The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras

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Abstract

Suppose that A is a C^*-algebra for which $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang–Su algebra: a unital, simple, stably finite, separable, nuclear, infinite dimensional C^*-algebra with the same Elliott invariant as the complex numbers. We show that:

(i) The Cuntz semigroup $W(A)$ of equivalence classes of positive elements in matrix algebras over A is almost unperforated[1].

(ii) If A is exact, then A is purely infinite if and only if A is traceless.

(iii) If A is separable and nuclear, then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is traceless.

(iv) If A is simple and unital, then the stable rank of A is one if and only if A is finite.

We also characterise when A is of real rank zero.

1 Introduction

Jiang and Su gave in their paper [12] a classification of simple inductive limits of direct sums of dimension drop C^*-algebras. (A dimension drop C^*-algebra is a certain sub-C^*-algebra of $M_n(C([0,1]))$, a precise definition of which is given in the next section.) They prove that inside this class there exists a unital, simple, infinite dimensional C^*-algebra $\mathcal{Z}$ whose Elliott invariant is isomorphic to the Elliott invariant of the complex numbers, that is,

$$(K_0(\mathcal{Z}), K_0(\mathcal{Z})^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \qquad K_1(\mathcal{Z}) = 0, \qquad T(\mathcal{Z}) = \{\tau\},$$

[1]Almost perforation is a natural extension of the notion of weak unperforation for *simple* ordered abelian (semi-)groups, see Section 3.

where τ is the unique tracial state on $\mathcal{Z}$. They proved that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \bigotimes_{n=1}^{\infty} \mathcal{Z}$, and that $A \otimes \mathcal{Z} \cong A$ if A is a simple, unital, infinite dimensional AF-algebra or if A is a unital Kirchberg algebra. Toms and Winter have in a paper currently under preparation extended the latter result by showing that $A \otimes \mathcal{Z} \cong A$ for all approximately divisible C^*-algebras. Toms and Winter note that upon combining results from [12] with [16, Theorem 8.2] one obtains that a separable C^*-algebra A is $\mathcal{Z}$-absorbing if and only if there is a unital embedding of $\mathcal{Z}$ into the relative commutant $\mathcal{M}(A)_\omega \cap A'$, where $\mathcal{M}(A)_\omega$ is the ultrapower, associated with a free filter ω on $\mathbb{N}$, of the multiplier algebra, $\mathcal{M}(A)$, of A. This provides a partial answer to the question raised by Gong, Jiang, and Su in [9] if one can give an intrinsic description of which (separable, nuclear) C^*-algebras absorb $\mathcal{Z}$.

Gong, Jiang, and Su prove in [9] that $(K_0(A), K_0(A)^+) \cong (K_0(A \otimes \mathcal{Z}), K_0(A \otimes \mathcal{Z})^+)$ if and only if $K_0(A)$ is weakly unperforated as an ordered group, when A is a simple C^*-algebra; and hence that A and $A \otimes \mathcal{Z}$ have isomorphic Elliott invariant if A is simple with weakly unperforated K_0-group. This result indicates that $A \cong A \otimes \mathcal{Z}$ whenever A is "classifiable" in the sense of Elliott (see Elliott, [8], or [23] by the author).

The results quoted above show on the one hand that surprisingly many C^*-algebras, including for example the irrational rotation C^*-algebras, absorb the Jiang–Su algebra, but on the other hand that not all simple, unital, nuclear, separable C^*-algebras are $\mathcal{Z}$-absorbing. Villadsen's example from [27] of a simple, unital AH-algebra whose K_0-group is not weakly unperforated cannot absorb $\mathcal{Z}$. The example by the author in [24] of a simple, unital, nuclear, separable C^*-algebra with a finite and an infinite projection is prime (i.e., is not the tensor product of two non type I C^*-algebras), and is hence not $\mathcal{Z}$-absorbing. Toms gave in [25] an example of a simple, unital ASH-algebra which is not $\mathcal{Z}$-absorbing, but which has weakly unperforated K_0-group. The latter two examples (by the author and by Toms) have the same Elliott invariant as, but are not isomorphic to, their $\mathcal{Z}$-absorbing counterparts; and so they serve as counterexamples to the classification conjecture of Elliott (as it is formulated in [23, Section 2.2]).

It appears plausible that the Elliott conjecture holds for all simple, unital, nuclear, separable $\mathcal{Z}$-absorbing C^*-algebras.

In the present paper we begin by showing that the Cuntz semigroup of equivalence classes of positive elements in a $\mathcal{Z}$-absorbing C^*-algebra is almost unperforated (a property that for simple ordered abelian (semi-)groups coincides with the weak unperforation property, see Section 3). We use this to show that the semigroup $V(A)$ of Murray–von Neumann equivalence classes of projections in a $\mathcal{Z}$-absorbing C^*-algebra A, and in some cases also $K_0(A)$, is almost (or weakly) unperforated. We show that the stable rank of A is one if A is a simple, finite, unital $\mathcal{Z}$-absorbing C^*-algebra, thus answering in the affirmative a question from [9]. In the last section we characterise when a simple unital $\mathcal{Z}$-absorbing C^*-algebra is of real rank zero.

2 Preliminary facts about the Jiang–Su algebra $\mathcal{Z}$

We establish a couple of results that more or less follow directly from Jiang and Su's paper [12] on their C^*-algebra $\mathcal{Z}$.

For each triple of natural numbers n, n_0, n_1, for which n_0 and n_1 divides n, the *dimension drop C^*-algebra* $I(n_0, n, n_1)$ is the sub-C^*-algebra of $C([0,1], M_n)$ consisting of all functions f such that $f(0) \in \varphi_0(M_{n_0})$ and $f(1) \in \varphi_1(M_{n_1})$, where $\varphi_j \colon M_{n_j} \to M_n$, $j = 0, 1$, are fixed unital *-homomorphisms. (The C^*-algebra $I(n_0, n, n_1)$ is—up to *-isomorphism—independent on the choice of the *-homomorphisms φ_j.) The dimension drop C^*-algebra $I(n_0, n, n_1)$ is said to be *prime* if n_0 and n_1 are relatively prime and $n = n_0 n_1$. If $I(n_0, n, n_1)$ is prime, then it has no projections other than the two trivial ones: 0 and 1, cf. [12].

The C^*-algebra $I(n, nm, m)$ can, and will in this paper, be realized as the sub-C^*-algebra of $C([0,1], M_n \otimes M_m)$ consisting of those functions f for which $f(0) \in M_n \otimes \mathbb{C}$ and $f(1) \in \mathbb{C} \otimes M_m$.

A unital *-homomorphism $\psi \colon I(n_0, n, n_1) \to \mathcal{Z}$ will here be said to be *standard*, if

$$\tau(\psi(f)) = \int_0^1 \operatorname{tr}(f(t))\, dm(t), \qquad f \in I(n_0, n, n_1), \tag{2.1}$$

where τ is the unique trace on $\mathcal{Z}$, and where tr is the normalised trace on M_n.

The following theorem is essentially contained in Jiang and Su's paper ([12]).

Theorem 2.1 (Jiang–Su) *Let n, n_0, n_1 be a triple of natural numbers where n_0 and n_1 are relatively prime and $n = n_0 n_1$. As above, let τ denote the unique trace on $\mathcal{Z}$.*

(i) *For each faithful tracial state τ_0 on $I(n_0, n, n_1)$ there exists a unital embedding $\psi \colon I(n_0, n, n_1) \to \mathcal{Z}$ such that $\tau \circ \psi = \tau_0$. In particular, there is a standard unital embedding of $I(n_0, n, n_1)$ into $\mathcal{Z}$.*

(ii) *Two unital embeddings $\psi_1, \psi_2 \colon I(n_0, n, n_1) \to \mathcal{Z}$ are approximately unitarily equivalent if and only if $\tau \circ \psi_1 = \tau \circ \psi_2$. In particular, ψ_1 and ψ_2 are approximately unitarily equivalent if they both are standard.*

Proof: For brevity, denote the prime dimension drop C^*-algebra $I(n_0, n, n_1)$ by I.

Find an increasing sequence $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ of sub-C^*-algebras of $\mathcal{Z}$ such that each B_k is (isomorphic to) a prime dimension drop algebra of the form $I(n_0(k), n(k), n_1(k))$, and such that $\bigcup_{k=1}^\infty B_k$ is dense in $\mathcal{Z}$. Simplicity of $\mathcal{Z}$ ensures that $n_0(k)$, $n(k)$, and $n_1(k)$ all tend to infinity as k tends to infinity.

It is shown in [12, Lemma 2.3] that $K_0(I)$ and $K_0(B_k)$ are infinite cyclic groups each generated by the class of the unit in the corresponding algebra, and $K_1(I)$ and $K_1(B_k)$ are both trivial. This entails that $KK(\psi_1) = KK(\psi_2)$ for any pair of *unital* *-homomorphisms $\psi_1, \psi_2 \colon I \to B_k$.

In both parts of the proof we shall apply the uniqueness theorem, [12, Corollary 5.6], in Jiang and Su's paper. For each $x \in [0,1]$ consider the extremal tracial state τ_x on a dimension drop C^*-algebra $I(m_0, m, m_1)$ given by $\tau_x(f) = \mathrm{tr}(f(x))$ (where tr is the normalised trace on M_m). Each self-adjoint element f in a dimension drop C^*-algebra $I(m_0, m, m_1)$ gives rise to a function $\hat{f} \in C_{\mathbb{R}}([0,1])$ defined by $\hat{f}(x) = \tau_x(f)$. If f is a self-adjoint element in the center of $I(m_0, m, m_1)$, then $\hat{f} = f$. Every *-homomorphism $\psi \colon I(m_0, m, m_1) \to I(m_0', m', m_1')$ between two dimension drop C^*-algebras induces a positive linear mapping $\psi_* \colon C_{\mathbb{R}}([0,1]) \to C_{\mathbb{R}}([0,1])$ given by $\psi_*(f) = \widehat{\psi(f)}$ (when we identify $C_{\mathbb{R}}([0,1])$ with the self-adjoint part of the center of $I(m_0, m, m_1)$). Let $h_{D,d} \in C_{\mathbb{R}}([0,1])$, $d = 1, 2, \ldots, D$, be the test functions defined in [12, (5.5)] (and previously considered by Elliott).

(i). Let τ_0 be a faithful trace on $I = I(n_0, n, n_1)$. Let $F_1 \subseteq F_2 \subseteq \cdots$ be an increasing sequence of finite subsets of I with dense union. By a one-sided approximate intertwining argument (after Elliott, see eg. [17, Theorem 1.10.14]) it suffices to find a sequence $1 \leq m(1) < m(2) < m(3) < \cdots$ of integers, a sequence $\psi_j \colon I \to B_{m(j)}$ of unital *-homomorphisms, and unitaries $u_j \in B_{m(j)}$ such that

$$\|u_{j+1}^* \psi_j(f) u_{j+1} - \psi_{j+1}(f)\| \leq 2^{-j}, \qquad |\tau(\psi_j(f)) - \tau_0(f)| \leq 1/j, \qquad f \in F_j,$$

for all $j \in \mathbb{N}$. It will then follow that there exist a *-homomorphism $\psi \colon I \to \mathcal{Z}$ and unitaries $v_j \in \mathcal{Z}$ such that $\|v_j^* \psi_j(f) v_j - \psi(f)\|$ tends to zero as j tends to infinity for all $f \in I$. This will imply that $\tau \circ \psi = \tau_0$.

For each j choose a natural number D_j such that $\|f(s) - f(t)\| \leq 2^{-j}$ for all $f \in F_j$ and for all $s, t \in [0,1]$ with $|s - t| \leq 6/D_j$. Let G_j be the finite set that contains F_j and the test functions $h_{D_j,d}$, $d = 1, \ldots, D_j$. Put

$$c_j = \tfrac{2}{3} \min \big\{ \tau_0\big(h_{D_j,d-1} - h_{D_j,d}\big) \mid d = 2, 3, \ldots, D_j \big\} > 0.$$

By [12, Corollary 4.4] — if $m(j)$ are chosen large enough — there exists for each j a unital *-homomorphism $\psi_j\colon I \to B_{m(j)}$ such that $|\tau'(\psi_j(f))-\tau_0(f)| < \min\{1/j, c_j/2, c_{j-1}/2\}$ for all tracial states τ' on $B_{m(j)}$ and for all $f \in G_j$. In particular, $|\tau(\psi_j(f)) - \tau_0(f)| < 1/j$ for $f \in F_j$, and

$$(\psi_j)_*(h_{D_j,d-1} - h_{D_j,d}) \geq c_j, \qquad (\psi_{j+1})_*(h_{D_j,d-1} - h_{D_j,d}) \geq c_j,$$
$$\|(\psi_{j+1})_*(h_{D_j,d}) - (\psi_j)_*(h_{D_j,d})\|_\infty < c_j,$$

for $d = (1), 2, 3, \dots, D_j$. It now follows from [12, Corollary 5.6] that there exists a unitary u_{j+1} in $B_{m(j+1)}$ such that $\|u_{j+1}^* \psi_j(f) u_{j+1} - \psi_{j+1}(f)\| \leq 2^{-j}$.

(ii). The "only if" part is trivial. Assume that $\tau \circ \psi_1 = \tau \circ \psi_2$. Take a finite subset F of I and let $\varepsilon > 0$.

It is shown in [12] (and in [6]) that the dimension drop C^*-algebra $I = I(n_0, n, n_1)$ is semiprojective. We can therefore, for some large enough k_0, find unital *-homomorphisms $\psi_1^{(k)}$, $\psi_2^{(k)}\colon I \to B_k$ for each $k \geq k_0$ such that

$$\lim_{k\to\infty} \|\psi_j(f) - \psi_j^{(k)}(f)\| = 0, \qquad f \in I, \;\; j = 1, 2. \tag{2.2}$$

We assert that

$$\lim_{k\to\infty} \|(\psi_j^{(k)})_*(h) - (\tau \circ \psi_j)(h)\mathbf{1}\|_\infty = 0, \qquad j = 1, 2, \quad h \in C_{\mathbb{R}}([0,1]), \tag{2.3}$$

when we identify $C_{\mathbb{R}}([0,1])$ with the self-adjoint portion of the center of $I = I(n_0, n, n_1)$. Indeed, because τ is the unique trace on $\mathcal{Z}$, the quantity

$$\sup_{\tau' \in T(B_k)} |\tau'(b) - \tau(b)|, \qquad b \in B_\ell,$$

tends to zero as k tends to infinity (with $k \geq \ell$). Hence, if we let $\iota_{k,\ell}$ denote the inclusion mapping $B_\ell \to B_k$, then $\|(\iota_{k,\ell})_*(h) - \tau(h)\mathbf{1}\|_\infty$ tends to zero as k tends to infinity ($k \geq \ell$). It follows from (2.2) that $\|(\iota_{k,\ell} \circ \psi_j^{(\ell)})_*(h) - (\psi_j^{(k)})_*(h)\|_\infty$ is small if ℓ is large (and $k \geq \ell$). The claim in (2.3) follows from these facts and the identity $(\iota_{k,\ell} \circ \psi_j^{(\ell)})_* = (\iota_{k,\ell})_* \circ (\psi_j^{(\ell)})_*$.

Choose an integer D such that $\|f(s) - f(t)\| < \varepsilon/9$ for all $f \in F$ and for all $s, t \in [0,1]$ with $|s - t| \leq 6/D$. Each $\psi_j(h_{D,d-1} - h_{D,d})$ is a non-zero and positive element in $\mathcal{Z}$, and we can therefore find $c > 0$ such that $(\tau \circ \psi_j)(h_{D,d-1} - h_{D,d}) \geq 2c$ for $d = 2, 3, \dots, D$ and $j = 1, 2$. Use (2.2) and the assumption $\tau \circ \psi_1 = \tau \circ \psi_2$ to find $k \geq k_0$ such that

$\|\psi_j(f) - \psi_j^{(k)}(f)\| < \varepsilon/3$ and

$$(\psi_j^{(k)})_*(h_{D,d-1} - h_{D,d}) \geq c, \qquad \|(\psi_1^{(k)})_*(h_{D,d}) - (\psi_2^{(k)})_*(h_{D,d})\|_\infty < c,$$

for all $f \in F$, for $d = (1), 2, \dots, D$, and for $j = 1, 2$. It then follows from [12, Corollary 5.6] that there is a unitary element u in B_k such that $\|\psi_2^{(k)}(f) - u^*\psi_1^{(k)}(f)u\| \leq \varepsilon/3$ for all $f \in F$; whence $\|\psi_2(f) - u^*\psi_1(f)u\| \leq \varepsilon$ for all $f \in F$. This proves that ψ_1 and ψ_2 are approximately unitarily equivalent. □

For any natural numbers n and m let $E(n,m)$ be the C^*-algebra that consists of all functions f in $C([0,1], M_{n^\infty} \otimes M_{m^\infty})$ for which $f(0) \in M_{n^\infty} \otimes \mathbb{C}$ and $f(1) \in \mathbb{C} \otimes M_{m^\infty}$.

Proposition 2.2 *There is a unital embedding of $E(n,m)$ into $\mathcal{Z}$ for every pair of natural numbers n, m that are relatively prime.*

Proof: For each k there is a unital embedding $\sigma_k \colon M_{n^k} \otimes M_{m^k} \to M_{n^{k+1}} \otimes M_{m^{k+1}}$ which satisfies

$$\sigma_k(M_{n^k} \otimes \mathbb{C}) \subseteq M_{n^{k+1}} \otimes \mathbb{C}, \qquad \sigma_k(\mathbb{C} \otimes M_{m^k}) \subseteq \mathbb{C} \otimes M_{m^{k+1}}.$$

Thus $f \mapsto \sigma_k \circ f$ defines a *-homomorphism $\rho_k \colon I(n^k, n^k m^k, m^k) \to I(n^{k+1}, n^{k+1}m^{k+1}, m^{k+1})$, and $E(n,m)$ is the inductive limit of the sequence

$$I(n,nm,m) \xrightarrow{\rho_1} I(n^2, n^2m^2, m^2) \xrightarrow{\rho_2} I(n^3, n^3m^3, m^3) \xrightarrow{\rho_3} \cdots \longrightarrow E(n,m).$$

Take standard unital embeddings $\psi_k \colon I(n^k, n^k m^k, m^k) \to \mathcal{Z}$ (cf. Theorem 2.1 (i)). Then ψ_k and $\psi_{k+1} \circ \rho_k$ are both standard unital embedding of $I(n^k, n^k m^k, m^k)$ into $\mathcal{Z}$, so they are approximately unitarily equivalent by Theorem 2.1 (ii). We obtain the desired embedding of $E(n,m)$ into $\mathcal{Z}$ from this fact combined with a one-sided approximate intertwining (after Elliott), see for example [17, Theorem 1.10.14]. □

3 Almost unperforation

Consider an ordered abelian semigroup $(W, +, \leq)$. An element $x \in W$ is called *positive* if $y + x \geq y$ for all $y \in W$, and W is said to be positive if all elements in W are positive. If W has a zero-element 0, then W is positive if and only if $0 \leq x$ for all $x \in W$. An abelian semigroup equipped with the *algebraic order*: $x \leq y$ iff $y = x + z$ for some $z \in W$, is positive.

Definition 3.1 A positive ordered abelian semigroup W is said to be *almost unperforated* if for all $x, y \in W$ and all $n, m \in \mathbb{N}$, with $nx \le my$ and $n > m$, one has $x \le y$.

Let W be a positive ordered abelian semigroup. Write $x \propto y$ if x, y are elements in W and $x \le ny$ for some natural number n (i.e., x belongs to the ideal in W generated by the element y). The element y is said to be an *order unit* for W if $x \propto y$ for all $x \in W$. For each positive element x in W let $S(W, x)$ be the set of order preserving additive maps $f \colon W \to [0, \infty]$ such that $f(x) = 1$. Although we shall not use this fact, we mention that $S(W, x)$ is non-empty if and only if for all natural numbers n and m, with $nx \le mx$, one has $n \le m$. This follows from [4, Corollary 2.7] and the following observation that also will be used in the proof of the proposition below. For any element $x \in W$ the set $W_0 = \{z \in W \mid z \propto x\}$ is an order ideal in W, and x is an order unit for W_0. Moreover, any state f in $S(W_0, x)$ extends to a state $\overline{f}$ in $S(W, x)$ by setting $\overline{f}(z) = \infty$ for $z \in W \setminus W_0$.

Proposition 3.2 *Let W be a positive ordered abelian semigroup. Then W is almost unperforated if and only if the following condition holds: For all elements x, y in W, with $x \propto y$ and $f(x) < f(y)$ for all $f \in S(W, y)$, one has $x \le y$.*

Proof: Following the argument above we can—if necessary by passing to an order ideal of W—assume that y is an order unit for W. The "only if" part now follows from [22, Proposition 3.1], which again uses Goodearl and Handelman's extension result [10, Lemma 4.1].

To prove the "if" part, take elements $x, y \in W$ and $n \in \mathbb{N}$ such that $(n+1)x \le ny$. Then $x \propto y$ because $x \le (n+1)x \le ny$; and $f(x) \le n(n+1)^{-1} < 1 = f(y)$ for all $f \in S(W, y)$, whence $x \le y$. □

Definition 3.3 An ordered abelian group (G, G^+) is said to be *almost unperforated* if for all $g \in G$ and for all $n \in \mathbb{N}$, with $ng, (n+1)g \in G^+$, one has $g \in G^+$.

Lemma 3.4 *Let (G, G^+) be an ordered abelian group. Then G is almost unperforated if and only if the positive semigroup G^+ is almost unperforated.*

Proof: Suppose that G is almost unperforated and that $x, y \in G^+$ satisfy $(n+1)x \le ny$ for some natural number n. Then $n(y - x) \ge x \ge 0$ and $(n+1)(y - x) \ge y \ge 0$, whence $y - x \ge 0$ and $y \ge x$. Conversely, suppose that G^+ is almost unperforated and that $ng, (n+1)g \in G^+$ for some $n \in \mathbb{N}$. Since $(n+1)ng = n(n+1)g$, we get $ng \le (n+1)g$, which implies that $g = (n+1)g - ng$ belongs to G^+. □

A *simple* ordered abelian group is almost unperforated if and only if it is weakly unperforated. Indeed, if $n \in \mathbb{N}$ and $g \in G$ are such that $ng \in G^+ \setminus \{0\}$, then, by simplicity of

G, there is a natural number k such that $kng \geq g$. Thus $(kn-1)g$ and kng are positive, so g is positive if G is almost unperforated (cf. Lemma 3.4). Conversely, if G is weakly unperforated and $ng, (n+1)g \in G^+$, then $g \in G^+$ if $ng \neq 0$, and $g = (n+1)g \in G^+$ if $ng = 0$.

Elliott considered in [7] a notion of what he called weak unperforation of (non-simple) ordered abelian groups with torsion. (We have refrained from using the term "weak unperforation" in Definition 3.3 to avoid conflict with Elliott's definition.) A torsion free group is weakly unperforated in the sense of Elliott if and only if it is unperforated: $ng \geq 0$ implies $g \geq 0$ for all group elements g and for all natural numbers n. The group $G = \mathbb{Z}^2$ with the positive cone generated by the three elements $(1,0), (0,1), (2,-2)$ is torsion free with perforation: $(2,-2) \in G^+$ but $(1,-1) \notin G^+$, so it is not weakly unperforated in the sense of Elliott. However, the group (G, G^+) is almost unperforated, as the reader can verify.

In the converse direction, any weakly unperforated group is almost unperforated. Indeed, if G is weakly unperforated and $g \in G$ and $n \in \mathbb{N}$ are such that $ng, (n+1)g$ are positive, then g is positive modulo torsion, i.e., $g+t$ is positive for some $t \in G_{\text{tor}}$. Let $k \in \mathbb{N}$ be the order of t, find natural numbers ℓ_1, ℓ_2 such that $N = \ell_1 n + \ell_2(n+1)$ is congruent with -1 modulo k. Then $Ng = Ng + (N+1)t$ is positive, whence $-t \leq N(g+t)$, which by the hypothesis of weak unperforation implies that $g = (g+t) + (-t)$ is positive.

4 Weak and almost unperforation of $\mathcal{Z}$-absorbing C^*-algebras

Cuntz associates in [5] to each C^*-algebra A a positive ordered abelian semigroup $W(A)$ as follows. Let $M_\infty(A)^+$ denote the (disjoint) union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \operatorname{diag}(a,b) \in M_{n+m}(A)^+$, and write $a \precsim b$ if there is a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Put $W(A) = M_\infty(A)^+/\sim$, and let $\langle a \rangle \in W(A)$ be the equivalence class containing a (so that $W(A) = \{\langle a \rangle \mid a \in M_\infty(A)^+\}$). Then $W(A)$ is a positive ordered abelian semigroup when equipped with the relations:

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle, \qquad \langle a \rangle \leq \langle b \rangle \iff a \precsim b, \qquad a, b \in M_\infty(A)^+.$$

Following the standard convention, for each positive element $a \in A$ and for each $\varepsilon \geq 0$, write $(a-\varepsilon)_+$ for the positive element in A given by $h_\varepsilon(a)$, where $h_\varepsilon(t) = \max\{t-\varepsilon, 0\}$. We recall below some facts about the comparison of two positive elements a, b in a C^*-algebra

A (see [5, Proposition 1.1] and [22, Section 2]):

(a) $a \precsim b$ if and only if $(a-\varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.

(b) $a \precsim b$ if and only if for each $\varepsilon > 0$ there exists $x \in A$ such that $x^*bx = (a-\varepsilon)_+$.

(c) If $\|a-b\| < \varepsilon$, then $(a-\varepsilon)_+ \precsim b$.

(d) $\big((a-\varepsilon_1)_+ - \varepsilon_2\big)_+ = \big(a-(\varepsilon_1+\varepsilon_2)\big)_+$.

(e) $a+b \precsim a \oplus b$; and if $a \perp b$, then $a+b \sim a \oplus b$.

If a belongs to the closed two-sided ideal, $\overline{AbA}$, generated by b, then $(a-\varepsilon)_+$ belongs to the algebraic two-sided ideal, AbA, generated by b for all $\varepsilon > 0$, in which case $(a-\varepsilon)_+ = \sum_{i=1}^n x_i^* b x_i$ for some $n \in \mathbb{N}$ and some $x_i \in A$. This shows that

$$a \in \overline{AbA} \iff \forall \varepsilon > 0 \exists n \in \mathbb{N} : \langle (a-\varepsilon)_+ \rangle \le n\langle b \rangle. \tag{4.1}$$

Lemma 4.1 *Let A and B be two C^*-algebras, let $a, a' \in A$ and $b, b' \in B$ be positive elements, and let n, m be natural numbers.*

(i) *If $n\langle a \rangle \le m\langle a' \rangle$ in $W(A)$, then $n\langle a \otimes b \rangle \le m\langle a' \otimes b \rangle$ in $W(A \otimes B)$.*

(ii) *If $n\langle b \rangle \le m\langle b' \rangle$ in $W(B)$, then $n\langle a \otimes b \rangle \le m\langle a \otimes b' \rangle$ in $W(A \otimes B)$.*

Proof: (i). Assume that $n\langle a \rangle \le m\langle a' \rangle$ in $W(A)$. Then there is a sequence $x_k = \{x_k(i,j)\}$ in $M_{m,n}(A)$ such that $x_k^*(a' \otimes 1_m)x_k \to a \otimes 1_n$ (or, equivalently, such that $\sum_{l=1}^m x_k(l,i)^* a' x_k(l,j) \to \delta_{ij} a$ for all $i,j = 1,\dots,n$). Let $\{e_k\}$ be a sequence of positive contractions in B such that $e_k b e_k \to b$. Put $y_k(i,j) = x_k(i,j) \otimes e_k \in A \otimes B$, and put $y_k = \{y_k(i,j)\} \in M_{m,n}(A \otimes B)$. Then $y_k^*((a' \otimes b) \otimes 1_m)y_k \to (a \otimes b) \otimes 1_n$ (or, equivalently, $\sum_{l=1}^m y_k(l,i)^*(a' \otimes b)y_k(l,j) \to \delta_{ij}(a \otimes b)$ for all $i,j = 1,\dots,n$). This shows that $n\langle a \otimes b \rangle \le m\langle a' \otimes b \rangle$.

(ii) follows from (i) by symmetry. □

Lemma 4.2 *For all natural numbers n there exists a positive element e_n in $\mathcal{Z}$ such that $n\langle e_n \rangle \le \langle 1_{\mathcal{Z}} \rangle \le (n+1)\langle e_n \rangle$.*

Proof: By Theorem 2.1 (a fact which follows easily from from Jiang and Su's paper [12]) the C^*-algebra $I = I(n, n(n+1), n+1)$ admits a unital embedding into $\mathcal{Z}$, so it suffices to find a positive element e_n in I such that $n\langle e_n \rangle \le \langle 1_I \rangle \le (n+1)\langle e_n \rangle$ in $W(I)$.

The idea of the proof is simple (but verifying the details requires some effort): There are positive functions $f_1, f_2, \dots, f_{n+1}$ in I such that

(i)

$$f_i(0) = \begin{cases} e_{ii}^{(n)} \otimes 1, & i = 1, \dots, n, \\ 0, & i = n+1, \end{cases} \qquad f_i(t) = 1 \otimes e_{ii}^{(n+1)}, \ \ t \in [1/2, 1],$$

(where $\{e_{ij}^{(m)}\}_{i,j=1}^m$ denotes the canonical set of matrix units for $M_m(\mathbb{C})$),

(ii) $f_1, f_2, \dots, f_n$ are pairwise orthogonal,

(iii) $\sum_{i=1}^{n+1} f_i = 1$, and

(iv) $f_{n+1} \precsim f_1 \sim f_2 \sim \cdots \sim f_n$.

It will follow from (iii) and (e) that $\sum_{i=1}^{n+1} \langle f_i \rangle \geq \langle 1 \rangle$; and (ii) and (e) imply that $\sum_{i=1}^{n} \langle f_i \rangle \leq \langle 1 \rangle$. It therefore follows from (iv) that $e_n = f_1$ has the desired property.

We proceed to construct the functions $f_1, \dots, f_{n+1}$. Put

$$W = \sum_{i,j=1}^{n} e_{ij}^{(n)} \otimes e_{ji}^{(n+1)} + 1 \otimes e_{n+1,n+1}^{(n+1)}. \tag{4.2}$$

Then W is a self-adjoint unitary element in $M_n \otimes M_{n+1}$, and

$$W(1 \otimes e_{ii}^{(n+1)})W^* = e_{ii}^{(n)} \otimes (1 - e_{n+1,n+1}^{(n+1)}) \leq e_{ii}^{(n)} \otimes 1, \tag{4.3}$$

for $1 \leq i \leq n$. Choose a continuous path of unitaries $t \mapsto V_t$ in $M_n \otimes M_{n+1}$, $t \in [0,1]$, such that $V_0 = 1$ and $V_t = W$ for $t \in [1/2, 1]$. Put $W_t = V_t W$. Choose a continuous path $t \mapsto \gamma_t \in [0,1]$ such that $\gamma_0 = 0$ and $\gamma_t = 1$ for $t \in [1/2, 1]$. Define $f_i \colon [0,1] \to M_n \otimes M_{n+1}$ by

$$\begin{aligned} f_i(t) &= \gamma_t W_t (1 \otimes e_{ii}^{(n+1)}) W_t^* + (1 - \gamma_t) V_t (e_{ii}^{(n)} \otimes 1) V_t^*, \qquad i = 1, \dots, n, \\ f_{n+1}(t) &= \gamma_t W_t (1 \otimes e_{n+1,n+1}^{(n+1)}) W_t^*, \end{aligned}$$

where $t \in [0,1]$. It is easy to check that (i) and (iii) above hold. From (i) we see that all f_i belong to I. Use (4.3) to see that $f_i(t) \leq V_t(e_{ii}^{(n)} \otimes 1)V_t^*$ for all t and all $i = 1, \dots, n$, and use again this to see that (ii) holds.

We proceed to show that (iv) holds. Let $S \in M_n$ and $T \in M_{n+1}$ be the permutation unitaries for which $S^{i-1} e_{11}^{(n)} S^{-(i-1)} = e_{ii}^{(n)}$ and $T^{i-1} e_{11}^{(n+1)} T^{-(i-1)} = e_{ii}^{(n+1)}$ for $i = 1, \dots, n, (n+1)$. Put

$$R_i(t) = V_t (S^{i-1} \otimes 1) V_t^*, \qquad t \in [0, 1/2], \ i = 1, \dots, n.$$

Brief calculations show that $R_i(0) = S^{i-1} \otimes 1 \in M_n \otimes \mathbb{C}$, $f_i(t) = R_i(t)f_1(t)R_i(t)^*$ for $t \in [0, 1/2]$, and

$$\begin{aligned} R_i(1/2)(1 \otimes e_{11}^{(n+1)})R_i(1/2)^* &= R_i(1/2)f_1(1/2)R_i(1/2)^* = f_i(1/2) = 1 \otimes e_{ii}^{(n+1)} \\ &= (1 \otimes T^{(i-1)})(1 \otimes e_{11}^{(n+1)})(1 \otimes T^{-(i-1)}), \end{aligned}$$

for $i = 1, \dots, n$. The unitary group of the relative commutant $M_n \otimes M_{n+1} \cap \{1 \otimes e_{11}^{(n+1)}\}'$ is connected, so we can extend the paths $t \mapsto R_i(t)$, $t \in [0, 1/2]$, to continuous paths $t \mapsto R_i(t)$, $t \in [0, 1]$, such that $R_i(1) = 1 \otimes T^{-(i-1)} \in \mathbb{C} \otimes M_{n+1}$ and $R_i(t)f_1(t)R_i(t)^* = 1 \otimes e_{ii}^{(n+1)} = f_i(t)$ for $t \in [1/2, 1]$ and for $i = 2, \dots, n$. Thus R_i is a unitary element in I and $R_i f_1 R_i^* = f_i$ for $i = 2, \dots, n$. This proves that $f_1 \sim f_2 \sim \cdots \sim f_n$.

We must also show that $f_{n+1} \precsim f_1$. Let $g_i \in I$ be given by $g_i(t) = \gamma_t W_t(1 \otimes e_{ii}^{(n+1)})W_t^*$ for $i = 1, \dots, n+1$ (so that $f_{n+1} = g_{n+1}$). A calculation then shows that $R_i g_1 R_i^* = g_i$ so that $g_1, \dots, g_n$ are unitarily equivalent. By symmetry, $f_{n+1} = g_{n+1}$ is unitarily equivalent to g_1, and as $g_1 \le f_1$, we conclude that $f_{n+1} \precsim f_1$. □

Lemma 4.3 *Let A be any C^*-algebra, and let a, a' be positive elements in A for which $(n+1)\langle a \rangle \le n\langle a' \rangle$ in $W(A)$ for some natural number n. Then $\langle a \otimes 1_{\mathcal{Z}} \rangle \le \langle a' \otimes 1_{\mathcal{Z}} \rangle$ in $W(A \otimes \mathcal{Z})$.*

Proof: Take e_n in $\mathcal{Z}$ as in Lemma 4.2. Then, by Lemma 4.1,

$$\langle a \otimes 1_{\mathcal{Z}} \rangle \le (n+1)\langle a \otimes e_n \rangle \le n\langle a' \otimes e_n \rangle \le \langle a' \otimes 1_{\mathcal{Z}} \rangle$$

in $W(A \otimes \mathcal{Z})$. □

Lemma 4.4 *Let A be a $\mathcal{Z}$-absorbing C^*-algebra. Then there is a sequence of isomorphisms $\sigma_n \colon A \otimes \mathcal{Z} \to A$ such that*

$$\lim_{n \to \infty} \|\sigma_n(a \otimes 1) - a\| = 0, \qquad a \in A.$$

Proof: It is shown in [12] that $\mathcal{Z}$ is isomorphic to $\bigotimes_{k=1}^{\infty} \mathcal{Z}$. We may therefore identify A with $A \otimes (\bigotimes_{k=1}^{\infty} \mathcal{Z})$. With this identification we define $\sigma_n \colon A \otimes (\bigotimes_{k=1}^{\infty} \mathcal{Z}) \otimes \mathcal{Z} \to A \otimes (\bigotimes_{k=1}^{\infty} \mathcal{Z})$ to be the isomorphism that fixes A and the first n copies of $\mathcal{Z}$, which sends the last copy of $\mathcal{Z}$ to the copy of $\mathcal{Z}$ at position "$n+1$", and which shifts the remaining copies of $\mathcal{Z}$ one place to the right. □

Theorem 4.5 *Let A be a $\mathcal{Z}$-absorbing C^*-algebra. Then $W(A)$ is almost unperforated.*

Proof: Let a, a' be positive elements in $M_\infty(A)$ for which $(n+1)\langle a\rangle \leq n\langle a'\rangle$. Upon replacing A by a matrix algebra over A (which still is $\mathcal{Z}$-absorbing) we may assume that a and a' both belong to A. Let $\varepsilon > 0$. It follows from Lemma 4.3 that $a \otimes 1 \precsim a' \otimes 1$ in $A \otimes \mathcal{Z}$, so there exists $x \in A \otimes \mathcal{Z}$ with $\|x^*(a' \otimes 1)x - a \otimes 1\| < \varepsilon$. Let $\sigma_k \colon A \otimes \mathcal{Z} \to A$ be as in Lemma 4.4, and put $x_k = \sigma_k(x)$. Then $\|x_k^*\sigma_k(a' \otimes 1)x_k - \sigma_k(a \otimes 1)\| < \varepsilon$, whence $\|x_k^* a' x_k - a\| < \varepsilon$ if k is chosen large enough. This shows that $\langle a\rangle \leq \langle a'\rangle$ in $W(A)$. $\square$

A C^*-algebra A, where $W(A)$ is almost unperforated, has nice comparability properties, as we shall proceed to illustrate in the remaining part of this section, and in the later sections of this paper.

We recall a few facts about *dimension function*, introduced by Cuntz in [5]. A dimension function on a C^*-algebra A is an additive order preserving function $d \colon W(A) \to [0,\infty]$. (We can also regard d as a function $M_\infty(A)^+ \to [0,\infty]$ that respects the rules $d(a \oplus b) = d(a) + d(b)$ and $a \precsim b \Rightarrow d(a) \leq d(b)$ for all $a, b \in M_\infty(A)^+$.) A dimension function d is said to be *lower semi-continuous* if $d = \overline{d}$, where

$$\overline{d}(a) \overset{\text{def}}{=} \lim_{\varepsilon \to 0+} d((a-\varepsilon)_+), \qquad a \in M_\infty(A)^+. \tag{4.4}$$

Moreover, $\overline{d}$ is a lower semi-continuous dimension function on A for each dimension function d, cf. [22, Proposition 4.1]. Note that $d((a-\varepsilon)_+) \leq \overline{d}(a) \leq d(a)$ for every dimension function d and for every $\varepsilon > 0$, and that $\overline{d}(p) = d(p)$ for every projection p.

By an *extended trace* on a C^*-algebra A we shall mean a function $\tau \colon A^+ \to [0,\infty]$ which is additive, homogeneous, and has the trace property: $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. If τ is an extended trace on A, then

$$d_\tau(a) \overset{\text{def}}{=} \lim_{\varepsilon \to 0+} \tau(f_\varepsilon(a)) \;\; (= \lim_{n \to \infty} \tau(a^{1/n})), \quad a \in M_\infty(A)^+, \tag{4.5}$$

where $f_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $f_\varepsilon(t) = \min\{\varepsilon^{-1}t, 1\}$, defines a lower semi-continuous dimension function on A. If A is exact, then every lower semi-continuous dimension function on A is of the form d_τ for some extended trace τ on A. (This follows from Blackadar and Handelman, [2, Theorem II.2.2], who show that one can lift d to a quasitrace, and from Haagerup, [11], and Kirchberg, [14], who show that quasitraces are traces on exact C^*-algebras).

With the characterization of lower semi-continuous dimension functions above and with Theorem 4.5 at hand one can apply the proof of [22, Theorem 5.2] to obtain:

Corollary 4.6 *Let A be a C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be a $\mathcal{Z}$-absorbing C^*-algebra), and suppose in addition that A is exact, simple and unital. Let a, b be positive elements in A. If $d_\tau(a) < d_\tau(b)$ for every tracial state τ on A, then $a \precsim b$.*

We also have the following "non-simple" version of the result above.

Corollary 4.7 *Let A be a C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be a $\mathcal{Z}$-absorbing C^*-algebra). Let a, b be positive elements in A. Suppose that a belongs to $\overline{AbA}$ and that $d(a) < d(b)$ for every dimension function d on A with $d(b) = 1$. Then $a \precsim b$.*

Proof: It follows from (4.1) that $\langle (a-\varepsilon)_+ \rangle \propto \langle b \rangle$ in $W(A)$ for each $\varepsilon > 0$; and by assumption, $d(\langle (a-\varepsilon)_+ \rangle) \le d(\langle a \rangle) < d(\langle b \rangle)$ for every $d \in S(W(A), \langle b \rangle)$. Thus $\langle (a-\varepsilon)_+ \rangle \le \langle b \rangle$ by Proposition 3.2, and this proves the corollary as $\varepsilon > 0$ was arbitrary. □

Gong, Jiang and Su proved in [9] that the K_0-group of a simple unital $\mathcal{Z}$-absorbing C^*-algebra is weakly unperforated. At the level of semigroups, we can extend this result to the non-simple case, as explained below.

Let $V(A)$ denote the semigroup of Murray-von Neumann equivalence classes of projections in matrix algebras over A equipped with the algebraic order: $x \le y$ if there exists z such that $y = x + z$. The relation "$\precsim$", defined in the beginning of this section, agrees with the usual comparison relation when applied to projections p and q, i.e., $p \precsim q$ if and only if p is equivalent to a subprojection of q. The corollary below is thus an immediate consequence of Theorem 4.5.

Corollary 4.8 *The semigroup $V(A)$ is almost unperforated for every $\mathcal{Z}$-absorbing C^*-algebra A.*

It follows from Lemma 3.4 that if A is a stably finite C^*-algebra with an approximate unit consisting of projections, then $K_0(A)$ is almost unperforated if and only if $K_0(A)^+$ is almost unperforated. It seems plausible that $K_0(A)^+$ is almost unperforated whenever $V(A)$ is almost unperforated; and this is trivially the case when $V(A)$ has the cancellation property. This implication also holds when $V(A)$ is simple. Indeed, let $\gamma\colon V(A) \to K_0(A)$ be the Grothendieck map, so that $K_0(A)^+ = \gamma(V(A))$. Take $x, y \in V(A)$ and $n \in \mathbb{N}$ such that $(n+1)\gamma(x) \le n\gamma(y)$. Then $(n+1)x + u \le ny + u$ for some $u \in V(A)$. Repeated use of this inequality yields $N(n+1)x + u \le Nny + u$ for all natural numbers N. That $V(A)$ is simple means that every non-zero element, and hence y, is an order unit for $V(A)$, so there is a natural number k with $u \le ky$. Now, $N(n+1)x \le N(n+1)x + u \le Nny + u \le (Nn+k)y$, which for $N \ge k+1$ yields $x \le y$ and hence $\gamma(x) \le \gamma(y)$.

We thus have the following result, that slightly extends [9, Theorem 1].

Corollary 4.9 *Let A be a stably finite $\mathcal{Z}$-absorbing C^*-algebra with an approximate unit consisting of projections. If $V(A)$ has the cancellation property or if $V(A)$ is simple, then $K_0(A)$ is almost unperforated.*

Corollary 4.10 *Let A be an exact C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be an exact $\mathcal{Z}$-absorbing C^*-algebra). Let p, q be projections in A such that p belongs to $\overline{AqA}$. Suppose that $\tau(p) < \tau(q)$ for every extended trace τ on A with $\tau(q) = 1$. Then $p \precsim q$.*

Proof: We show that $d(p) < d(q)$ for every dimension function d on A with $d(q) = 1$, and the result will then follow from Corollary 4.7. Let $\overline{d}$ be the lower semi-continuous dimension function associated with d in (4.4). As remarked above, by Haagerup's theorem on quasitraces, $\overline{d} = d_\tau$ for some extended trace τ, cf. (4.5). Because d, $\overline{d}$ and τ agree on projections, we have $\tau(q) = d(q) = 1$ and $d(p) = \tau(p) < \tau(q) = d(q)$ as desired. □

5 Applications to purely infinite C^*-algebras

In this short section we derive two results that say when a $\mathcal{Z}$-absorbing C^*-algebra is purely infinite and $\mathcal{O}_\infty$-absorbing. Similar results were obtained in [16] for approximately divisible C^*-algebras. An exact C^*-algebra is said to be *traceless* when it admits no extended trace (see Section 3) that takes values other than 0 and ∞.

Corollary 5.1 *Let A be an exact C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be an exact $\mathcal{Z}$-absorbing C^*-algebra). Then A is purely infinite if and only if A is traceless.*

Proof: Note first that A, being traceless, can have no abelian quotients. Each lower semi-continuous dimension function on A arises from an extended trace on A (as remarked above Corollary 4.6), and must therefore take values in $\{0, \infty\}$, again because A is traceless.

Take positive elements a, b in A such that a belongs to $\overline{AbA}$. We must show that $a \precsim b$ (cf. [15]), and it suffices to show that $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$. Take $\varepsilon > 0$. Let d be a dimension function on A such that $d(b) = 1$ (if such a dimension function exists), and let $\overline{d}$ be its associated lower semi-continuous dimension function, cf. (4.4). Then $\overline{d}(b) = 0$ (as remarked above). Use (4.1) to see that $\overline{d}((a-\varepsilon/2)_+) = 0$, and hence that $d((a-\varepsilon)_+) = 0$. Thus $(a-\varepsilon)_+ \precsim b$ by Corollary 4.7. □

Theorem 5.2 *Let A be a nuclear separable $\mathcal{Z}$-absorbing C^*-algebra. Then A absorbs $\mathcal{O}_\infty$ (i.e., $A \cong A \otimes \mathcal{O}_\infty$) if and only if A is traceless.*

Proof: If A absorbs $\mathcal{O}_\infty$, then A is traceless (see [16, Theorem 9.1]). Suppose conversely that A is traceless. We then know from Corollary 5.1 that A is purely infinite. It follows from [16, Theorem 9.1] (in the unital or the stable case) and from [13, Corollary 8.1] (in the general case) that A absorbs $\mathcal{O}_\infty$ if A is *strongly purely* infinite, cf. [16, Definition 5.1].

We need therefore only show that any $\mathcal{Z}$-absorbing purely infinite C^*-algebra is strongly purely infinite. Let

$$\begin{pmatrix} a & x \\ x^* & b \end{pmatrix} \in M_2(A)^+,$$

and let $\varepsilon > 0$ be given. Take pairwise orthogonal non-zero positive contractions h_1, h_2 in the simple C^*-algebra $\mathcal{Z}$. Then

$$a \otimes 1 \in \overline{(A\otimes\mathcal{Z})(a\otimes h_1)(A\otimes\mathcal{Z})}, \qquad b\otimes 1 \in \overline{(A\otimes\mathcal{Z})(b\otimes h_2)(A\otimes\mathcal{Z})}.$$

Since $A \otimes \mathcal{Z} \cong A$ is purely infinite there are elements c_1, c_2 in $A \otimes \mathcal{Z}$ such that

$$\|c_1^*(a\otimes h_1)c_1 - a\otimes 1\| < \varepsilon, \qquad \|c_2^*(b\otimes h_2)c_2 - b\otimes 1\| < \varepsilon.$$

Let $\sigma_n \colon A\otimes\mathcal{Z} \to A$ be as in Lemma 4.4, and put $d_{1,n} = \sigma_n((1\otimes h_1^{1/2})c_1)$ and $d_{2,n} = \sigma_n((1\otimes h_2^{1/2})c_2)$. Then

$$\|d_{1,n}^*\sigma_n(a\otimes 1)d_{1,n} - \sigma_n(a\otimes 1)\| < \varepsilon, \qquad \|d_{2,n}^*\sigma_n(b\otimes 1)d_{2,n} - \sigma_n(b\otimes 1)\| < \varepsilon,$$
$$d_{2,n}^*\sigma_n(x\otimes 1)d_{1,n} = 0.$$

The norm of $d_{j,n}$ does not dependent on n. Thus, if we take $d_1 = d_{1,n}$ and $d_2 = d_{2,n}$ for some large enough n, then we obtain the desired estimates: $\|d_1^*ad_1 - a\| < \varepsilon$, $\|d_2^*bd_2 - b\| < \varepsilon$, and $\|d_2^*xd_1\| < \varepsilon$. □

6 The stable rank of $\mathcal{Z}$-absorbing C^*-algebras

We shall in this section show that simple, finite $\mathcal{Z}$-absorbing C^*-algebras have stable rank one.

Definition 6.1 A unital C^*-algebra A is said to be *strongly K_1-surjective* if the canonical mapping $\mathcal{U}(A_0 + \mathbb{C}1_A) \to K_1(A)$ is surjective for every full hereditary sub-C^*-algebra A_0 of A. If the canonical mapping $\mathcal{U}(A_0 + \mathbb{C}1_A)/\mathcal{U}_0(A_0 + \mathbb{C}1_A) \to K_1(A)$ is injective for every full hereditary sub-C^*-algebra A_0, then we say that A is *strongly K_1-injective.*

Note that we do not assume simplicity in the two next lemmas.

Lemma 6.2 *Every full hereditary sub-C^*-algebra in a unital approximately divisible C^*-algebra contains a full projection.*

Proof: Let B be a full hereditary sub-C^*-algebra of a unital approximately divisible C^*-algebra A. Take a full positive element b in B. Then $n\langle b\rangle \geq \langle 1_A\rangle$ in $W(A)$ for some natural number n (by (4.1)). Since A is approximately divisible, there is a unital embedding of $M_{n+1} \oplus M_{n+2}$ into A, and, as shown in [26], A is $\mathcal{Z}$-absorbing. (We shall only apply this lemma in the case where A is the tensor product of a unital C^*-algebra with a UHF-algebra, and in this case we can conclude that A is $\mathcal{Z}$-absorbing by the result in [12] that any non-elementary simple AF-algebra, and in particular, every UHF-algebra, is $\mathcal{Z}$-absorbing.)

Let e and f be one-dimensional projections in M_{n+1} and M_{n+2}, respectively, and let $p \in A$ be the image of (e, f) under the inclusion mapping $M_{n+1} \oplus M_{n+2} \to A$. Then p is a full projection that satisfies $(n+1)\langle p\rangle \leq \langle 1\rangle \leq n\langle b\rangle$. Hence, by Theorem 4.5, $\langle p\rangle \leq \langle b\rangle$, i.e., $p \precsim b$. It follows that $p = x^*bx$ for some $x \in A$. Put $v = b^{1/2}x$. Then $p = v^*v$ and $p \sim vv^* = b^{1/2}xx^*b^{1/2} \in B$, so vv^* is a full projection in B. □

Lemma 6.3 *Every unital approximately divisible C^*-algebra is strongly K_1-surjective.*

Proof: Let B be a full hereditary sub-C^*-algebra of A. We must show that the canonical map $\mathcal{U}(B+\mathbb{C}1_A) \to K_1(A)$ is surjective. Use Lemma 6.2 to find a full projection p in B. It suffices to show that the canonical map $\mathcal{U}(pAp + \mathbb{C}(1_A - p)) \to K_1(A)$ is surjective. Take an element g in $K_1(A)$, and represent g as the class of a unitary element u in $M_n(A)$ for some large enough natural number n. Upon replacing $M_n(A)$ by A we can assume that $n = 1$.

Let $\mathcal{P}$ be the set of projections $q \in A$ such that there exists a unitary element $v \in qAq$ for which $g = [v + (1_A - q)]_1$ in $K_1(A)$. We must show that $\mathcal{P}$ contains all full projections in A. Note first that if q_1, q_2 are projections in A with $q_1 \precsim q_2$ and $q_1 \in \mathcal{P}$, then $q_2 \in \mathcal{P}$. Indeed, if $v_1 \in q_1Aq_1$ is unitary with $g = [v_1 + (1_A - q_1)]_1$ and if $s^*s = q_1$, $ss^* \leq q_2$, then v_2 given by $sv_1s^* + (q_2 - ss^*)$ is a unitary element in q_2Aq_2, and $[v_1 + (1_A - q_1)]_1 = [v_2 + (1_A - q_2)]_1$.

Let $p \in A$ be a full projection. Then $(n-1)\langle p\rangle \geq \langle 1_A\rangle$ in $W(A)$ for some large enough natural number n, cf. (4.1). By approximate divisibility of A there is a unitary element

$u_0 \in A$ with $\|u - u_0\| < 2$, and a unital embedding $M_n \oplus M_{n+1} \to A \cap \{u_0\}'$; in other words, there are matrix units $\{e_{ij}\}_{i,j=1}^{n}$ and $\{f_{ij}\}_{i,j=1}^{n+1}$ in $A \cap \{u_0\}'$ such that $\sum_i e_{ii} + \sum_i f_{ii} = 1_A$. Note that $g = [u_0]_1$. Put $q = e_{11} + f_{11}$ and put $v = u_0^n e_{11} + u_0^{n+1} f_{11}$, so that v is a unitary element in qAq. It follows from the Whitehead lemma that u_0 is homotopic to $v + (1_A - q)$, whence q belongs to $\mathcal{P}$. As $n\langle q \rangle \leq \langle 1_A \rangle \leq (n-1)\langle p \rangle$, it follows from Theorem 4.5 that $q \precsim p$, whence $p \in \mathcal{P}$ by the result in the second paragraph of the proof. $\square$

Rieffel proved in [19] that if A is a unital C^*-algebra of stable rank one, then the canonical map $\mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ is an isomorphism, and hence injective. Rieffel also showed for any such C^*-algebra A and any hereditary sub-C^*-algebra B of A (full or not) that the stable rank of $B + \mathbb{C}1_A$ is one.

Take now a full hereditary sub-C^*-algebra B of A, where A is unital and of stable rank one. Then

$$\mathcal{U}(B + \mathbb{C}1_A)/\mathcal{U}_0(B + \mathbb{C}1_A) \to K_1(B) \to K_1(A)$$

is an isomorphism (the second map is an isomorphism by Brown's theorem, which guarantees that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$).

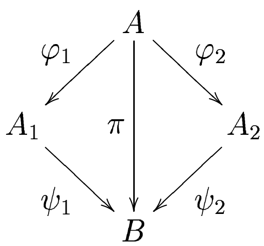
This shows that any unital C^*-algebra of stable rank one is strongly K_1-injective.

For each element x in a C^*-algebra A we can write $x = v|x|$, where v is a partial isometry in A^{**}. The element $x_\varepsilon \stackrel{\text{def}}{=} v(|x| - \varepsilon)_+$ belongs to A for every $\varepsilon \geq 0$, and $\|x - x_\varepsilon\| \leq \varepsilon$. If x is positive, then $x_\varepsilon = (x - \varepsilon)_+$.

Lemma 6.4 *Let A be a unital C^*-algebra, let a be a positive element in A, let $0 < \varepsilon' < \varepsilon$ be given, and set $A^0 = \overline{g_\varepsilon(a) A g_\varepsilon(a)}$, where $g_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $g_\varepsilon(t) = \max\{1 - t/\varepsilon, 0\}$. Then $wa_\varepsilon = a_\varepsilon$ for every $w \in A^0 + 1_A$; and if w is a unitary element in A that satisfies $wa_{\varepsilon'} = a_{\varepsilon'}$, then w belongs to $A^0 + 1_A$.*

Proof: The first claim follows from the fact that $xa_\varepsilon = 0$ for every $x \in A^0$. Suppose that w is a unitary element in A with $wa_{\varepsilon'} = a_{\varepsilon'}$. Then $a_{\varepsilon'} w = a_{\varepsilon'}$, so $w - 1_A$ is orthogonal to $a_{\varepsilon'}$. But the orthogonal complement of $a_{\varepsilon'}$ is contained in A^0. $\square$

Proposition 6.5 *Given a pull-back diagram*



(6.1)

with surjective *-homomorphisms ψ_1 and ψ_2. Suppose that A, A_1, A_2 and B are unital C^*-algebras and that $a \in A$ are such that

(i) *A_1 and A_2 are strongly K_1-surjective,*

(ii) *B is strongly K_1-injective,*

(iii) *$\mathrm{Im}(K_1(\psi_1)) + \mathrm{Im}(K_1(\psi_2)) = K_1(B)$,*

(iv) *a^*a is non-invertible in every non-zero quotient of A.*

Then a belongs to the closure of $\mathrm{GL}(A)$ if and only if $\varphi_j(a)$ belongs to the closure of $\mathrm{GL}(A_j)$ for $j = 1, 2$.

The pull-back diagram (6.1) can, given $\psi_j \colon A_j \to B$, $j = 1, 2$, be realized with $A = \{(a_1, a_2) \in A_1 \oplus A_2 \mid \psi_1(a_1) = \psi_2(a_2)\}$ and with $\varphi_j(a_1, a_2) = a_j$.

Proof: The "only if" part is trivial. Assume now that $\varphi_j(a)$ belongs to the closure of $\mathrm{GL}(A_j)$ for $j = 1, 2$. Let $\varepsilon > 0$ be given. It then follows from [20, Theorem 2.2] that there are unitary elements u_j in A_j such that $\varphi_j(a_{\varepsilon/2}) = u_j|\varphi_j(a_{\varepsilon/2})|$ for $j = 1, 2$. We show below that there are unitary elements v_j in A_j, $j = 1, 2$, such that $\varphi_j(a_\varepsilon) = v_j|\varphi_j(a_\varepsilon)|$, $j = 1, 2$, and $\psi_1(v_1) = \psi_2(v_2)$. It follows that $v = (v_1, v_2)$ is a unitary element in A and that $a_\varepsilon = v|a_\varepsilon|$. This shows that a belongs to the closure of the invertibles in A (because $\varepsilon > 0$ was arbitrary).

Let $g_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ be as in Lemma 6.4, and put $A^0 = \overline{g_\varepsilon(|a|)Ag_\varepsilon(|a|)}$. Put

$$A_j^0 = \varphi_j(A^0) = \overline{g_\varepsilon(|\varphi_j(a)|)A_j g_\varepsilon(|\varphi_j(a)|)}, \qquad B^0 = \pi(A^0) = \overline{g_\varepsilon(|\pi(a)|)Bg_\varepsilon(|\pi(a)|)}.$$

Assumption (iv) implies that A^0 is full in A. It follows that the hereditary subalgebras A_1^0, A_2^0, B^0 are full in A_1, A_2 and B, respectively.

It follows from the identity

$$\pi(a_{\varepsilon/2}) = \psi_1(u_1)|\pi(a_{\varepsilon/2})| = \psi_2(u_2)|\pi(a_{\varepsilon/2})|,$$

that $\psi_2(u_2)^*\psi_1(u_1)|\pi(a_{\varepsilon/2})| = |\pi(a_{\varepsilon/2})|$, and so $z \overset{\text{def}}{=} \psi_2(u_2)^*\psi_1(u_1)$ belongs to $B^0 + 1_B$ (cf. Lemma 6.4). We show below that $z = \psi_2(w_2)\psi_1(w_1^*)$ for some unitaries w_j in $A_j^0 + \mathbb{C}1_{A_j}$, $j = 1, 2$.

Use conditions (i) and (iii) to find unitaries $y_j \in A_j^0 + \mathbb{C}1_{A_j}$ such that $[\psi_2(y_2)\psi_1(y_1)^*]_1 = [z]_1$ in $K_1(B)$. By condition (ii), the unitary element $(z_0 =)\ z\psi_1(y_1)\psi_2(y_2^*)$ is homotopic to

1 in the unitary group of $B^0 + \mathbb{C}1_B$. Hence $z_0 = \psi_2(y_0)$ for some unitary y_0 in $A_2^0 + \mathbb{C}1_{A_2}$. Now, $w_1 = y_1$ and $w_2 = y_0 y_2$ are as desired.

Upon replacing w_1 and w_2 by λw_1 and λw_2 for a suitable $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we can assume that $w_j \in A_j^0 + 1_{A_j}$. Then, by Lemma 6.4, $w_j|\varphi_j(a_\varepsilon)| = |\varphi_j(a_\varepsilon)|$, $j = 1, 2$. It follows that $(v_j =)\ u_j w_j$ is a unitary in A_j, that $v_j|\varphi_j(a_\varepsilon)| = u_j|\varphi_j(a_\varepsilon)| = \varphi_j(a_\varepsilon)$ for $j = 1, 2$, and $\psi_1(v_1) = \psi_1(u_1)\psi_1(w_1) = \psi_2(u_2)\psi_2(w_2) = \psi_2(v_2)$, as desired. $\square$

Lemma 6.6 *Let A be a simple, unital, finite C^*-algebra. Then $a \otimes 1$ belongs to the closure of the invertibles in $A \otimes \mathcal{Z}$ for every $a \in A$.*

Proof: Let $E_{2,3}$ be the C^*-algebra which in Proposition 2.2 is shown to have a unital embedding into $\mathcal{Z}$. It suffices to show that $a \otimes 1$ belongs to the invertibles in $A \otimes E_{2,3}$. If $a^*a \otimes 1$ is invertible in some non-zero quotient of $A \otimes E_{2,3}$, then a^*a is invertible in A by simplicity of A, which again implies that a is invertible, because A is finite. The claim of the lemma is trivial in this case. Suppose now that there is no non-zero quotient of $A \otimes E_{2,3}$ in which $a^*a \otimes 1$ is invertible.

Identify $A \otimes E_{2,3}$ with

$$\{f \in C([0,1], A \otimes B) \mid f(0) \in A \otimes B_1,\ f(1) \in A \otimes B_2\},$$

where B, B_1, B_2 are UHF algebras of type 6^∞, 2^∞, and 3^∞, respectively, with $B_j \subseteq B$. We have a pull-back diagram

$$\begin{array}{ccccc} & & A \otimes E_{2,3} & & \\ & \overset{\varphi_1}{\swarrow} & \Big\downarrow & \overset{\varphi_2}{\searrow} & \\ D_1 & & \mathrm{ev}_{\frac{1}{2}} & & D_2 \\ & \underset{\mathrm{ev}_{\frac{1}{2}}}{\searrow} & \Big\downarrow & \underset{\mathrm{ev}_{\frac{1}{2}}}{\swarrow} & \\ & & A \otimes B & & \end{array} \qquad (6.2)$$

where

$$D_1 = \{f \in C([0, \tfrac{1}{2}], A \otimes B) \mid f(0) \in A \otimes B_1\}, \quad D_2 = \{f \in C([\tfrac{1}{2}, 1], A \otimes B) \mid f(1) \in A \otimes B_2\},$$

and where φ_1 and φ_2 are the restriction mappings.

We shall now use Proposition 6.5 to prove that $a \otimes 1$ belongs to the closure of the invertibles in $A \otimes E_{2,3}$. Condition (iv) of Proposition 6.5 is satisfied by the assumption on a made in the first paragraph of the proof. The C^*-algebras D_1 and D_2 are approximately divisible because $D_j \cong D_j \otimes B_j$. It thus follows from Lemma 6.3 that condition (i) of Proposition 6.5 is satisfied. The C^*-algebra $A \otimes B$ is of stable rank one (cf. [21]), whence $A \otimes B$ is strongly K_1-injective (cf. the remarks below Lemma 6.3).

We have natural inclusions $A \otimes B_j \subseteq D_j$ (identifying an element in $A \otimes B_j$ with a constant function), and the composition $A \otimes B_j \to D_j \to A \otimes B$ is the inclusion mapping. Hence, to prove that (iii) of Proposition 6.5 is satisfied, it suffices to show that $K_1(A \otimes B)$ is generated by the images of the two mappings $K_1(A \otimes B_j) \to K_1(A \otimes B)$, $j = 1, 2$. We have natural identifications:

$$K_1(A \otimes B_1) = K_1(A) \otimes \mathbb{Z}[1/2], \qquad K_1(A \otimes B_2) = K_1(A) \otimes \mathbb{Z}[1/3],$$
$$K_1(A \otimes B) = K_1(A) \otimes \mathbb{Z}[1/6].$$

The desired identity now follows from the elementary fact that $\mathbb{Z}[1/2] + \mathbb{Z}[1/3] = \mathbb{Z}[1/6]$.

Retaining the inclusion $A \otimes B_j \subseteq D_j$ from the previous paragraph, $\varphi_j(a \otimes 1_{E_{2,3}}) = a \otimes 1_{B_j}$. Following [21], $a \otimes 1_{B_j}$ belongs to the closure of the invertibles in $A \otimes B_j$ (and hence to the closure of the invertibles in D_j) if (and only if) $\alpha_s(a) = 0$; and $\alpha_s(a) = 0$ for every element a in any unital, finite, simple C^*-algebra A. It thus follows that $\varphi_j(a)$ belongs to the closure of $\mathrm{GL}(D_j)$, $j = 1, 2$. (If we had assumed that A is stably finite, then we could have used [21, Corollary 6.6] to conclude that the stable rank of $A \otimes B_j$ is one, which would have given us a more direct route to the conclusion above.) □

Theorem 6.7 *Every simple, unital, finite $\mathcal{Z}$-absorbing C^*-algebra has stable rank one.*

Proof: Let A be a simple, unital, finite C^*-algebra such that A is isomorphic to $A \otimes \mathcal{Z}$. Let $a \in A$ and $\varepsilon > 0$ be given. It follows from Lemma 6.6 that there is an invertible element $b \in A \otimes \mathcal{Z}$ such that $\|a \otimes 1 - b\| < \varepsilon/2$. Let $\sigma_n \colon A \otimes \mathcal{Z} \to A$ be as in Lemma 4.3 and choose n such that $\|\sigma_n(a \otimes 1) - a\| < \varepsilon/2$. Then $\|a - \sigma_n(b)\| < \varepsilon$, and $\sigma_n(b)$ is an invertible element in A. □

7 The real rank of $\mathcal{Z}$-absorbing C^*-algebras

We conclude this paper with a result that describes when a simple $\mathcal{Z}$-absorbing C^*-algebra is of real rank zero. A simple *approximately divisible* C^*-algebra is of real rank zero if and only if projections separate quasitraces, as shown in [3] (and, as remarked earlier, each quasitrace on an exact C^*-algebra is a trace by [11] and [14]). It is not true that any $\mathcal{Z}$-absorbing C^*-algebra, where quasitraces are being separated by projections, is of real rank zero. The Jiang–Su $\mathcal{Z}$ itself is a counterexample. We must therefore require some further properties, for example that the K_0-group is *weakly divisible*: for each $g \in K_0^+$ and for each $n \in \mathbb{N}$ there are $h_1, h_2 \in K_0^+$ such that $g = nh_1 + (n+1)h_2$.

Let A be a unital C^*-algebra. Let $T(A)$ denote the simplex of tracial states on A, and let $\mathrm{Aff}(T(A))$ denote the normed space of real valued affine continuous functions on $T(A)$. Let $\rho\colon K_0(A) \to \mathrm{Aff}(T(A))$ be the canonical map defined by $\rho(g)(\tau) = K_0(\tau)(g)$.

The result below is essentially contained in [22, Theorem 7.2] (see also [3]).

Proposition 7.1 *Let A be an exact unital simple C^*-algebra of stable rank one for which $W(A)$ is almost unperforated. Then A is of real rank zero if and only if $\rho(K_0(A))$ is uniformly dense in the normed space* $\mathrm{Aff}(T(A))$.

The proof of [22, Theorem 7.2] applies almost verbatim. (At the point where we have a positive element $x \in A$ and $\delta > 0$ such that $d_\tau(\langle f_{\delta/2}(x)\rangle) < d_\tau(\langle f_{\delta/4}(x)\rangle)$ for all $\tau \in T(A)$, then, because $\tau \mapsto d_\tau(\langle y\rangle)$ defines an element in $\mathrm{Aff}(T(A))$ for every $y \in A^+$, it follows by density of $\rho(K_0(A))$ in $\mathrm{Aff}(T(A))$ that there is an element $g \in K_0(A)$ such that $d_\tau(\langle f_{\delta/2}(x)\rangle) < K_0(\tau)(g) < d_\tau(\langle f_{\delta/4}(x)\rangle)$ for all $\tau \in T(A)$. One next uses weak unperforation of $K_0(A)$ ([9] or Corollary 4.9) to conclude that g is positive, i.e., that $g = [q]$ for some projection q in a matrix algebra over A.)

Theorem 7.2 *The following conditions are equivalent for each unital, simple, exact, finite, $\mathcal{Z}$-absorbing C^*-algebra A.*

(i) $\mathrm{RR}(A) = 0$,

(ii) $\rho(K_0(A))$ *is uniformly dense in* $\mathrm{Aff}(T(A))$,

(iii) $K_0(A)$ *is weakly divisible and projections in A separate traces on A,*

If A is a simple, *infinite*, $\mathcal{Z}$-absorbing C^*-algebra, then A is purely infinite by [9]; and purely infinite C^*-algebras are of real rank zero by [28].

Proof: It follows from Theorem 4.5 that $W(A)$ is almost unperforated, and from Theorem 6.7 that the stable rank of A is one. We therefore get (ii) $\Rightarrow$ (i) from Proposition 7.1. If (iii) holds, then for each element g in $K_0(A)^+$ and for each natural number n there exists $h \in K_0(A)^+$ such that $nh \le g \le (n+1)h$. This implies that the uniform closure of $\rho(K_0(A))$ in $\mathrm{Aff}(T(A))$ is a closed subspace which separates points. Thus, by Kadison's Representation Theorem (see [1, II.1.8]), $\rho(K_0(A))$ is uniformly dense in $\mathrm{Aff}(T(A))$, so (ii) holds. If (i) holds, then A is weakly divisible by [18] and traces on A are separated by projections. $\square$

A C^*-algebra is said to have property (SP) ("small projections") if each non-zero hereditary sub-C^*-algebra contains a non-zero projection.

Corollary 7.3 *Let A be a simple, unital, exact $\mathcal{Z}$-absorbing C^*-algebra with a* unique *trace τ. Then the following conditions are equivalent:*

(i) $\mathrm{RR}(A) = 0$,

(ii) $K_0(\tau)(K_0(A))$ *is dense in* $\mathbb{R}$,

(iii) $K_0(A)$ *is weakly divisible,*

(iv) A *has property (SP).*

Proof: The equivalence of (i), (ii), and (iii) follows immediately from Theorem 7.2. The implication (i) $\Rightarrow$ (iv) is trivial, and one easily sees that (iv) implies (ii). □

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