CLASSIFICATION OF CERTAIN NON-SIMPLE C^*-ALGEBRAS

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ABSTRACT. It is proved that the lattice of closed, two-sided ideals in a C^*-algebra classifies the class of unital C^*-algebras, which are inductive limits of sequences of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$ and have totally ordered lattice of ideals, up to *-isomorphism.

Furthermore it is proved that if the lattice of ideals of a separable, unital C^*-algebra is totally ordered, then it is compact, metrizable and has an isolated maximum in the order topology. Conversely each totally ordered space (containing at least two points) which is compact, metrizable and has an isolated maximum in the order topology appears as the lattice of ideals of a C^*-algebra which is an inductive limit of a sequence of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$.
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0. INTRODUCTION

Consider the class of unital C^*-algebras which can be realized as inductive limits of sequences of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$, where $\mathcal{O}_2$ is the Cuntz algebra with two generators i.e. the universal C^*-algebra generated by two isometries, s_1 and s_2, satisfying the relation $s_1 s_1^* + s_2 s_2^* = 1$. This C^*-algebra was introduced in [3] where it was also proved to be simple. It follows by continuity of K_0 and K_1 and [4, Theorem 2.3] that a C^*-algebra in the above class must have trivial K-theory. It also follows that any closed, two-sided ideal in such a C^*-algebra must have trivial K-theory. Furthermore by definition C^*-algebras in this class can have no tracial states. This indicates that this class of C^*-algebras could be classified up to *-isomorphism by the lattice of ideals (i.e. the set of closed, two-sided ideals ordered by inclusion). The purpose of this paper is to prove that this is the case for C^*-algebras in the above class which have totally ordered lattice of ideals.

The following properties of $\mathcal{O}_2$ will be used frequently in this paper. Any pair of non-zero projections in $\mathcal{O}_2$ are unitarily equivalent. This implies that the unit of $\mathcal{O}_2$ can be split into any finite number of mutually orthogonal projections, that if e is any non-zero projection in $\mathcal{O}_2$, then $\mathcal{O}_2$ is isomorphic to $e\mathcal{O}_2 e$ and finally that $M_n(\mathcal{O}_2)$ is isomorphic to $\mathcal{O}_2$ for all $n \in \mathbb{N}$.

1. THE LATTICE OF IDEALS

1.1. Let $\mathcal{A}$ be a C^*-algebra. Denote by $\mathcal{I}(\mathcal{A})$ the set of closed, two-sided ideals in $\mathcal{A}$. It is a distributive lattice when ordered by inclusion. The infimum of two ideals, I_1 and I_2, is their intersection and the supremum is $\{a_1 + a_2 \mid a_1 \in I_1 \text{ and } a_2 \in I_2\}$. Furthermore $\mathcal{I}(\mathcal{A})$ is a complete lattice i.e. it has the property that any family of ideals has an infimum (their intersection) and a supremum (the intersection of the ideals which contain all ideals in the family). In the following proposition a topology is put on $\mathcal{I}(\mathcal{A})$, however it should be emphasised that in the classification theorem 5.1.1 the invariant will be $\mathcal{I}(\mathcal{A})$ considered as an ordered set.

Proposition 1.1.1.

(i) *If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{I}(\mathcal{A})$ is a compact Hausdorff space when given the weak topology induced by the maps $\hat{a} : \mathcal{I}(\mathcal{A}) \to \mathbb{R}_+$ defined by $\hat{a}(I) = \|a + I\|$, $a \in \mathcal{A}$. If $\mathcal{A}$ is unital, then $\mathcal{A}$ is an isolated point in $\mathcal{I}(\mathcal{A})$ and if $\mathcal{A}$ is assumed to be separable, then $\mathcal{I}(\mathcal{A})$ is metrizable.*

(ii) *If $\varphi : \mathcal{A} \to \mathcal{B}$ is *-homomorphism between C^*-algebras, then the induced map $\hat{\varphi} : \mathcal{I}(\mathcal{B}) \to \mathcal{I}(\mathcal{A})$ defined by $\hat{\varphi}(I) = \varphi^{-1}(I)$ is continuous and infimum preserving.*

Proof. For part (i) only compactness will be proved. Let I_λ be a universal net in $\mathcal{I}(\mathcal{A})$ and let $a \in \mathcal{A}$. Since $\hat{a}(I_\lambda)$ is a universal net in $[0, \|a\|]$, which is compact, it follows that $\hat{a}(I_\lambda)$ is convergent. Put $I = \{a \in \mathcal{A} \mid \lim \hat{a}(I_\lambda) = 0\}$. Then I is a closed, two-sided ideal in $\mathcal{A}$. Define a map, $\gamma : \mathcal{A} \to \mathbb{R}_+$, by setting $\gamma(a) = \lim \hat{a}(I_\lambda)$. This is a C^*-semi-norm on $\mathcal{A}$ and $\gamma^{-1}(0) = I$. Hence γ induces a C^*-norm on $\mathcal{A}/I$. Then by uniqueness of the C^*-norm $\hat{a}(I) = \gamma(a) = \lim \hat{a}(I_\lambda)$. This proves that $\mathcal{I}(\mathcal{A})$ is compact.

To prove (ii) it is enough to show that if $a \in \mathcal{A}$, then $\hat{a} \circ \hat{\varphi}$ is continuous. Let I be a closed, two-sided ideal in $\mathcal{B}$. Let $\pi_I : \mathcal{B} \to \mathcal{B}/I$ and $\pi_{\hat{\varphi}(I)} : \mathcal{A} \to \mathcal{A}/\hat{\varphi}(I)$ be the quotient maps. Since $\hat{\varphi}(I)$ is the kernel of the map $\pi_I \circ \varphi$, it follows from the first isomorphism theorem that there exists an injective *-homomorphism $\psi : \mathcal{A}/\hat{\varphi}(I) \to \mathcal{B}/I$ such that $\psi \circ \pi_{\hat{\varphi}(I)} = \pi_I \circ \varphi$. Hence $\|a + \hat{\varphi}(I)\| = \|\varphi(a) + I\|$. This proves that $\hat{a} \circ \hat{\varphi} = \widehat{\varphi(a)}$ so that $\hat{a} \circ \hat{\varphi}$ is continuous. $\square$

Hence taking $\mathcal{A}$ to $\mathcal{I}(\mathcal{A})$ and φ to $\hat{\varphi}$ defines a contravariant functor from the category where the objects are C^*-algebras and the morphisms are *-homomorphisms, to the category where the objects are distributive lattices (with the additional property that any family of elements has an infimum and a supremum) which are also compact Hausdorff spaces and the morphisms are infimum preserving, continuous maps. Note that if φ is onto, then $\hat{\varphi}$ preserves supremum of finite sets.

In the remaining part of this section the topology on $\mathcal{I}(\mathcal{A})$ introduced in 1.1.1 is described in some special cases. First for inductive limit C^*-algebras where it is also proved that the functor defined above is well-behaved with respect to taking inductive limits.

Proposition 1.1.2. *Let $(\mathcal{A}_n, \varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1})_{n=1}^{\infty}$ be a sequence of C^*-algebras and *-homomorphisms with inductive limit $\mathcal{A}$. Let $\mu_n : \mathcal{A}_n \to \mathcal{A}$ be the natural *-homomorphism. Then*

(i) *the weak topology on the set of closed, two-sided ideals in $\mathcal{A}$ defined in 1.1.1 coincides with the weak topology induced by the maps $\hat{\mu}_n$ (here $\mathcal{I}(\mathcal{A}_n)$ is given the topology from 1.1.1),*

(ii) *$\mathcal{I}(\mathcal{A})$ is an inverse limit of the sequence $(\mathcal{I}(\mathcal{A}_n), \hat{\varphi}_n : \mathcal{I}(\mathcal{A}_{n+1}) \to \mathcal{I}(\mathcal{A}_n))_{n=1}^{\infty}$ in the category described above.*

Proof. Since $\mathcal{I}(\mathcal{A})$ is compact when given the weak topology from 1.1.1 and Hausdorff when given the weak topology induced by the $\hat{\mu}_n$'s it is enough to show that the identity map between these two spaces is continuous. This will follow if $\hat{\mu}_n$ is continuous and this is the case by 1.1.1 (ii). This proves (i).

Assume that $\mathcal{L}$ is a lattice in the category introduced above and that $\pi_n : \mathcal{L} \to \mathcal{I}(\mathcal{A}_n)$, $n \in \mathbb{N}$, are continuous, infimum preserving maps such that $\pi_n = \hat{\varphi}_n \circ \pi_{n+1}$ for all $n \in \mathbb{N}$. Define a map $\Gamma : \mathcal{L} \to \mathcal{I}(\mathcal{A})$ by sending $x \in \mathcal{L}$ to the ideal in $\mathcal{A}$ determined by the sequence of ideals $(\pi_n(x))_{n=1}^{\infty}$. Clearly $\hat{\mu}_n \circ \Gamma = \pi_n$ and Γ is the only map with this property. It is also continuous and infimum preserving $\square$

Next it is proved that in a special case there is a particularly nice metric which induces the topology defined in 1.1.1.

Proposition 1.1.3. *Let $I_1, I_2, \ldots, I_n$ be a finite family of mutually disjoint, closed intervals contained in $[0,1]$ each containing more than one point. Let d_H be the Hausdorff metric on the set of non-empty, closed subsets of $[0,1]$. Extend it to set of all closed subsets of $[0,1]$ by defining $d_H(\emptyset, F) = d_H(F, \emptyset) = 1$ (any number bigger than $\frac{1}{2}$ could be used here) and $d_H(\emptyset,\emptyset) = 0$. Put*

$$\mathcal{A} = C(\bigcup_{j=1}^{n} I_j, \mathcal{O}_2).$$

Let I and J be closed, two-sided ideals in $\mathcal{A}$ with corresponding closed subsets F and G of $\bigcup_{j=1}^{n} I_j$. Define a metric, d, on $\mathcal{I}(\mathcal{A})$ by setting

$$d(I,J) = d_H(F,G).$$

This metric has the following properties:

(i) *It induces the topology on $\mathcal{I}(\mathcal{A})$ defined in 1.1.1.*
(ii) *If $I, J \in \mathcal{I}(\mathcal{A}) \setminus \{\mathcal{A}\}$, then $d(I,J) \leq 1$.*
(iii) *If $I \in \mathcal{I}(\mathcal{A}) \setminus \{\mathcal{A}\}$, then $d(I,\mathcal{A}) = 1$.*
(iv) *If I, J and K are closed, two-sided ideals in $\mathcal{A}$ and $I \subseteq J \subseteq K$, then $d(J,K) \leq d(I,K)$ and $d(I,J) \leq d(I,K)$.*

Proof. The statements (ii), (iii) and (iv) follow directly from the definition of d.

Since the set of closed subsets of $\bigcup_{j=1}^{n} I_j$ equipped with the metric d_H is compact and $\mathcal{I}(\mathcal{A})$ is Hausdorff it is enough to show that the bijective map, which maps a closed subset, F, to the corresponding ideal, $I_F = \{f \in C(\bigcup_{j=1}^{n} I_j, \mathcal{O}_2) \mid f|_F = 0\}$, is continuous. This is the case since the map, $F \mapsto \|a + I_F\|$, is continuous for all $a \in \mathcal{A}$. $\square$

Remark 1.1.4. Let $\mathcal{A} = \bigoplus_{i=1}^{n} C([0,1]) \otimes \mathcal{O}_2$. If $I_1, I_2, \ldots, I_n$ is any choice of mutually disjoint, closed intervals contained in $[0,1]$ each containing more than one point, then $\mathcal{A}$ is *-isomorphic to $C(\bigcup_{i=1}^{n} I_i, \mathcal{O}_2)$. Hence there is a metric (depending on the choice of the intervals) on $\mathcal{I}(\mathcal{A})$ with the properties stated in the proposition.

Finally it is proved that if $\mathcal{I}(\mathcal{A})$ is totally ordered, then the order topology coincides with the topology defined in 1.1.1.

Proposition 1.1.5. *Let $\mathcal{A}$ be a C^*-algebra and assume that $\mathcal{I}(\mathcal{A})$ is totally ordered. Denote by $(\mathcal{I}(\mathcal{A}), \tau)$ the set of closed, two-sided ideals in $\mathcal{A}$ equipped with the order topology. Then the identity map, $\mathrm{id} : \mathcal{I}(\mathcal{A}) \to (\mathcal{I}(\mathcal{A}), \tau)$, is a homeomorphism. Hence the the weak topology defined in 1.1.1 coincides with the order topology.*

Proof. Since $\mathcal{I}(\mathcal{A})$ is compact and $(\mathcal{I}(\mathcal{A}), \tau)$ is Hausdorff it is enough to show that id is continuous. Assume that $I_1 \subset I \subset I_2$ are three ideals in $\mathcal{A}$. Put $\mathcal{B} = I_2/I_1$ and $J = I/I_1$. Then J is an ideal in $\mathcal{B}$. Let $\pi : \mathcal{B} \to \mathcal{B}/J$ be the quotient map. Since $J \neq \{0\}$ it follows that there exists $b \in \mathcal{B} \setminus J$ such that

$$0 \neq \|\pi(b)\| < \|b\|.$$

The map, $\Gamma : I_2/I \to \mathcal{B}/J$, defined by $\Gamma(a + I) = \pi(a + I_1)$ is a *-isomorphism. Hence there exists $a \in I_2$ such that $\Gamma(a + I) = \pi(b) \neq 0$. It follows that $a \in I_2 \setminus I$ and that

$$0 < \|a + I\| = \|\Gamma(a + I)\| = \|\pi(b)\| < \|b\| = \|a + I_1\|.$$

Hence $\hat{a}^{-1}(]0, \hat{a}(I_1)[)$ is an open set in $\mathcal{I}(\mathcal{A})$ contained in $]I_1, I_2[$ and it contains I.

Using similar methods it can be proved that sets of the form $[0, I[$ and $]I, \mathcal{A}]$ are also open sets. This proves that id is continuous. $\square$

Corollary 1.1.6. *If $\mathcal{A}$ is separable C^*-algebra and $\mathcal{I}(\mathcal{A})$ is totally ordered, then $\mathcal{I}(\mathcal{A})$ is order isomorphic to a compact subset of the real line.*

Proof. Since any totally ordered space, which is compact and metrizable in the order topology, is order isomorphic to a compact subset of the real line, this follows from 1.1.1 and 1.1.5. $\square$

1.2. This section gives a converse to the last corollary using building blocks of the form $\bigoplus_{i=1}^{n} C([0,1]) \otimes \mathcal{O}_2$. Let K be a compact subset of $\mathbb{R}$ containing at least three points and having an isolated maximum. Put $K' = K \setminus \{\max K\}$. Then K' is compact and contains at least two points. Set

$$K'_n = \{s \in \mathbb{R} \mid \min K' \leq s \leq \max K' \text{ and } \exists t \in K' : |s - t| \leq \frac{1}{n}\}.$$

Then K'_n is a disjoint union of a finite number of intervals all containing more than one point, $K'_{n+1} \subseteq K'_n$ and $K' = \bigcap_{n=1}^{\infty} K'_n$. Put

$$\mathcal{A}_n = C(K'_n, M_{2^{n-1}}(\mathcal{O}_2)) \ \ (\cong C(K'_n, \mathcal{O}_2)).$$

Let $(q_n)_{n=1}^{\infty}$ be a dense sequence in K' with the property that any tail of the sequence is also dense in K' (i.e. each isolated point in K' appears an infinite number of times in the sequence). Define a continuous function, $\lambda_n : K'_{n+1} \to K'_{n+1}$ $(\subseteq K'_n)$, by setting

$$\lambda_n(t) = \begin{cases} q_n & , \quad t \leq q_n \\ t & , \quad t \geq q_n \end{cases}.$$

Define a unital *-homomorphism,

$$\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1},$$

by

$$\varphi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f \circ \lambda_n(t) \end{pmatrix}$$

where $t \in K'_{n+1}$. Put

$$\mathcal{A} = \varinjlim(\mathcal{A}_n, \varphi_n),$$

and let $\mu_n : \mathcal{A}_n \to \mathcal{A}$, $n \in \mathbb{N}$, be the natural *-homomorphisms.

Theorem 1.2.1. *The C^*-algebra, $\mathcal{A}$, constructed above is a non-simple C^*-algebra with the following properties:*

(i) *If I is an ideal in $\mathcal{A}$, $I \neq \mathcal{A}$, then there exists a unique $t \in K'$ such that for all $n \in \mathbb{N}$ the ideal $\hat{\mu}_n(I)$ corresponds to the closed set $[t, \infty[\, \cap K'$. Conversely for any $t \in K'$ the sequence $([t, \infty[\, \cap K')_{n=1}^{\infty}$ determines an ideal in $\mathcal{A}$. Hence mapping the ideal determined by the sequence $([t, \infty[\, \cap K')_{n=1}^{\infty}$ to t and $\mathcal{A}$ to $\max K$ defines an order isomorphism between $\mathcal{I}(\mathcal{A})$ and K.*

(ii) *If $\mathcal{I}(\mathcal{A})$ is identified with K and $\mathcal{I}(\mathcal{A}_n)$ is identified with the set of closed subsets of K'_n, then the natural map, $\pi_n : \mathcal{I}(\mathcal{A}) \to \mathcal{I}(\mathcal{A}_n)$, is given by*

$$\pi_n(t) = \begin{cases} [t, \infty[\cap K' & , \quad t < \max K \\ \emptyset & , \quad t = \max K \end{cases}.$$

In particular π_n is continuous and descending.

(iii) *The connecting maps $\hat{\varphi}_n : \mathcal{I}(\mathcal{A}_{n+1}) \to \mathcal{I}(\mathcal{A}_n)$ are contractions with respect to the metrics induced by the Hausdorff metric on the set of non-empty, closed subsets of $[\min K', \max K']$ (see 1.1.3).*

Proof. Let I be an ideal in $\mathcal{A}$ which is not $\mathcal{A}$ itself. Put $I_n = \hat{\mu}_n(I)$ and let F_n be the non-empty, closed subset of K'_n corresponding to I_n. Since $\hat{\varphi}_n(I_{n+1}) = I_n$ it follows that

$$F_n = F_{n+1} \cup \lambda_n(F_{n+1}).$$

Hence F_{n+1} must be contained in F_n so that $\min F_n \leq \min F_{n+1}$. On the other hand since $\lambda_n(t) \geq t$ for all $t \in K'_{n+1}$ it follows that $\min F_{n+1} \leq \min F_n$. This proves that $\min F_n = \min F_m$ for all n and m in $\mathbb{N}$. Let t be this common minimum.

Now fix $N \in \mathbb{N}$. By construction of the connecting *-homomorphism

$$F_N = F_{N+1} \cup (\{q_N\} \cap [t, \infty[\,).$$

Iterating this it follows that for all $j \in \mathbb{N}$

$$F_N = F_{N+j} \cup (\{q_N, q_{N+1}, \ldots, q_{N+j-1}\} \cap [t, \infty[\,).$$

Hence $F_N \subseteq K'_{N+j}$ for all $j \in \mathbb{N}$ so that $F_N \subseteq K'$. This proves that $F_N \subseteq [t, \infty[\, \cap K'$.

If $s \in [t, \infty[\, \cap K'$, $s > t$, then by the choice of the sequence $(q_n)_{n=1}^{\infty}$ it has a subsequence contained in $[t, \infty[\, \cap K'$ with limit s and such that all elements in the subsequence have index larger than N. By the above all elements in the subsequence belong to F_N. Since F_N is closed s must be contained in F_N. Hence $F_N = [t, \infty[\, \cap K'$.

Let $t \in K'$. Since

$$([t, \infty[\cap K') \cup \lambda_n([t, \infty[\cap K') = [t, \infty[\cap K'$$

the sequence $([t, \infty[\cap K')_{n=1}^{\infty}$ defines an ideal in $\mathcal{A}$. The statement about the order isomorphism now follows. This proves (i), and (ii) follows directly from (i). The statement (iii) follows from the observation that if $s, t \in K'_{n+1}$, then $|\lambda_n(s) - \lambda_n(t)| \leq |s - t|$. $\square$

The following theorem summarizes the results in 1.1.6 and 1.2.1 and takes into account the fact that a simple C^*-algebra can be obtained in the limit using these building blocks.

Theorem 1.2.2. *If $\mathcal{A}$ is unital C^*-algebra which is an inductive limit of a sequence of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$ and $\mathcal{I}(\mathcal{A})$ is totally ordered, then $\mathcal{I}(\mathcal{A})$ is compact, metrizable and has an isolated maximum in the order topology. Conversely each such totally ordered space containing at least two points appears as the lattice of ideals of such a C^*-algebra.*

Remark 1.2.3. Let $\mathcal{A}$ be a C^*-algebra which can be realized as an inductive limit of a sequence of finite direct sums of $\mathcal{O}_2$. It follows from 1.1.2 (i) that $\mathcal{I}(\mathcal{A})$ is totally disconnected hence the class of C^*-algebras considered in the classification theorem 5.1.1 can not be obtained in this way. It can be proved that every totally ordered set which is totally disconnected, compact, metrizable and has an isolated maximum in the order topology appears as the lattice of ideals of a unital C^*-algebra which is an inductive limit of a sequence of finite direct sums of $\mathcal{O}_2$.

1.3. Let $\mathcal{A}$ be a C^*-algebra which is an inductive limit of a sequence, $\mathcal{A}_n$, $n \in \mathbb{N}$, of C^*-algebras. In this section the natural map, $\pi_n : \mathcal{I}(\mathcal{A}) \to \mathcal{I}(\mathcal{A}_n)$, is studied. It is proved that in a special case it has certain properties if n is large enough.

Lemma 1.3.1. *Suppose that I and J are ideals in a C^*-algebra, $\mathcal{A}$, such that $I \subseteq J$ and $aIa = aJa$ for some positive $a \in \mathcal{A}$ which generates J as an ideal. Then $I = J$.*

Proof. Let b be a positive element in J and let $\varepsilon > 0$. Since $b^{\frac{1}{4}} \in J$ and J is the ideal generated by a it follows that for every $\delta > 0$ there are finitely many elements, x_i and y_i, in A such that

$$\|b^{\frac{1}{4}} - \sum_i x_i a y_i\| < \delta.$$

Put $c = \sum_{i=1}^{k} b^{\frac{1}{8}} x_i a y_i b^{\frac{1}{8}}$. Then $\|b^{\frac{1}{2}} - c\| \leq \|b^{\frac{1}{8}}\|^2 \|b^{\frac{1}{4}} - \sum_i x_i a y_i\|$ and

$$\begin{aligned} \|b - cc^*\| &\leq \|b^{\frac{1}{2}}\|\|b^{\frac{1}{2}} - c\| + 3\|c\|\|b^{\frac{1}{2}} - c\| \\ &\leq \|b^{\frac{1}{2}}\|\|b^{\frac{1}{2}} - c\| + 3(\|c - b^{\frac{1}{2}}\| + \|b^{\frac{1}{2}}\|)\|b^{\frac{1}{2}} - c\| \end{aligned}$$

By choosing δ small enough it then follows that $\|b - cc^*\| < \varepsilon$. Furthermore

$$cc^* = \sum_j \sum_i b^{\frac{1}{8}} x_i a y_i b^{\frac{1}{4}} y_j^* a x_j^* b^{\frac{1}{8}}$$

and since

$$a y_i b^{\frac{1}{4}} y_j^* a \in aJa = aIa \subseteq I,$$

it follows that $cc^* \in I$. $\square$

The rest of this section contains the proof of the following proposition.

Proposition 1.3.2. *For all $n \in \mathbb{N}$ let*

$$\mathcal{A}_n = \bigoplus_{j=1}^{k_n} \mathcal{A}_{n,j}$$

*where $\mathcal{A}_{n,j} = C([0,1]) \otimes \mathcal{O}_2$. Let $\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ be a unital *-homomorphism and put*

$$\mathcal{A} = \varinjlim(\mathcal{A}_n, \varphi_n).$$

*Let $\mu_n : \mathcal{A}_n \to \mathcal{A}$ be the natural *-homomorphism. Assume that $\mathcal{I}(\mathcal{A})$ is totally ordered*

Let $I_1 \subset I_2 \subset \cdots \subset I_k \subset J$ be ideals in $\mathcal{A}$ such that $I_1, I_2, \ldots, I_k$ all belong to the same component of $\mathcal{I}(\mathcal{A})$ and J belongs to some other component. Then there is $n \in \mathbb{N}$ such that for all $m \geq n$ the following holds: There are indices $j_1, j_2, \ldots, j_{l_m} \in \{1, 2, \ldots, k_m\}$ so that

$$p_{m,j_i} \hat{\mu}_m(J) p_{m,j_i} = \mathcal{A}_{m,j_i}$$

for all i (where p_{m,j_i} is the projection onto the j_i'th summand of $\mathcal{A}_m$),

$$p_{m,j_i} \hat{\mu}_m(I_l) p_{m,j_i} \subset \mathcal{A}_{m,j_i}$$

for all i and l and so that if $p_m = p_{m,j_1} + p_{m,j_2} + \cdots + p_{m,j_{l_m}}$, then

$$p_m \hat{\mu}_m(I_1) p_m \subset \cdots \subset p_m \hat{\mu}_m(I_k) p_m.$$

For all $n \in \mathbb{N}$ and all $j \in \{1, 2, \ldots, k_n\}$ put

$$K_{n,j} = \{I \in \mathcal{I}(\mathcal{A}) \mid \hat{\mu}_n(I) \cap \mathcal{A}_{n,j} = \mathcal{A}_{n,j}\}.$$

Since $\mathcal{A}_{n,j}$ is a unital C^*-algebra it follows from 1.1.1 that $\{\mathcal{A}_{n,j}\}$ is a clopen set in $\mathcal{I}(\mathcal{A}_{n,j})$. Let $\iota_{n,j} : \mathcal{A}_{n,j} \to \mathcal{A}_n$ be the natural inclusion. Then $K_{n,j}$ is the pre-image of of $\{\mathcal{A}_{n,j}\}$ under the continuous map $\hat{\iota}_{n,j} \circ \hat{\mu}_n$. Hence $K_{n,j}$ is a clopen set in $\mathcal{I}(\mathcal{A})$. Put

$$J_{n,j} = \min K_{n,j},$$

then $K_{n,j} = \{I \in \mathcal{I}(\mathcal{A}) \mid J_{n,j} \subseteq I\}$. Fix in the following $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, k_n\}$.

Lemma 1.3.3. *If $m > n$, then there is $i \in \{1, 2, \ldots k_m\}$ such that $J_{m,i} = J_{n,j}$.*

Proof. Let F be the pre-image of $\{A_{n,j}\}$ under the continuous map $\hat{\iota}_{n,j} \circ \hat{\varphi}_{m,n}$. Then F is clopen and since $\mu_n = \mu_m \circ \varphi_{m,n}$ it follows that the pre-image of F under $\hat{\mu}_m$ is $K_{n,j}$.

Since F is clopen in $\mathcal{I}(\mathcal{A}_m)$ it follows that the pre-image of F under $\hat{\mu}_m$ can be written as a finite intersection, $\bigcap F_l$, where each F_l is either $K_{m,i}$ or $K_{m,i}^{\complement}$ for some i. Since $\bigcap F_l = K_{n,j}$ it follows that at least one of the F_l's must have the form $K_{m,i}$ for some i. By the above description of the $K_{m,i}$'s it follows that the minimum of $\bigcap F_l$ must be attained on such an F_l. $\square$

For all $m \geq n$ put

$$p_m = \sum p_{m,i}$$

where $p_{m,i}$ is the projection onto the i'th summand in $\mathcal{A}_m$ and the summation is over the set $\{i \mid \mathcal{A}_{m,i} \subseteq \hat{\mu}_m(J_{n,j})\}$. By the previous lemma this set is non-empty. Let I_m be the ideal generated by the projection $\mu_m(p_m)$ in $\mathcal{A}$. From the definition of p_m it follows that $\mu_m(p_m) \in J_{n,j}$ so that $I_m \subseteq J_{n,j}$. Use the above lemma to choose $K_{m,i}$ such that $J_{m,i} = J_{n,j}$. Then note that $\hat{\mu}_m(I_m)$ is an ideal in $\mathcal{A}_m$ containing p_m. Hence I_m belongs to $K_{m,i}$. By minimality of $J_{n,j}$, $J_{n,j} \subseteq I_m$ so that $J_{n,j} = I_m$ for all $m \geq n$.

Lemma 1.3.4. *If $n \leq k < m$, then $\varphi_{m,k}(p_k) \leq p_m$.*

Proof. Put $F = \{i \mid \varphi_{m,k}(p_k)p_{m,i} \neq 0\}$. Since $\varphi_{m,k}(p_k)p_{m,i} \leq p_{m,i}$ it follows that

$$\varphi_{m,k}(p_k) = \sum_{i=1}^{k_m} \varphi_{m,k}(p_k)p_{m,i} = \sum_{F} \varphi_{m,k}(p_k)p_{m,i} \leq \sum_{F} p_{m,i}.$$

Hence it is enough to prove that F is contained in $\{i \mid \mathcal{A}_{m,i} \subseteq \hat{\mu}_m(J_{n,j})\}$. By the above $J_{n,j}$ is the ideal generated by the projection $\mu_m(\varphi_{m,k}(p_k))$. This implies that the ideal generated by $\varphi_{m,k}(p_k)$ in $\mathcal{A}_m$ is contained in $\hat{\mu}_m(J_{n,j})$. If $\varphi_{m,k}(p_k)p_{m,i} \neq 0$, then this projection is equivalent to $p_{m,i}$ (all non-zero projections in $\mathcal{A}_{m,i}$ are equivalent). Hence in this case the cut down of the ideal generated by $\varphi_{m,k}(p_k)p_{m,i}$, by the projection $p_{m,i}$, is $\mathcal{A}_{m,i}$ i.e. $\mathcal{A}_{m,i} \subseteq \hat{\mu}_m(J_{n,j})$. $\square$

Now let $I_1 \subseteq I_2 \subset J$ be ideals in $\mathcal{A}$. Assume that I_1 and I_2 belong to the same connected component of $\mathcal{I}(\mathcal{A})$ and that J belongs to some other component of $\mathcal{I}(\mathcal{A})$. Then there is $n \in \mathbb{N}$ such that for all $m \geq n$ there is $j \in \{1, 2, \ldots, k_m\}$ (depending on m) so that $J \in K_{m,j}$ while $I_1, I_2 \notin K_{m,j}$ (this uses that if I and J belong to different components of $\mathcal{I}(\mathcal{A})$, then eventually $\hat{\mu}_m(I)$ and $\hat{\mu}_m(J)$ belong to different components of $\mathcal{I}(\mathcal{A}_m)$). Hence $I_1 \subseteq I_2 \subset J_{n,j}$ and, as proved above, $J_{n,j} = I_m$ for all $m \geq n$.

Lemma 1.3.5. *If $I_1 \subset I_2$ ($\subset J_{n,j} \subseteq J$), then there exists $k \in \mathbb{N}$, $k \geq n$, such that for $m \geq k$,*

$$p_m \hat{\mu}_m(I_1) p_m \subset p_m \hat{\mu}_m(I_2) p_m.$$

Proof. If not, then there is a sequence $n \leq m_1 < m_2 < \ldots$, such that

$$p_{m_i} \hat{\mu}_{m_i}(I_1) p_{m_i} = p_{m_i} \hat{\mu}_{m_i}(I_2) p_{m_i}.$$

By 1.3.4, $\varphi_{m_i,n}(p_n) \leq p_{m_i}$ so that

$$\varphi_{m_i,n}(p_n) p_{m_i} \hat{\mu}_{m_i}(I_1) p_{m_i} \varphi_{m_i,n}(p_n) = \varphi_{m_i,n}(p_n) \hat{\mu}_{m_i}(I_1) \varphi_{m_i,n}(p_n)$$

and the same with I_1 replaced by I_2. Hence for all i,

$$\mu_n(p_n) \mu_{m_i}(\hat{\mu}_{m_i}(I_1)) \mu_n(p_n) = \mu_n(p_n) \mu_{m_i}(\hat{\mu}_{m_i}(I_2)) \mu_n(p_n)$$

so that

$$\mu_n(p_n)(\bigcup_{i=1}^{\infty}\mu_{m_i}(\hat{\mu}_{m_i}(I_1)))\mu_n(p_n) = \mu_n(p_n)(\bigcup_{i=1}^{\infty}\mu_{m_i}(\hat{\mu}_{m_i}(I_2)))\mu_n(p_n).$$

This implies that

$$\mu_n(p_n)I_1\mu_n(p_n) = \mu_n(p_n)I_2\mu_n(p_n).$$

By 1.3.1 this implies that $I_1 = I_2$ which is a contradiction. □

Proposition 1.3.2 now follows.

2. The uniqueness theorem

2.1. This section is devoted to proving that if h_1 and h_2 are self-adjoint elements in $C([0,1],\mathcal{O}_2)$ with spectra contained in $[0,1]$ such that $\operatorname{sp} h_1(t)$ and $\operatorname{sp} h_2(t)$ are close with respect to the Hausdorff metric for all $t \in [0,1]$, then there is a unitary, u, in $C([0,1],\mathcal{O}_2)$ such that uh_1u^* is close to h_2. In fact something slightly more general will be proved.

Lemma 2.1.1. *Let $I_1, I_2, \dots, I_n$ and $J_1, J_2, \dots, J_m$ be two families of mutually disjoint, closed and bounded sub-intervals of $\mathbb{R}$ each containing more than one point. Let $\varepsilon > 0$. Let d be the Hausdorff metric on the set of closed, non-empty subsets of $\bigcup_{j=1}^m J_j$. If π is a continuous function from $\bigcup_{i=1}^n I_i$ into the set of closed, non-empty subsets of $\bigcup_{j=1}^m J_j$, then there exist continuous functions, $\lambda_1, \lambda_2, \dots, \lambda_N$, from $\bigcup_{i=1}^n I_i$ into $\bigcup_{j=1}^m J_j$, such that*

$$\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_N(t)$$

and

$$d(\pi(t), \{\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)\}) < \varepsilon$$

for all $t \in \bigcup_{i=1}^n I_i$.

Proof. Assume first that π is a continuous map from $[0,1]$ into the set of non-empty, closed subsets of $[0,1]$. Since π is uniformly continuous there is a $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow d(\pi(s), \pi(t)) < \frac{\varepsilon}{2}.$$

for all $s, t \in [0,1]$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $|t_{i+1} - t_i| < \delta$. For each $i \in \{0, 1, \dots, n\}$ choose a finite subset, F_i, of $\pi(t_i)$ which is $\frac{\varepsilon}{2}$-dense in $\pi(t_i)$.

Let $\mathcal{F}$ be the set of piecewise, linear functions such that if $\lambda \in \mathcal{F}$, then $\lambda(t_i) \in F_i$ and $|\lambda(t_i) - \lambda(t_{i+1})| < \varepsilon$ for $i = 0, 1, \dots, n-1$.

If $r \in F_i$, then by the choice of δ there is $s \in \pi(t_{i+1})$ so that $|r - s| < \frac{\varepsilon}{2}$. By the choice of F_{i+1} there is $t \in F_{i+1}$ such that $|s - t| < \frac{\varepsilon}{2}$ i.e. $|r - t| < \varepsilon$. This proves that $F_i = \{\lambda(t_i) \mid \lambda \in \mathcal{F}\}$. Since each F_i is finite $\mathcal{F}$ is finite. It is not hard to check that this family of functions has the property that

$$d(\pi(t), \{\lambda(t) \mid \lambda \in \mathcal{F}\}) < \varepsilon$$

for all $t \in [0,1]$. The lemma now follows in this special case.

Let $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Define a continuous map $\pi_{i,j}$ from I_i into the set of closed subsets of J_j by setting

$$\pi_{i,j}(t) = \pi(t) \cap J_j.$$

Since π is continuous either $\pi_{i,j}(t) = \emptyset$ or $\pi_{i,j}(t) \neq \emptyset$ for all $t \in I_i$. In the last case use the above to chose a finite family of continuous functions from I_i into J_j with the desired properties. In this way finite families of functions are defined on I_i for $i = 1, 2, \ldots, n$. It can be assumed that these families have the same number of elements. Note that for each i there is $j \in \{1, 2, \ldots m\}$ such that $\pi_{i,j}(t) \neq \emptyset$ for all $t \in I_i$. □

If h is a self-adjoint element in $C([0,1], \mathcal{O}_2)$ with spectrum contained in $[0,1]$, then the lemma states that the map $t \mapsto \operatorname{sp} h(t)$ can be approximated by a map determined by a finite family of continuous functions from $[0,1]$ into $[0,1]$.

Recall that a C^*-algebra is said to have *real rank zero* (this was defined in [2]) if the set of self-adjoint elements with finite spectrum is dense in the set of self-adjoint elements. By [11], $\mathcal{O}_2$ has real rank zero. Using this the following lemma can be obtained

Lemma 2.1.2. *Let h be a self-adjoint element in $\mathcal{O}_2$ and let F be a finite subset of $\mathbb{R}$. Then for every $\delta > 0$ there is a self-adjoint element, h', in $\mathcal{O}_2$ with spectrum F and $\|h - h'\| < d(F, \operatorname{sp} h) + \delta$, where d is the Hausdorff metric.*

The next lemma is a corollary to [1, Lemma 7.1] stated for $\mathcal{O}_2$.

Lemma 2.1.3. *For every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: If u is a unitary in $\mathcal{O}_2$, h is a self-adjoint element in $\mathcal{O}_2$ with spectrum contained in $[0,1]$ and*

$$\|uh - hu\| < \delta,$$

then there is a continuous path of unitaries, u_t, $t \in [0,1]$, in $\mathcal{O}_2$ such that $u_0 = 1$, $u_1 = u$ and

$$\|u_t h - h u_t\| < \varepsilon$$

for all $t \in [0,1]$.

Proposition 2.1.4. *Let $I_1, I_2, \ldots, I_n$ be mutually disjoint, closed and bounded intervals contained in $\mathbb{R}$ and each containing more than one point. For every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: If h is a self-adjoint element in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ with spectrum contained in $[0,1]$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are continuous functions from $\bigcup_{i=}^n I_i$ into $[0,1]$ such that*

$$d(\operatorname{sp} h(t), \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}) < \delta$$

for all $t \in \bigcup_{i=1}^n I_i$, then there are mutually orthogonal projections, $e_1, e_2, \ldots, e_N$, with sum 1 in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ such that

$$\|h(t) - \sum_{j=1}^N \lambda_j(t) e_j(t)\| < \varepsilon$$

for all $t \in \bigcup_{i=1}^n I_i$.

Proof. It is enough to consider the case where $n = 1$ and the interval is $[0, 1]$. Let $\varepsilon > 0$ and let $\delta' > 0$ be the number corresponding to $\frac{\varepsilon}{3}$ in 2.1.3. Put

$$\delta = \min\{\frac{\delta'}{4}, \frac{\varepsilon}{6}\}.$$

Assume that h is a self-adjoint element in $C([0,1], \mathcal{O}_2)$ with spectrum contained in $[0, 1]$ and that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are continuous functions from $[0, 1]$ into $[0, 1]$ such that

$$d(\operatorname{sp} h(t), \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}) < \delta$$

for all $t \in [0, 1]$.

Choose a partition, $0 = t_0 < t_1 < \cdots < t_n = 1$, of $[0, 1]$ with the following properties:

$$|\lambda_j(s) - \lambda_j(t)| < \min\{\frac{\delta'}{4N}, \frac{\varepsilon}{3N}\} \quad \text{and} \quad \|h(s) - h(t)\| < \delta$$

for all $s, t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$ and $j = 1, 2, ..., N$. By 2.1.2 there is a self-adjoint element, $h'(t_i)$, in $\mathcal{O}_2$ such that

$$\|h(t_i) - h'(t_i)\| < \delta$$

and

$$\operatorname{sp} h'(t_i) = \{\lambda_1(t_i), \lambda_2(t_i), \ldots, \lambda_N(t_i)\}.$$

Then $h'(t_i)$ can be written

$$h'(t_i) = \sum_{j=1}^{N} \lambda_j(t_i) e_j^i$$

where $e_1^i, e_2^i, \ldots, e_N^i$ is a family of mutually orthogonal projections in $\mathcal{O}_2$ with sum 1. Find a unitary, u^i, in $\mathcal{O}_2$ such that

$$e_j^{i+1} = u^i e_j^i (u^i)^*$$

for $j = 1, 2, \ldots N$. Then

$$\begin{aligned} \|h'(t_i) - u^i h'(t_i)(u^i)^*\| \;\; \leq \;\; & \|h'(t_i) - h(t_i)\| + \|h(t_i) - h(t_{i+1})\| + \|h(t_{i+1}) - h'(t_{i+1})\| + \\ & \|\sum_{j=1}^{N} (\lambda_j(t_{i+1}) - \lambda_j(t_i)) e_j^{i+1}\| < \delta'. \end{aligned}$$

Hence by 2.1.3 there is a continuous path of unitaries, u_t^i, $t \in [t_i, t_{i+1}]$, in $\mathcal{O}_2$ such that $u_{t_i}^i = 1$, $u_{t_{i+1}}^i = u^i$ and

$$\|h'(t_i) - u_t^i h'(t_i)(u_t^i)^*\| < \frac{\varepsilon}{3}$$

for all $t \in [t_i, t_{i+1}]$. Putting $e_j(t) = u_t^i e_j^i (u_t^i)^*$ for $t \in [t_i, t_{i+1}]$ defines a projection, e_j, in $C([0, 1], \mathcal{O}_2)$. Set

$$h'(t) = \sum_{j=1}^{N} \lambda(t) e_j(t).$$

If $t \in [t_i, t_{i+1}]$,then

$$\begin{aligned}\|h(t) - h'(t)\| &\leq \|h(t) - h(t_i)\| + \|h(t_i) - h'(t_i)\| + \|h'(t_i) - u_t^i h'(t_i)(u_t^i)^*\| + \\ &\quad \|u_t^i h'(t_i)(u_t^i)^* - h'(t)\| \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} + \sum_{j=1}^{N} |\lambda_j(t_i) - \lambda_j(t)| < \varepsilon.\end{aligned}$$

□

It follows from 2.1.1 and 2.1.4 that if $\varepsilon > 0$ and h is a self-adjoint element in $C([0,1], \mathcal{O}_2)$ with spectrum contained in $[0,1]$, then there is a finite family of continuous functions, $\lambda_1, \lambda_2, \ldots, \lambda_N$, from $[0,1]$ into $[0,1]$ and mutually orthogonal projections, $e_1, e_2, \ldots, e_N$, in $C([0,1], \mathcal{O}_2)$ such that $\|h(t) - \sum_{j=1}^{N} \lambda_j(t) e_j(t)\| < \varepsilon$ for all $t \in [0,1]$.

Theorem 2.1.5. *Let $I_1, I_2, \ldots, I_n$ be mutually disjoint, closed and bounded sub-intervals of $\mathbb{R}$ each containing more than one point. Let d be the Hausdorff metric on the set of non-empty, closed subsets of $[0,1]$. For every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: If h_1 and h_2 are self-adjoint elements in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ with spectra in $[0,1]$ and*

$$d(\operatorname{sp} h_1(t), \operatorname{sp} h_2(t)) < \delta$$

for all t in $\bigcup_{i=1}^n I_i$, then there is a unitary, u, in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ such that

$$\|uh_1u^* - h_2\| < \varepsilon.$$

Proof. Again is is enough to consider the case with only one interval which can be assumed to be $[0,1]$. Let $\delta' > 0$ be the number corresponding to $\frac{\varepsilon}{2}$ in 2.1.4. Assume that h_1 and h_2 are self-adjoint elements in $C([0,1], \mathcal{O}_2)$ such that

$$d(\operatorname{sp} h_1(t), \operatorname{sp} h_2(t)) < \frac{\delta'}{2}$$

for all t in $[0,1]$.

Use 2.1.1 to find continuous functions, $\lambda_1, \lambda_2, \ldots, \lambda_N$, from $[0,1]$ into $[0,1]$ such that

$$d(\operatorname{sp} h_1(t), \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}) < \frac{\delta'}{2}$$

for all $t \in [0,1]$. Then

$$d(\operatorname{sp} h_2(t), \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}) < \delta'$$

for all $t \in [0,1]$. By 2.1.4 there are two families of mutually orthogonal projections, $e_1, e_2, \ldots, e_N$ and $f_1, f_2, \ldots, f_N$, in $C([0,1], \mathcal{O}_2)$ with sum 1 such that

$$\|h_1(t) - \sum_{j=1}^{N} \lambda_j(t) e_j(t)\| < \frac{\varepsilon}{2}$$

and

$$\|h_2(t) - \sum_{j=1}^{N} \lambda_j(t) f_j(t)\| < \frac{\varepsilon}{2}$$

for all $t \in [0,1]$. There is a unitary, u, in $C([0,1], \mathcal{O}_2)$ such that $e_j = u f_j u^*$ for $j = 1, 2, \ldots, N$. Then

$$\begin{aligned} \|u(t)h_2(t)u(t)^* - h_1(t)\| &\leq \|u(t)h_2(t)u(t)^* - u(t)(\sum_{j=1}^{N} \lambda_j(t) f_j(t))u(t)^*\| + \|\sum_{j=1}^{N} \lambda_j(t) e_j(t) - h_1(t)\| \\ &< \varepsilon. \end{aligned}$$

□

2.2. Let $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ be unital *-homomorphisms between unital C^*-algebras. Then φ and ψ are said to be *approximately unitarily equivalent* if for all $\varepsilon > 0$ and all finite subsets, $\mathcal{F}$, of $\mathcal{A}$ there is a unitary, u, in $\mathcal{B}$ such that

$$\|u\varphi(x)u^* - \psi(x)\| < \varepsilon$$

for all $x \in \mathcal{F}$.

In the following unital *-homomorphisms between C^*-algebras which are finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$ will be studied. From [7, Theorem 5.1] it is know that any pair of unital *-homomorphisms from $\mathcal{O}_2$ into $\bigoplus_{i=1}^{n} C([0,1]) \otimes \mathcal{O}_2$ are approximately unitarily equivalent. Reducing to the case with only one interval and using that Rørdam's proof is constructive the following lemma can be obtained by repeating the proof in [7] with some extra bookkeeping.

Lemma 2.2.1. *For every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: Let s_1 and s_2 be the canonical generators of $\mathcal{O}_2$ and let $I_1, I_2, \ldots, I_n$ be mutually disjoint, closed and bounded intervals all containing more than one point. If φ and ψ are two unital *-homomorphisms from $\mathcal{O}_2$ into $C(\bigcup_{i=1}^{n} I_i, \mathcal{O}_2)$, h is a self-adjoint element in $C(\bigcup_{i=1}^{n} I_i, \mathcal{O}_2)$ with spectrum contained in $[0,1]$,*

$$\|[\varphi(s_j), h]\| < \delta$$

and

$$\|[\psi(s_j), h]\| < \delta$$

for $j = 1, 2$, then there exists a unitary, u, in $C(\bigcup_{i=1}^{n} I_i, \mathcal{O}_2)$ such that

$$\|u\varphi(s_j)u^* - \psi(s_j)\| < \varepsilon$$

for $j = 1, 2$ and

$$\|uhu^* - h\| < \epsilon.$$

The proof of the following lemma consists of applying the previous lemma to the restrictions of φ and ψ to $\mathcal{O}_2$ and the self-adjoint element $\psi(h \otimes 1)$. The details are left to the reader.

Lemma 2.2.2. *Let $I_1, I_2, \ldots, I_n$ be mutually disjoint, closed and bounded intervals contained in $[0,1]$ each containing more than one point and let h be the canonical generator of $C(\bigcup_{i=1}^n I_i)$ (in particular the spectrum of h is contained in $[0,1]$). Let s_1 and s_2 be the canonical generators of $\mathcal{O}_2$. For every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: Let $J_1, J_2, \ldots, J_m$ be another family of mutually disjoint, closed and bounded intervals each containing more than one point. If φ and ψ are two unital *-homomorphisms from $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$ into $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ such that*

$$\|\varphi(h \otimes 1) - \psi(h \otimes 1)\| < \delta,$$

then there exists a unitary, u, in $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ such that

$$\|u\varphi(1 \otimes s_l)u^* - \psi(1 \otimes s_l)\| < \varepsilon$$

for $l = 1, 2$ and

$$\|u\phi(h \otimes 1)u^* - \psi(h \otimes 1)\| < \varepsilon.$$

The following theorem is the uniqueness theorem.

Theorem 2.2.3. *Let $I_1, I_2, \ldots, I_n$ be a family of mutually disjoint, closed intervals contained in $[0,1]$ each containing more than one point. For all $\varepsilon > 0$ and all finite subsets, $\mathcal{F}$, of $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$, there exists a $\delta > 0$ with the following property: If $J_1, J_2, \ldots, J_m$ is another family of mutually disjoint, closed intervals all contained in $[0,1]$ and each containing more than one point, d is the Hausdorff metric on the set of closed, non-empty subsets of $[0,1]$, h is the canonical generator of $C(\bigcup_{i=1}^n I_i)$ (in particular $h \otimes 1 \in C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$ has spectrum contained in $[0,1]$) and*

$$\varphi, \psi : C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2 \to C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$$

*are unital *-homomorphisms with*

$$d(\operatorname{sp} \varphi(h \otimes 1)(t), \operatorname{sp} \psi(h \otimes 1)(t)) < \delta$$

for all $t \in \bigcup_{j=1}^m J_j$, then there is a unitary, u, in $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ such that

$$\|u\varphi(x)u^* - \psi(x)\| < \varepsilon$$

for all $x \in \mathcal{F}$.

Proof. Let s_1 and s_2 be the canonical generators of $\mathcal{O}_2$. Since $\mathcal{F}$ is finite and $h \otimes 1$, $1 \otimes s_1$ and $1 \otimes s_2$ generate $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$ it is enough to prove the theorem with $\mathcal{F} = \{h \otimes 1, 1 \otimes s_1, 1 \otimes s_2\}$. Let $\delta' > 0$ be the number corresponding to ε in 2.2.2. Then let δ be the number corresponding to taking ε equal to δ' in 2.1.5. Assume that φ and ψ are two unital *-homomorphisms from $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$ into $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ with

$$d(\operatorname{sp} \varphi(h \otimes 1)(t), \operatorname{sp} \psi(h \otimes 1)(t)) < \delta$$

for all $t \in \bigcup_{j=1}^m J_j$. By 2.1.5 there is a unitary, v, in $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ such that

$$\|v\varphi(h \otimes 1)v^* - \psi(h \otimes 1)\| < \delta'.$$

Then by 2.2.2 there is a unitary, w, in $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ such that

$$\|wv\varphi(1 \otimes s_l)v^*w^* - \psi(1 \otimes s_l)\| < \varepsilon$$

for $l = 1, 2$ and such that

$$\|wv\varphi(h \otimes 1)v^*w^* - \psi(h \otimes 1)\| < \varepsilon.$$

Hence $u = wv$ is the desired unitary. $\square$

Let φ and h be as in the theorem and recall that if I is an ideal in $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$, then $\hat{\varphi}(I)$ denotes the ideal $\varphi^{-1}(I)$ in $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$. If F is a closed subset of either $\bigcup_{i=1}^n I_i$ or $\bigcup_{j=1}^m J_j$, then let I_F denote the corresponding ideal in either $C(\bigcup_{i=1}^n I_i) \otimes \mathcal{O}_2$ or $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$. If $f \in C(\bigcup_{i=1}^n I_i) = C(\operatorname{sp} h)$, then $f \otimes 1 = f(h \otimes 1) \in \hat{\varphi}(I_{\{t\}})$ if and only if $f|_{\operatorname{sp}\varphi(h\otimes 1)(t)} = 0$. Hence

$$\hat{\varphi}(I_{\{t\}}) = \{f \in C(\bigcup_{i=1}^n I_i) \mid f \otimes 1 \in \hat{\varphi}(I_{\{t\}})\} \otimes \mathcal{O}_2 = I_{\operatorname{sp}\varphi(h\otimes 1)(t)}$$

and it follows that

$$\hat{\varphi}(I_F) = \hat{\varphi}(I_{\bigcup_{t\in F}\{t\}}) = \hat{\varphi}(\bigcap_{t\in F} I_{\{t\}}) = \bigcap_{t\in F} \hat{\varphi}(I_{\{t\}}) = \bigcap_{t\in F} I_{\operatorname{sp}\varphi(h\otimes 1)(t)} = I_{\bigcup_{t\in F}\operatorname{sp}\varphi(h\otimes 1)(t)}$$

(since the map $t \mapsto \operatorname{sp}\varphi(h \otimes 1)(t)$ is continuous and F is compact the set $\bigcup_{t\in F}\operatorname{sp}\varphi(h \otimes 1)(t)$ is automatically closed). From these observations the following corollary can be obtained. It will be used in the proof of the classification theorem.

Corollary 2.2.4. *Let $I_1, I_2, \ldots, I_n$ be a family of mutually disjoint, closed intervals all contained in $[0,1]$ and each containing more than one point. For all $\varepsilon > 0$ and all finite subsets, $\mathcal{F}$, of $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$, there is a $\delta > 0$ with the following property: If $J_1, J_2, \ldots, J_m$ is another family of mutually disjoint, closed intervals all contained in $[0,1]$, d is the metric on $\mathcal{I}(C(\bigcup_{i=1}^n I_i, \mathcal{O}_2))$ coming from 1.1.3 and*

$$\varphi, \psi : C(\bigcup_{i=1}^n I_i, \mathcal{O}_2) \to C(\bigcup_{j=1}^m J_j, \mathcal{O}_2)$$

*are unital *-homomorphisms with*

$$d(\hat{\varphi}(I), \hat{\psi}(I)) < \delta$$

for all ideals, I, in $C(\bigcup_{j=1}^m J_j, \mathcal{O}_2)$, then there exists a unitary, u, in $C(\bigcup_{j=1}^m J_j, \mathcal{O}_2)$ such that

$$\|u\varphi(x)u^* - \psi(x)\| < \varepsilon$$

for all $x \in \mathcal{F}$.

Remark 2.2.5. A consequence of the above is the following. Two unital *-homomorphism, φ and ψ, between two finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$ are approximately unitarily equivalent if and only if $\hat{\varphi} = \hat{\psi}$.

3. The existence theorem

3.1. This section contains the existence theorem.

Lemma 3.1.1. *Let $I_1, I_2, \ldots, I_n$ and $J_1, J_2, \ldots, J_m$ be two families of mutually disjoint, closed intervals contained in $[0,1]$ such that each of the intervals contain more than one point. If π is a continuous map from $\bigcup_{i=1}^n I_i$ into the set of non-empty, closed subsets of $\bigcup_{j=1}^m J_j$ equipped with the Hausdorff metric, then there is a self-adjoint element $h \in C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ such that*

$$\operatorname{sp} h(t) = \pi(t)$$

*for all t in $\bigcup_{i=1}^n I_i$ and such that $\mathcal{O}_2$ can be embedded into the commutant of h via a unital *-homomorphism.*

Proof. Since $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2$, [8], it is enough to find a self-adjoint element, h, such that

$$\operatorname{sp} h(t) = \pi(t)$$

for all t in $\bigcup_{i=1}^n I_i$.

Let d be the Hausdorff metric on the set of non-empty, closed subsets of $[0,1]$. For all r in $\mathbb{N}$ use 2.1.1 to find continuous functions, $\lambda_1^r, \lambda_2^r, \ldots, \lambda_{k_r}^r$, from $\bigcup_{i=1}^n I_i$ into $\bigcup_{j=1}^m J_j$ such that $\lambda_1^r(t) < \lambda_2^r(t) < \cdots < \lambda_{k_r}^r(t)$ and

$$d(\pi(t), \{\lambda_1^r(t), \lambda_2^r(t), \ldots, \lambda_{k_r}^r(t)\}) < 2^{-r}$$

for all t in $\bigcup_{i=1}^n I_i$. Then choose non-zero, orthogonal projections, $e_1^r, e_2^r, \ldots, e_{k_r}^r$, with sum 1 in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$. Define h_r in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ by setting

$$h_r(t) = \sum_{j=1}^{k_r} \lambda_j^r(t) e_j^r(t).$$

Then $(\operatorname{sp} h_r(t))_{n=1}^\infty$ is uniformly Cauchy in the set of non-empty, closed subsets of $\bigcup_{j=1}^m J_j$.

Take $\varepsilon = 2^{-1}$ in 2.1.5 and let $\delta_1 > 0$ be the corresponding δ. Find r_1 in $\mathbb{N}$ such that

$$k, l \geq r_1 \Rightarrow d(\operatorname{sp} h_k(t), \operatorname{sp} h_l(t)) < \delta_1$$

for all t in $\bigcup_{i=1}^n I_i$. Next take $\varepsilon = 2^{-2}$ in 2.1.5 to get $\delta_2 > 0$. Find r_2 in $\mathbb{N}$ such that $r_2 > r_1$ and

$$k, l \geq r_2 \Rightarrow d(\operatorname{sp} h_k(t), \operatorname{sp} h_l(t)) < \delta_2$$

for all t in $\bigcup_{i=1}^n I_i$. Continue by induction to find δ_j and r_j for all j in $\mathbb{N}$.

By 2.1.5 there are unitaries, u_j, in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ such that

$$\|h_{r_j} - u_j^* h_{r_{j+1}} u_j\| < 2^{-j}.$$

Put $h'_{r_1} = h_{r_1}$ and $h'_{r_j} = u_1^* u_2^* \cdots u_{j-1}^* h_{r_j} u_{j-1} u_{j-2} \cdots u_1$ for $j \geq 2$. Then

$$\operatorname{sp} h'_{r_j}(t) = \operatorname{sp} h_{r_j}(t)$$

for all t in $\bigcup_{i=1}^n I_i$ and $\|h'_{r_j} - h'_{r_{j+1}}\| < 2^{-j}$. It follows that $(h'_{r_j})_{j=1}^\infty$ is Cauchy. Let h be the limit of this sequence, then $\operatorname{sp} h(t) = \pi(t)$ for all t in $\bigcup_{i=1}^n I_i$. $\square$

The following is the existence theorem.

Theorem 3.1.2. *Let $I_1, I_2, \ldots, I_n$ and $J_1, J_2, \ldots, J_m$ be two families of mutually disjoint, closed intervals contained in $[0,1]$ each of the intervals containing more than one point. Let $\mathcal{D}_n$ (respectively $\mathcal{D}_m$) be the set of non-empty, closed subsets of $\bigcup_{i=1}^n I_i$ (respectively $\bigcup_{j=1}^m J_j$) equipped with the Hausdorff metric.*

Let $\Lambda : \mathcal{D}_n \to \mathcal{D}_m$ be a continuous map such that

$$\Lambda(F \cup G) = \Lambda(F) \cup \Lambda(G)$$

*for all $F, G \in \mathcal{D}_n$. Then there is a unital *-homomorphism*

$$\varphi : C(\bigcup_{j=1}^m J_j, \mathcal{O}_2) \to C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$$

with the following property: If I_F is a closed, two-sided ideal in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ with corresponding closed subset, F, in $\mathcal{D}_n$, then

$$\hat{\varphi}(I_F) = I_{\Lambda(F)}$$

*i.e. the closed, two-sided ideal $\hat{\varphi}(I_F)$ corresponds to the closed set $\Lambda(F)$ in $\mathcal{D}_m$. Furthermore if $\Lambda(\bigcup_{i=1}^n I_i) = \bigcup_{j=1}^m J_j$, then any unital *-homomorphism, which induces Λ in the above sense, is injective.*

Proof. Define $\pi : \bigcup_{i=1}^n I_i \to \mathcal{D}_m$ by setting $\pi(t) = \Lambda(\{t\})$. Then π is continuous. By 3.1.1 there is a self-adjoint element, h', in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ such that $\operatorname{sp} h'(t) = \pi(t)$ and $\mathcal{O}_2$ can be embedded into the commutant of h' via a unital *-homomorphism. Then, using that Λ is continuous and preserves finite union, it follows that

$$\Lambda(F) = \bigcup_{t \in F} \pi(t) = \bigcup_{t \in F} \operatorname{sp} h'(t)$$

(the right hand side is automatically closed). Let s_1 and s_2 be the canonical generators of $\mathcal{O}_2$ and let h be the canonical generator of $C(\bigcup_{j=1}^m J_j)$. Then $h \otimes 1$, $1 \otimes s_1$ and $1 \otimes s_2$ generates $C(\bigcup_{j=1}^m J_j) \otimes \mathcal{O}_2$ ($\cong C(\bigcup_{j=1}^m J_j, \mathcal{O}_2)$). By the above there is a self-adjoint element, h', in $C(\bigcup_{i=1}^n I_i, \mathcal{O}_2)$ and a copy of $\mathcal{O}_2$ commuting with h' such that

$$\Lambda(F) = \bigcup_{t \in F} \operatorname{sp} h'(t).$$

Mapping $h \otimes 1$ to h' and $1 \otimes s_1$ and $1 \otimes s_2$ to the canonical generators of the copy of $\mathcal{O}_2$ commuting with h' defines a unital *-homomorphism, φ. Since $\hat{\varphi}(I_F) = I_{\cup_{t \in F} \operatorname{sp} \varphi(h \otimes 1)(t)}$ it follows that φ has the desired property. The condition $\Lambda(\bigcup_{i=1}^n I_i) = \bigcup_{j=1}^m J_j$ translates to the statement that $\varphi^{-1}(0) = \{0\}$ i.e. that φ is injective. $\square$

4. Intertwining the invariant

Throughout this section the following conventions will be used. If K is a compact subset of $\mathbb{R}$, then $\mathcal{D}(K)$ will denote the set of non-empty, closed subsets of K equipped with the Hausdorff metric which will be denoted, d. If K and L are non-empty, compact subsets of $\mathbb{R}$, then a map, $\Lambda : \mathcal{D}(K) \to \mathcal{D}(L)$, will be called *liftable* if it is continuous and

$$\Lambda(F \cup G) = \Lambda(F) \cup \Lambda(G)$$

for all $F, G \in \mathcal{D}(K)$. This condition should be compared with the condition in the existence theorem 3.1.2.

4.1. Given non-empty, closed subsets, $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$, of $[0,1]$ one can ask if there is a continuous function, f, from $[0,1]$ into $[0,1]$ such that $f(F_i) = G_i$ for $i = 1, 2, \ldots, N$. In the following it is proved that under certain conditions such a function exists. The proof of the next lemma is left to the reader.

Lemma 4.1.1. *Let $\pi : [0,1] \to \mathcal{D}([0,1])$ be a continuous, descending map. For each $t \in \pi(0)$ there is a continuous function, $\gamma_t : [0,1] \to [0,1]$, such that $\gamma_t(0) = t$ and $\gamma_t(s) \in \pi(s)$ for all $s \in [0,1]$.*

Proposition 4.1.2. *Let $t_0, t_1, \ldots, t_n$ be points in $[0,1]$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$ and let $F_0, F_1, \ldots, F_n$ be non-empty, closed subsets of $[0,1]$. Assume that:*

(i) $F_n \subset F_{n-1} \subset \cdots \subset F_0$.
(ii) *There is a continuous, descending function, $\pi : [0,1] \to \mathcal{D}([0,1])$, which starts in F_0, passes through the F_j's and ends in F_n.*

Then there exists a continuous function, $f : [0,1] \to [0,1]$, such that $f(F_j) = [t_j, 1]$.

Proof. If F is a closed subset of $[0,1]$ and $t \in [0,1]$, then put

$$\operatorname{dist}(F, t) = \inf\{|s - t| \mid s \in F\}.$$

Choose $x_j \in F_j \setminus F_{j+1}$ for $j = 0, 1, ..., n-1$. Define continuous functions, $f_j : [0,1] \to [0, t_{j+1} - t_j]$, $j = 0, 1, ..., n-1$, by

$$f_j(t) = \frac{\operatorname{dist}(\{x_0, x_1, ... x_j\}, t)}{\operatorname{dist}(\{x_0, x_1, \ldots, x_j\}, t) + \operatorname{dist}(F_{j+1}, t)}(t_{j+1} - t_j).$$

By the choice of the x_j's and assumption (i) the denominator is never 0. Since $f_j(x_0) = f_j(x_1) = \cdots = f_j(x_j) = 0$ and $f_j(t) = t_{j+1} - t_j$ whenever $t \in F_i$ for some i strictly larger than j, it follows that the image of $[0,1]$ under f_j is $[0, t_{j+1} - t_j]$. Put $f = \sum_{j=0}^{n-1} f_j$, then f is continuous and the image of f is contained in $[0,1]$ since

$$f(t) = \sum_{j=0}^{n-1} f_j(t) \leq \sum_{j=0}^{n-1} (t_{j+1} - t_j) = 1.$$

If $t \in F_n$, then

$$f(t) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) = 1.$$

This proves that the image of F_n under f is $\{t_n\}$.

Let $i \in \{0, 1, ..., n-1\}$. Then $f(F_0) \subseteq [0,1]$ and if $i > 0$ and $t \in F_i$, then

$$f(t) = \sum_{j=0}^{n-1} f_j(t) = \sum_{j=0}^{i-1}(t_{j+1} - t_j) + \sum_{j=i}^{n-1} f_j(t) \geq \sum_{j=0}^{i-1}(t_{j+1} - t_j) = t_i.$$

This proves that $f(F_i)$ is contained in $[t_i, 1]$. To show that $f(F_i)$ is all of $[t_i, 1]$ choose $s \in [0,1]$, $s < 1$, such that $\pi(s) = F_i$. Now 4.1.1 gives a continuous function, $\gamma_{x_i} : [s, 1] \to [0, 1]$, such that $\gamma_{x_i}(s) = x_i$ and $\gamma_{x_i}(t) \in \pi(t)$ for all $t \in [s, 1]$. In particular $\gamma_{x_i}(1) \in F_n$. Note that $\gamma_{x_i}(t) \in F_i$ for all $t \in [s, 1]$. Consider the continuous function $f \circ \gamma_{x_i} : [s, 1] \to [0, 1]$. Since $F_i \subset F_{i-1} \subset \cdots \subset F_0$, it follows that

$$f \circ \gamma_{x_i}(t) = \sum_{j=0}^{n-1} f_j(\gamma_{x_i}(t)) = t_i + \sum_{j=i}^{n-1} f_j(\gamma_{x_i}(t)).$$

By construction $f_i(x_i) = f_{i+1}(x_i) = \cdots = f_{n-1}(x_i) = 0$ so that $f \circ \gamma_{x_i}(s) = t_i$. Since $\gamma_{x_i}(1) \in F_n$ it follows that $f \circ \gamma_{x_i}(1) = 1$. Hence the image of $f \circ \gamma_{x_i}$ is $[t_i, 1]$ and since the image of γ_{x_i} is contained in F_i, it follows that the image of F_i under f is $[t_i, 1]$. □

Applying the proposition on each of the intervals $I_1, I_2, \ldots I_m$ the following corollary is obtained.

Corollary 4.1.3. *Let $I_1, I_2, \ldots, I_m$ be a finite family of mutually disjoint, closed and bounded sub-intervals of $\mathbb{R}$ such that each of them contain more than one point. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be points in $[0,1]$ and let $F_0, F_1, \ldots, F_n$ be non-empty, closed subsets of $\bigcup_{j=1}^m I_j$. Assume that:*

(i) *$F_i \cap I_j \neq \emptyset$ for all i and j.*
(ii) *$F_n \subset F_{n-1} \subset \cdots \subset F_0$.*
(iii) *There is a continuous, descending function, $\pi : [0,1] \to \mathcal{D}(\bigcup_{j=1}^m I_j)$, which starts in F_0, passes through the F_j's and ends in F_n.*

Then there is a continuous function, $f : \bigcup_{j=1}^m I_j \to [0,1]$, such that $f(F_j) = [t_j, 1]$.

4.2. Let K be a compact subset of $\mathbb{R}$ containing at least two points and let $\varepsilon > 0$. Find a finite subset, G, of K containing at least two points such that G is $\frac{\varepsilon}{6}$-dense in K and has the property that if $t \in K \setminus G$, then there are t_1 and t_2 in G such that $t_1 < t < t_2$ and $\max\{t - t_1, t_2 - t\} < \frac{\varepsilon}{6}$.

Assume that $I_1, I_2, \ldots, I_m$ is a finite family of mutually disjoint, closed and bounded sub-intervals of $\mathbb{R}$ so that each of them contains more than one point, and assume that

$$\pi : K \to \mathcal{D}(\bigcup_{j=1}^m I_j)$$

is a continuous, descending map with the following properties:

(i) If $s, t \in G$, $s \neq t$, then $\pi(s) \neq \pi(t)$.

(ii) If C is a connected component of K and $C \cap G = \{s_1, s_2, \ldots, s_r\}$, $s_i < s_{i+1}$, and $s_r < \max G$, then there exist $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, m\}$ (both k and the indices depending on the component C and on the points $s_1, s_2, \ldots, s_r$) such that

$$\pi(s_i) \cap I_{j_l} \neq \emptyset$$

for all i and l,

$$\pi(s) \cap I_{j_l} = \emptyset$$

for all l and all $s \in G$ with $s > s_r$, and

$$\pi(s_r) \cap (\bigcup_{l=1}^{k} I_{j_l}) \subset \cdots \subset \pi(s_1) \cap (\bigcup_{l=1}^{k} I_{j_l}).$$

Theorem 4.2.1. *There is a liftable map,*

$$\Lambda : \mathcal{D}(\bigcup_{j=1}^{m} I_j) \to \mathcal{D}(K),$$

such that

$$d(\Lambda \circ \pi(t), \{s \in K \mid s \geq t\}) < \varepsilon$$

for all $t \in K$, where d is the Hausdorff metric.

Remark 4.2.2. For all $n \in \mathbb{N}$, put

$$K_n = \{s \in \mathbb{R} \mid \exists t \in K : |s - t| \leq \frac{1}{n} \text{ and } \min K \leq s \leq \max K\}.$$

Then K_n is the disjoint union of finitely many intervals, $K_{n+1} \subseteq K_n$ and $K = \bigcap_{n=1}^{\infty} K_n$. Let $\pi_n : K \to \mathcal{D}(K_n)$ be the continuous, descending map defined by $\pi_n(t) = \{s \in K \mid s \geq t\}$ (hence π_n maps into $\mathcal{D}(K)$ which is contained in $\mathcal{D}(K_n)$). In the theorem K should be thought of as the set of ideals in a C^*-algebra (without the C^*-algebra itself) of the form

$$\mathcal{A} = \varinjlim(\bigoplus_{j=1}^{l_m} C([0,1]) \otimes \mathcal{O}_2, \psi_m).$$

The form of π_n should be compared with that of the natural map in 1.2.1. The theorem then states that there is a liftable map, $\Lambda : \mathcal{D}(\bigcup_{j=1}^{m} I_j) \to \mathcal{D}(K_n)$, such that $d(\Lambda \circ \pi(t), \pi_n(t)) < \varepsilon$ for all $t \in K$. By 3.1.2 Λ can be "lifted" to a unital *-homomorphism from $C(K_n, \mathcal{O}_2)$ into $C(\bigcup_{j=1}^{m} I_j, \mathcal{O}_2)$. It follows from 1.3.2 that the natural map from $\mathcal{I}(\mathcal{A})$ to $\mathcal{I}(\mathcal{A}_m)$ will satisfy the requirements put on π in the theorem (for a large enough m which will depend on the choice of G).

Example 4.2.3. Let $K = \{0, 1\}$ and let $\pi : K \to \mathcal{D}([0,1])$ be the continuous, descending map defined by $\pi(0) = [0,1]$ and $\pi(1) = F$, where F is any proper, non-empty, closed subset of $[0,1]$. Since $\mathcal{D}([0,1])$ is connected (it is contractible) any continuous function from $\mathcal{D}([0,1])$ into $\mathcal{D}(K)$ must be constant. This proves that a condition on the map π in the theorem is needed.

Proof. Let $n \in \mathbb{N}$. Write $G = \{t_0, t_1, \ldots, t_N\}$ where $t_i < t_{i+1}$. Find i_1 as large as possible such that $t_0, t_1, \ldots, t_{i_1}$ belong to the same component, C, of K. There are now three cases to be considered $i_1 = 0$, $0 < i_1 < N$ and $i_1 = N$.

If $i_1 = 0$, then by the assumptions there is an interval, I_j, such that $\pi(t_0) \cap I_j \neq \emptyset$ and $\pi(t_i) \cap I_j = \emptyset$ for all $i > 0$. Choose a finite subset, F, of $\{s \in K \mid s \geq t_0\}$ which is $\frac{\varepsilon}{6}$-dense in this set. Define for $s \in F$ a function, $f_s : I_j \to K_n$, by putting $f_s(t) = s$ for all $t \in I_j$. Then $\bigcup_{s \in F} f_s(\pi(t_0) \cap I_j)$ is $\frac{\varepsilon}{6}$-dense in $\{s \in K \mid s \geq t_0\}$.

If $0 < i_1 < N$, then by the assumptions there are indices, $j_1, j_2, \ldots, j_k$ in $\{1, 2, \ldots, m\}$ (depending on C and the points $s_1, s_2, \ldots, s_r$), such that $\pi(t_0), \pi(t_1), \ldots, \pi(t_{i_1})$ all have non-empty intersections with all of the I_{j_l}'s while

$$\pi(t_{i_1+1}) \cap (\bigcup_{l=1}^{k} I_{j_l}) = \emptyset$$

and such that

$$\pi(t_{i_1}) \cap (\bigcup_{l=1}^{k} I_{j_l}) \subset \cdots \subset \pi(t_0) \cap (\bigcup_{l=1}^{k} I_{j_l}).$$

Use 4.1.3 to find a continuous function,

$$f : \bigcup_{l=1}^{k} I_{j_l} \to [t_0, t_{i_1}],$$

such that the image of $\pi(t_i) \cap (\bigcup_{l=1}^{k} I_{j_l})$ under this function is $[t_i, t_{i_1}]$ (which is contained in K) for all $i \leq i_1$. Next choose a finite set, F, of $\{s \in K \mid s \geq t_{i_1}\}$ which is $\frac{\varepsilon}{6}$-dense in this set. Define for all $s \in F$ a continuous function, $f_s : \bigcup_{l=1}^{k} I_{j_l} \to K_n$, by setting $f_s(t) = s$ for all $t \in \bigcup_{l=1}^{k} I_{j_l}$. Then the union of the images of $\pi(t_i) \cap (\bigcup_{l=1}^{k} I_{j_l})$ under these functions is $\frac{\varepsilon}{6}$-dense in $\{s \in K \mid s \geq t_i\}$ for all $i \leq i_1$.

If $i_1 = N$, then there are indices $j_1, j_2, \ldots, j_k$ in $\{1, 2, \ldots, m\}$ such that

$$\pi(t_i) \cap I_{j_l} \neq \emptyset$$

for all i and l and such that

$$\pi(t_n) \cap (\bigcup_{l=1}^{k} I_{j_l}) \subset \cdots \subset \pi(t_0) \cap (\bigcup_{l=1}^{k} I_{j_l}).$$

Use 4.1.3 to find a continuous function,

$$f : \bigcup_{l=1}^{k} I_{j_l} \to [t_0, t_N],$$

such that the image of $\pi(t_i) \cap (\bigcup_{l=1}^{k} I_{j_l})$ under f is $[t_i, t_N]$. Note that this interval is $\frac{\varepsilon}{6}$-dense in $\{s \in K \mid s \geq t_i\}$.

Next if $i_1 < N$, then choose $i_2 > i_1$ as large as possible such that $t_{i_1+1}, t_{i_1+2}, \ldots, t_{i_2}$ belong to the same component of K. Again there are three cases to be considered $i_2 = i_1 + 1$,

$i_1+1 < i_2 < N$, and $i_2 = N$. Find, in the same way as above, a finite family of continuous functions defined on some of the intervals $I_1, I_2, \ldots, I_m$ such that they have the following property: If $J_1, J_2, \ldots, J_s$ are the intervals on which the functions are defined and $i \in \{i_1+1, i_1+2, \ldots, i_2\}$, then the union of the images of $\pi(t_i) \cap (\bigcup_{j=1}^{s} J_j)$ under these functions is $\frac{\varepsilon}{6}$-dense in $\{s \in K \mid s \geq t_i\}$. Next i_3 is determined and so on until G is exhausted.

If f is one of the finitely many continuous functions defined above, then extend it to $\bigcup_{j=1}^{m} I_j$ by defining it to take the value $\max K$ on the intervals where it is not already defined. This gives finitely many continuous functions, $f_1, f_2, \ldots, f_M$, such that

$$d(\bigcup_{j=1}^{M} f_j(\pi(t)), \{s \in K \mid s \geq t\}) < \frac{\varepsilon}{6}$$

for all $t \in G$. Define $\Lambda : \mathcal{D}(\bigcup_{j=1}^{m} I_j^l) \to \mathcal{D}(K_n)$ by setting

$$\Lambda(F) = \bigcup_{j=1}^{M} f_j(F).$$

Then Λ is liftable and

$$d(\Lambda \circ \pi(t), \{s \in K \mid s \geq t\}) < \frac{\varepsilon}{6}$$

for all $t \in G$.

If $t \in K \setminus G$, then by the choice of G there is $i \in \{0, 1, \ldots, n-1\}$ such that $t_i < t < t_{i+1}$ and $\max\{t - t_i, t_{i+1} - t\} < \frac{\varepsilon}{6}$. Then, writing $\gamma(t)$ for the set $\{s \in K \mid s \geq t\}$,

$$d(\gamma(t), \Lambda \circ \pi(t)) \leq d(\gamma(t), \gamma(t_i)) + d(\gamma(t_i), \Lambda \circ \pi(t_i)) + d(\Lambda \circ \pi(t_i), \Lambda \circ \pi(t)).$$

Since $\Lambda \circ \pi(t_{i+1}) \subseteq \Lambda \circ \pi(t) \subseteq \Lambda \circ \pi(t_i)$, it follows that the last term is less than $d(\Lambda \circ \pi(t_i), \Lambda \circ \pi(t_{i+1}))$. Hence

$$\begin{aligned} d(\gamma(t), \Lambda \circ \pi(t)) \;\; &\leq \;\; d(\gamma(t), \gamma(t_i)) + d(\gamma(t_i), \Lambda \circ \pi(t_i)) + \\ &\quad\;\; d(\Lambda \circ \pi(t_i), \gamma(t_i)) + d(\gamma(t_i), \gamma(t)) \\ &\quad\;\; d(\gamma(t), \gamma(t_{i+1})) + d(\gamma(t_{i+1}), \Lambda \circ \pi(t_{i+1})) < \varepsilon, \end{aligned}$$

where the fact that $d(\gamma(t'), \gamma(t)) = |t' - t|$ and the above estimate of $d(\gamma(t), \Lambda \circ \pi(t))$ on the set G have been used. $\square$

4.3. This section gives the proof of the following theorem which is analogous to 4.2.1.

Theorem 4.3.1. *Let K be a compact subset of $\mathbb{R}$ containing at least two points. For all $n \in \mathbb{N}$, put*

$$K_n = \{s \in \mathbb{R} \mid \exists t \in K : |s - t| \leq \frac{1}{n} \text{ and } \min K \leq s \leq \max K\}$$

(then K_n is a finite, disjoint union of intervals, $K_{n+1} \subseteq K_n$ and $K = \bigcap_{n=1}^{\infty} K_n$). Let $\pi_n : K \to \mathcal{D}(K_n)$ be the continuous, descending map defined by

$$\pi_n(t) = \{s \in K \mid s \geq t\}$$

(hence π_n maps into $\mathcal{D}(K)$ which is contained in $\mathcal{D}(K_n)$). Let $I_1, I_2, \dots, I_m$ be a finite family of mutually disjoint, closed and bounded intervals, each containing more than one point. Finally let $\pi : K \to \mathcal{D}(\bigcup_{l=1}^{l} I_l)$ be a continuous, descending map. Then there is $n \in \mathbb{N}$ and a liftable map, $\Lambda : \mathcal{D}(K_n) \to \mathcal{D}(\bigcup_{l=1}^{m} I_l)$, such that

$$\Lambda \circ \pi_n(t) = \pi(t)$$

for all $t \in K$.

Proof. Define a map, $\Lambda : \mathcal{D}(K) \to \mathcal{D}(\bigcup_{j=1}^{l} I_j)$, by

$$\Lambda(F) = \bigcup_{t \in F} \pi(t)$$

(the right hand side is automatically closed since π is continuous and F is compact). Then Λ is liftable. Note that since π is descending $\Lambda(F) = \pi(\min F)$ and hence $\Lambda \circ \pi_n = \pi$ for any $n \in \mathbb{N}$. The aim of the proof will be to find n such that Λ extends to $\mathcal{D}(K_n)$. If π has a continuous (not necessarily descending) extension, $\tilde{\pi}$, to K_n for some n, then Λ can be extended (as a liftable map) to $\mathcal{D}(K_n)$ by defining

$$\Lambda(F) = \bigcup_{t \in F} \tilde{\pi}(t).$$

In the following π is extended to K_n for a large enough n.

Let $\mathcal{P}(\{1, 2, \dots, m\})$ be the set of non-empty subsets of $\{1, 2, \dots, m\}$. Define a continuous map, $g : K \to \mathcal{P}(\{1, 2, \dots, m\})$, by

$$g(t) = \{j \in \{1, 2, \dots, m\} \mid \pi(t) \cap I_j \neq \emptyset\}.$$

Choose $\delta > 0$ such that if $P_1, P_2 \in \mathcal{P}(\{1, 2, \dots, m\})$, $P_1 \neq P_2$, then $|s - t| \geq \delta$ for all $s \in g^{-1}(P_1)$ and all $t \in g^{-1}(P_2)$. Then choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\delta}{2}$.

By definition K_n is a disjoint union of finitely many closed intervals. Next it is proved that if I is one of these intervals, then $g^{-1}(P) \cap I \neq \emptyset$ for exactly one subset P of $\{1, 2, \dots, m\}$.

Since $I \cap K \neq \emptyset$, it follows that there is at least one subset, P, of $\{1, 2, \dots, m\}$ such that $g^{-1}(P) \cap I \neq \emptyset$. For $P \in \mathcal{P}(\{1, 2, \dots, m\})$, put

$$I_P = \{t \in I \mid \exists s \in g^{-1}(P) : |s - t| \leq \frac{1}{n}\}.$$

Then I_P is closed and I can be written as the finite union

$$I = \bigcup_{P \in \mathcal{P}(\{1,2,\dots,m\})} I_P.$$

Since I is a closed interval, it is enough to prove that the I_P's are disjoint. Let $t \in I_P$ and $t' \in I_{P'}$. Find $s' \in g^{-1}(P)$ and $s' \in g^{-1}(P')$ such that $|t - s| \leq \frac{1}{n}$ and $|t' - s'| \leq \frac{1}{n}$. By the choice of δ, $|s - s'| \geq \delta$. Hence

$$|t - t'| \geq \delta - \frac{2}{n} > 0,$$

so that $s \neq t$. This proves that $I_P \cap I_{P'} = \emptyset$.

Let I be one of the finitely many intervals which K_n consists of and let P be the unique subset of $\{1, 2, \ldots, m\}$ for which $g^{-1}(P) \cap I \neq \emptyset$. It is enough to extend the restriction of π to $g^{-1}(P) \cap I$ to I. In order to do this, consider for each $j \in P$ the continuous map from $g^{-1}(P) \cap I$ into $\mathcal{D}(I_j)$ defined by

$$t \mapsto \pi(t) \cap I_j.$$

Using that $\mathcal{D}(I_j)$ is path connected it follows that each of these maps has a continuous extension to I. $\square$

5. The classification theorem

5.1. The following is the main theorem of this paper.

Theorem 5.1.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be unital C^*-algebras which can be realized as inductive limits of sequences of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$. Let $\mathcal{I}(\mathcal{B}_1)$ (respectively $\mathcal{I}(\mathcal{B}_2)$) be the lattice of closed, two-sided ideals in $\mathcal{B}_1$ (respectively $\mathcal{B}_2$). Assume that $\mathcal{I}(\mathcal{B}_1)$ and $\mathcal{I}(\mathcal{B}_2)$ are totally ordered and order isomorphic via an order isomorphism,*

$$\hat{\Psi} : \mathcal{I}(\mathcal{B}_2) \to \mathcal{I}(\mathcal{B}_1).$$

*Then there is a *-isomorphism, $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$, which induces $\hat{\Psi}$ i.e. $\Psi^{-1}(I) = \hat{\Psi}(I)$ for all $I \in \mathcal{I}(\mathcal{B}_2)$.*

Remark 5.1.2. The proof given below does not apply in the simple case (basically because the construction in 1.2.1 does not give a simple C^*-algebra). However, it is not difficult to make a proof for the simple case. That the theorem is true in the simple case also follows from [6]. It also follows from this paper that the C^*-algebra obtained in the simple case is $\mathcal{O}_2$.

5.2. This section contains the proof of 5.1.1. For all $m \in \mathbb{N}$, let $\mathcal{B}_m = \bigoplus_{j=1}^{l_m} C([0,1]) \otimes \mathcal{O}_2$ and let $\psi_m : \mathcal{B}_m \to \mathcal{B}_{m+1}$ be a unital *-homomorphism. Put

$$\mathcal{B} = \varinjlim(\mathcal{B}_m, \psi_m).$$

Recall that $\mathcal{B}_m$ is *-isomorphic to $C(\bigcup_{j=1}^{l_m} I_j^m, \mathcal{O}_2)$, where $I_1^m, I_2^m, \ldots, I_{l_m}^m$ is a family of mutually disjoint, closed intervals contained in $[0,1]$ each containing more than one point. Denote the set of non-empty, closed subsets of $\bigcup_{j=1}^{l_m} I_j^m$ equipped with the Hausdorff metric by $\mathcal{D}(\bigcup_{j=1}^{l_m} I_j^m)$. Identifying $\mathcal{I}(\mathcal{B}_m) \setminus \{\mathcal{B}_m\}$ with $\mathcal{D}(\bigcup_{j=1}^{l_m} I_j^m)$ gives a metric on the set of closed, two-sided ideals which induces the topology defined in 1.1.1 on $\mathcal{I}(\mathcal{B}_m) \setminus \{\mathcal{B}_m\}$ (see 1.1.3).

Assume that $\mathcal{B}$ is non-simple and that $\mathcal{I}(\mathcal{B})$ is totally ordered. By 1.1.6 it follows that $\mathcal{I}(\mathcal{B})$ is order isomorphic (in particular homeomorphic) to a compact subset, K, of $[0,1]$. Since $\mathcal{B}$ is unital $\max K$ is an isolated point in K.

Let $\mathcal{A}$ be the C^*-algebra constructed in 1.2.1 with $\mathcal{I}(\mathcal{A})$ order isomorphic to K via the canonical order isomorphism constructed in 1.2.1. In particular this gives an order isomorphism

$$\hat{\Psi} : \mathcal{I}(\mathcal{B}) \to \mathcal{I}(\mathcal{A}).$$

Put $K' = K \setminus \{\max K\}$ and for $n \in \mathbb{N}$

$$K'_n = \{t \in \mathbb{R} \mid \min K' \leq t \leq \max K' \text{ and } \exists s \in K' : |s-t| \leq \frac{1}{n}\}.$$

Then K'_n is a disjoint union of finitely many intervals, $K'_{n+1} \subseteq K'_n$ and $K' = \bigcap_{n=1}^{\infty} K'_n$. Set $\mathcal{A}_n = C(K'_n) \otimes \mathcal{O}_2$, then

$$\mathcal{A} = \varinjlim(\mathcal{A}_n, \varphi_n),$$

where the φ_n's are unital *-homomorphisms (for the definition of the φ_n's see 1.2.1).

In the following it is proved that $\mathcal{A}$ and $\mathcal{B}$ are *-isomorphic via a *-isomorphism which induces $\hat{\Psi}$. The special properties of $\mathcal{A}$ will be used (see 1.2.1). Below the natural map from the lattice of ideals of an inductive limit C^*-algebra to the lattice of ideals of one of the building blocks will be denoted by π (with a subscript if needed). In diagrams the natural maps will just appear as arrows as will the maps induced by the connecting *-homomorphisms.

Lemma 5.2.1. *Let $\varepsilon > 0$ and $\rho > 0$. Assume that there is a unital *-homomorphism, $\xi : \mathcal{B}_m \to \mathcal{A}_n$, such that*

$$\begin{array}{ccc} \mathcal{I}(\mathcal{A}_n) & \longleftarrow & \mathcal{I}(\mathcal{A}) \\ {\scriptstyle \hat{\xi}}\downarrow & & \uparrow{\scriptstyle \hat{\Psi}} \\ \mathcal{I}(\mathcal{B}_m) & \longleftarrow & \mathcal{I}(\mathcal{B}) \end{array}$$

*commutes. Then there is $k \in \mathbb{N}$, $k > m$, and a unital *-homomorphism, $\eta : \mathcal{A}_n \to \mathcal{B}_k$, such that*

$$\begin{array}{ccc} \mathcal{I}(\mathcal{A}_n) & \longleftarrow & \mathcal{I}(\mathcal{A}) \\ {\scriptstyle \hat{\eta}}\uparrow & & \uparrow{\scriptstyle \hat{\Psi}} \\ \mathcal{I}(\mathcal{B}_k) & \longleftarrow & \mathcal{I}(\mathcal{B}) \end{array}$$

commutes within ε (with respect to the metric defined in 1.1.3 on $\mathcal{I}(\mathcal{A}_n)$) and

$$\begin{tikzcd} & \mathcal{I}(\mathcal{A}_n) \arrow[dl, "\hat{\xi}"'] & \\ \mathcal{I}(\mathcal{B}_m) & & \mathcal{I}(\mathcal{B}_k) \arrow[ll] \arrow[ul, "\hat{\eta}"'] \end{tikzcd}$$

commutes within ρ (with respect to the metric defined in 1.1.3 on $\mathcal{I}(\mathcal{B}_m)$).

Proof. Choose $\delta_1 > 0$ such that

$$d(I,J) < \delta_1 \Rightarrow d(\hat{\xi}(I), \hat{\xi}(J)) < \frac{\rho}{3}$$

for all I and J in $\mathcal{I}(\mathcal{A}_n)$.

As in the previous lemma identify $\mathcal{I}(\mathcal{A}_n) \setminus \{A_n\}$ with $\mathcal{D}(K'_n)$ and $\mathcal{I}(\mathcal{B}_k) \setminus \{B_k\}$ with $\mathcal{D}(\bigcup_{j=1}^{l_k} I_j^k)$ for all $k \in \mathbb{N}$. Choose a finite subset, G, of K' such that for all $t \in K' \setminus G$, there exist t_1 and t_2 in G so that $t_1 < t < t_2$ and $\max\{t - t_1, t_2 - t\} < \frac{\min\{\delta_1,\varepsilon\}}{6}$. It follows from 1.3.2, 4.2.1 and 3.1.2 that there is $k' \in \mathbb{N}$, $k' > m$, and a unital *-homomorphism, $\eta' : \mathcal{A}_n \to \mathcal{B}_{k'}$, such that

$$\begin{array}{ccc} \mathcal{I}(\mathcal{A}_n) & \longleftarrow & \mathcal{I}(\mathcal{A}) \\ {\scriptstyle \hat{\eta}'}\big\uparrow & & \big\uparrow{\scriptstyle \hat{\Psi}} \\ \mathcal{I}(\mathcal{B}_{k'}) & \longleftarrow & \mathcal{I}(\mathcal{B}) \end{array}$$

commutes within $\min\{\delta_1, \varepsilon\}$. If $I \in \mathcal{I}(\mathcal{B})$, then by the choice of δ_1 it follows that

$$\begin{aligned} d(\hat{\xi} \circ \hat{\eta}' \circ \pi_{k'}(I), \hat{\psi}_{k',m} \circ \pi_{k'}(I)) \;\; &\leq \;\; d(\hat{\xi} \circ \hat{\eta}' \circ \pi_{k'}(I), \hat{\xi} \circ \pi_n \circ \hat{\Psi}(I)) + \\ &\;\; d(\hat{\xi} \circ \pi_n \circ \hat{\Psi}(I)), \hat{\psi}_{k',m} \circ \pi_{k'}(I)) < \frac{\rho}{3}. \end{aligned}$$

Consider the two uniformly continuous maps

$$\hat{\psi}_{k',m}, \hat{\xi} \circ \hat{\eta}' : \mathcal{I}(\mathcal{B}_{k'}) \to \mathcal{I}(\mathcal{B}_m).$$

Choose $\delta_2 > 0$ such that

$$d(I, J) < \delta_2 \Rightarrow \max\{d(\hat{\psi}_{k',m}(I), \hat{\psi}_{k',m}(J)), d(\hat{\xi} \circ \hat{\eta}'(I), \hat{\xi} \circ \hat{\eta}'(J))\} < \frac{\rho}{3}$$

for all I and J in $\mathcal{I}(\mathcal{B}_{k'})$. Find $k \in \mathbb{N}$, $k > k'$, with the property that for all $I \in \mathcal{I}(\mathcal{B}_k)$ there is $J \in \mathcal{I}(\mathcal{B})$ such that $d(\hat{\psi}_{k,k'}(I), \pi_{k'}(J)) < \delta_2$.

Put $\eta = \psi_{k,k'} \circ \eta'$, then η is a unital *-homomorphism. Let $I \in \mathcal{B}_k$. Choose $J \in \mathcal{I}(\mathcal{B})$ with the above property. Then by the choice of δ_2 and the above estimate it follows that

$$\begin{aligned} d(\hat{\xi} \circ \hat{\eta}(I), \hat{\psi}_{k,m}(I)) \;\; &\leq \;\; d(\hat{\xi} \circ \hat{\eta}' \circ \hat{\psi}_{k,k'}(I), \hat{\xi} \circ \hat{\eta}' \circ \pi_{k'}(J)) + \\ &\;\; d(\hat{\xi} \circ \hat{\eta}' \circ \pi_{k'}(J), \hat{\psi}_{k',m} \circ \pi_{k'}(J)) + \\ &\;\; d(\hat{\psi}_{k',m} \circ \pi_{k'}(J), \hat{\psi}_{k',m} \circ \hat{\psi}_{k,k'}(I)) < \rho. \end{aligned}$$

By definition η makes

$$\begin{array}{ccc} \mathcal{I}(\mathcal{A}_n) & \longleftarrow & \mathcal{I}(\mathcal{A}) \\ {\scriptstyle \hat{\eta}}\big\uparrow & & \big\uparrow{\scriptstyle \hat{\Psi}} \\ \mathcal{I}(\mathcal{B}_k) & \longleftarrow & \mathcal{I}(\mathcal{B}) \end{array}$$

commute within ε. $\square$

The proof of the next lemma is very similar to the proof of the previous lemma (in fact it is easier) except that is uses 4.3.1 instead of 4.2.1. The details are left to the reader.

Lemma 5.2.2. *Let $\varepsilon > 0$. Assume that there is a unital *-homomorphism, $\eta : \mathcal{A}_n \to \mathcal{B}_m$, such that the induced map, $\hat{\eta} : \mathcal{I}(\mathcal{B}_m) \to \mathcal{I}(\mathcal{A}_n)$, makes the diagram*

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_n) & \mathcal{I}(\mathcal{A}) \arrow[l] \\ \mathcal{I}(\mathcal{B}_m) \arrow[u, "\hat{\eta}"] & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"'] \end{tikzcd}$$

*commute within $\frac{\varepsilon}{3}$. Then there is a $k \in \mathbb{N}$, $k > n$, and a unital *-homomorphism, $\xi : \mathcal{B}_m \to \mathcal{A}_k$, such that*

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_k) \arrow[d, "\hat{\xi}"'] & \mathcal{I}(\mathcal{A}) \arrow[l] \\ \mathcal{I}(\mathcal{B}_m) & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"'] \end{tikzcd}$$

commutes and

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_n) & & \mathcal{I}(\mathcal{A}_k) \arrow[ll] \arrow[dl, "\hat{\xi}"] \\ & \mathcal{I}(\mathcal{B}_m) \arrow[ul, "\hat{\eta}"] & \end{tikzcd}$$

commutes within ε with respect to the metric defined in 1.1.3 on $\mathcal{I}(\mathcal{A}_n)$.

Proposition 5.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be as above, then there is a *-isomorphism, $\Psi : \mathcal{A} \to \mathcal{B}$, which induces $\hat{\Psi}$.*

Proof. The proof consists of constructing an approximate intertwining in the sense of [5, Theorem 2.1]. See also [9].

Choose for all $n \in \mathbb{N}$ a finite set of generators, X_n, of $\mathcal{A}_n$. Put Set $F_1 = X_1$ and $F_n = \bigcup_{i=1}^{n} \varphi_{n,i}(X_i)$ whenever $n \geq 2$ ($\varphi_{n,n}$ is defined to be the identity map on $\mathcal{A}_n$). Then F_n is a finite set of generators for $\mathcal{A}_n$. For each $m \in \mathbb{N}$ choose in the same way a finite set of generators, G_m, for $\mathcal{B}_m$.

Step 0: Identifying $\mathcal{I}(\mathcal{A}_n) \setminus \{A_n\}$ with $\mathcal{D}(K'_n)$ for all $n \in \mathbb{N}$ and $\mathcal{I}(\mathcal{B}_1) \setminus \{B_1\}$ with $\mathcal{D}(\bigcup_{j=1}^{l_1} I_j^1)$ it follows from 4.3.1 and 3.1.2 that there is $n_1 \in \mathbb{N}$, and a unital *-homomorphism, $\xi_1 : \mathcal{B}_1 \to \mathcal{A}_{n_1}$, such that

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_{n_1}) \arrow[d, "\hat{\xi}_1"'] & \mathcal{I}(\mathcal{A}) \arrow[l] \\ \mathcal{I}(\mathcal{B}_1) & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"'] \end{tikzcd}$$

commutes. Set $F'_{n_1} = F_{n_1} \cup \xi_1(G_1)$.

Step 1: Let $\delta_1 > 0$ be the number corresponding to taking $\varepsilon = \frac{1}{2}$ and the finite set equal to G_1 in 2.2.4. Next take $\varepsilon = \frac{1}{2}$ and the finite set equal to F'_{n_1} in 2.2.4 to get $\delta_2 > 0$.

Use 5.2.1 with $\rho = \delta_1$ and $\varepsilon = \frac{\delta_2}{3}$ to find $m_2 \in \mathbb{N}$, $m_2 > 1$, and a unital *-homomorphism, $\eta_1 : \mathcal{A}_{n_1} \to \mathcal{B}_{m_2}$, such that

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_{n_1}) & \mathcal{I}(\mathcal{A}) \arrow[l] \\ \mathcal{I}(\mathcal{B}_{m_2}) \arrow[u, "\hat{\eta}_1"] & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"'] \end{tikzcd}$$

commutes within $\frac{\delta_2}{3}$ and

$$\begin{tikzcd} & \mathcal{I}(\mathcal{A}_{n_1}) \arrow[dl, "\hat{\xi}_1"'] & \\ \mathcal{I}(\mathcal{B}_1) & & \mathcal{I}(\mathcal{B}_{m_2}) \arrow[ll] \arrow[ul, "\hat{\eta}_1"'] \end{tikzcd}$$

commutes within δ_1. By 2.2.4 there is a unitary, u, in $\mathcal{B}_{m_2}$ such that

$$\|\mathrm{Ad}(u) \circ \eta_1 \circ \xi_1(x) - \psi_{m_2,1}(x)\| < \frac{1}{2}$$

for all $x \in G_1$. Replace η_1 with $\mathrm{Ad}(u) \circ \eta_1$. Then the above diagrams will still commute within the given values. Put $G'_{m_2} = G_{m_2} \cup \eta_1(F'_{n_1})$.

Step 2: Use 5.2.2 with $\varepsilon = \delta_2$ to find $n_2 \in \mathbb{N}$, $n_2 > n_1$, and a unital *-homomorphism, $\xi_2 : \mathcal{B}_{m_2} \to \mathcal{A}_{n_2}$, such that

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_{n_2}) \arrow[d, "\hat{\xi}_2"'] & \mathcal{I}(\mathcal{A}) \arrow[l] \\ \mathcal{I}(\mathcal{B}_{m_2}) & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"'] \end{tikzcd}$$

commutes and

$$\begin{tikzcd} \mathcal{I}(\mathcal{A}_{n_1}) & & \mathcal{I}(\mathcal{A}_{n_2}) \arrow[ll] \arrow[dl, "\hat{\xi}_2"] \\ & \mathcal{I}(\mathcal{B}_{m_2}) \arrow[ul, "\hat{\eta}_1"] & \end{tikzcd}$$

commutes within δ_2. From 2.2.4 it follows that there is a unitary, u, in $\mathcal{A}_{n_2}$ such that

$$\|\mathrm{Ad}(u) \circ \xi_2 \circ \eta_1(x) - \varphi_{n_2,n_1}(x)\| < \frac{1}{2}$$

for all x in F'_{n_1}. Replace ξ_2 with $\mathrm{Ad}(u) \circ \xi_2$. Again the above diagrams will still commute within the given values. Set $F'_{n_2} = \xi_2(G'_{m_2}) \cup F_{n_2} \cup \varphi_{n_2,n_1}(F'_{n_1})$.

Continue by induction and obtain after passing to subsequences and renumbering the following diagram of *-homomorphism and C^*-algebras

$$\begin{tikzcd}
& \mathcal{A}_1 \arrow[rr, "\varphi_1"] \arrow[dr, "\eta_1"] & & \mathcal{A}_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\eta_2"] & & \mathcal{A}_3 \arrow[r] & \cdots \arrow[r] & \mathcal{A} \\
\mathcal{B}_1 \arrow[ur, "\xi_1"] \arrow[rr, "\psi_1"'] & & \mathcal{B}_2 \arrow[ur, "\xi_2"] \arrow[rr, "\psi_2"'] & & \mathcal{B}_3 \arrow[ur, "\xi_3"] \arrow[r] & \cdots \arrow[rr] & & \mathcal{B},
\end{tikzcd}$$

which is an approximate intertwining. This gives a *-isomorphism $\rho : \mathcal{B} \to \mathcal{A}$. Let $n \in \mathbb{N}$ and $k \geq n$. Set $\rho_n^k = \mu_k^{\mathcal{A}} \circ \xi_k \circ \psi_{k,n}$, where $\mu_k^{\mathcal{A}}$ is the natural *-homomorphism from $\mathcal{A}_k$ into $\mathcal{A}$, and $\rho_n(b) = \lim \rho_n^k(b)$. Then

$$\hat{\rho}_n(I) = \lim \hat{\rho}_n^k(I)$$

for all $I \in \mathcal{I}(\mathcal{A})$. Note that by construction the following diagram

$$\begin{tikzcd}
\mathcal{I}(\mathcal{A}_n) \arrow[d, "\hat{\xi}_n"'] & \mathcal{I}(\mathcal{A}) \arrow[l] \\
\mathcal{I}(\mathcal{B}_n) & \mathcal{I}(\mathcal{B}) \arrow[l] \arrow[u, "\hat{\Psi}"']
\end{tikzcd}$$

commutes. It follows that if $I \in \mathcal{I}(\mathcal{A})$, then $\hat{\rho}_n^k(I) = \pi_n \circ \hat{\Psi}^{-1}(I)$. Hence

$$\pi_n \circ \hat{\rho} = \hat{\rho}_n = \pi_n \circ \hat{\Psi}^{-1}$$

for all $n \in \mathbb{N}$ so that $\hat{\rho} = \hat{\Psi}^{-1}$. Putting $\Psi = \rho^{-1}$ completes the proof. $\square$

Theorem 5.1.1 follows from the above.

5.3. The following theorem is the homomorphism version of 5.1.1

Theorem 5.3.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be unital C^*-algebras which can be realized as inductive limits of sequences of finite direct sums of $C([0,1]) \otimes \mathcal{O}_2$. Let $\mathcal{I}(\mathcal{B}_1)$ (respectively $\mathcal{I}(\mathcal{B}_2)$) be the lattice of closed, two-sided ideals in $\mathcal{B}_1$ (respectively $\mathcal{B}_2$). Assume that $\mathcal{I}(\mathcal{B}_1)$ and $\mathcal{I}(\mathcal{B}_2)$ are totally ordered and that*

$$\hat{\Psi} : \mathcal{I}(\mathcal{B}_2) \to \mathcal{I}(\mathcal{B}_1)$$

*is an order preserving map which is continuous with respect to the order topologies and has the property that $\hat{\Psi}^{-1}(\{\mathcal{B}_1\}) = \{\mathcal{B}_2\}$. Then there is a unital *-homomorphism, $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$, which induces $\hat{\Psi}$ i.e. $\Psi^{-1}(I) = \hat{\Psi}(I)$ for all $I \in \mathcal{I}(\mathcal{B}_2)$.*

Proof. By 5.1.1 it can be assumed that $\mathcal{B}_2$ is on standard form i.e. it is as in 1.2.1. Using 4.3.1, the existence theorem 3.1.2 and the uniqueness theorem 2.2.4 a one-sided approximate intertwining can be constructed (the unital *-homomorphisms going from the building blocks of $\mathcal{B}_1$ into the building blocks of $\mathcal{B}_2$). This uses that $\mathcal{B}_2$ is on standard form and gives a unital *-homomorphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$. To see that Ψ induces $\hat{\Psi}$ the same argument as in the proof of 5.2.3 can be used. $\square$

5.4. This final section gives an application of the the classification theorem 5.1.1. Let $\mathcal{B}$ be a non-simple, unital C^*-algebra, which can be realized as an inductive limit of a sequence of $C([0,1]) \otimes \mathcal{O}_2$. It can be assumed that the connecting *-homomorphisms in such a realization are unital. Assume that $\mathcal{I}(\mathcal{B})$ is totally ordered. Then $\mathcal{I}(\mathcal{B})$ is compact and metrizable in the order topology and has an isolated maximum. Since the space of non-empty, closed subsets of $[0,1]$ is connected (it is even contractible) it follows that $\mathcal{I}(\mathcal{B}) \setminus \{\mathcal{B}\}$ is connected. Hence $\mathcal{I}(\mathcal{B}) \setminus \{\mathcal{B}\}$ is a compact, connected Hausdorff space with more than one point. By [10, Theorem 28.8] it follows that $\mathcal{I}(\mathcal{B}) \setminus \{\mathcal{B}\}$ has at least two non-cut points (for the definition see [10, Definition 28.5]). It is not difficult to see that there are exactly two non-cut points in $\mathcal{I}(\mathcal{B}) \setminus \{\mathcal{B}\}$; the maximum and the minimum of this set. Imitating the proof of [10, Theorem 28.13] it follows that $\mathcal{I}(\mathcal{B}) \setminus \{\mathcal{B}\}$ is order isomorphic to $[0,1]$. Hence $\mathcal{I}(\mathcal{B})$ is order isomorphic to $[0,1] \cup \{2\}$. Applying 5.1.1 then gives the following theorem.

Theorem 5.4.1. *Up to *-isomorphism there is exactly one unital, non-simple C^*-algebra with totally ordered lattice of ideals which can be realized as an inductive limit a of sequence of $C([0,1]) \otimes \mathcal{O}_2$.*

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