open invariant subsets of the G-space βG, and their minimal closed invariant subspaces. These are locally compact free G-spaces, and the latter are also minimal. We examine the properties of these G-spaces with emphasis on their universal properties.

As an example of our results, we use combinatorial methods to show that each countable infinite group admits a free minimal action on the locally compact non-compact Cantor set.

1. Introduction

Ellis proved in [2] that every group G admits a free minimal action on a compact Hausdorff space, and he proved that universal minimal G-spaces exist and are unique. (See also [3].) A (minimal) compact G-space is universal if any other (minimal) compact G-space is the image of the universal space by a continuous G-map. Ellis proved, more specifically, that each minimal closed invariant subset of the G-space βG is a free minimal G-space which is universal. The number of minimal closed invariant subsets of βG is very large, see [5], but they are all isomorphic by Ellis' uniqueness theorem.

Hjorth and Molberg, [6], established the existence of a free minimal action of any countable infinite group on the Cantor set (this is obtained from Ellis' results through a standard reduction argument). They also obtained a free action of an arbitrary countable group on the Cantor set admitting an invariant Borel probability measure.

The goal of this paper is to extend Ellis' results to the locally compact, non-compact setting, and in particular to study properties of the locally compact G-spaces that arise as *open* invariant subsets of βG, and their minimal *closed* invariant subsets (when they exist). We are particularly interested in those open invariant subsets of βG that give rise to co-compact G-spaces (they always contain minimal closed invariant subsets). These turn out to be of the form $X_A = \bigcup_{g \in G} K_{gA}$ for some subset A of G, where K_{gA} denotes the closure of the set gA in βG, and they are thus "indexed" by the set A. If A and B are subsets of G, then $X_A = X_B$ if and only if A is "B-bounded" and B is "A-bounded", or, equivalently, if the Hausdorff distance between A and B with respect to (any) proper right-invariant metric on G is finite.

The minimal closed invariant subspaces of a co-compact open invariant subspace of βG provide examples of locally compact free minimal G-spaces. We prove universality and uniqueness results for these spaces (explained in more detail below).

Kellerhals, Monod and the second named author studied actions of *supramenable* groups on locally compact spaces in [7]. By definition, a group is supramenable if it contains no (non-empty) paradoxical subset. It was shown in [7] that a group is supramenable if and only if whenever it acts *co-compactly* on a locally compact Hausdorff space, then there is a non-zero invariant Radon measure. As a step towards proving this result it was shown that there is a non-zero invariant Radon measure on the G-space X_A if and only if A is non-paradoxical. This is an example where a property of the G-space X_A is reflected in a property of the set A. We shall exploit such connections further in this paper. The condition that the action be co-compact in the characterization of supramenable groups from [7] cannot be removed as shown in Section 4.

It was further shown in [7] that if A is paradoxical, then any minimal closed invariant subset of X_A is a *purely infinite*[1] free minimal G-space. This was used to prove that any countable non-supramenable group admits a free minimal purely infinite action on the locally compact non-compact Cantor set. It was left open in [7] if *all* countable infinite groups admit a free minimal action on the locally compact non-compact Cantor set. In Section 8 we answer this question affirmatively for all countable infinite groups.

We show that the G-spaces X_A are universal with respect to the class of locally compact G-spaces that have the same *type* as X_A. The (base point dependent) type is defined for each pair (X, x_0), where X is a locally compact (co-compact) G-space such that $G.x_0$ is dense in X, and it is defined to be the collection of sets $B \subseteq G$ for which $B.x_0$ is relatively compact in X. In the co-compact case this information can be compressed into the equivalence class, $[A]$, of a single set $A \subseteq G$, and we say that the type of (X, x_0) is $[A]$ in this case. The type of the G-space (X_A, e) is $[A]$, which in particular shows that all (equivalence classes of) subsets of G are realized as a type. We show that there is a (necessarily unique and surjective) proper continuous G-map $\varphi\colon X_A \to X$ with $\varphi(e) = x_0$ if and only if the type of X is $[A]$, thus providing a universal property of the space X_A, see Example 7.12, Case III.

If $A \subseteq G$ is the type of a minimal G-space (with respect to some base point x_0), then all minimal closed invariant subsets of X_A are also of type $[A]$, and we show that such minimal G-spaces are universal among all minimal locally compact G-spaces (X, x_0) of type $[A]$. Moreover, relying heavily on the ideas from a new proof of Ellis' uniqueness theorem by Gutman and Li, [4], we prove that all minimal closed invariant subspaces of X_A are pairwise isomorphic as G-spaces whenever A is of "minimal type". If A is not of minimal type, then X_A may have non-isomorphic, even non-homeomorphic, closed invariant subsets.

The minimal closed invariant subspaces of the co-compact G-spaces X_A is a source of examples of free minimal locally compact G-spaces. These spaces are always totally disconnected. One can obtain a free minimal action of the given (countable infinite) group on the locally compact non-compact Cantor set from any of these minimal closed invariant subspaces of X_A using a standard reduction, provided that the minimal G-space is neither compact nor discrete.

[1] An action of a group on a totally disconnected space is purely infinite if all compact-open subsets are paradoxical relatively to the compact-open subsets of the space.

It was shown in [7] that a minimal G-space of X_A is never compact if A is not equivalent to an absorbing subset of G (see [7, Definition 3.5] or Definition 7.1). It was observed in [7] that a minimal G-space of X_A can be discrete, even when A is infinite (in which case X_A itself is non-discrete), but it was left open precisely for which subsets A of G this can happen. We show here that X_A contains no discrete minimal closed invariant subspace if and only if A is *infinitely divisible* (see Definition 7.3). Accordingly, if A is infinitely divisible and not equivalent to an absorbing set, then all minimal closed invariant subsets of X_A are non-compact and non-discrete. We show that each countable infinite group G contains an infinitely divisible subset A which is not equivalent to an absorbing set, and we conclude that each countable infinite group admits a free minimal action on a non-compact non-discrete locally compact Hausdorff space, which, moreover, can be taken to be the locally compact non-compact Cantor set.

As an illustration of the problems we are attempting to address, consider the following concrete question: Let X_1 and X_2 be minimal locally compact G-spaces. When does there exist a minimal locally compact G-space Z and proper continuous (necessarily surjective) G-maps $Z \to X_j$, $j = 1, 2$? If Z exists and if one of X_1 or X_2 is compact, respectively, discrete, then the other must also be compact, respectively, discrete (and, conversely, in these cases Z clearly does exist). We shall give a sufficient and necessary condition for the existence of the G-space Z in the general case, where X_1 and X_2 are locally compact G-spaces in terms of the (base point free) type of X_1 and X_2 (Corollary 9.15). The base point free type is developed in Section 9, where it is also used to give an alternative characterization of subsets A of G of minimal type, as well as to make more precise the meaning of universal minimal locally compact G-spaces.

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2. Preliminaries

This section contains some background material, primarily from [7].

Throughout this paper G will denote a (discrete) group with neutral element $e \in G$. Most of the time G will be assumed to be countable and infinite. We shall be studying locally compact (typically non-compact) G-spaces, and we shall primarily be interested in G-spaces with the following property:

Definition 2.1 (Co-compact actions)**.** Let G be a group acting on a locally compact Hausdorff space X. The action of G on X (or the G-space X) is said to be *co-compact* if there is a compact subset K of X such that

$$\bigcup_{g \in G} g.K = X.$$

Every *minimal* locally compact G-space is co-compact (take K to be any compact set with non-empty interior).

In each locally compact co-compact Hausdorff G-space X there is a compact subset K of X such that $X = \bigcup_{g \in G} g.K^{\circ}$. A compact subset K with these properties will be called G-*regular*.

Here is a useful property of co-compact G-spaces:

Proposition 2.2. *Each locally compact co-compact G-space X contains a minimal closed G-invariant subset. Moreover, one can find such a minimal closed G-invariant subset inside every non-empty closed G-invariant subset of X.*

Proof. We pass to complements and show that each G-invariant open proper subset U of X is contained in a maximal G-invariant open proper subset of X. Use Zorn's lemma to find a maximal linearly ordered collection $\{U_\alpha\}_{\alpha \in I}$ of G-invariant open proper subset of X each containing U, and set $V = \bigcup_{\alpha \in I} U_\alpha$. Then V is an open G-invariant subset of X which contains U and is not properly contained in any G-invariant open proper subset of X. We must show that $V \neq X$.

Let K be a compact subset of X that witnesses the co-compactness. If $V = X$, then $K \subseteq U_\alpha$ for some $\alpha \in I$. But this would imply that $X = \bigcup_{g \in G} g.K \subseteq U_\alpha$, contradicting that $U_\alpha \neq X$ for all $\alpha \in I$. □

The following definition (also considered in [7]) plays a central role in this paper:

Definition 2.3. Let G be a group. Let A and B be non-empty subsets of G. Write $A \propto B$ if A is B-*bounded*, i.e., if $A \subseteq FB = \bigcup_{g \in F} gB$ for some finite subset F of G. Write $A \approx B$ if $A \propto B$ and $B \propto A$.

Denote by $P_{\approx}(G)$ the set of equivalence classes $P(G)/\approx$.

It is easy to see that $\propto$ is a pre-order relation on the power set $P(G)$ of G, and $\approx$ is an equivalence relation on $P(G)$. The relation $\propto$ defines a partial order relation on $P_{\approx}(G)$, which again is denoted by $\propto$. When, in the sequel, we call two subsets of a group *equivalent*, we shall have the equivalence relation $\approx$ defined above in mind.

Example 2.4. (i). Let G be a group and let H be a subgroup of G. Then $G \approx H$ if and only if $|G : H| < \infty$.

(ii). More generally, if $H_1 \subseteq H_2 \subseteq G$ are subgroups of a group G, then $H_1 \approx H_2$ if and only if $|H_2 : H_1| < \infty$.

(iii). Let $F \subseteq G$ be non-empty. Then $F \approx \{e\}$ if and only if F is finite.

We have the following geometric interpretation of the relations defined above.

Lemma 2.5. *Let G be a group equipped with a proper right-invariant metric*[2] *d, and let A and B be non-empty subsets of G. Then:*

(i) *$A \propto B$ if and only if there exists $R < \infty$ such that $d(g, B) \leq R$ for all $g \in A$.*

(ii) *$A \approx B$ if and only if there exists $R < \infty$ such that $d(g, B) \leq R$ and $d(h, A) \leq R$ for all $g \in A$ and $h \in B$, i.e., if the Hausdorff distance between A and B with respect to d is finite.*

Note that the conclusion of the lemma does not depend on the choice of proper right-invariant metric.

Proof. (i). Suppose that $A \subseteq FB$ for some finite subset F of G. Set

$$R = \max\{d(g, e) \mid g \in F\} < \infty.$$

Let $g \in A$ be given. Then $g = th$ for some $t \in F$ and $h \in B$, so $d(g, B) \leq d(g, h) = d(t, e) \leq R$.

Assume conversely that $d(g, B) \leq R$ for all $g \in A$. Let F be the set of elements $t \in G$ such that $d(t, e) \leq R$. Let $g \in A$. Then $d(g, h) \leq R$ for some $h \in B$, whence $gh^{-1} \in F$, so $g \in FB$. This proves that $A \subseteq FB$.

Part (ii) follows from (i). □

It is perhaps of interest to note that the relations from Definition 2.3 also can be interpreted at the level of C^*-algebras, more precisely, in terms of the Roe algebra $\ell^\infty(G) \rtimes_{\mathrm{red}} G$. The Roe algebra is the sub-C^*-algebra of $B(\ell^2(G))$ generated by the natural copy of $\ell^\infty(G)$ inside $B(\ell^2(G))$ and the image, $\lambda(G)$, of the left-regular representation, λ, of G on $\ell^2(G)$. Denote the unitary operator $\lambda(g)$ by u_g. Then

$$(u_g f u_g^*)(h) = f(g^{-1}h), \qquad u_g \, 1_A \, u_g^* = 1_{gA},$$

for all $f \in \ell^\infty(G)$, all $g, h \in G$, and all subsets A of G.

Let $E\colon \ell^\infty(G) \rtimes_{\mathrm{red}} G \to \ell^\infty(G)$ be the canonical conditional expectation.

Lemma 2.6. *Let G be a discrete group and let A and B be non-empty subsets of G. Then:*

(i) *$A \propto B$ if and only if 1_A is in the closed two-sided ideal in $\ell^\infty(G) \rtimes_{\mathrm{red}} G$ generated by 1_B.*

(ii) *$A \approx B$ if and only if 1_A and 1_B generate the same closed two-sided ideal in $\ell^\infty(G) \rtimes_{\mathrm{red}} G$.*

Proof. (i). Each element $y \in \ell^\infty(G) \rtimes_{\mathrm{red}} G$ can be written as a (formal) sum $y = \sum_{g \in G} f_g \, u_g$; and $E(y) = f_e$. Let $\mathrm{Supp}(y)$ be the set of those $g \in G$ such that $f_g \neq 0$. The set of elements y for which $\mathrm{Supp}(y)$ is finite is a dense *-subalgebra of $\ell^\infty(G) \rtimes_{\mathrm{red}} G$.

[2] A metric d on G is said to be *right-invariant* if $d(hg, h'g) = d(h, h')$ for all $g, h, h' \in G$. It is *proper* if the set $\{g \in G : d(g, h) \leq R\}$ is finite for all $h \in G$ and all $R < \infty$. Every countable group admits a proper right-invariant metric.

If $f \in \ell^\infty(G)$, then let $\mathrm{supp}(f)$ be the set of those $g \in G$ such that $f(g) \neq 0$. An easy calculation shows that for $y \in \ell^\infty(G) \rtimes_{\mathrm{red}} G$, with $\mathrm{Supp}(y)\,(= F)$ finite and $A \subseteq G$, one has

$$\mathrm{supp}\, E(y\, 1_A\, y^*) \subseteq FA. \tag{2.1}$$

Suppose that 1_A belongs to the closed two-sided ideal in $\ell^\infty(G) \rtimes_{\mathrm{red}} G$ generated by 1_B. Then there exist $x_1, \dots, x_n \in \ell^\infty(G) \rtimes_{\mathrm{red}} G$ such that $1_A = \sum_{j=1}^n x_j\, 1_B\, x_j^*$. We can approximate each x_j with an element y_j in $\ell^\infty(G) \rtimes_{\mathrm{red}} G$, with $\mathrm{Supp}(y_j)\,(= F_j)$ finite, and such that

$$\|1_A - \sum_{j=1}^n y_j\, 1_B\, y_j^*\| < 1.$$

Then $\|1_A - \sum_{j=1}^n E(y_j\, 1_B\, y_j^*)\| < 1$, so

$$A \;\subseteq\; \bigcup_{j=1}^n \mathrm{supp}(E(y_j\, 1_B\, y_j^*)) \;\subseteq\; \bigcup_{j=1}^n F_j B \;=\; FB,$$

by (2.1), when $F = \bigcup_{j=1}^n F_j$. This shows that $A \propto B$.

Suppose now that $A \propto B$, and let $F \subseteq G$ be a finite set such that $A \subseteq FB$. Then

$$1_A \leq 1_{FB} \leq \sum_{g \in F} 1_{gB} = \sum_{g \in F} u_g\, 1_B\, u_g^*.$$

The element on the right-hand side belongs to the closed two-sided ideal in the Roe algebra generated by 1_B, and hence so does 1_A.

Part (ii) clearly follows from (i). □

The G-space βG and its open invariant subsets.

Any (discrete) group G acts on itself by left multiplication. By the universal property of the β-compactification this action extends to a continuous action of G on βG. The action of G on βG is free; and it is amenable if and only if G is exact (see [1, Theorem 5.1.6]). The G-space βG is never minimal (unless G is finite). In fact, βG has an abundance of open invariant subsets whenever G is infinite (see [5]).

Note that βG has a dense orbit (for example $G = G.e \subseteq \beta G$, where $e \in G$ is the neutral element). Each non-empty open invariant subset of βG contains G as an open and dense subset, and is thus a locally compact G-space with a dense orbit. In Proposition 2.8 below we shall say more about G-spaces with a dense orbit.

We proceed to describe the open invariant subsets of βG. Let A be a non-empty subset of G, and, following the notation of [7], let K_A denote the closure of A in βG. Then K_A is a compact-open subset of βG (since βG is a Stonean space). Put

$$X_A = \bigcup_{g \in G} g.K_A = \bigcup_{g \in G} K_{gA} \subseteq \beta G. \tag{2.2}$$

Then X_A is an open and invariant subset of βG, and is hence a locally compact G-space, which is also co-compact and has a dense orbit $G = G.e$.

It is observed in [7, Lemma 2.5] that $X_A = X_B$ if and only if $A \approx B$, and $X_A \subseteq X_B$ if and only if $A \propto B$. In particular, $X_A = G$ if and only if A is finite and non-empty, and $X_A = \beta G$ $(= X_G)$ if and only if $A \approx G$.

Proposition 2.7. *Each G-invariant open subset of βG on which G acts co-compactly is equal to X_A for some non-empty $A \subseteq G$.*

Proof. As remarked below Definition 2.1 there is a G-regular compact subset K of X. Let L be the closure of K^{o}. Then $L \subseteq K \subseteq X$, and L is compact-open (being equal to the closure in βG of the open set K^{o}). Hence $L = K_A$, when $A = L \cap G$, cf. [7, Lemma 2.4]. Since $K_A \subseteq X$ and X is G-invariant, it follows that $X_A \subseteq X$. Conversely,

$$X = \bigcup_{g \in G} g.K^{\mathrm{o}} \subseteq \bigcup_{g \in G} g.K_A = X_A.$$

□

We conclude from Proposition 2.7 and the previous remarks that the map $A \mapsto X_A$ induces an order isomorphism from the partially ordered set $(P_{\approx}(G), \propto)$ onto the set of G-invariant open co-compact subsets of βG.

Not all open invariant subsets of βG are co-compact as G-spaces as will be shown in Proposition 4.3. However, they can still be classified in terms of *left G-ideals*, see Proposition 3.16.

We end this section with a description of G-spaces with a dense orbit. The proposition below is probably well-known to experts. The second named author thanks Zhuang Niu for pointing out that the implication (iii) ⇒ (vi) holds (in the second countable case).

A G-space X is said to be *topologically transitive* if for every pair of non-empty open sets U and V there is $g \in G$ such that $g.U \cap V \neq \emptyset$. An action of G on X is said to have the *intersection property* if each non-zero ideal in $C_0(X) \rtimes_{\mathrm{red}} G$ has non-zero intersection with $C_0(X)$. It is well-known, and follows for example from [8, Lemma 7.1] (as well as from the work of Elliott and Kishimoto), that an action of G on X has the intersection property if it is topologically free. (If the action of G on X is topologically free, then the associated action, $g \mapsto \alpha_g$, of G on $C_0(X)$ consists of properly outer automorphisms, α_g, for $g \neq e$, and then we can apply [8, Lemma 7.1] to conclude that the action has the intersection property.)

Proposition 2.8. *Consider the following conditions on a countable group G acting on a locally compact Hausdorff space X:*

(i) *$C_0(X) \rtimes_{\mathrm{red}} G$ is prime.*
(ii) *The action of G on X is topologically transitive.*
(iii) *Each non-empty open G-invariant subset of X is dense in X.*
(iv) *Each proper closed G-invariant subset of X has empty interior.*
(v) *There exists $x \in X$ such that $G.x$ is dense in X.*
(vi) *The set of points $x \in X$ for which $G.x$ is dense in X is a dense G_δ-set.*

Then

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftarrow \text{(v)} \Leftarrow \text{(vi)};$$

and (iv) $\Rightarrow$ (vi) *if* X *is second countable. Moreover,* (ii) $\Rightarrow$ (i) *if the action of* G *on* X *has the intersection property.*

In particular, all six conditions are equivalent if X *is second countable and the action of* G *on* X *is topologically free.*

Proof. The implications (iii) $\Leftrightarrow$ (iv) and (vi) $\Rightarrow$ (v) are trivial.

(i) $\Rightarrow$ (ii). Let U and V be open non-empty subsets of X, and choose non-zero positive elements $a \in C_0(U) \subseteq C_0(X)$ and $b \in C_0(V) \subseteq C_0(X)$. Since $C_0(X) \rtimes_{\text{red}} G$ is prime it follows that

$$a\Big(C_0(X) \rtimes_{\text{red}} G\Big)b \neq 0.$$

This implies that $au_gb \neq 0$ for some $g \in G$, where $g \mapsto u_g$ is the unitary representation of G in (the multiplier algebra of) $C_0(X) \rtimes_{\text{red}} G$.

The support of the non-zero element $au_gbu_g^* \in C_0(X)$ is contained in $U \cap g.V$, so $U \cap g.V \neq \emptyset$.

(ii) $\Rightarrow$ (iii). Let U be a non-empty open invariant subset of X. If U is not dense, then there is an open (not necessarily invariant) subset V of $X \setminus U$. As $g.U = U$ for all $g \in G$ there is no $g \in G$ such that $g.U \cap V \neq \emptyset$.

(iii) $\Rightarrow$ (ii). Let U and V be non-empty open subsets of X. Then $\widetilde{U} = \bigcup_{g \in G} g.U$ is a (non-empty) invariant open subset of X. If (iii) holds, then $\widetilde{U} \cap V \neq \emptyset$. This implies that $g.U \cap V \neq \emptyset$ for some $g \in G$. Hence (ii) holds.

(v) $\Rightarrow$ (iii). Let $x \in X$ be such that $G.x$ is dense in X, and let U be a non-empty invariant open subset of X. Then $g.x \in U$ for some $g \in G$. As U is invariant it follows that $G.x \subseteq U$. Hence U must be dense.

We prove (iii) $\Rightarrow$ (vi) assuming that X is second countable. Let $\{U_n\}_{n=1}^\infty$ be a basis for the topology on X consisting of (non-empty) open sets, and put $\widetilde{U}_n = \bigcup_{g \in G} g.U_n$. Then each $\widetilde{U}_n$ is non-empty, G-invariant and open, and therefore dense in X. It follows that

$$X_0 := \bigcap_{n \in \mathbb{N}} \widetilde{U}_n$$

is a dense G_δ set in X. If $x \in X_0$, then $G.x \cap U_n \neq \emptyset$ for all $n \in \mathbb{N}$. This shows that $G.x$ is dense in X for all $x \in X_0$.

Finally, we prove (iii) $\Rightarrow$ (i) assuming that the action of G on $C_0(X)$ has the intersection property. Let I and J be closed two-sided non-zero ideals in $C_0(X) \rtimes_{\text{red}} G$. Then $I_0 = I \cap C_0(X)$ and $J_0 = J \cap C_0(X)$ are non-zero. Hence $I_0 = C_0(U)$ and $J_0 = C_0(V)$ for some non-empty open invariant subsets U and V of X. It follows that $U \cap V \neq \emptyset$. Hence $I_0 J_0 \neq 0$, so also $IJ \neq 0$. □

3. The type of a group action on a locally compact space

In Section 5 we will show that the locally compact G-space X_A associated to a subset A of G, considered in the previous section, has a universal property in a similar way as the space βG itself is universal among all compact G-spaces possessing a dense orbit. In Section 6 we will determine the universal properties of the minimal closed G-invariant subspaces of X_A among minimal locally compact G-spaces. It turns out that X_A is universal relatively to a subclass of the locally compact G-spaces (with a dense orbit), namely those that have the same *type* as X_A itself.

In this section we shall define and prove basic properties of the type of an action of a group on a locally compact Hausdorff space as alluded to above. The type keeps track of which subsets of the group gives rise to relatively compact subsets of the space. To make sense of this we must specify a base point of the space in a dense orbit, and we talk about a *pointed* locally compact G-space. In Section 9 we shall define the type of a locally compact G-space without reference to a base point.

The type will be defined in terms of a left G-ideal in the power set of the group.

Definition 3.1. Let G be a group. A collection $\mathcal{M}$ of subsets of G is called a *left G-ideal* if it contains at least one non-empty set and

(i) $A \in P(G)$, $B \in \mathcal{M}$, and $A \subseteq B$ implies $A \in \mathcal{M}$,

(ii) $A, B \in \mathcal{M}$ implies $A \cup B \in \mathcal{M}$,

(iii) $A \in \mathcal{M}$ and $g \in G$ implies $gA \in \mathcal{M}$.

A left G-ideal $\mathcal{M}$ is said to be *compact* if it is compact in the standard hull-kernel topology on the set of all left G-ideals. In other words, $\mathcal{M}$ is compact if and only if whenever $\{\mathcal{M}_\alpha\}_{\alpha\in I}$ is an upwards directed net of left G-ideals such that $\mathcal{M} \subseteq \bigcup_{\alpha\in I} \mathcal{M}_\alpha$, then $\mathcal{M} \subseteq \mathcal{M}_\alpha$ for some $\alpha \in I$.

Recall the definition of the order relation "$\propto$" from Definition 2.3. For emphasis we mention the following (trivial) fact about this order relation and left G-ideals:

Lemma 3.2. *Let G be a group, let A, B be non-empty subsets of G, and let $\mathcal{M}$ be a left G-ideal. If $A \propto B$ and $B \in \mathcal{M}$, then $A \in \mathcal{M}$.*

Example 3.3. (a). For each fixed non-empty subset A of G set

$$\mathcal{M}_A = \{B \in P(G) \mid B \propto A\}.$$

Then $\mathcal{M}_A$ is a left G-ideal. By transitivity of the relation "$\propto$", we see that if A and B are subsets of G, then $\mathcal{M}_A = \mathcal{M}_B$ if and only if $A \approx B$.

(b). The collection $\mathcal{M} = P(G)$ of all subsets of G is a left G-ideal. It is equal to $\mathcal{M}_G$, and is the largest left G-ideal.

(c). The collection $\mathcal{M}_{\text{fin}}$ of all finite subsets of G is a left G-ideal. It is equal to $\mathcal{M}_{\{e\}}$, cf. Example 2.4 (iii), and it is the smallest left G-ideal.

Proposition 3.4. *Let $\mathcal{M}$ be a left G-ideal. Then $\mathcal{M}$ is compact if and only if $\mathcal{M} = \mathcal{M}_A$ for some $A \subseteq G$.*

Proof. Let $\emptyset \neq A \subseteq G$. If $\mathcal{M}_A \subseteq \bigcup_{\alpha\in I} \mathcal{M}_\alpha$ for some increasing net of left G-ideals, then $A \in \mathcal{M}_\alpha$ for some $\alpha \in I$, whence $\mathcal{M}_A \subseteq \mathcal{M}_\alpha$ by Lemma 3.2. This shows that $\mathcal{M}_A$ is compact.

Suppose that $\mathcal{M}$ is compact. The family $\{\mathcal{M}_A\}_{A\in\mathcal{M}}$ is an upward directed net of left G-ideals which satisfies $\mathcal{M} = \bigcup_{A\in\mathcal{M}} \mathcal{M}_A$. It follows by compactness that $\mathcal{M} = \mathcal{M}_A$ for some $A \in \mathcal{M}$. □

Definition 3.5. Let G be a group. By a *pointed locally compact G-space* we shall mean a pair (X, x_0) consisting of a locally compact Hausdorff space X on which the group G acts, and a point $x_0 \in X$ such that $G.x_0$ is dense in X.

To each pointed locally compact G-space (X, x_0) associate the set

$$\mathcal{M}(G, X, x_0) = \{A \in P(G) \mid \overline{A.x_0} \text{ is compact}\}.$$

It is easy to verify that the invariant $\mathcal{M}(G, X, x_0)$ is a left G-ideal.

If X is a G-space, then associate to each $x \in X$ and to each subset $V \subseteq X$ the following subset of G:

$$O_X(V, x) = \{g \in G \mid g.x \in V\}. \tag{3.1}$$

The results of the following two lemmas will be used very often. The easy proof of the first lemma is omitted.

Lemma 3.6. *Let X be a G-space, let $V \subseteq X$, let $x \in X$, and set $A = O_X(V, x)$.*

(i) *If F is a subset of G, then $O_X(F.V, x) = FA$.*

(ii) *$G.x \cap V = A.x$. In particular, $A.x \subseteq V$.*

Recall the definition of a G-regular compact set from below Definition 2.1.

Lemma 3.7. *Suppose that (X, x_0) is a pointed locally compact G-space. Let $V \subseteq X$ and set $A = O_X(V, x_0)$. Then:*

(i) *$V^{\mathrm{o}} \subseteq \overline{A.x_0}$.*

(ii) *$A \in \mathcal{M}(G, X, x_0)$ if V is relatively compact.*

(iii) *If X is a co-compact G-space and if K is a G-regular compact subset of X, then for any compact subset L of X and for any $x \in X$, we have*

$$O_X(L, x) \;\propto\; O_X(K^{\mathrm{o}}, x) \;\subseteq\; O_X(K, x).$$

Proof. (i) follows from Lemma 3.6 (ii) and the fact that $G.x_0 \cap V^{\mathrm{o}}$ is dense in V^{o} when $G.x_0$ is dense in X. (ii) follows Lemma 3.6 (ii) and from the definition of $\mathcal{M}(G, X, x_0)$.

(iii). By G-regularity of K and compactness of L we find that $L \subseteq \bigcup_{g \in F} g.K^{\mathrm{o}} = F.K^{\mathrm{o}}$ for some finite subset F of G. Hence (iii) holds by Lemma 3.6 (i). □

The invariant $\mathcal{M}(G, X, x_0)$ depends on x_0 in a subtle way (see more about this in Section 9). Some properties of this invariant, however, are independent of the choice of base point, such as when the left G-ideal $\mathcal{M}(G, X, x_0)$ is compact.

Proposition 3.8. *Let G be a countable group and let (X, x_0) be a pointed locally compact G-space. It follows that G acts co-compactly on X if and only if $\mathcal{M}(G, X, x_0)$ is compact and X is σ-compact.*

Proof. Suppose first that G acts co-compactly on X. Then X is σ-compact (because G is assumed to be countable). Choose a G-regular compact subset K of X, and set $A = O_X(K, x_0)$. We show that $\mathcal{M}(G, X, x_0) = \mathcal{M}_A$, from which we can conclude that $\mathcal{M}(G, X, x_0)$ is compact by Proposition 3.4.

First, $A \in \mathcal{M}(G, X, x_0)$ by Lemma 3.7 (ii), so $\mathcal{M}_A \subseteq \mathcal{M}(G, X, x_0)$. Suppose next that $B \in \mathcal{M}(G, X, x_0)$, and let K' be the closure of $B.x_0$. Then K' is compact, and we deduce from Lemma 3.6 (ii) and Lemma 3.7 (iii) that

$$B \subseteq O_X(K', x_0) \propto O_X(K, x_0) = A,$$

so $B \in \mathcal{M}_A$. This shows that $\mathcal{M}(G, X, x_0) \subseteq \mathcal{M}_A$.

Suppose now that $\mathcal{M}(G, X, x_0)$ is compact and X is σ-compact. Then $\mathcal{M}(G, X, x_0) = \mathcal{M}_A$ for some $A \subseteq G$ by Proposition 3.4, and $K := \overline{A.x_0}$ is compact. We show that K witnesses co-compactness of the action of G on X. Find a sequence $\{L_n\}_{n=1}^{\infty}$ of compact subsets of X such that $L_n \subseteq L_{n+1}^{\circ}$ for all n, and $X = \bigcup_{n=1}^{\infty} L_n$. Put $B_n = O_X(L_n, x_0)$. Then $B_n \in \mathcal{M}(G, X, x_0) = \mathcal{M}_A$, so $B_n \subseteq F_n A$ for some finite subset F_n of G.

We now have:

$$L_n^{\circ} \subseteq \overline{B_n.x_0} \subseteq \overline{F_n A.x_0} = \bigcup_{g \in F_n} \overline{gA.x_0} = \bigcup_{g \in F_n} g.K \subseteq \bigcup_{g \in G} g.K$$

for all $n \geq 1$, where the first inclusion follows from Lemma 3.7 (i). This shows that $\bigcup_{g \in G} g.K = X$. □

Definition 3.9 (The type of a pointed locally compact G-space)**.** Let G be a group, let $P_{\approx}(G)$ be as in Definition 2.3, and for each $A \subseteq G$ let $[A] \in P_{\approx}(G)$ denote the equivalence class containing A.

To each pointed locally compact co-compact G-space (X, x_0) set

$$\mathrm{T}(G, X, x_0) = [A] \in P_{\approx}(G),$$

whenever $\emptyset \neq A \subseteq G$ is such that $\mathcal{M}(G, X, x_0) = \mathcal{M}_A$. This is well-defined by Proposition 3.8 and by the fact that $\mathcal{M}_A = \mathcal{M}_B$ if and only if $A \approx B$.

We call $[A]$ the *type* of (X, x_0), and we say that A *represents the type of* (X, x_0).

One can define the type of a not necessarily co-compact pointed locally compact G-space (X, x_0) to be the left G-ideal $\mathcal{M}(G, X, x_0)$. However, we shall not consider this general case systematically in this paper. In the following two propositions we give conditions on a pointed locally compact co-compact G-space to be of a given type. The results sharpen the statement in Proposition 3.8.

Proposition 3.10. *Let G be a countable group, let (X, x_0) be a pointed locally compact co-compact G-space.*

(i) *Let A be a non-empty subset of G. Then A represents the type of (X, x_0) if and only if*
  (a) *$\overline{A.x_0}$ is compact, and*
  (b) *$O_X(K', x_0) \propto A$ for all compact subset $K' \subseteq X$.*

(ii) *For each subset A of G that represents the type of (X, x_0), the set $K = \overline{A.x_0}$ is compact and*
  (c) *$\bigcup_{g \in G} g.K = X$.*

(iii) *For some subset A of G that represents the type of (X, x_0), the set $K = \overline{A.x_0}$ is G-regular, i.e., K is compact and*

(d) $\displaystyle\bigcup_{g \in G} g.K^{\circ} = X$.

Proof. (i). Suppose that $\mathrm{T}(G, X, x_0) = [A]$, i.e., that $\mathcal{M}(G, X, x_0) = \mathcal{M}_A$. As $A \in \mathcal{M}_A$, we see that (a) holds. Let K' be a compact subset of X. Then $O_X(K', x_0)$ belongs to $\mathcal{M}_A$ (by Lemma 3.7 (ii)), so (b) holds.

Suppose, conversely, that (a) and (b) hold. Then $A \in \mathcal{M}(G, X, x_0)$ by (a) and the definition of $\mathcal{M}(G, X, x_0)$, and so $\mathcal{M}_A \subseteq \mathcal{M}(G, X, x_0)$. If B is in $\mathcal{M}(G, X, x_0)$, then $K' = \overline{B.x_0}$ is compact, whence $B \subseteq O_X(K', x_0) \propto A$ by (b), so $B \in \mathcal{M}_A$. We conclude that $\mathcal{M}(G, X, x_0) = \mathcal{M}_A$, so $\mathrm{T}(G, X, x_0) = [A]$.

(ii). Let $x \in X$ and find a relatively compact open neighborhood V of x. Then $B := O_X(V, x_0)$ belongs to $\mathcal{M}_A$ by Lemma 3.7 (ii), so $B \propto A$. There is a finite subset F of G such that $B \subseteq FA$. It follows from Lemma 3.7 (i) that

$$x \in V \subseteq \overline{B.x_0} = \bigcup_{g \in F} \overline{gA.x_0} = \bigcup_{g \in F} g.K \subseteq \bigcup_{g \in G} g.K.$$

This shows that (c) holds.

(iii). Let $B \subseteq G$ be any representative of $\mathrm{T}(G, X, x_0)$ and choose a G-regular compact subset L of X. Put $A_0 = O_X(L, x_0)$. Then $A_0 \propto B$ by (b). Put $A = A_0 \cup B$ and let K be the closure of $A.x_0$. Then $A \approx B$, so $\mathrm{T}(G, X, x_0) = [A]$. It follows from Lemma 3.7 (i) that $L^{\circ} \subseteq \overline{A_0.x_0} \subseteq K$. This implies that $L^{\circ} \subseteq K^{\circ}$, so K is G-regular. □

Proposition 3.11. *Let G be a countable group, let (X, x_0) be a pointed locally compact co-compact G-space and let K be a G-regular compact subset of X. Then*

$$\mathrm{T}(G, X, x_0) = [O_X(K^{\circ}, x_0)] = [O_X(K, x_0)].$$

Proof. Let $A \subseteq G$ be a representative of the type of (X, x_0). Then

$$O_X(K^{\circ}, x_0) \subseteq O_X(K, x_0) \propto A$$

by Proposition 3.10 (i). Let L be the closure of $A.x_0$, which is a compact set by Proposition 3.10 (i). Then $A \subseteq O_X(L, x_0) \propto O_X(K^{\circ}, x_0)$ by Lemma 3.7 (iii). This completes the proof. □

Each element in $P_{\approx}(G)$ is the type of some, in fact a canonical, pointed locally compact co-compact G-space:

Proposition 3.12. *Let A be a non-empty subset of a group G, and let $X_A \subseteq \beta G$ be the locally compact co-compact G-space defined in* (2.2)*. Then* $\mathrm{T}(G, X_A, e) = [A]$.

Proof. We must show that $\mathcal{M}(G, X_A, e) = \mathcal{M}_A$. Let $B \subseteq G$. The closure of $B = B.e$ in X_A is equal to $K_B \cap X_A$ (where K_B is the closure of B in βG). We now have

$$\begin{aligned} B \in \mathcal{M}(G, X_A, e) \quad &\Longleftrightarrow \quad K_B \cap X_A \text{ is compact} \\ &\overset{(1)}{\Longleftrightarrow} \quad K_B \subseteq X_A \overset{(2)}{\Longleftrightarrow} B \propto A \Longleftrightarrow B \in \mathcal{M}_A. \end{aligned}$$

The "⇒" part of (1) follows because K_B is compact-open and X_A is open and dense in βG, so $K_B \cap X_A$ is dense in K_B.

(2) follows from [7, Lemma 2.5(i)]. □

Suppose that (X, x_0) and (Y, y_0) are pointed locally compact G-spaces and that $\varphi\colon X \to Y$ is a continuous proper G-map such that $\varphi(x_0) = y_0$. Then φ is necessarily surjective because $\varphi(G.x_0) = G.y_0$ and because any continuous proper map between locally compact Hausdorff spaces maps closed sets to closed sets.

Proposition 3.13. *Let G be a group and let (X, x_0) and (Y, y_0) be pointed locally compact G-spaces. Suppose there exists a continuous proper G-map $\varphi\colon X \to Y$ such that $\varphi(x_0) = y_0$. Then*

$$\mathcal{M}(G, X, x_0) = \mathcal{M}(G, Y, y_0).$$

In particular, if one of the G-spaces X and Y is co-compact, then so is the other, in which case

$$\mathrm{T}(G, X, x_0) = \mathrm{T}(G, Y, y_0).$$

Proof. Let $A \subseteq G$. Observe that

$$\varphi(A.x_0) = A.y_0, \qquad A.x_0 \subseteq \varphi^{-1}(A.y_0).$$

If $A \in \mathcal{M}(G, X, x_0)$, then $A.y_0$ is contained in the compact set $\varphi(\overline{A.x_0})$, so $A \in \mathcal{M}(G, Y, y_0)$. Similarly, if $A \in \mathcal{M}(G, Y, y_0)$, then $A.x_0$ is contained in the compact set $\varphi^{-1}(\overline{A.y_0})$, so $A \in \mathcal{M}(G, X, x_0)$.

Because φ necessarily is surjective, it follows that if one of the spaces X and Y is σ-compact, then so is the other. The last claim therefore follows from the the former together with Proposition 3.8. □

The two extreme values of the type are treated in the following two propositions. We omit the easy proof of the former.

Proposition 3.14. *Let G be a countable group and let (X, x_0) be a pointed locally compact G-space. The following conditions are equivalent:*

(i) $\mathcal{M}(G, X, x_0) = P(G)$,
(ii) *X is co-compact and* $\mathrm{T}(G, X, x_0) = [G]$,
(iii) *X is compact.*

Proposition 3.15. *Let G be a countable group and let (X, x_0) be a pointed locally compact G-space. Suppose that the isotropy group of $\{x_0\}$ is trivial, i.e., the map $g \mapsto g.x_0$ is injective. Then the following conditions are equivalent:*

(i) $\mathcal{M}(G, X, x_0) = \mathcal{M}_{\mathrm{fin}}$,
(ii) *X is co-compact and* $\mathrm{T}(G, X, x_0) = [\{e\}]$,
(iii) *X is discrete,*
(iv) *(X, x_0) is isomorphic to (G, e).*

Proof. (ii) ⇒ (i) and (iv) ⇒ (iii) are trivial.

(i) ⇒ (iv). Let $x \in X$ and find a relatively compact open neighborhood V of x. Then $A := O_X(V, x_0)$ belongs to $\mathcal{M}(G, X, x_0)$ (by Lemma 3.7 (ii)), so A

is finite. Since $A.x_0 = G.x_0 \cap V$ is dense in V (by Lemma 3.7 (i)) and since finite sets are closed, we must have $A.x_0 = V$. This shows that $x \in V \subseteq G.x_0$. As $x \in X$ was arbitrary we conclude that $X = G.x_0$. The set $A.x_0$ is a finite and open, so each point in this set must be isolated. Hence every point in X is isolated. The map $g \mapsto g.x_0$ is assumed to be injective, and it is therefore a G-isomorphism from (G, e) onto (X, x_0).

(iii) ⇒ (ii). Since $G.x_0$ is dense in X and X is discrete we have $X = G.x_0$, so X is co-compact. The claim about the type follows from the assumption that the isotropy group of x_0 is trivial together with the fact that the compact subsets of a discrete set precisely are the finite subsets. □

We end this section with a result stating that there is a one-to-one correspondence between (not necessarily compact) left G-ideals in $P(G)$ and (not necessarily co-compact) open invariant subsets of βG, hence classifying the latter; see also Proposition 2.7. In particular, all left G-ideals arise from a pointed locally compact G-space. The result also extends Proposition 3.12 to the non-compact case.

Proposition 3.16. *For each left G-ideal $\mathcal{M}$ put*

$$X_{\mathcal{M}} = \bigcup_{A \in \mathcal{M}} K_A \subseteq \beta G. \tag{3.2}$$

Then $X_{\mathcal{M}}$ is open, G-invariant, and $e \in X_{\mathcal{M}}$. In particular, $(X_{\mathcal{M}}, e)$ is a pointed locally compact G-space. Moreover,

(i) $\mathcal{M}(G, X_{\mathcal{M}}, e) = \mathcal{M}$.

(ii) *Every open invariant subset of βG is equal to $X_{\mathcal{M}}$ for some left G-ideal $\mathcal{M}$.*

Proof. It is clear that $X_{\mathcal{M}}$ is open. Invariance of $X_{\mathcal{M}}$ follows from the fact that $g.K_A = K_{gA}$ (see [7, Lemma 2.4]) and the assumption that $\mathcal{M}$ is a left G-ideal. Each non-empty open invariant subset of βG contains $G = G.e$ as a dense orbit, so $(X_{\mathcal{M}}, e)$ is a pointed locally compact G-space.

(i). The proof follows the same idea as the proof of Proposition 3.12. Let $A \subseteq G$. Then:

$$\begin{aligned} A \in \mathcal{M}(G, X_{\mathcal{M}}, e) \quad &\overset{(1)}{\iff} \quad K_A \cap X_{\mathcal{M}} \text{ is compact} \\ &\overset{(2)}{\iff} \quad K_A \subseteq X_{\mathcal{M}} \overset{(3)}{\iff} A \in \mathcal{M}. \end{aligned}$$

(1) holds because the closure of $A = A.e$ in $X_{\mathcal{M}}$ is equal to $K_A \cap X_{\mathcal{M}}$; and (2) is established as in the proof of Proposition 3.12. Let us look at (3): "⇐" holds by definition. Suppose that $K_A \subseteq X_{\mathcal{M}}$. Then, by compactness of K_A, there exist $A_1, \dots, A_n \in \mathcal{M}$ such that

$$K_A \subseteq K_{A_1} \cup K_{A_2} \cup \cdots \cup K_{A_n} = K_{\bigcup_{i=1}^n A_i}.$$

Hence $A \subseteq \bigcup_{i=1}^n A_i$, and so $A \in \mathcal{M}$.

(ii). Let X be an open G-invariant subset of βG, and let $\mathcal{M}$ be the set of all $A \subseteq G$ such that $K_A \subseteq X$. Then $\mathcal{M}$ is a left G-ideal, and $X_{\mathcal{M}} \subseteq X$ by (3.2). To prove the reverse inclusion, let $x \in X$. Since βG, and hence X, are totally disconnected locally compact Hausdorff spaces, and since the compact-open sets in βG are of the form K_A for some $A \subseteq G$, there exists $A \subseteq G$ with $x \in K_A \subseteq X$. Hence $A \in \mathcal{M}$, so $x \in K_A \subseteq X_{\mathcal{M}}$. ☐

4. Non co-compact actions

A group G is said to be *supramenable* if it has no paradoxical subsets, cf. [10]. It was shown in [7] that a group is supramenable if and only if any co-compact action of the group on a locally compact Hausdorff space admits a non-zero invariant Radon measure. We show here, using some of the machinery of the previous section, that the assumption that the action be co-compact cannot be removed. In fact, any infinite countable group admits an action on a locally compact Hausdorff space which does not admit a non-zero invariant Radon measure (Proposition 4.3). We also prove the existence of non-compact left G-ideals and non co-compact open G-invariant subsets of βG for every infinite countable group (Proposition 4.4).

We start by giving an obstruction to having a non-zero invariant Radon measure:

Lemma 4.1. *Let G be a countable group acting on a locally compact Hausdorff space X. Suppose that there is a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of X such that:*

(i) *For each n there is an infinite subset J_n of G such that $\{g.K_n\}_{g \in J_n}$ are pairwise disjoint subsets of K_{n+1},*

(ii) $X = \bigcup_{n=1}^{\infty} \bigcup_{g \in G} g.K_n.$

Then X admits no non-zero invariant Radon measure.

Proof. Suppose that λ is an invariant Radon measure on X. For each $n \in \mathbb{N}$ we have that $\lambda(K_{n+1}) < \infty$, which by (i) and invariance of λ entails that $\lambda(K_n) = 0$. It then follows from (ii) and invariance that $\lambda(X) = 0$. ☐

We thank the referee for suggesting the proof of the following lemma (which allowed us to remove the condition that G contains an element of infinite order).

Lemma 4.2. *Let G be an infinite countable group. Then there is a sequence $I_1, I_2, I_3, \ldots$ of infinite subsets of G such that the product maps*

$$\Phi_n \colon I_n \times \cdots \times I_2 \times I_1 \to G, \quad \Phi_n(g_n, \ldots, g_2, g_1) = g_n \cdots g_2 g_1, \quad g_j \in I_j,$$

are injective for each $n \in \mathbb{N}$, and such that $e \in I_n$ for all $n \in \mathbb{N}$.

Proof. If G has an element g of infinite order, then the proof is very easy: Choose mutually distinct natural numbers $\{k(n,j)\}$ for all integers $n \geq 1$ and $j \geq 2$, put $m(n,j) = 2^{k(n,j)}$, and put

$$I_n = \{e, g^{m(n,2)}, g^{m(n,3)}, g^{m(n,4)}, \cdots\}.$$

Then $I_1, I_2, I_3, \ldots$ have the desired properties.

Consider now the general case, and denote the center of the group G by $Z(G)$. For $g \in G$, we denote the centralizer of g in G by $C_G(g)$. First we prove the lemma in the case where $C_G(g)$ is finite for every $g \in G \setminus Z(G)$ (this implies either $Z(G)$ is finite or $G = Z(G)$, i.e., G is abelian). It is easy to see that for any $h \in G$ and any finite subset $F \subset G$, the set of $g \in G$ such that $\{ghg^{-1}\} \cap F$ is not contained in $Z(G)$ is finite. Hence for any finite subsets $F_0, F_1 \subset G$, the set of $g \in G$ such that $gF_0g^{-1} \cap F_1 \subset Z(G)$ is infinite.

We will define each of the sets I_n as an increasing union $I_n = \bigcup_{j=0}^{\infty} I_n^j$, where each I_n^j is finite, and $I_n^0 = \{e\}$. Fix a sequence $\{n_j\}$ of natural numbers with $n_j \leq j$, in which every natural number appears infinitely often. At stage j of the construction we will add a single element to $I_{n_j}^{j-1}$ to obtain $I_{n_j}^j$ (and take $I_n^j = I_n^{j-1}$ for $n \neq n_j$).

The sets I_n^j are defined inductively as follows. For $j = 1$ we have $n_j = 1$. Take any $g_1 \neq e$, set $I_1^1 = I_0^1 \cup \{g_1\}$, and set $I_n^1 = I_n^0$ for $n \neq 1$. Let now $j \geq 2$ be given and assume that the sets I_n^{j-1}, $n \in \mathbb{N}$, are defined such that the restriction of Φ_{j-1} to $I_{j-1}^{j-1} \times \cdots \times I_2^{j-1} \times I_1^{j-1}$ is injective. For $n \neq n_j$, define $I_n^j = I_n^{j-1}$. Let

$$F_0 = I_{n_j-1}^j I_{n_j-2}^j \dots I_1^j, \qquad F_1 = I_j^j I_{j-1}^j \dots I_{n_j+1}^j.$$

Since $F_0F_0^{-1}$ and $F_1^{-1}F_1$ are finite, the assumption on G implies that the set A, of all group elements $g \in G$ such that $gF_0F_0^{-1}g^{-1} \cap F_1^{-1}F_1 \subseteq Z(G)$, is infinite. Let g_j be any element of $A \setminus F_1^{-1}F_1I_{n_j}^{j-1}F_0F_0^{-1}$ and define $I_{n_j}^j = I_{n_j}^{j-1} \cup \{g_j\}$. We claim that the restriction of Φ_j to $I_j^j \times \cdots \times I_2^j \times I_1^j$ is injective. By induction (and since $I_j^{j-1} = \{e\}$) the restriction of Φ_j to $I_j^{j-1} \times \cdots \times I_2^{j-1} \times I_1^{j-1}$ is injective. The choice of $g_j \notin F_1^{-1}F_1I_{n_j}^{j-1}F_0F_0^{-1}$ ensures that

$$F_1g_jF_0 \cap F_1I_{n_j}^{j-1}F_0 = \emptyset,$$

so it remains to show that the restriction of Φ_j to the set

$$I_j^j \times I_{j-1}^j \times \cdots \times I_{n_j+1}^j \times \{g_j\} \times I_{n_j-1}^j \times I_{n_j-2}^j \times \cdots \times I_1^j$$

is injective. Suppose that

$$h_j \dots h_{n_j+1}g_jh_{n_j-1} \dots h_1 = k_j \dots k_{n_j+1}g_jk_{n_j-1} \dots k_1,$$

where $h_i, k_i \in I_i^j$. Then

$$g_jk_{n_j-1} \dots k_1(h_{n_j-1} \dots h_1)^{-1}g_j^{-1} = (k_j \dots k_{n_j+1})^{-1}h_j \dots h_{n_j+1},$$

which is in the set $g_jF_0F_0^{-1}g_j^{-1} \cap F_1^{-1}F_1 \subseteq Z(G)$. Thus

$$k_j \dots k_{n_j+1}k_{n_j-1} \dots k_1 = h_j \dots h_{n_j+1}h_{n_j-1} \dots h_1$$

so by the induction hypothesis, $h_i = k_i$ for all i, which finishes the proof in this case.

If there exists an infinite subgroup of G which satisfies the hypothesis above, then we can apply the argument above to this subgroup. So, we may assume that G does not have such a subgroup. Then there exists $g_1 \in G \setminus Z(G)$ such that $C_G(g_1)$ is infinite. By considering the subgroup $C_G(g_1)$, we can find $g_2 \in C_G(g_1) \setminus Z(C_G(g_1))$ such that $C_{C_G(g_1)}(g_2) = C_G(g_1) \cap C_G(g_2)$ is infinite. In the same way, there exists $g_3 \in (C_G(g_1) \cap C_G(g_2)) \setminus Z(C_G(g_1) \cap C_G(g_2))$ such that $C_G(g_1) \cap C_G(g_2) \cap C_G(g_3)$ is infinite. Repeating this argument, we obtain elements $g_1, g_2, g_3, \ldots$, which generate an infinite abelian subgroup. This is a contradiction. □

Proposition 4.3. *Let G be an infinite countable group. Then there is a locally compact σ-compact Hausdorff space X on which G acts freely and with a dense orbit, such that there is no non-zero invariant Radon measure on X. If G is supramenable, then the action is necessarily non co-compact.*

The space X can be chosen to be an open invariant subset of βG.

Proof. Let $\{I_n\}_{n\geq 1}$ be as in Lemma 4.2, and put

$$A_n = I_n I_{n-1} \cdots I_1 = \Phi_n(I_n, I_{n-1}, \ldots, I_1)$$

for each $n \geq 1$. Then $\{A_n\}_{n=1}^{\infty}$ is increasing, because $e \in I_n$ for all $n \geq 1$. Put

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_{A_n},$$

and put $X = X_{\mathcal{M}}$, cf. (3.2). Then X is an open invariant subset of βG. We show that conditions (i) and (ii) of Lemma 4.1 are satisfied with $K_n = K_{A_n}$.

It follows from injectivity of Φ_{n+1} that the sequence of sets $\{gA_n\}_{g\in I_{n+1}}$ are pairwise disjoint. By construction, $gA_n \subseteq A_{n+1}$ for all $g \in I_{n+1}$. As $K_{gA} = g.K_A$ for all $g \in G$ and all $A \subseteq G$, and since $K_A \cap K_B = \emptyset$ whenever $A \cap B = \emptyset$, we conclude that $\{g.K_{A_n}\}_{g\in I_{n+1}}$ is a sequence of pairwise disjoint subsets of $K_{A_{n+1}}$. Hence Lemma 4.1 (i) holds.

By (3.2),

$$X = X_{\mathcal{M}} = \bigcup_{A\in\mathcal{M}} K_A.$$

Let $A \in \mathcal{M}$ be given. Then $A \in \mathcal{M}_{A_n}$ for some $n \in \mathbb{N}$, which means that $A \propto A_n$, so $A \subseteq \bigcup_{g\in F} gA_n$ for some finite subset F of G. This entails that

$$K_A \subseteq \bigcup_{g\in F} g.K_{A_n}.$$

As $A \in \mathcal{M}$ was arbitrary, we see that condition (ii) in Lemma 4.1 holds. Hence there is no non-zero invariant Radon measure on X. Since condition (ii) in Lemma 4.1 holds, X is σ-compact. Each non-empty open G-invariant subset of βG contains the dense set $G = G.e$, so X contains a dense orbit.

If G is supramenable, then the action must be non co-compact, by [7, Theorem 1]. □

For completeness we show that non-compact left G-ideals and locally compact G-spaces, that are not co-compact, exist for every infinite countable group:

Proposition 4.4. *In each infinite countable group G there exists a left G-ideal that is not compact; and there exists a locally compact σ-compact Hausdorff space X on which G acts freely with a dense orbit, but not co-compactly.*

Proof. We construct an increasing sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of G such that $A_{n+1} \not\propto A_n$ for all n. It will then follow that

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_{A_n}$$

is non-compact. Indeed, if it were compact, then $\mathcal{M} = \mathcal{M}_{A_n}$ for some n, which would entail that $A_{n+1} \propto A_n$ contrary to the construction. The G-space $X = X_{\mathcal{M}}$ will therefore be non co-compact by Proposition 3.16 and Proposition 3.8, and G acts freely and with a dense orbit on X.

To construct the sets $A_1, A_2, A_3, \dots$ take a proper right-invariant metric d on G. Choose inductively a sequence $\{g_n\}_{n=1}^{\infty}$ of elements in G such that $d(g_n, \{g_1, \dots, g_{n-1}\}) \geq n$ for $n \geq 2$. Let each A_n be a subset of $\{g_1, g_2, g_3, \dots\}$ such that $\{A_n\}$ is increasing and $A_{n+1} \setminus A_n$ is infinite for all n. It then follows from Lemma 2.5 that $A_{n+1} \not\propto A_n$ for all n. □

5. A universal property of the G-space X_A

The G-space βG of a (discrete) group G is universal among all compact G-spaces with a dense orbit in the following sense: for each compact G-space X with a dense orbit $G.x_0$ there is a surjective continuous G-map $\varphi \colon \beta G \to X$ (which is unique if we also require that $\varphi(e) = x_0$).

We show in this section that the pointed locally compact G-space (X_A, e) (associated with a non-empty subset $A \subseteq G$) is universal among pointed locally compact co-compact G-spaces of type $[A]$, see also Proposition 3.13. The morphisms between locally compact G-spaces are *proper* continuous G-maps.

For each $f \in C(\beta G)$ denote by $\hat{f} \in \ell^{\infty}(G)$ the restriction of f to G.

Lemma 5.1. *Let A be a non-empty subset of a group G and let $f \in C(\beta G)$. It follows that $f \in C_0(X_A)$ if and only if for each $\varepsilon > 0$ there exists $B \subseteq G$ with $B \propto A$ such that*

$$g \in G \setminus B \implies |\hat{f}(g)| < \varepsilon.$$

Proof. Suppose first that $f \in C_0(X_A)$ and let $\varepsilon > 0$ be given. Then there is a compact subset L of X_A such that $|f(x)| < \varepsilon$ for all $x \in X_A \setminus L$. Take a finite subset F of G such that $L \subseteq \bigcup_{g \in F} g.K_A$ and put $B = FA \propto A$. If $g \in G \setminus B$, then $g \notin L$, so $|\hat{f}(g)| = |f(g)| < \varepsilon$.

To prove the "if"-part, let $f \in C(\beta G)$ and suppose that f has the stipulated property. Let $\varepsilon > 0$, and let $B \subseteq G$ with $B \propto A$ be such that $|\hat{f}(g)| < \varepsilon$ for all $g \in G \setminus B$. Then K_B is a compact-open subset of X_A and if $g \in G \cap (X_A \setminus K_B)$, then $g \notin B$, so $|f(g)| = |\hat{f}(g)| < \varepsilon$. As $G \cap (X_A \setminus K_B)$ is dense in $X_A \setminus K_B$, we conclude that $|f(x)| \leq \varepsilon$ for all $x \in X_A \setminus K_B$. This proves that $f \in C_0(X_A)$. □

The theorem below says that the pointed locally compact G-space (X_A, e) is universal for the class of pointed locally compact co-compact G-spaces of type $[A]$.

Theorem 5.2. *Let A be a non-empty subset of a countable group G. Then for each pointed locally compact G-space (X, x_0), there is a (necessarily unique and surjective) proper continuous G-map $\varphi \colon X_A \to X$ with $\varphi(e) = x_0$ if and only if X is co-compact and $\mathrm{T}(G, X, x_0) = [A]$.*

Proof. The "only if" part follows from Propositions 3.13 and 3.12, and the fact that X is co-compact if it is the image of the co-compact space X_A under a continuous G-map.

Suppose that (X, x_0) is a pointed locally compact co-compact G-space of type $[A]$. Define a homomorphism $\hat{\pi}\colon C_0(X) \to \ell^\infty(G)$ by

$$\hat{\pi}(f)(g) = f(g.x_0), \qquad f \in C_0(X), \quad g \in G.$$

Composing π with the canonical isomorphism $\ell^\infty(G) \to C(\beta G)$ we obtain a homomorphism $\pi\colon C_0(X) \to C(\beta G)$ which satisfies $\widehat{\pi(f)} = \hat{\pi}(f)$ for all $f \in C_0(X)$. The homomorphisms π and $\hat{\pi}$ are injective because $G.x_0$ is dense in X.

We claim that $\pi(f) \in C_0(X_A)$ for all $f \in C_0(X)$. Let $\varepsilon > 0$, and let L be a compact subset of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus L$. Put $B = O_X(L, x_0)$. Then $B \propto A$ by Proposition 3.10. If $g \in G \setminus B$, then $g.x_0 \notin L$, so $|\hat{\pi}(f)(g)| = |f(g.x_0)| < \varepsilon$. It therefore follows from Lemma 5.1 that $\pi(f) \in C_0(X_A)$.

We have now obtained a homomorphism $\pi\colon C_0(X) \to C_0(X_A)$ satisfying $\pi(f)(g) = f(g.x_0)$. Since G is dense in X_A we see that π is G-equivariant. We show that the image of π is full in $C_0(X_A)$. To this end we must show that for each $y \in X_A$ there exists $f \in C_0(X)$ such that $\pi(f)(y) \neq 0$. Take $g \in G$ such that $y \in g.K_A = K_{gA}$. Put $K = \overline{A.x_0}$, which is a compact subset of X by Proposition 3.10. Let $f \in C_0(X)$ be such that the restriction of f to $g.K$ is 1. For each $h \in gA$ we have $h.x_0 \in g.K$, so $\pi(f)(h) = f(h.x_0) = 1$. Hence $\pi(f)(x) = 1$ for all x in the closure of gA in X_A, and this closure is precisely K_{gA}. This entails that $\pi(f)(y) = 1$.

In conclusion we obtain a continuous proper G-equivariant epimorphism $\varphi\colon X_A \to X$ such that $\pi(f) = f \circ \varphi$. In particular, $\varphi(e) = x_0$. □

It was shown in [7, Proposition 2.6] that there is a non-zero invariant Radon measure on the G-space X_A if and only if A is non-paradoxical. Using the theorem above we can extend this result as follows:

Corollary 5.3. *Let G be a group and let A be a non-empty subset of G. Then A is non-paradoxical if and only if every pointed locally compact co-compact G-space (X, x_0) of type $[A]$ admits a non-zero invariant Radon measure.*

Proof. The "if" part follows from [7, Proposition 2.6] with $(X, x_0) = (X_A, e)$. Suppose that A is non-paradoxical. Use Theorem 5.2 to find a proper continuous G-map $\varphi\colon X_A \to X$ (such that $\varphi(e) = x_0$). Let μ be a non-zero invariant Radon measure on X_A, cf. [7, Proposition 2.6], and let $\tilde{\mu} = \mu \circ \varphi^{-1}$ be the image measure on X. Then $\tilde{\mu}$ is a non-zero invariant Radon measure on X. □

The existence of a non-zero invariant Radon measure is of course independent on the choice of base point, and, indeed, a "base point free" version of this results holds, see Proposition 9.13.

One can alternatively obtain Corollary 5.3 from [7, Lemma 2.2] in combination with Proposition 3.11 without making reference to Theorem 5.2.

6. Universal locally compact minimal G-spaces

In the previous section we showed that the co-compact open G-invariant subspaces of βG, which we know are of the form X_A for some $A \subseteq G$, are universal among a certain class of locally compact G-spaces. In this section we shall show that the minimal closed G-invariant subsets of X_A are universal among certain minimal locally compact G-spaces (provided A is of *minimal type*), and that any two minimal closed G-invariant subsets of X_A are isomorphic as G-spaces.

Ellis proved in [2] that the minimal closed invariant subsets of the G-space βG are universal and pairwise isomorphic. Y. Gutman and H. Li gave in [4] a short and elegant new proof of this theorem. We shall mimic their proof in our proof of Theorem 6.7 below.

Definition 6.1. For a group G, let $P_{\approx}^{\min}(G)$ be the set of equivalence classes $[A] \in P_{\approx}(G)$ that arise as the type of a minimal pointed locally compact G-space. A subset $A \subseteq G$ for which $[A] \in P_{\approx}^{\min}(G)$ is said to be of minimal type.

In other words, a subset A of G is of minimal type if and only if there is a pointed locally compact minimal G-space (X, x_0) such that $\mathrm{T}(G, X, x_0) = [A]$. Recall that minimal locally compact G-spaces automatically are co-compact.

Example 6.2. Each subgroup H of a group G is of minimal type. This can be seen as follows: If H is finite, then $[H]$ is the type of the trivial (minimal) G-space (G, e), cf. Proposition 3.15.

Suppose next that H is an infinite subgroup of G. Then H admits a free minimal action on a compact Hausdorff space X, see for example [2]. This induces a free minimal action of G on $Y = X \times (G/H)$, where G/H is the left coset, see [7, Lemma 7.1]. The set $K = X \times \{e\}$ is a G-regular compact subset of Y. Choose $x_0 \in X$, and put $y_0 = (x_0, e)$. Then $g.y_0 \in K$ if and only if $g \in H$, so $O_Y(K, y_0) = H$. It follows from Proposition 3.11 that $\mathrm{T}(G, Y, y_0) = [H]$, so $[H]$ is of minimal type.

In particular, $[G]$ and $[\{e\}]$ are of minimal types representing compact minimal G-spaces, which always exist, respectively, the trivial discrete G-space G.

A priori, it is not clear if every infinite group G contains minimal types other than $[G]$ and $[\{e\}]$. In other words, does every infinite group G act minimally on some locally compact non-compact and non-discrete space. We shall answer this question affirmatively in Section 8.

In Section 9 we shall give an intrinsic description of the subsets of a group that are of minimal type. Far from all subsets of a group are of minimal type. If $A \subseteq G$ is of minimal type, then we shall show that the minimal closed G-invariant subsets of X_A have the following nice universal property:

Definition 6.3. Let G be a group and let A be a non-empty subset of G of minimal type.

A locally compact minimal G-space Z is said to be a *universal minimal G-space of type A* if for each pointed minimal locally compact G-space (X, x_0) with $\mathrm{T}(G, X, x_0) = [A]$ there is a surjective proper continuous G-map $\varphi\colon Z \to X$.

Lemma 6.4. *Let G be a group and let A be a non-empty subset of G of minimal type. Suppose that Z is a universal minimal G-space of type A. Then $\mathrm{T}(G, Z, z_0) = [A]$ for some $z_0 \in Z$.*

Proof. Since A is of minimal type there is a pointed locally compact G-space (X, x_0) with $\mathrm{T}(G, X, x_0) = [A]$. By the universal property of Z there is a surjective proper continuous G-map $\varphi\colon Z \to X$. Choose $z_0 \in Z$ such that $\varphi(z_0) = x_0$. It then follows from Proposition 3.13 that $\mathrm{T}(G, Z, z_0) = \mathrm{T}(G, X, x_0) = [A]$. □

Proposition 6.5. *Let G be a group and let $A \subseteq G$ be of minimal type. Then any closed minimal G-invariant subset Z of X_A is a universal minimal G-space of type A.*

Proof. Let (X, x_0) be any pointed minimal locally compact G-space of type $[A]$. By Theorem 5.2 there is a continuous proper G-epimorphism $\varphi\colon X_A \to X$ with $\varphi(e) = x_0$. Since a proper continuous map between locally compact Hausdorff spaces maps closed sets to closed sets, it follows that $\varphi(Z)$ is a closed G-invariant subset of X. Hence $\varphi(Z) = X$, so the restriction of φ to Z has the desired properties. □

We proceed to prove that universal minimal G-spaces of a given type are essentially unique. First we need to know that we can take projective limits in the class of pointed locally compact G-spaces of a given type:

Lemma 6.6. *Let G be a group and let $A \subseteq G$ be of minimal type. Let I be an upwards directed totally ordered set, let $(X_\alpha, x_\alpha)_{\alpha \in I}$ be a family of pointed minimal locally compact G-spaces of type $[A]$ equipped with surjective proper continuous G-maps $\varphi_{\beta,\alpha}\colon X_\alpha \to X_\beta$, for $\alpha > \beta$, satisfying*

(i) *$\varphi_{\beta,\alpha}(x_\alpha) = x_\beta$ for all $\alpha > \beta$,*

(ii) *$\varphi_{\gamma,\beta} \circ \varphi_{\beta,\alpha} = \varphi_{\gamma,\alpha}$ for all $\alpha > \beta > \gamma$.*

Then there is a pointed minimal locally compact G-space (X, x_0) of type $[A]$ and surjective proper continuous G-maps $\varphi_\alpha\colon X \to X_\alpha$ satisfying $\varphi_\alpha(x_0) = x_\alpha$ for all $\alpha \in I$, and $\varphi_{\beta,\alpha} \circ \varphi_\alpha = \varphi_\beta$ for all $\alpha > \beta$.

The G-space (X, x_0) with the mappings $\varphi_\alpha\colon X \to X_\alpha$ is the *projective limit* of the family $(X_\alpha, x_\alpha)_{\alpha \in I}$.

Proof. Put

$$X = \Big\{ (z_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \;\big|\; \varphi_{\gamma,\beta}(z_\beta) = z_\gamma \text{ for all } \gamma < \beta \Big\},$$

put $x_0 = (x_\alpha)_{\alpha \in I} \in X$, and let $\varphi_\alpha\colon X \to X_\alpha$ be the (restriction to X of the) projection map. Equip the space $\prod_{\alpha \in I} X_\alpha$ with the product topology, and equip $X \subseteq \prod_{\alpha \in I} X_\alpha$ with the subspace topology. Note that the product space is not locally compact if I is an infinite set and the spaces X_α are non-compact.

The projection maps φ_α are continuous, also when restricted to X. The maps φ_α are also proper (when restricted to X). Indeed, if $K \subseteq X_\alpha$ is compact, then $\varphi_\alpha^{-1}(K)$ is a closed (and hence compact) subset of the compact set $\prod_{\beta \in I} K_\beta$, where

$$K_\beta = \begin{cases} K, & \beta = \alpha, \\ \varphi_{\beta,\alpha}(K), & \beta < \alpha, \\ \varphi_{\alpha,\beta}^{-1}(K), & \beta > \alpha. \end{cases}$$

It is straightforward to show that the family of sets of the form $\varphi_\alpha^{-1}(U)$, where $\alpha \in I$ and $U \subseteq X_\alpha$ is open, is closed under intersections, and hence forms a basis for the topology on X. Using this, and the fact, established above, that each φ_α is proper, we conclude that X is locally compact (and Hausdorff).

Equip X with the natural G-action. Then each φ_α becomes a continuous, proper G-map. We show that each orbit in X is dense. Take $z \in X$ and take a non-empty open subset V of X of the form $\varphi_\alpha^{-1}(U)$, where $\alpha \in I$ and where $U \subseteq X_\alpha$ is open, cf. the comment above. We must show that $G.z \cap V \neq \emptyset$. Write $z = (z_\alpha)$ and use that X_α is a minimal G-space to find $g \in G$ with $g.z_\alpha \in U$. Then $\varphi_\alpha(g.z) = g.z_\alpha \in U$, so $g.z \in \varphi_\alpha^{-1}(U) = V$ as desired.

It finally follows from Proposition 3.13 that

$$\mathrm{T}(G, X, x_0) = \mathrm{T}(G, X_\alpha, x_\alpha) = [A].$$

□

The proof of the next theorem closely follows the ideas from [4]:

Theorem 6.7. *Let G be a group and let $A \subseteq G$ be a subset of minimal type. Let Z_1 and Z_2 be universal minimal G-spaces of type A. Then Z_1 and Z_2 are isomorphic as G-spaces.*

Proof. It follows from Lemma 6.4 that there exist points $z_i \in Z_i$ such that $\mathrm{T}(G, Z_i, z_i) = [A]$ for $i = 1, 2$. Since Z_1 and Z_2 are universal spaces of type A we can find surjective proper continuous G-maps $\varphi_1 \colon Z_1 \to Z_2$ and $\varphi_2 \colon Z_2 \to Z_1$. (We do not expect $\varphi_1(z_1) = z_2$ and $\varphi_2(z_2) = z_1$.) Thus $\varphi_2 \circ \varphi_1 \colon Z_1 \to Z_1$ and $\varphi_1 \circ \varphi_2 \colon Z_2 \to Z_2$ are surjective proper continuous G-maps.

It therefore suffices to show that if Z is a universal minimal G-space of type A, then every surjective continuous proper G-map $\rho \colon Z \to Z$ is injective. Suppose that ρ is not injective. We apply the argument from [4] to reach a contradiction.

Let α be an ordinal with $|\alpha| > |Z^2|$. We shall construct a pointed locally compact G-space (X_β, x_β) of type $[A]$ for each ordinal $\beta \leq \alpha$ and proper continuous G-maps $\psi_{\gamma,\beta} \colon X_\beta \to X_\gamma$ for $\gamma < \beta \leq \alpha$ with the following properties:

- $\psi_{\gamma,\beta}(x_\beta) = x_\gamma$.
- $\psi_{\delta,\gamma} \circ \psi_{\gamma,\beta} = \psi_{\delta,\beta}$ for any $\delta < \gamma < \beta \leq \alpha$.
- For each ordinal $\beta < \alpha$, there exist distinct elements $y_\beta, z_\beta \in X_{\beta+1}$ with $\psi_{\beta,\beta+1}(y_\beta) = \psi_{\beta,\beta+1}(z_\beta)$.

Suppose the above G-spaces and G-maps have been constructed. By the universal property of Z one has an epimorphism $Z \to X_\alpha$. This implies that $|X_\alpha| \leq |Z|$. Let $\beta < \alpha$ be given, and consider the surjective G-maps:

$$X_\beta \xleftarrow{\psi_{\beta,\beta+1}} X_{\beta+1} \xleftarrow{\psi_{\beta+1,\alpha}} X_\alpha$$

Find $\tilde{y}_\beta, \tilde{z}_\beta \in X_\alpha$ with $\psi_{\beta+1,\alpha}(\tilde{y}_\beta) = y_\beta$ and $\psi_{\beta+1,\alpha}(\tilde{z}_\beta) = z_\beta$.

For any $\gamma < \beta < \alpha$, one has

$$\begin{aligned} \psi_{\gamma+1,\alpha}(\tilde{y}_\beta) &= (\psi_{\gamma+1,\beta} \circ \psi_{\beta,\beta+1})(y_\beta) = (\psi_{\gamma+1,\beta} \circ \psi_{\beta,\beta+1})(z_\beta) \\ &= \psi_{\gamma+1,\alpha}(\tilde{z}_\beta) \end{aligned}$$

whereas $\psi_{\gamma+1,\alpha}(\tilde{y}_\gamma) = y_\gamma \neq z_\gamma = \psi_{\gamma+1,\alpha}(\tilde{z}_\gamma)$. Hence $(\tilde{y}_\beta, \tilde{z}_\beta) \neq (\tilde{y}_\gamma, \tilde{z}_\gamma)$ whenever $\beta \neq \gamma$, so that the map

$$\{\beta \mid 0 \leq \beta < \alpha\} \to X_\alpha \times X_\alpha, \qquad \beta \mapsto (\tilde{y}_\beta, \tilde{z}_\beta),$$

is injective. This implies that $|\alpha| \leq |X_\alpha^2|$, which leads to the contradiction $|Z^2| < |\alpha| \leq |X_\alpha^2| \leq |Z^2|$.

The construction of the G-spaces and the G-maps above is carried out through transfinite induction. For $\beta = 0$, let $X_0 = Z$, and choose $x_0 \in X_0$ such that the type of the pointed locally compact G-space (X_0, x_0) is $[A]$, cf. Lemma 6.4.

If β is a successor ordinal, set $X_\beta = Z$ and use the universal property of the space Z to find a surjective continuous proper G-map $\varphi_\beta \colon X_\beta \to X_{\beta-1}$. Set $\psi_{\beta-1,\beta} = \varphi_\beta \circ \rho$, and for $\gamma < \beta - 1$, set $\psi_{\gamma,\beta} = \psi_{\gamma,\beta-1} \circ \psi_{\beta-1,\beta}$. Let $x_\beta \in X_\beta$ be a preimage of $x_{\beta-1}$ by $\psi_{\beta-1,\beta}$. Then (X_β, x_β) is of type $[A]$ by Proposition 3.13. Since ρ is not injective we can find distinct elements $y_\beta, z_\beta \in X_\beta$ such that $\psi_{\beta,\beta+1}(y_\beta) = \psi_{\beta,\beta+1}(z_\beta)$.

If β is a limit ordinal, then define (X_β, x_β) to be the projective limit of $(X_\gamma)_{\gamma<\beta}$, cf. Lemma 6.6; and for $\gamma < \beta$, define $\psi_{\gamma,\beta} \colon X_\beta \to X_\gamma$ to be the epimorphism coming from the projective limit. □

Corollary 6.8. *Let G be a group and let A be a subset of G of minimal type. Then any two minimal closed G-invariant subsets of X_A are isomorphic as G-spaces.*

Proof. This follows immediately from Proposition 6.5 and Theorem 6.7. □

7. Minimal closed invariant subsets of X_A

Fix a group G and a non-empty subset A of G, and let X_A be the open G-invariant subset of βG associated with A. Consider the following questions:

- Are any two minimal closed G-invariant subsets of X_A isomorphic as G-spaces?
- For which A is it possible to find a minimal closed G-invariant subset of X_A which is compact, respectively, discrete, respectively, neither compact nor discrete?

The answer to first question is affirmative when A is of minimal type (Corollary 6.8). In general this question has negative answer (see Example 7.12, Case III), two minimal closed G-invariant subsets of X_A need not even be homeomorphic.

The second question, in the case of compact minimal subsets, was answered in [7]:

Definition 7.1 (cf. [7, Definition 3.5]). A non-empty subset A of a group G is said to be *absorbing* if all finite subset F of G there exists $g \in G$ such that $Fg \subseteq A$,

Equivalently, A is absorbing if $\bigcap_{g\in F} gA \neq \emptyset$ for all finite subsets F of G.

Proposition 7.2 (cf. [7, Proposition 5.5 (iv)]). *Let A be a non-empty subset of a group G. Then X_A contains a compact minimal closed G-invariant subset if and only if there is no absorbing subset B of G such that $A \approx B$.*

If A is equivalent to G, then $X_A = \beta G$ is compact and, of course, *all* closed subsets of X_A are compact. Any infinite group contains many absorbing subsets that are not equivalent to the group itself (see Example 7.12, Case III). For a quick example, take $G = \mathbb{Z}$ and $A = \mathbb{N}$.

This section is devoted to answer the second question in the case of discrete minimal subsets (Theorem 7.10). At the same time we will obtain a necessary and sufficient condition on the set A ensuring that all minimal closed G-invariant subsets of X_A are neither compact nor discrete.

If A is finite, then $X_A = G$, which is a minimal discrete G-space. More generally, it was shown in [7, Example 5.6] that *every* minimal closed G-invariant subset of X_A is discrete if $|gA \cap A| < \infty$ for all $g \in G \setminus \{e\}$. It was also shown in [7] that each infinite group G possesses an *infinite* subset A with this property. A complete answer to the question of when X_A contains a discrete minimal closed G-invariant subset is phrased in terms of a divisibility property of the set A:

Definition 7.3. Let G be a group and let A be a non-empty subset of G. For each $n \geq 1$ we say that A is *n-divisible* if there are pairwise disjoint subsets $\{A_j\}_{j=1}^n$ of A such that $A_j \approx A$ for all j.

We say that A is *infinitely divisible* if A is n-divisible for all integers $n \geq 1$.

If $A \subseteq G$ is non-empty and finite, then A is n-divisible if and only if $|A| \geq n$, cf. Example 2.4 (ii). In Example 7.7 below we show that each infinite group is infinitely divisible.

Recall from (3.1) the definition of the subset $O_X(V, x)$ of G associated with a G-space X, a subset V of X, and an element x in X. In the case where $X = \beta G$ and $V = K_A$ we denote this set by $O(A, x)$. In other words,

$$O(A, x) = \{g \in G \mid g.x \in K_A\}.$$

Observe that $O(A, e) = A$ and that $e \in O(A, x)$ whenever $x \in K_A$. Here are some further properties of finitely and infinitely divisible sets, and of the set $O(A, x)$:

Lemma 7.4. *Let A and B be non-empty subsets of a group G and let $x \in \beta G$. Then:*

- (i) *$O(A \cup B, x) = O(A, x) \cup O(B, x)$; and if A and B are disjoint, then so are $O(A, x)$ and $O(B, x)$.*
- (ii) *If $A \propto B$, then $O(A, x) \propto O(B, x)$.*
- (iii) *If $A \approx B$, then $O(A, x) \neq \emptyset \Leftrightarrow O(B, x) \neq \emptyset$.*
- (iv) *If $A \approx B$, then $|O(A, x)| = \infty \Leftrightarrow |O(B, x)| = \infty$.*
- (v) *If A is n-divisible, then $|O(A, x)| \geq n$ for all $x \in K_A$.*
- (vi) *If A is infinitely divisible, then $|O(A, x)| = \infty$ for all $x \in K_A$.*

Proof. (i) follows from the identity $K_{A \cup B} = K_A \cup K_B$, and from the fact that $A \cap B = \emptyset$ implies that $K_A \cap K_B = \emptyset$, see [7, Lemma 2.4(i)]. To see (ii), if $A \propto B$, then $A \subseteq FB$ for some finite subset F of G. Now use that $F.K_B = K_{FB}$ for all $B \subseteq G$, cf. [7, Lemma 2.4(v)], to conclude that $O(A, x) \subseteq F.O(B, x)$.

(iii) and (iv) follow from (ii). (v) follows from (i), (iii) and (iv) and from the fact that $O(A,x) \neq \emptyset$ whenever $x \in K_A$. (vi) follows from (v). □

Lemma 7.5. *Let G be a discrete group and let A be a non-empty subset of G. Then A is 2-divisible if and only if for all $x \in K_A$ there exists $g \neq e$ in G such that $g.x \in K_A$.*

In the language of Lemma 7.4 this lemma says that A is 2-divisible if and only if $|O(A,x)| \geq 2$ for all $x \in K_A$.

Proof. The "only if" part follows from Lemma 7.4 (v). Let us prove the "if" part of the lemma. We show first that there are clopen subsets $U_1, U_2, \dots, U_n$ of K_A and elements $g_1, g_2, \dots, g_n \in G$ such that

- $K_A = U_1 \cup U_2 \cup \cdots \cup U_n$,
- $g_j.U_j \cap U_j = \emptyset$ for all j,
- $g_j.U_j \subseteq K_A$ for all j.

For each $x \in K_A$ there exists an element $g_x \neq e$ in G such that $g_x.x \in K_A$. Since K_A is totally disconnected there is a clopen subset U_x of K_A such that $g_x.U_x \subseteq K_A$, $g_x.U_x \cap U_x = \emptyset$, and $x \in U_x$. We can now take the clopen subsets $U_1, U_2, \dots, U_n$ of K_A to be a subset of the open cover $\{U_x\}_{x \in K_A}$ of K_A which still covers K_A, and we take $g_1, \dots, g_n$ to be their associated group elements.

Put $V_1 = U_1$, put

$$V_j = U_j \setminus \Big(\bigcup_{i=1}^{j-1} (g_i.V_i \cup g_j^{-1}.V_i) \Big), \qquad j = 2, 3, \dots, n,$$

and put

$$K_1 = V_1 \cup V_2 \cup \cdots \cup V_n, \qquad K_2 = g_1.V_1 \cup g_2.V_2 \cup \cdots \cup g_n.V_n.$$

Then K_1 and K_2 are pairwise disjoint clopen subsets of K_A. Let F be the finite subset of G that consists of the elements $g_1, \dots, g_n$, their inverses, and the neutral element e. Then $K_A \subseteq F.K_1$ and $K_A \subseteq F^2.K_2$.

To prove that $K_A \subseteq F.K_1$, it suffices to show that $U_j \subseteq F.K_1$ for all j. Each V_j is contained in K_1, which again is contained in $F.K_1$. Hence $g_i.V_i$ and $g_j^{-1}.V_i$ are contained in $F.K_1$ for all i, j. This implies that U_j is contained in $F.K_1$ for all j.

As $K_1 \subseteq F.K_2$ it follows that $K_A \subseteq F^2.K_2$.

Put $A_j = G \cap K_j$. Then A_1 and A_2 are disjoint subsets of A, and $K_{A_j} = K_j$ by [7, Lemma 2.4(ii)]. Using [7, Lemma 2.4(v)] we further conclude that $K_A \subseteq F.K_{A_1} = K_{FA_1}$, which by [7, Lemma 2.4(iii)] entails that $A \subseteq FA_1$, i.e., $A \propto A_1$. In a similar way we see that $A \propto A_2$. As A_1 and A_2 are subsets of A we conclude that $A \approx A_1 \approx A_2$, so A is 2-divisible. □

Lemma 7.6. *The following three conditions are equivalent for each non-empty subset A of a discrete group G:*

(i) *A is infinitely divisible,*
(ii) *$|O(A,x)| = \infty$ for all $x \in K_A$,*

(iii) *there are pairwise disjoint subsets* $\{A_j\}_{j=1}^{\infty}$ *of* A *such that* $A_j \approx A$ *for all* j.

Moreover, if A *and* B *are non-empty subsets of* G *with* $A \approx B$*, then* A *is infinitely divisible if and only if* B *is infinitely divisible.*

Proof. (i) $\Rightarrow$ (ii) is contained in Lemma 7.4 (vi). (iii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). If (ii) holds, then it follows from Lemma 7.5 that A is 2-divisible, so there exist pairwise disjoint subsets A_1 and A_2 of A such that $A \approx A_1 \approx A_2$. Lemma 7.4 (iv) tells us that $|O(A_1, x)| = |O(A_2, x)| = \infty$ for all $x \in K_A$. Hence A_1 and A_2 are 2-divisible by Lemma 7.5. We can therefore keep dividing, and we arrive at (iii).

The last statement of the lemma follows from Lemma 7.4 (iv) and the equivalence of (i) and (ii). □

Example 7.7. Any infinite group G is infinitely divisible (as a subset of itself). This follows immediately from Lemma 7.6 above, since $O(G, x) = G$ for all $x \in K_G = \beta G$.

More generally, every infinite subgroup H of G is infinitely divisible. Indeed, we just saw that H is infinitely divisible viewed as a subset of itself, and hence it is infinitely divisible relatively to G.

We shall use the following standard result about crossed product C^*-algebras.

Lemma 7.8. *Let* G *be a discrete group acting on a locally compact Hausdorff space* X*. For each* $x \in X$*, let* φ_x *be the state on* $C_0(X) \rtimes_{\mathrm{red}} G$ *given by* $\varphi_x = \rho_x \circ E$*, where* $E\colon C_0(X) \rtimes_{\mathrm{red}} G \to C_0(X)$ *is the canonical conditional expectation, and where* $\rho_x\colon C_0(X) \to \mathbb{C}$ *is point evaluation at* x.

Let K *be a compact-open subset of* X*, let* $x \in K$*, and suppose that* $g.x \notin K$ *for all* $g \neq e$*. The restriction of* φ_x *to* $1_K\big(C_0(X) \rtimes_{\mathrm{red}} G\big)1_K$ *is then a character.*[3]

Proof. Observe that $\varphi_x(1_K) = 1_K(x) = 1$, so $\varphi_x \neq 0$. To show that φ_x is a character, it suffices to show that $\varphi_x(a^2) = \varphi_x(a)^2$ for all a in the corner algebra $1_K\big(C_0(X) \rtimes_{\mathrm{red}} G\big)1_K$. Each such a is a formal sum $a = \sum_{g \in G} f_g u_g$, where $f_g \in C_0(X)$ and $g \mapsto u_g$ is the unitary representation (in the multiplier algebra of $C_0(X) \rtimes_{\mathrm{red}} G$) of the action of G on $C_0(X)$. Since $E(a) = f_e$, we see that $\varphi_x(a) = f_e(x)$.

The condition that $a = 1_K a 1_K$ implies that the support of each f_g is contained in K. In particular, $f_h(g.x) = 0$ for all $h \in G$ and for all $g \neq e$. Now, $E(a^2) = \sum_{g \in G} f_g \alpha_g(f_{g^{-1}})$. Hence

$$\varphi_x(a^2) = \sum_{g \in G} f_g(x) f_{g^{-1}}(g.x) = f_e(x)^2 = \varphi_x(a)^2,$$

as desired. □

Proposition 7.9. *Let* G *be a discrete group and let* A *be a non-empty subset of* G*. Then the following are equivalent:*

(i) $1_A\big(\ell^\infty(G) \rtimes_{\mathrm{red}} G\big)1_A$ *has a character.*

[3]A character on a unital C^*-algebra A is a non-zero homomorphism from A into $\mathbb{C}$.

(ii) *A is not 2-divisible.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (ii) does not hold, i.e., A is the disjoint union of A_1 and A_2 where $A \approx A_1 \approx A_2$. Then 1_{A_1} and 1_{A_2} are full and pairwise orthogonal projections in $1_A\big(\ell^\infty(G) \rtimes_{\rm red} G\big)1_A$, cf. Lemma 2.6. However, no unital C^*-algebra containing two full pairwise orthogonal projections can admit a character, so (i) does not hold.

(ii) $\Rightarrow$ (i). It follows from Lemma 7.5 that K_A contains an element x such that $g.x \notin K_A$ for all $g \neq e$. Lemma 7.8 then says that φ_x is a character on

$$1_{K_A}\big(C(\beta G) \rtimes_{\rm red} G\big)1_{K_A} \cong 1_A\big(\ell^\infty(G) \rtimes_{\rm red} G\big)1_A.$$

□

Theorem 7.10. *Let G be a discrete group and let A be a non-empty subset of G. Then the following are equivalent:*

(i) *Some minimal closed G-invariant subset of X_A is discrete.*

(ii) *$1_A\big(\ell^\infty(G) \rtimes_{\rm red} G\big)1_A$ has a non-zero finite-dimensional quotient.*

(iii) *A is not infinitely divisible.*

Proof. Set

$$\mathcal{A} = 1_A\big(\ell^\infty(G) \rtimes_{\rm red} G\big)1_A \cong 1_{K_A}\big(C(\beta G) \rtimes_{\rm red} G\big)1_{K_A} = 1_{K_A}\big(C_0(X_A) \rtimes_{\rm red} G\big)1_{K_A},$$

and observe that 1_{K_A} is a full projection in $C_0(X_A) \rtimes_{\rm red} G$.

(i) $\Rightarrow$ (ii). Suppose that Z is a discrete minimal closed invariant subset of X_A. Then Z is a discrete minimal and free G-space, whence Z is isomorphic to G (viewed as a G-space with respect to left multiplication). The inclusion mapping $Z \hookrightarrow X_A$, which is a G-map, induces a G-equivariant surjection $C_0(X_A) \to C_0(Z)$, which again induces a surjective *-homomorphism

$$C_0(X_A) \rtimes_{\rm red} G \to C_0(Z) \rtimes_{\rm red} G \cong C_0(G) \rtimes_{\rm red} G \cong \mathcal{K},$$

where $\mathcal{K}$ is the algebra of compact operators (on $\ell^2(G)$). Hence there is a surjective *-homomorphism from $\mathcal{A}$ onto a unital corner of $\mathcal{K}$, which is a finite dimensional matrix algebra.

(ii) $\Rightarrow$ (i). If (ii) holds, then there is a proper closed two-sided ideal I in $C_0(X_A) \rtimes_{\rm red} G$ such that the quotient is stably isomorphic to a finite-dimensional C^*-algebra. Upon replacing I with a larger ideal, we may assume that I is a maximal proper ideal and that the quotient is isomorphic to the algebra of compact operators $\mathcal{K}(H)$ on a Hilbert space H.

The intersection $C_0(X_A) \cap I$ is an invariant ideal in $C_0(X_A)$ and therefore equal to $C_0(U)$ for some open invariant subset U of X_A. Since I is a maximal ideal in $C_0(X_A) \rtimes_{\rm red} G$ it follows that U is a maximal proper open invariant subset of X_A, so $Z = X_A \setminus U$ is a minimal closed invariant subset of X_A. The kernel of the map $C_0(X_A) \to \big(C_0(X_A) \rtimes_{\rm red} G\big)/I$ is equal to $C_0(U)$, so we have an injective *-homomorphism

$$C_0(Z) \to \big(C_0(X_A) \rtimes_{\rm red} G\big)/I \cong \mathcal{K}(H).$$

Hence Z must be discrete.

(ii) $\Rightarrow$ (iii). Suppose that A is infinitely divisible. Find a sequence $\{A_j\}_{j=1}^\infty$ of pairwise disjoint subsets of A with $A \approx A_j$ for all j. It follows from Lemma 2.6

that the projections 1_{A_j} are full in $\mathcal{A}$, and they are also pairwise orthogonal. However, no unital C^*-algebra which contains a sequence of pairwise orthogonal full projections can have a (non-zero) finite dimensional quotient. Hence (ii) does not hold.

(iii) $\Rightarrow$ (ii). If A is not infinitely divisible, then there exists a subset B of A such that B is not 2-divisible and $A \approx B$. (Indeed, let $n \geq 1$ be the maximal number for which there exist pairwise disjoint subsets $\{A_j\}_{j=1}^n$ of A with $A_j \approx A$ for all j. Put $B = A_1$.)

Let $p = 1_B \in \mathcal{A}$, and use Lemma 2.6 to see that p is a full projection in $\mathcal{A}$. It follows from Proposition 7.9 that $p\mathcal{A}p$ admits a character. In other words, $(p+I)(\mathcal{A}/I)(p+I)$ is one-dimensional for some closed two-sided ideal I in $\mathcal{A}$. Thus $\mathcal{A}/I$ is a unital C^*-algebra which is Morita equivalent to $\mathbb{C}$, which entails that $\mathcal{A}/I$ is finite dimensional. □

It is easy to decide when subsets of *minimal* type of a group are infinitely divisible, respectively, equivalent to an absorbing set:

Corollary 7.11. *Let G be an infinite group and let A be a subset of G of minimal type. Then*

(i) *A is infinitely divisible if and only if A is infinite.*
(ii) *A is equivalent to an absorbing set if and only if A is equivalent to G.*

Proof. (i). "Only if" is clear. Suppose that A is infinite. For each minimal closed invariant subset Z of X_A, there is $z_0 \in Z$ such that $\mathrm{T}(G, Z, z_0) = [A]$, cf. Lemma 6.4. As $[A] \neq [\{e\}]$, it follows from Proposition 3.15 that Z is not discrete. Hence A must be infinitely divisible by Theorem 7.10.

(ii). "Only if" is clear. Suppose that A is equivalent to an absorbing set. Then there is a compact minimal closed invariant subset Z of X_A by [7, Proposition 5.5 (iv)], and there is $z_0 \in Z$ such that $\mathrm{T}(G, Z, z_0) = [A]$, cf. Lemma 6.4. But then $A \approx G$ by Proposition 3.14. □

There is an abundance of subsets A of G which are infinite but not infinitely divisible, see Example 7.15 below for a method to construct such sets. None of these sets are of minimal type by Corollary 7.11.

Example 7.12 (Four classes of subsets of a group). Let G be an infinite group. We describe here four classes of subsets A of G depending on whether or not A is infinitely divisible, respectively, equivalent to an absorbing set. All four cases occur in all infinite groups. The stated properties of the minimal closed G-invariant subsets of X_A follow from Theorem 7.10 and from [7, Proposition 5.5 (iv)] (see also Proposition 7.2).

Case I: *A is infinitely divisible and absorbing.* This is satisfied for example with $A = G$.

In this case no minimal closed G-invariant subset of X_A is discrete, while at least some minimal closed G-invariant subsets of X_A are compact (they are all compact if $A = G$).

Case II: *A is infinitely divisible and not equivalent to an absorbing set.* Examples of such sets in an arbitrary infinite countable group G is given in Theorem 8.2 in the next section.

In this case every minimal closed G-invariant subset of X_A is non-compact and non-discrete. The existence of such subsets A therefore gives the existence of a free minimal action of G on a non-compact and non-discrete locally compact Hausdorff space. We can take this space to be the non-compact locally compact Cantor set $\mathbf{K}^*$, cf. Theorem 8.6.

Case III: *A is absorbing but not infinitely divisible.* Examples of such sets in an arbitrary infinite countable group G are given in Example 7.15 below.

In this case some minimal closed G-invariant subset of X_A is compact while some other minimal closed G-invariant subset of X_A is discrete. The existence of such sets A shows that Corollary 6.8 does not hold in general (when A is not of minimal type) and that the first question posed in the beginning of this section has a negative answer in general.

Case IV: *A is not infinitely divisible and not equivalent to an absorbing set.* This is satisfied for example with $A = \{e\}$, and also for any (finite or infinite) subset A of G such that $|A \cap gA| < \infty$ for all $g \in G \setminus \{e\}$. Each infinite group contains infinite subsets A with this property, see [7, Lemma 3.8] and its proof.

In this case no minimal closed G-invariant subset of X_A is compact, while at least some minimal closed G-invariant subset of X_A is discrete. If A satisfies $|A \cap gA| < \infty$ for all $g \in G \setminus \{e\}$, then *all* minimal closed G-invariant subsets of X_A are discrete by [7, Example 5.6].

Remark 7.13. We present here a general method to construct an infinitely divisible subset in an arbitrary group G.

Choose first an infinite sequence $\{a_n\}_{n=1}^\infty$ of elements in G so that the set

$$F_n = \{b_1 b_2 \dots b_n \mid b_i = e \text{ or } b_i = a_i\}$$

has cardinality 2^n for each n. Observe that $F_1 = \{e, a_1\}$ and that $F_{n+1} = F_n \cup F_n\, a_{n+1}$ for $n \geq 1$. It follows that $|F_n| = 2^n$ for all n if the a_n's are chosen such that $F_n \cap F_n\, a_{n+1} = \emptyset$ for all n, i.e., such that

$$a_{n+1} \in G \setminus F_n^{-1} F_n.$$

Take now another infinite sequence $\{c_n\}_{n=1}^\infty$ of elements in G so that the sets $F_n c_n$, $n \in \mathbb{N}$, are mutually disjoint. Then

$$A = \bigcup_{n=1}^{\infty} F_n c_n \tag{7.1}$$

is infinitely divisible. Let us check this. Fix an integer $n \geq 1$. For $m > n$, define

$$G_m = \{b_{n+1} b_{n+2} \dots b_m \mid b_i = e \text{ or } b_i = a_i\}.$$

Note that $F_n G_m = F_m$. Set

$$B = \bigcup_{m=n+1}^{\infty} G_m c_m.$$

Then $\{aB\}_{a \in F_n}$ are pairwise disjoint subsets of A. Since $A \setminus F_n B$ is a finite set, we see that $B \approx A$, and hence that $aB \approx A$ for all $a \in F_n$. Therefore A is 2^n-divisible. As $n \geq 1$ was arbitrary we conclude that A is infinitely divisible.

The lemma below, whose proof easily follows from Lemma 2.5, contains an easy criterion for not being 2-divisible.

Lemma 7.14. *Let G a countable group and let d be a proper right-invariant metric on G. Suppose that A is a subset of G such that for each $C < \infty$ there exists $g \in A$ with*

$$d(g, A \setminus \{g\}) > C.$$

Then A is not 2-divisible.

Example 7.15. We give here examples of subsets A of any infinite countable group G which are absorbing but not infinitely divisible, cf. Example 7.12 (Case III).

For a first easy example of such a set, take $G = \mathbb{Z}$ and

$$A = \{-n^2 \mid n \in \mathbb{N}\} \cup \mathbb{N}.$$

Then A is absorbing, cf. Definition 7.1. Use Lemma 7.14 to see that A is not 2-divisible.

Consider now an arbitrary infinite countable group G and choose a proper right-invariant metric d on G. Write $G = \bigcup_{n=1}^\infty F_n$, where $\{F_n\}_{n=1}^\infty$ is an increasing sequence of finite subsets of G. We claim that there exist sequences $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ in G such that if

$$A = \{g_1, g_2, g_3, \dots\} \cup \bigcup_{n=1}^\infty F_n h_n,$$

then $d(g_n, A \setminus \{g_n\}) \geq n$ for all n. It will then follow from Lemma 7.14 that A is not 2-divisible; and we see directly from Definition 7.1 that A is absorbing.

Note first that if F, F' are finite subsets of the infinite group G and if $C < \infty$, then there exists $g \in G$ such that $d(Fg, F') \geq C$. We can now construct $\{g_n\}$ and $\{h_n\}$ as follows. Put $g_1 = e$ and choose h_1 such that $d(F_1 h_1, g_1) \geq 1$. Next, choose g_2 and then h_2 such that

$$d\big(g_2, \{g_1\} \cup F_1 h_1\big) \geq 2, \qquad d\big(F_2 h_2, \{g_1, g_2\}\big) \geq 2.$$

Continue in this way. At stage n we find $g_n, h_n \in G$ such that

$$d\big(g_n, \{g_1, g_2, \dots, g_{n-1}\} \cup F_1 h_1 \cup \cdots \cup F_{n-1} h_{n-1}\big) \geq n,$$

$$d\big(F_n h_n, \{g_1, g_2, \dots, g_n\}\big) \geq n.$$

Then A has the desired properties.

8. Minimal G-spaces that are non-discrete and non-compact

We show here that each countable infinite group contains an infinite divisible subset which is not equivalent to an absorbing set. This, in turn, gives the existence of a minimal free action of the group on a non-compact and non-discrete locally compact Hausdorff space. This space can, moreover, be taken to be the non-compact locally compact Cantor set.

Before constructing the general examples, let us look at a concrete example:

Example 8.1. Let G be a group and let H be a subgroup of G. We know from Example 6.2 that H is of minimal type. It follows from Lemma 6.4 and Corollary 6.8 that each minimal closed invariant subset of X_H is of type $[H]$ (with respect to a suitable base point), and that all such minimal closed invariant sets are isomorphic as G-spaces.

We know from Example 7.7 that each infinite subgroup is infinitely divisible. We show below that no subgroup of infinite index is equivalent to an absorbing set. It follows that if H is an infinite subgroup of G of infinite index, then each minimal closed G-invariant subset of X_H is non-compact and non-discrete. Each of these invariant subsets of X_H is then a free minimal G-spaces, which is neither compact nor discrete. Not all infinite groups have an infinite subgroup of infinite index. The Tarski monsters, whose existence was proved by Olshanskii, have no proper infinite subgroups.

Let us show that no subgroup H of G of infinite index is equivalent to an absorbing set. We must show that FH is not absorbing for any finite subset F of G, i.e., that $\bigcap_{s\in S} sFH = \emptyset$ for some finite subset S of G. Write $F = \{f_1, f_2, \dots, f_n\}$. Since FH is contained in the union of finitely many left-cosets of H there exists $x \in G$ such that $FH \cap xH = \emptyset$. Put

$$S = \{e, f_1x^{-1}, f_2x^{-1}, \dots, f_nx^{-1}\}.$$

Note that $f_jx^{-1}FH \cap f_jH = \emptyset$. Hence

$$\bigcap_{s\in S} sFH \;\subseteq\; FH \cap \bigcap_{j=1}^{n} f_jx^{-1}FH \;\subseteq\; \bigcup_{j=1}^{n} \big(f_jH \cap f_jx^{-1}FH\big) \;=\; \emptyset,$$

as desired.

Theorem 8.2. *In any countably infinite group G there exists an infinitely divisible subset A which is not equivalent to an absorbing subset of G.*

Proof. Let $\{T_n\}_{n=1}^{\infty}$ be an increasing sequence of finite subsets of G whose union is G. Following the recipe of Remark 7.13 we will construct sequences $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ of elements in G, and sequences $\{A_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ of finite subsets of G as follows. For $n = 1$ choose a_1, c_1, d_1, F_1 and A_1 such that

(a) $a_1 \neq e$, $F_1 = \{e, a_1\}$, $c_1 = e$, $A_1 = F_1$, $d_1 \notin T_1A_1A_1^{-1}T_1^{-1}$.

This is clearly possible. For all $n \geq 1$, the elements $a_{n+1}, b_{n+1}, c_{n+1}$ and the sets A_{n+1} and F_{n+1} are constructed inductively so that they satisfy:

(b) $F_{n+1} = F_n \cup F_n\, a_{n+1}$ (disjoint union).

(c) $A_{n+1} = A_n \cup F_{n+1}\, c_{n+1} = \bigcup_{k=1}^{n+1} F_k\, c_k$ (disjoint unions).

(d) $d_k \notin T_kA_{n+1}A_{n+1}^{-1}T_k^{-1}$ for $1 \leq k \leq n+1$.

Given $n \geq 1$, and let us show how to find $a_{n+1}, c_{n+1}, d_{n+1}, F_{n+1}$ and A_{n+1} (when the previous elements and sets have been found).

Choose $a_{n+1} \in G$ outside the finite set

$$F_n^{-1}F_n \;\cup\; \bigcup_{k=1}^{n} F_n^{-1}T_k^{-1}\{d_k, d_k^{-1}\}T_kF_n,$$

and set $F_{n+1} = F_n \cup F_n\, a_{n+1}$. Then (b) holds. Moreover

$$d_k \notin T_k F_n \{a_{n+1}, a_{n+1}^{-1}\} F_n^{-1} T_k^{-1}, \quad 1 \le k \le n. \tag{8.1}$$

Since $F_n c_n$ is a subset of A_n, it follows from (d) that

$$d_k \notin T_k F_n F_n^{-1} T_k^{-1}, \quad 1 \le k \le n. \tag{8.2}$$

By Equations (8.1) and (8.2) and the definition of F_{n+1} it follows that

$$d_k \notin T_k F_{n+1} F_{n+1}^{-1} T_k^{-1}, \quad 1 \le k \le n. \tag{8.3}$$

Choose $c_{n+1} \in G$ outside the finite set

$$F_{n+1}^{-1} A_n \cup \bigcup_{k=1}^{n} \big(A_n^{-1} T_k^{-1} d_k T_k F_{n+1} \cup F_{n+1}^{-1} T_k^{-1} d_k T_k A_n \big),$$

and set $A_{n+1} = A_n \cup F_{n+1} c_{n+1}$. Then (c) holds, and

$$d_k \notin T_k F_{n+1} c_{n+1} A_n^{-1} T_k^{-1} \cup T_k A_n c_{n+1}^{-1} F_{n+1}^{-1} T_k^{-1}, \quad 1 \le k \le n. \tag{8.4}$$

It follows from Equations (8.2), (8.3) and (8.4) that (d) is satisfied for $1 \le k \le n$. Choose $d_{n+1} \in G$ outside the finite set $T_{n+1} A_{n+1} A_{n+1}^{-1} T_{n+1}^{-1}$. Then (d) holds also for $k = n+1$. This completes the construction of the sequences $\{a_n\}_{n=1}^\infty$, $\{c_n\}_{n=1}^\infty$, $\{d_n\}_{n=1}^\infty$, $\{A_n\}_{n=1}^\infty$ and $\{F_n\}_{n=1}^\infty$.

Set

$$A = \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty F_n\, c_n.$$

Then A is infinitely divisible by Remark 7.13. Condition (d) (and the fact that the A_n's are increasing) imply that d_k does not belong to $T_k A A^{-1} T_k^{-1}$. Hence, $T_k A \cap d_k T_k A = \emptyset$. This means that $T_k A$ is not absorbing. As every finite subset F of G is contained in T_k for some k we can conclude that A is not equivalent to an absorbing set. □

Corollary 8.3. *Every countable infinite group G admits a free minimal action on a totally disconnected locally compact Hausdorff space, which is neither discrete nor compact. The action can, moreover, be assumed to be amenable if G is exact.*

Proof. Let G be a given countable infinite group, and let $A \subseteq G$ be as in Theorem 8.2. Let Z be any minimal closed invariant subset of X_A. Then Z is non-compact by Proposition 7.2 (cf. [7, Proposition 5.5(iv)]), since A is not equivalent to an absorbing set; and Z is non-discrete by Theorem 7.10.

It follows from [7, Proposition 5.5] that Z is a locally compact totally disconnected Hausdorff space, and that G acts freely on Z. It further follows from [7, Proposition 5.5] that the action of G on Z is amenable if G is exact (essentially by Ozawa's result, that G acts amenably on βG when G is exact). □

Recall the definition of minimal types from Definition 6.1.

Corollary 8.4. *Each countable infinite group G contains minimal types different from $[G]$ (the compact type) and $[\{e\}]$ (the discrete type).*

Proof. Let Z be a locally compact Hausdorff space, which is neither discrete nor compact, on which G acts minimally (and freely). Pick any $z_0 \in Z$ and set $[A] = \mathrm{T}(G, Z, z_0)$. Then $[A]$ is of minimal type, and $[A] \neq [G]$ (because Z is non-compact) and $[A] \neq [\{e\}]$ (because Z is non-discrete). □

We wish to replace Z above with a second countable G-space, and need for this the following lemma:

Lemma 8.5. *Let G be a countable group, and let Z be a minimal locally compact free G-space. Let $\mathcal{A}$ be a non-zero G-invariant closed sub-C^*-algebra of $C_0(Z)$, and let $Y = \widehat{\mathcal{A}}$ denote the spectrum of $\mathcal{A}$ (so that $\mathcal{A} = C_0(Y)$).*

Then Y is a locally compact G-space, and the inclusion $\mathcal{A} \subseteq C_0(Z)$ arises from a surjective proper continuous G-map $\varphi \colon Z \to Y$. Moreover:

(i) *Y is non-compact if Z is non-compact.*
(ii) *The action of G on Y is minimal.*
(iii) *Y is non-discrete if the action of G on Y is free and Z is non-discrete.*

Proof. The spectrum Y of the commutative C^*-algebra $\mathcal{A}$ is a locally compact Hausdorff space; and the action of G on $\mathcal{A}$ comes from an action of G on Y. Since $\mathcal{A}$ is non-zero and G-invariant, and $C_0(Z)$ by assumption is G-simple, it follows that $\mathcal{A}$ is not contained in a proper ideal of $C_0(Z)$ (i.e., for each $z \in Z$ there exists $f \in \mathcal{A}$ such that $f(z) \neq 0$). The inclusion $\mathcal{A} \to C_0(Z)$ is therefore induced by a surjective proper continuous G-map $\varphi \colon Z \to Y$.

(i). If Y is compact, then $Z = \varphi^{-1}(Y)$ is compact because φ is proper.

(ii). If Y_0 is a closed G-invariant subspace of Y, then $\varphi^{-1}(Y_0)$ is a closed G-invariant subspace of Z. Hence $\varphi^{-1}(Y_0)$ is either empty or equal to Z, so Y_0 is either empty or equal to Y.

(iii). Suppose that Y is discrete and that G acts freely on Y. Let $z \in Z$. We claim that $G.z$ is closed in Z. Indeed, suppose that z_0 belongs to the closure of $G.z$. Then $g_\alpha.z \to z_0$ for some net $\{g_\alpha\}$ in G, whence $g_\alpha.\varphi(z) \to \varphi(z_0)$ in Y. As Y is discrete, this entails that $g_\alpha.\varphi(z)$ is eventually constant; and as G acts freely on Y we conclude that $\{g_\alpha\}$ is eventually constant. This, of course, implies that $z_0 \in G.z$.

Since Z is minimal, $G.z = Z$ for (some/any) $z \in Z$. Hence Z must have an isolated point by Baire's theorem, being a countable locally compact Hausdorff space, and therefore each point of Z is isolated. □

The Cantor set $\mathbf{K}$ is the unique compact Hausdorff space which is second countable, totally disconnected and without isolated points. There is also a unique *locally compact, non-compact* Hausdorff space which is second countable, totally disconnected and without isolated points, denoted $\mathbf{K}^*$ and referred to as the non-compact locally compact Cantor set. It arises, for example, by removing one point from $\mathbf{K}$, or as $\mathbf{K} \times \mathbb{N}$, or as the p-adic numbers.

It was shown in [7] that "most" countable infinite groups admit a free minimal action on $\mathbf{K}^*$. It remained in [7] an open question if *all* countable infinite groups admit such an action. We can now answer this question in the affirmative:

Theorem 8.6. *Every countable infinite group G admits a free minimal action on the non-compact locally compact Cantor set $\mathbf{K}^*$. If G is exact, then the action can, moreover, be taken to be amenable.*

For each $A \subseteq G$ of minimal type, such that $[A] \neq [G]$ and $[A] \neq [\{e\}]$, there is a free minimal action of G on $\mathbf{K}^*$ and $x_0 \in \mathbf{K}^*$ such that $\mathrm{T}(G, \mathbf{K}^*, x_0) = [A]$. Again, if G is exact, then the action can be taken to be amenable.

Proof. Let A be a subset of G of minimal type such that $[A] \neq [G]$ and $[A] \neq [\{e\}]$, whose existence is ensured by Corollary 8.4, and let Z be a minimal closed invariant subset of X_A. Then $\mathrm{T}(G, Z, z_0) = [A]$ for some $z_0 \in Z$ by Lemma 6.4. Observe that Z is non-compact and non-discrete by the assumption that $[A] \neq [G]$ and $[A] \neq [\{e\}]$, cf. Proposition 3.14 and Proposition 3.15.

This G-space, Z, satisfies all the axioms of the locally compact non-compact Cantor set, except it is not second countable. We shall replace Z by a G-equivariant quotient of Z which is second countable, and which retains all the other properties of the G-space Z.

We construct the quotient space of Z at the level of algebras. At this level we seek a separable G-invariant subalgebra $\mathcal{A}$ of $C_0(Z)$. The spectrum, $\widehat{\mathcal{A}}$, of $\mathcal{A}$ is then a second countable G-space. We arrange, moreover, that $\mathcal{A}$ is generated by a set of projections (this will imply that $\widehat{\mathcal{A}}$ is totally disconnected); and that the action of G on $\widehat{\mathcal{A}}$ is free, and also amenable if G is exact. It will then follow from Lemma 8.5 that $\widehat{A}$ is non-compact, non-discrete and minimal as a G-space.

We can therefore, once $\mathcal{A}$ has been constructed, identify $\widehat{\mathcal{A}}$ with $\mathbf{K}^*$, and we get a G-action on $\mathbf{K}^*$ (arising from the action of G on $\mathcal{A}$) with the desired properties. The quotient mapping $\varphi\colon Z \to \mathbf{K}^*$, which induces the inclusion $\mathcal{A} \subseteq C_0(Z)$, cf. Lemma 8.5, will be a proper continuous surjective G-map. It follows from Proposition 3.13 that $\mathrm{T}(G, \mathbf{K}^*, x_0) = \mathrm{T}(G, Z, z_0) = [A]$ when $x_0 = \varphi(z_0)$.

We construct the separable subalgebra $\mathcal{A}$ of $C_0(Z)$ following the ideas of [9] and [7]. The set-up here is a little different because we seek a quotient of the space Z, whereas in the mentioned references quotients of X_A were constructed.

Observe that $C_0(Z)$ has a countable approximate unit consisting of projections, call it $\{p_n\}_{n=1}^{\infty}$. Indeed, since Z is totally disconnected and Hausdorff it contains a compact-open subsets K. By minimality we must have $Z = \bigcup_{g \in G} g.K = \bigcup_{n=1}^{\infty} K_n$, when $K_n = \bigcup_{g \in F_n} g.K$ for some increasing sequence $\{F_n\}$ of finite subsets of G with $\bigcup F_n = G$. We can thus take $p_n = 1_{K_n} \in C_0(Z)$.

We use [7, Lemma 6.1] to construct $\mathcal{A}$ so the action of G on $\widehat{\mathcal{A}}$ becomes free. Since G acts freely on Z, we can apply the proof of [7, Lemma 6.1] (with respect to the approximate unit $\{p_n\}$) to obtain a countable subset M' of $C_0(Z)$ such that whenever $\mathcal{A}$ is a G-invariant sub-C^*-algebra of $C_0(Z)$ which contains $\{p_n\}$ and M', then G acts freely on $\widehat{\mathcal{A}}$.

Assume that G is exact. As in the proof of [7, Lemma 6.2], and using the fact that the action of G on Z is amenable, there is a countable subset M'' of $C_0(Z)$ such that the action of G on $\widehat{\mathcal{A}}$ is amenable whenever $\mathcal{A}$ contains M''.

It follows from [7, Proposition 6.6] or [9, Lemma 6.7] that there exists a countable G-invariant family, P, of projections in $C_0(Z)$, such that $\mathcal{A} := C^*(P)$ contains $\{p_n\} \cup M' \cup M''$ if G is exact, and $\{p_n\} \cup M'$ otherwise. This completes the proof. □

9. Removing the base point from the invariant

In this last section we investigate universal properties of locally compact G-spaces without making reference to a base point. This leads to a new order and equivalence relations on the power set of G, and we define the (base point free) type of a locally compact G-space in terms of this new equivalence relation. It turns out that a subset is of minimal type (it represents a minimal G-space) if and only if it is minimal with respect to this new ordering on $P(G)$. As a byproduct of these efforts we give in Corollary 9.15 an answer to the following question: Suppose that X_1 and X_2 are minimal locally compact G-spaces. When does there exists a minimal locally compact G-space X with surjective proper continuous G-maps $X \to X_j$ for $j = 1, 2$?

To motivate the discussion below, consider a (countable) group G and a locally compact G-space X with dense orbits. Let $K \subseteq X$ be a G-regular compact set, and let $x \in X$ be such that $G.x$ is dense in X. Then

$$\mathrm{T}(G, X, x) = [A], \quad \text{where } A = O_X(K, x),$$

by Proposition 3.11. If we replace the base point x with $g.x$, then $O_X(K, g.x) = O_X(K, x)g$, so $\mathrm{T}(G, X, g.x) = [Ag]$.

If $y \in X$ is an arbitrary point, then $g_n.x \to y$ for some sequence (or net) $\{g_n\}_{n=1}^\infty$ in G. We show below that $O_X(K, y)$ is (equivalent) to the limit of the sets $\{Ag_n\}$.

Let G be an infinite countable group. A sequence (or a net) $\{A_n\}$ of subsets of G converges to a subset $B \subseteq G$ if the functions 1_{A_n} tend to 1_B pointwise. In other words, $A_n \to B$ if and only if the identity $A_n \cap F = B \cap F$ holds eventually for any finite set F. We may identify 1_A with a point in $\{0, 1\}^G$, which is a Cantor set when equipped with the product topology. The pointwise convergence is exactly the same as the convergence with respect to this topology. In particular, any sequence (or net) $\{A_n\}_{n=1}^\infty$ has a convergent subsequence (or subnet), because $\{0, 1\}^G$ is a (metrizable) compact space.

Lemma 9.1. *Let G be a countable infinite group, let A and B be subsets of G, and let $\{g_n\}_{n=1}^\infty$ be a sequence in G. Suppose that $\{Ag_n\}_{n=1}^\infty$ converges to B.*

(i) *For any finite subset $F \subseteq G$, the sequence $\{FAg_n\}_{n=1}^\infty$ converges to FB.*
(ii) *Let A' and B' be subsets of G. If $A \approx A'$ and $\{A'g_n\}_{n=1}^\infty$ converges to B', then $B \approx B'$.*

Proof. (i). It is easy to see that hAg_n converges to hB for any $h \in G$. As

$$1_{FA} = 1_G \wedge \sum_{h \in F} 1_{hA}$$

for all finite subsets F of G and for all subsets A of G (where $\wedge$ denotes taking minimum), we see that

$$\lim_{n\to\infty} 1_{FAg_n} = \lim_{n\to\infty} \big(1_G \wedge \sum_{h\in F} 1_{hAg_n}\big) = 1_G \wedge \sum_{h\in F} 1_{hB} = 1_{FB}.$$

(ii). There exists a finite set $F \subseteq G$ such that $A' \subseteq FA$ and $A \subseteq FA'$. By (i), we have $FAg_n \to FB$. This, together with $1_{A'} \le 1_{FA}$, implies that $1_{B'} \le 1_{FB}$. Thus $B' \subseteq FB$. In the same way, we get $B \subseteq FB'$. Hence $B \approx B'$. $\square$

Definition 9.2. Let G be a countable infinite group, and let A and B be subsets of G. Write $A \succsim B$ if there exist a sequence $\{g_n\}_{n=1}^\infty$ in G and a subset B' of G such that $Ag_n \to B'$ and $B \approx B'$.

When $A \succsim B$ and $B \succsim A$, we write $A \sim B$.

Lemma 9.3. *Let A, B, C be subsets of G. If $A \succsim B$ and $B \succsim C$, then $A \succsim C$. In particular, the relation $\sim$ is an equivalence relation.*

Proof. Suppose that $A \succsim B$ and $B \succsim C$. Then there exist a sequence $\{g_n\}_{n=1}^\infty$ and a subset B' such that $Ag_n \to B'$ and $B \approx B'$; and there exist a sequence $\{h_n\}_{n=1}^\infty$ and a subset C' such that $Bh_n \to C'$ and $C \approx C'$.

The sequence $\{B'h_n\}_{n=1}^\infty$ has a convergent subsequence. By taking a subsequence of $\{h_n\}_{n=1}^\infty$, we may assume that $B'h_n \to C''$. By Lemma 9.1 we have $C' \approx C''$.

Write $G = \bigcup_{k=1}^\infty F_k$, where $\{F_k\}$ is an increasing sequence of finite subsets of G. Let $k \ge 1$. There exists $i \ge 1$ such that $B'h_i \cap F_k = C'' \cap F_k$. Since $Ag_n \to B'$, we have $Ag_nh_i \to B'h_i$, so there exists $j \ge 1$ such that $Ag_jh_i \cap F_k = B'h_i \cap F_k$. It follows that $Af_k \cap F_k = C'' \cap F_k$ when $f_k = g_jh_i$. Hence $Af_k \to C'' \approx C$, so $A \succsim C$. $\square$

Definition 9.4. For each countable infinite group G let $P_\sim(G)$ be the quotient space $P(G)/\sim$. For each $A \subseteq G$ denote by $\langle A\rangle$ its equivalence class in $P_\sim(G)$. Note that $P_\sim(G)$ becomes a partially ordered set when equipped with the order $\precsim$.

The lemma below follows easily from Definition 9.2 and from Lemma 9.1.

Lemma 9.5. *Let A and B be subsets of G.*

(i) *If $A \approx A'$ and $B \approx B'$, then $A \succsim B \implies A' \succsim B'$.*

(ii) $A \approx B \implies A \sim B$.

It follows from the lemma above that we have a natural surjection at the level of sets:

$$P_\approx(G) \to P_\sim(G), \qquad [A] \mapsto \langle A\rangle, \quad A \subseteq G.$$

This map is *not order preserving* because $A \propto B$ does not imply that $A \precsim B$. It is not even true that $A \subseteq B$ implies $A \precsim B$. If $A, B \subseteq G$, then we write $[A] \sim [B]$ if $A \sim B$.

Lemma 9.6. *Let X be a locally compact Hausdorff G-space, let $x \in X$, let $U \subseteq X$ be a relatively compact open subset such that $\bigcup_{g\in G} g.U = X$, and let $A := O_X(U, x)$ be as defined in (3.1). Assume that A is non-empty.*

(i) *If* $B \subseteq G$, *if* $\{g_\alpha\}$ *is a net in* G, *and if* $y \in X$ *satisfies* $g_\alpha.x \to y$ *and* $Ag_\alpha^{-1} \to B$, *then* $B \approx O_X(U, y)$.

(ii) *If* $y \in \overline{G.x}$, *then* $O_X(U, y) \precsim O_X(U, x)$.

(iii) *If* $B \subseteq G$ *is non-empty and* $B \precsim O_X(U, x)$, *then there exists* $y \in \overline{G.x}$ *such that* $O_X(U, y) \approx B$.

Proof. (i). First we show that $O_X(U, y) \subseteq B$. Let $h \in O_X(U, y)$. Then $h.y \in U$, so $hg_\alpha.x \in U$ eventually. Hence $hg_\alpha \in A$ eventually, so $h \in Ag_\alpha^{-1}$ eventually. This implies that $h \in B$.

Conversely, let $h \in B$. Then $hg_\alpha \in A$ eventually. It follows that $hg_\alpha.x \in U$ eventually, so $h.y$ is in $\overline{U}$. This shows that $B \subseteq O_X(\overline{U}, y)$. Since $\overline{U}$ is compact, and by the assumptions on U, there is a finite set $F \subseteq G$ such that $\overline{U}$ is contained in $\bigcup_{g \in F} g.U = F.U$. Hence, by Lemma 3.6,

$$B \subseteq O_X(\overline{U}, y) \subseteq O_X(F.U, y) = F.O_X(U, y) \propto O_X(U, y).$$

(ii). There exists a net $\{g_\alpha\}$ in G such that $g_\alpha.x \to y$. By taking a subsequence if necessary, we may assume that $Ag_\alpha^{-1} \to B$ for some subset B of G. It follows from (i) that $B \approx O_X(U, y)$. Hence (ii) holds.

(iii). There exist a net $\{g_\alpha\}$ in G and $B' \subseteq G$ such that $Ag_\alpha \to B' \approx B$. Since B is non-empty, B' is also non-empty. Take $h \in B'$. Then hg_α^{-1} is in A eventually, and so $g_\alpha^{-1}.x \in h.U$ eventually. By passing to a subnet of $\{g_\alpha\}$ we may assume that $g_\alpha^{-1}.x \to y$ for some $y \in h.\overline{U} \subseteq X$. It now follows from (i) that $B' \approx O_X(U, y)$, which completes the proof. ☐

Theorem 9.7. *Let* G *be a countable infinite group and let* X *be a locally compact Hausdorff space on which* G *acts co-compactly. Suppose that* $x, y \in X$ *are such that* $G.x$ *and* $G.y$ *are dense in* X. *It follows that*

$$\mathrm{T}(G, X, x) \sim \mathrm{T}(G, X, y).$$

Proof. Choose a G-regular compact subset $K \subseteq X$. Then

$$\mathrm{T}(G, X, x) = [O_X(K^{\mathrm{o}}, x)], \qquad \mathrm{T}(G, X, y) = [O_X(K^{\mathrm{o}}, y)].$$

by Proposition 3.11. By the assumption that $G.x$ and $G.y$ are dense in X, it follows from Lemma 9.6 (ii) that $O_X(K^{\mathrm{o}}, x) \sim O_X(K^{\mathrm{o}}, y)$. ☐

Because of Theorem 9.7 we can make the following definition:

Definition 9.8. Let X be a locally compact Hausdorff space on which a countable infinite group G acts co-compactly, and such that $G.x$ is dense in X for some $x \in X$. Then set

$$\mathscr{T}(G, X) = \langle A \rangle \in P_\sim(G),$$

where $A \subseteq G$ is such that $\mathrm{T}(G, X, x) = [A]$.

Proposition 9.9. *Let* G *be a countable infinite group and let* X *and* Y *be locally compact Hausdorff spaces on which* G *acts co-compactly with a dense orbit. If there is a surjective proper continuous* G*-map* $\varphi\colon X \to Y$, *then* $\mathscr{T}(G, X) = \mathscr{T}(G, Y)$.

Proof. Let $x_0 \in X$ be such that $G.x_0$ is dense in X and set $y_0 = \varphi(x_0)$. Being continuous and surjective, φ maps dense sets onto dense sets, it follows that $G.y_0 = \varphi(G.x_0)$ is dense in Y. By Proposition 3.13 we have $\mathrm{T}(G, X, x_0) = \mathrm{T}(G, Y, y_0)$. This implies that $\mathscr{T}(G, X) = \mathscr{T}(G, Y)$. □

Recall that $A \subseteq G$ is of minimal type if $[A]$ is the type of some pointed minimal locally compact G-space. We give an intrinsic description of subsets of minimal type in terms of the ordering $\precsim$ on $P(G)$:

Proposition 9.10. *Let G be a countable infinite group and let $A \subseteq G$. Then A is of minimal type if and only if $\langle A \rangle$ is a minimal point in the partially ordered set $(P_\sim(G), \precsim)$.*

In other words, A is of minimal type if and only if whenever $B \subseteq G$ satisfies $B \precsim A$, then $B \sim A$.

Proof. Suppose first that A is of minimal type and let (X, x_0) be a minimal pointed locally compact G-space with $\mathrm{T}(G, X, x_0) = [A]$. Let $K \subseteq X$ be a G-regular compact subset of X.

Suppose that $B \subseteq G$ and $B \precsim A$. It then follows from Lemma 9.6 (iii) that $O_X(K^{\mathrm{o}}, y) \approx B$ for some $y \in X$. Since X is minimal we know that $G.y$ is dense in X. It therefore follows from Theorem 9.7 and Proposition 3.11 that

$$[A] = \mathrm{T}(G, X, x_0) \sim \mathrm{T}(G, X, y) = [O_X(K^{\mathrm{o}}, y)] = [B].$$

Suppose, conversely, that $\langle A \rangle$ is a minimal point in $(P_\sim(G), \precsim)$. Consider the G-space $X_A \subseteq \beta G$, and take a minimal closed invariant subset Z of X_A. Take $z \in Z$. Set $U = K_A \cap Z$, which is a compact open subset of Z. Since the G-orbit of e is dense in X_A, it follows from Lemma 9.6 (ii) that

$$A = O_{X_A}(K_A, e) \succsim O_{X_A}(K_A, z) = O_Z(U, z).$$

Hence $A \sim O_Z(U, z)$, because A is minimal. By Lemma 9.6 (iii), we can find $w \in Z$ such that $O_Z(U, w) \approx A$. It follows from Proposition 3.11 that $\mathrm{T}(G, Z, w) = [A]$, so A is of minimal type. □

Each element of $P_\sim(G)$ dominates a minimal element, and the order relation "$\precsim$" on $P_\sim(G)$ corresponds to the order by inclusion on invariant subsets of any locally compact G-space:

Corollary 9.11. *Let G be a countable group, let A be a non-empty subset of G, and let X be a locally compact space on which G acts co-compactly with a dense orbit such that $\mathscr{T}(G, X) = \langle A \rangle$.*

It follows that the map $Y \mapsto \mathscr{T}(G, Y)$ is an order preserving surjection from the set of closed G-invariant singly generated [4] *subsets of X, ordered by inclusion, onto the ordered set*

$$\{\alpha \in \mathscr{T}(G, X) \mid \alpha \precsim \langle A \rangle\}.$$

In particular, for each non-empty subset A of G there exists a non-empty subset B of G of minimal type such that $B \precsim A$.

[4] We say that a closed invariant subset Y of a G-space X is singly generated if $Y = \overline{G.y}$ for some $y \in X$, i.e., if there is a dense orbit.

Proof. Let $G.x$ be a dense orbit in X. Let K be a G-regular compact subset of X. Then $\mathrm{T}(G,X,x) = [O_X(K^{\mathrm{o}},x)]$ by Proposition 3.11, so $O_X(K^{\mathrm{o}},x) \sim A$.

Suppose that Y is a singly generated closed invariant subsets of X, with dense orbit $G.y$, where $y \in X = \overline{G.x}$. Then $O_X(K^{\mathrm{o}},y) \precsim O_X(K^{\mathrm{o}},x)$ by Lemma 9.6 (ii). Moreover, $K \cap Y$ is a G-regular compact subset of Y whose relative interior is $K^{\mathrm{o}} \cap Y$; and

$$O_X(K^{\mathrm{o}},y) = O_X(K^{\mathrm{o}} \cap Y, y)$$

by G-invariance of Y. Hence $\mathrm{T}(G,Y,y) = [O_X(K^{\mathrm{o}},y)]$ by Proposition 3.11, so $\mathscr{T}(G,Y) = \langle O_X(K^{\mathrm{o}},y)\rangle \precsim \langle A\rangle$.

If $Y_1 \subseteq Y_2$ are singly generated closed invariant subsets of X, then $\mathscr{T}(G,Y_1) \precsim \mathscr{T}(G,Y_2)$ by the first part of this proof (with Y_2 in the place of X).

If $B \precsim A$, then there exists $y \in X$ such that $O_X(K^{\mathrm{o}},y) \approx B$ by Lemma 9.6 (iii). Arguing as above, using that $K \cap \overline{G.y}$ is G-regular in $\overline{G.y}$, we see that

$$\mathrm{T}(G,\overline{G.y},y) = [O_X(K^{\mathrm{o}} \cap \overline{G.y}, y)] = [O_X(K^{\mathrm{o}},y)] \approx B.$$

Hence $\mathscr{T}(G,\overline{G.y}) = \langle B\rangle$.

To prove the last claim, let A be any non-empty subset of G. Then there exists a co-compact locally compact G-space X with $\mathscr{T}(G,X) = \langle A\rangle$, for example $X = X_A$. The co-compact locally compact G-space X has a minimal closed invariant subset Y. Let $B \subseteq G$ be such that $\mathscr{T}(G,Y) = \langle B\rangle$. Then B is of minimal type and $B \precsim A$. □

The example and the proposition below illustrate some properties of the the relation "$\precsim$".

Example 9.12. Let G be a countable group.

(i). If $A \subseteq G$, then $G \precsim A$ if and only if A is equivalent to an absorbing set.

It is not difficult to prove this directly. It also follows from Corollary 9.11. Indeed, consider the G-space X_A which is of type $\langle A\rangle$. Then $G \precsim A$ if and only if there is a closed G-invariant singly generated subset Y of X_A of type $\langle G\rangle$. Now, Y is of type $\langle G\rangle$ if and only if Y is compact, cf. Proposition 3.14. It follows from [7, Proposition 5.5] that X_A has a compact minimal closed invariant subspace if and only if A is equivalent to an absorbing set.

(ii). If $A \subseteq G$, then $\{e\} \precsim A$ if and only if A is not infinitely divisible.

This follows from Corollary 9.11 in a similar way as above. A minimal closed G-invariant subset $Z \subseteq X_A$ is of type $\langle\{e\}\rangle$ if and only if Z is discrete, cf. Proposition 3.15; and X_A has a discrete minimal closed invariant subset if and only if A is not infinitely divisible by Theorem 7.10.

It follows from the proposition below that if A and B are non-empty subsets of G such that $A \sim B$, then A is paradoxical if and only if B is paradoxical.

Proposition 9.13. *Let G be a countable infinite group, and let A be a non-empty subset of G. Then the following conditions are equivalent:*

(i) *A is non-paradoxical.*

(ii) *$B \precsim A$ for some non-paradoxical non-empty subset B of G.*

(iii) *Any locally compact Hausdorff space X, on which G acts co-compactly and with $\mathscr{T}(G,X) = \langle A\rangle$, admits a non-zero G-invariant Radon measure.*

Proof. (i) $\Rightarrow$ (ii) is trivial (take $B = A$).

(i) $\Leftrightarrow$ (iii) follows from Corollary 5.3 and Definition 9.8.

(ii) $\Rightarrow$ (i). Note first that if C and D are subsets of G such that $C \approx D$, then C is non-paradoxical if and only if D is non-paradoxical. Indeed, if μ is any finitely additive left-invariant measure on $P(G)$, then

$$0 < \mu(C) < \infty \iff 0 < \mu(D) < \infty,$$

whenever $C \approx D$.

It therefore suffices to show that if $\{g_n\}$ is a sequence in G such that $Ag_n \to B$, and if B is non-paradoxical, then A is non-paradoxical. Let μ be a left-invariant measure on $P(G)$ such that $\mu(B) = 1$.

Choose a free ultrafilter ω on $\mathbb{N}$. We can then take the limit of any sequence $\{C_n\}_{n=1}^{\infty}$ of subsets of G along ω as follows:

$$\lim_{\omega} C_n = C \iff \forall x \in G : \lim_{\omega} 1_{C_n}(x) = 1_C(x).$$

If $C_n \to C$, then $\lim_\omega C_n = C$.

Define $\phi\colon P(G) \to P(G)$ by $\phi(E) = \lim_\omega Eg_n$. In other words,

$$\forall E \subseteq G \ \forall x \in G : 1_{\phi(E)}(x) = \lim_{\omega} 1_{Eg_n}(x). \tag{9.1}$$

Then $\phi(A) = B$. One can use (9.1) to verify that the following holds:

- $\phi(\emptyset) = \emptyset$.
- $\phi(C \cup D) = \phi(C) \cup \phi(D)$ for all $C, D \subseteq G$.
- $\phi(C \cap D) = \phi(C) \cap \phi(D)$ for all $C, D \subseteq G$.
- $\phi(gC) = g\,\phi(C)$ for all $g \in G$ and $C \subseteq G$.

It follows that $\nu := \mu \circ \phi$ is a finitely additive left-invariant measure on $P(G)$ with $\nu(A) = \mu(\phi(A)) = \mu(B) = 1$. Hence (i) holds. $\square$

With the base point free notion of type we can reinterpret Definition 6.3 as follows:

Proposition 9.14. *Let G be a countable infinite group and let A be a non-empty subset of G of minimal type. A locally compact minimal G-space Z is then a universal minimal G-space of type A (in the sense of Definition 6.3) if and only if for each minimal locally compact G-space X with $\mathscr{T}(G,X) = \mathscr{T}(G,Z)$ there is a surjective proper continuous G-map $\varphi\colon Z \to X$.*

Because of this result it makes sense to refer to Z as being a *universal minimal G-space* (without reference to the subset A of G), as the class of this space is contained in Z.

Proof. Choose $x_0 \in X$, and let $B \subseteq G$ be such that $\mathrm{T}(G,X,x_0) = [B]$. Then $A \sim B$ because $\mathscr{T}(G,X) = \langle A\rangle$. Let K be a G-regular compact subset of X. Then $[B] = [O_X(K^{\circ},x_0)]$ by Proposition 3.11. It follows in particular that $A \precsim O_X(K^{\circ},x_0)$. We can therefore use Lemma 9.6 (iii) to find x_1 in X such that $O_X(K^{\circ}, x_1) \approx A$. Minimality of X ensures that $G.x_1$ is dense in X. Hence $\mathrm{T}(G, X, x_1) = [A]$ by Proposition 3.11. The existence of a surjective proper continuous G-map $\varphi\colon Z \to X$ now follows from the universal property of Z.

Conversely, if $\varphi\colon Z \to X$ is a proper continuous G-map, then $\mathscr{T}(G, X) = \mathscr{T}(G, Z) = \langle A \rangle$ by Proposition 9.9. □

Recall from Proposition 6.5 that every closed minimal G-invariant subset of X_A is a universal minimal G-space of type A, whenever A is a subset of G of minimal type. Hence universal minimal G-spaces always exist, and they are also unique (by Theorem 6.7).

Corollary 9.15. *Let $(X_i)_{i \in I}$ be a family of minimal locally compact G-spaces. Then the following conditions are equivalent:*

(i) $\mathscr{T}(G, X_i) = \mathscr{T}(G, X_j)$ *for all $i, j \in I$.*

(ii) *There exist a minimal locally compact G-space Z and a (surjective) proper continuous G-map $\varphi_i\colon Z \to X_i$ for each $i \in I$.*

(iii) *There exists $x_i \in X_i$ for each $i \in I$ such that*

$$\mathrm{T}(G, X_i, x_i) = \mathrm{T}(G, X_j, x_j)$$

for all $i, j \in I$.

Proof. (i) ⇒ (ii). Fix $i_0 \in I$ and let A be a (necessarily minimal) non-empty subset of G such that $\mathscr{T}(G, X_{i_0}) = \langle A \rangle$. Take a universal minimal G-space Z of type A, cf. Definition 6.3 and Proposition 6.5. Then $\mathscr{T}(G, Z) = \langle A \rangle = \mathscr{T}(G, X_i)$ for all $i \in I$ by Lemma 6.4 and Definition 9.8, and so it follows from Proposition 9.14 that Z satisfies the conditions in (ii).

(ii) ⇒ (iii). Suppose that Z and $\varphi_i\colon Z \to X_i$ are as in (ii). Pick any point $z_0 \in Z$, and set $x_i = \varphi(z_0)$. Then

$$\mathrm{T}(G, X_i, x_i) = \mathrm{T}(G, Z, z_0)$$

by Proposition 3.13.

(iii) ⇒ (i) follows from Definition 9.8 (together with Theorem 9.7). □

The corollary above is stated in a rather general form, but is perhaps of particular interest when the family $(X_i)_{i \in I}$ has just two elements. Consider the relation on minimal locally compact G-spaces X_1 and X_2, that there exists a third G-space Z that maps surjectively onto X_1 and X_2 by continuous G-maps. By the corollary, this happens if and only if $\mathscr{T}(G, X_1) = \mathscr{T}(G, X_2)$. In particular, this defines an equivalence relation on the class of minimal locally compact G-spaces. A priori, it was not clear to the authors that this relation is transitive.

We end by stating two questions:

Question 9.16. Given a non-empty subset A (not necessarily of minimal type) of a (countable infinite) group G. Is it true that each minimal closed G-invariant subset of X_A is a universal minimal G-space (in the sense defined below Proposition 9.14)?

In other words, are all minimal closed G-invariant subsets of any co-compact open G-invariant subset of βG a universal space? If yes, then these spaces are classified by their type by Theorem 6.7.

Question 9.17. Given a non-empty subset A (not necessarily of minimal type) of a (countable infinite) group G. Classify the (base point free) types of the minimal closed G-invariant subsets of X_A in terms of the set A.

If A is of minimal type, then each minimal closed G-invariant subsets of X_A also has type A (by Proposition 6.5). We also know when X_A has minimal closed G-invariant subspaces that are discrete (of type $\langle\{e\}\rangle$), respectively, compact (of type $\langle G\rangle$).

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Graduate School of Science, Chiba University, Inage-ku, Chiba 263-8522, Japan

E-mail address: matui@math.s.chiba-u.ac.jp

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark

E-mail address: rordam@math.ku.dk