On Stability of C^*-algebras

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Abstract

Let A be a σ-unital C^*-algebra, i.e. A admits a countable approximate unit. It is proved that A is stable, i.e. A is isomorphic to $A \otimes \mathcal{K}$ where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space, if and only if for each positive element $a \in A$ and each $\varepsilon > 0$ there exists a positive element $b \in A$ such that $\|ab\| < \varepsilon$ and $x^*x = a$, $xx^* = b$ for some x in A.

Using this characterization it is proved among other things that the inductive limit of any sequence of σ-unital stable C^*-algebras is stable, and that the crossed product of a σ-unital stable C^*-algebra by a discrete group is again stable.

1 Introduction

One can characterize stable AF-algebras as being precisely those AF-algebras that do not admit a bounded densely defined trace. This can be seen by using the classification of AF-algebras by their ordered K_0-group (see also Section 5). One motivation for this paper is if a similar strong characterization of stable C^*-algebras might hold in general (see Section 5). Another motivation is to decide whether stability is closed under some natural operations such as the ones mentioned in the abstract, if an extension of two stable C^*-algebras always is stable, and if one can conclude that A is stable if $M_2(A)$ is stable. These question would be easy to answer in the affirmative, if a characterization of stability, like the one that holds for AF-algebras, were valid in general.

We give in this paper a (weaker) characterization of stable C^*-algebras, as described in the abstract (cf. Theorem 2.1 and the remarks at the end of Section 2). With this characterization it is easy to prove the claims in the second paragraph of the abstract (see Section 4). However, our methods do not in an obvious way provide an answer to the other questions stated above.

Our characterization result can be viewed as a generalization of a theorem of Shuang Zhang [9, Thm. 1.2], that every non-unital purely infinite simple C^*-algebra is stable (cf. Prop. 5.1). In Section 5 we also discuss how stability of C^*-algebras is related to the structure problem for simple C^*-algebras: if every simple, unital C^*-algebra is either stably finite or purely infinite.

2 Characterization of stable C^*-algebras

The main result of this section (Theorem 2.1 below) gives a characterization of stable C^*-algebras. The C^*-algebras in question are assumed to be σ-unital, i.e. they admit a countable approximate unit. Recall that an element a in a C^*-algebra A is *strictly positive* if $\varphi(a) > 0$ for every non-zero positive linear functional φ on A, and that A contains a strictly positive element if and only if A is σ-unital (see [6, Prop. 3.10.5]). Recall also that every separable C^*-algebra is σ-unital.

Denote by A^+ the positive cone of A. For positive elements a, b in A define:

$$\begin{aligned} a \sim b \quad &\Leftrightarrow \quad \exists x \in A : x^*x = a \quad \text{and} \quad xx^* = b, \\ a \perp b \quad &\Leftrightarrow \quad ab = 0 \quad (= ba). \end{aligned}$$

The following functions will be used throughout this paper. For each $\varepsilon > 0$ define (continuous) functions $h_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $f_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \varepsilon \\ t - \varepsilon & \text{if} \quad t \ge \varepsilon \end{cases}, \qquad f_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \varepsilon \\ \varepsilon^{-1}t - 1 & \text{if} \quad \varepsilon \le t \le 2\varepsilon \\ 1 & \text{if} \quad t \ge 2\varepsilon \end{cases}.$$

Let $F(A)$ denote the set of positive elements which have a multiplicative identity, i.e.

$$F(A) = \big\{ a \in A^+ \mid \exists\, b \in A^+ : ab = a \big\}.$$

Note that if $a \in A^+$ then $h_\varepsilon(a)$ and $f_\varepsilon(a)$ belong to $F(A)$ for all $\varepsilon > 0$ because $f_{\varepsilon/2}(a)$ is a multiplicative identity for these elements. For all $\varepsilon > 0$ the function h_ε has the property that $\|a - h_\varepsilon(a)\| \le \varepsilon$ for all $a \in A^+$, and hence $F(A)$ is dense in A^+.

If $a \in F(A)$, then there is $e \in F(A)$ with $ea = a$ $(= ae)$ and $\|e\| = 1$. To see this note first that if $a, b \in A^+$ satisfy $ab = a$, then for each continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $f(1) = 1$ we have $f(b)a = a$ (this is clearly true, if f is a polynomial with $f(1) = 1$). Hence, $e = f_{\frac{1}{2}}(b)$ will have the desired properties.

Theorem 2.1 *Let A be a C^*-algebra which is σ-unital. The following three statements are equivalent:*

(a) A is stable.

(c) For all $a \in F(A)$ there exists $b \in A^+$ such that $a \sim b$ and $a \perp b$.

(e) There is a sequence of mutually orthogonal and equivalent projections $(E_n)_{n=1}^{\infty}$ in $M(A)$, the multiplier algebra of A such that the infinite sum $\sum E_n$ converges to the unit 1 in the strict topology on $M(A)$.

For the proof of the theorem we need some preliminary results. Denote by $\mathcal{U}_0(\tilde{A})$ the connected component of the group of unitary elements in $\tilde{A}$ that contains the unit. We begin by rephrasing condition (c):

Proposition 2.2 *Let A be a C^*-algebra. The following three statements are equivalent:*

(b) For all $a \in F(A)$ and all $\varepsilon > 0$ there are $b, c \in A^+$ such that $\|a - b\| < \varepsilon$, $b \sim c$ and $\|bc\| < \varepsilon$.

(c) For all $a \in F(A)$ there exists $b \in A^+$ such that $a \sim b$ and $a \perp b$.

(d) For all $a \in F(A)$ there exists a unitary $u \in \mathcal{U}_0(\tilde{A})$ such that $uau^* \perp a$.

For the proof of the proposition we need some lemmas.

Lemma 2.3 *Let A be a C^*-algebra and assume $b, c \in A^+$ satisfy $b \sim c$. Then for each $\varepsilon > \|b^{\frac{1}{2}}c^{\frac{1}{2}}\|^{\frac{1}{4}}$ there exists a unitary $u \in \mathcal{U}(\tilde{A})$ such that $uf(b)u^* = f(c)$ for each continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ being zero on $[0, \varepsilon]$.*

Proof: Let $\varepsilon > 0$ be given. Assume $x \in A$ satisfies $x^*x = b$ and $xx^* = c$. Let $x = vb^{\frac{1}{2}}$ be the polar decomposition of x, where v is a partial isometry in A^{**}. Notice that $x = c^{\frac{1}{2}}v$. By assumption, $\|x^2\|^{\frac{1}{2}} = \|vb^{\frac{1}{2}}c^{\frac{1}{2}}v\|^{\frac{1}{2}} = \|b^{\frac{1}{2}}c^{\frac{1}{2}}\|^{\frac{1}{2}} < \varepsilon^2$. Since $\|x^2\|^{\frac{1}{2}} \geq \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(x)\}$, the spectral radius of x, we obtain that $\mathrm{dist}(x, \mathrm{GL}(A)) < \varepsilon^2$, i.e. the distance from x to the invertibles of A is less than ε^2.

For each $t \in \mathbb{R}^+$ set $E_t = 1_{[0,t]}(|x|) \in A^{**}$, the spectral projection corresponding to the interval $[0,t]$ for $|x|$. By [7, Thm. 2.2] there is a unitary u in $\mathcal{U}(\tilde{A})$ such that $v(1 - E_{\varepsilon^2}) = u(1 - E_{\varepsilon^2})$. By this identity we obtain for all continuous functions $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ being zero on $[0, \varepsilon^2]$ that $vg(|x|) = ug(|x|)$. Since $vg(|x|)v^* = g(|x^*|)$, we get that $g(|x^*|) = ug(|x|)u^*$.

Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be any continuous function which is zero on $[0, \varepsilon]$, and set $g(t) = f(t^2)$. Then $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, and g is zero on $[0, \varepsilon^2]$. It follows that

$$f(c) = f(|x^*|^2) = g(|x^*|) = ug(|x|)u^* = uf(|x|^2)u^* = uf(b)u^*,$$

as desired. □

The lemma below can be proved by approximating the square root $a^{1/2}$ of a positive element a in a C^*-algebra by elements $p(a)$ for suitable polynomials p with vanishing constant term.

Lemma 2.4 *For each $\varepsilon > 0$ and for each $K < \infty$ there is $\delta > 0$ so that for every C^*-algebra A and for every pair of positive elements b, c in A, if $\|b\| \le K$, $\|c\| \le K$, and $\|bc\| \le \delta$, then $\|b^{1/2}c^{1/2}\| \le \varepsilon$, and $\|b^{1/2}c\| \le \varepsilon K^{1/2}$.*

Lemma 2.5 *If A is a C^*-algebra satisfying property (b) of Proposition 2.2, then for each $a \in F(A)$ and each $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(\tilde{A})$ such that $\|auau^*\| < \varepsilon$.*

Proof: Let $a \in F(A)$ and $\varepsilon > 0$ be given. We may without loss of generality assume that $\|a\| \le 1$. Find $\delta > 0$ such that $7\delta + 4\delta^2 < \varepsilon$. By Lemma 2.4 and by the assumption that property (b) of Proposition 2.2 holds we can find $b, c \in A^+$ satisfying $\|b - a\| < \delta$, $b \sim c$, $\|bc\| < \delta$ and $\|b^{1/2}c^{1/2}\| < \delta^4$. By Lemma 2.3 there is a unitary $u \in \mathcal{U}(\tilde{A})$ such that $h_\delta(c) = uh_\delta(b)u^*$. Notice that $\|a - h_\delta(b)\| < 2\delta$ (because $\|a - b\| < \delta$ and $\|b - h_\delta(b)\| \le \delta$) and notice also that $\|b\|$ and $\|c\|$ are less than $1 + \delta$. We can now make the following estimate:

$$
\begin{aligned}
&\|auau^*\| \\
\leq\ & \|auau^* - h_\delta(b)uau^*\| + \|h_\delta(b)uau^* - h_\delta(b)uh_\delta(b)u^*\| + \|h_\delta(b)h_\delta(c) - bc\| + \|bc\| \\
<\ & 2\delta + \|h_\delta(b)\|2\delta + \delta(\|b\| + \|h_\delta(c)\|) + \delta \\
\leq\ & 2\delta + (1+\delta)2\delta + \delta(2+2\delta) + \delta \ = \ 7\delta + 4\delta^2 \ < \ \varepsilon.
\end{aligned}
$$

□

Proof of Proposition 2.2:

(b) $\Rightarrow$ (c): Let $a \in F(A)$ and find $e \in F(A)$ such that $ae = ea = a$. By lemmas 2.4 and 2.5 there is a unitary $u \in \mathcal{U}(\tilde{A})$ such that $(\delta =)\ \|(ueu^*)^{\frac{1}{2}} e (ueu^*)^{\frac{1}{2}}\| < 1$. Set $x = ue^{\frac{1}{2}}$, set $y = (1-e)^{\frac{1}{2}} x a^{\frac{1}{2}}$, and observe that $y^*y \perp yy^*$. Let $v|x|$ be the polar decomposition for x, where v is a partial isometry in A^{**}.

Notice that $|x^*| = (xx^*)^{\frac{1}{2}} = (ueu^*)^{\frac{1}{2}}$. Hence $x^*ex = v^*|x^*|e|x^*|v = v^*(ueu^*)^{\frac{1}{2}} e (ueu^*)^{\frac{1}{2}} v$ which shows that $\|x^*ex\| = \delta$. Therefore $y^*y = a - a^{\frac{1}{2}} x^*exa^{\frac{1}{2}} \geq (1-\delta)a$. By [6, Prop. 1.4.5] there is $r \in A$ such that $a = r^*(y^*y)^{\frac{1}{2}} r$. Let $w|y|$ be the polar decomposition of y and put $z = w|y|^{\frac{1}{2}} r$. Then $z^*z = r^*|y|^{\frac{1}{2}} w^*w|y|^{\frac{1}{2}} r = r^*|y|r = a$ and $zz^* = w|y|^{\frac{1}{2}} rr^*|y|^{\frac{1}{2}} w^* = |y^*|^{\frac{1}{2}} wrr^*w^*|y^*|^{\frac{1}{2}}$. Since $|y^*| \perp a$ it follows that $zz^* \perp a$, and we may set $b = zz^*$.

(c) $\Rightarrow$ (d): Let $a \in F(A)$ and find $e \in F(A)$ with $\|e\| = 1$ and $ea = ae = a$. By (c) there are $f \in F(A)$ orthogonal to e and $x \in A$ such that $x^*x = e$ and $xx^* = f$. Because $x + x^*$ is a self-adjoint element of norm ≤ 1, and $(x+x^*)^2 = xx^* + x^*x$, it follows that

$$
u = x + x^* + i(1 - xx^* - x^*x)^{1/2} \in \mathcal{U}_0(\tilde{A}).
$$

Also, $uau^* = xax^* \perp a$ as desired.

(d) $\Rightarrow$ (b): Take $b = a$ and $c = uau^*$. □

For every strictly positive element a in A define

$$
F_a(A) = \big\{ b \in A^+ \mid \exists\ \varepsilon > 0 : f_\varepsilon(a)b = b \big\}.
$$

Notice that $F_a(A) \subseteq F(A)$.

Lemma 2.6 *Let A be a σ-unital C^*-algebra which satisfies property (c) of Theorem 2.1. For every strictly positive element $a \in A^+$ it follows that:*

(i) For all $b \in F_a(A)$ there exists $c \in F_a(A)$ with $b \sim c$ and $b \perp c$.

(ii) For all $\varepsilon > 0$ there is a projection $G \in M(A)$ satisfying $1 - G \perp f_\varepsilon(a)$, $G \sim 1$, and $1 - G \gtrsim 1$.

In order to prove Lemma 2.6 we need the some facts about properly infinite projections summarized in the remarks and in the lemma below. A C^*-subalgebra B of a C^*-algebra A is said to be *full* if it is not contained in any proper two-sided closed ideal of A. A projection p in A is *full* if the (hereditary) C^*-subalgebra pAp is full in A. A projection p is said to be *properly infinite* if there exist two projections p_1 and p_2, each Murray-von Neumann equivalent to p, such that $p_1 + p_2 \leq p$ (in particular $p_1 \perp p_2$).

If p is a properly infinite, full projection and if $p \leq q$, then q is properly infinite and full. If p and q are properly infinite, full projections, then $p \lesssim q$ and $q \lesssim p$. It can be deduced from [4, Section 1] that any two properly infinite full projections in a C^*-algebra A are Murray-von Neumann equivalent if they define the same element of $K_0(A)$, and that if A contains at least one properly infinite, full projection, then every element of $K_0(A)$ is represented by a properly infinite, full projection. The lemma below follows easily from these facts:

Lemma 2.7 *Let A be a unital C^*-algebra. If e and f are projections in A such that $f \leq e$ and $e - f$ dominates a properly infinite projection which is full in A, then there is a projection q in A such that*

(i) $f \leq q \leq e$, and

(ii) $q \sim 1$ and $e - q \gtrsim 1$.

Proof of Lemma 2.6:

(i). Suppose $a \in A^+$ is strictly positive. Let $b \in F_a(A)$ and find $\varepsilon > 0$ such that $bf_\varepsilon(a) = b = f_\varepsilon(a)b$. Since $f_\varepsilon(a) \in F(A)$ there exists $y \in A$ such that $f_\varepsilon(a) = yy^*$ and $f_\varepsilon(a) \perp y^*y$. Because a is strictly positive, there exists a $\delta > 0$ such that $(\delta_0 =)$ $\|yf_\delta(a)^2y^* - f_\varepsilon(a)\| < \frac{1}{2}$.

There is $r \in A$ such that $ryf_\delta(a)^2y^*r^* \geq f_{\frac{1}{2}}(f_\varepsilon(a))$ (one may take $r = (\frac{1}{2} - \delta_0)^{\frac{1}{2}} f_{\frac{1}{2}}(f_\varepsilon(a))^{\frac{1}{2}}$, cf. [8, Prop. 2.2]), and by [6, Prop. 1.4.5] there is an $s \in A$ such that

$$s(ryf_\delta(a)^2y^*r^*)^{\frac{1}{2}}s^* = f_{\frac{1}{2}}(f_\varepsilon(a)).$$

Observe that $f_{\frac{1}{2}}(f_\varepsilon(a))b = b = bf_{\frac{1}{2}}(f_\varepsilon(a))$. Let $v|ryf_\delta(a)|$ be the polar decomposition of $ryf_\delta(a)$, where $v \in A^{**}$ is a partial isometry, and put $x = b^{\frac{1}{2}}sv|ryf_\delta(a)|^{\frac{1}{2}} \in A$. Then

$$\begin{aligned} xx^* &= b^{\frac{1}{2}}sv|ryf_\delta(a)|v^*s^*b^{\frac{1}{2}} = b^{\frac{1}{2}}s|(ryf_\delta(a))^*|s^*b^{\frac{1}{2}} \\ &= b^{\frac{1}{2}}s(ryf_\delta(a)^2y^*r^*)^{\frac{1}{2}}s^*b^{\frac{1}{2}} = b^{\frac{1}{2}}f_{\frac{1}{2}}(f_\varepsilon(a))b^{\frac{1}{2}} = b. \end{aligned}$$

Since $f_{\delta/2}(a)f_\delta(a) = f_\delta(a)$ we see that $f_{\delta/2}(a)x^*x = x^*x$, and so $x^*x \in F_a(A)$. Also,

$$|ryf_\delta(a)|^2 f_\varepsilon(a) = f_\delta(a)y^*r^*ryf_\delta(a)f_\varepsilon(a) = f_\delta(a)y^*r^*ryf_\varepsilon(a)f_\delta(a) = 0,$$

from which we see that $x^*x \perp f_\varepsilon(a)$, and hence $x^*x \perp b$. We may therefore set $c = x^*x$.

(ii). Suppose $a \in A$ is a strictly positive element of norm 1. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$ let $g_n\colon [0,1] \to \mathbb{R}^+$ be piecewise linear functions satisfying

(α) g_1 is zero on $[0, \frac{1}{2}]$,

(β) g_n is zero outside the interval $[\frac{1}{n+1}, \frac{1}{n-1}]$ for all $n \geq 2$,

(γ) $\sum_{n=1}^\infty g_n(t) = 1$ for all $t \in (0,1]$.

Then the infinite sum $\sum g_n(a)$ converges strictly to the unit 1 in $M(A)$.

We shall inductively construct sequences $(b_n)_{n=1}^\infty$, $(c_n)_{n=1}^\infty$ in $F_a(A)$ and $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in A such that the set $\{b_n \mid n \in \mathbb{N}\} \cup \{c_n \mid n \in \mathbb{N}\}$ consists of mutually orthogonal elements,

$$b_n = x_n^*x_n, \quad c_n = y_n^*y_n, \quad g_n(a) = x_nx_n^* = y_ny_n^*,$$

and such that b_n is orthogonal to $f_\varepsilon(a)$, and b_n and c_n are orthogonal to $f_{1/n}(a)$.

Let $n \in \mathbb{N}$ be given, and suppose, if $n \geq 2$, that elements b_j, c_j, x_j, y_j, $j \leq n-1$, with the desired properties have been constructed. Choose δ with $0 < \delta < \frac{1}{2}\min\{\varepsilon, 1/(n+1)\}$ and such that $f_\delta(a)$ is a (two-sided) multiplicative identity for the elements $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}$ in $F_a(A)$. By (i), there are $d \in F_a(A)$ and $z \in A$ such that $d \perp f_\delta(a)$, $z^*z = d$ and

$zz^* = f_\delta(a)$. Set $x_n = g_n(a)^{\frac{1}{2}} z$. Then $x_n^* x_n = z^* g_n(a) z$ and $x_n x_n^* = g_n(a)$. By letting $b_n = x_n^* x_n$ we obtain that $b_n \in F_a(A)$ and $b_n \perp f_\delta(a)$, because b_n lies in the hereditary C^*-subalgebra of A generated by d. Hence b_n is orthogonal to $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}$, $f_\varepsilon(a)$ and $f_{1/n}(a)$. By the same argument we can construct c_n in $F_a(A)$ and y_n in A.

Because a is strictly positive, it follows that $(f_{1/n}(a))_{n=1}^\infty$ is an approximate unit for A. Since each of x_n, y_n, b_n, c_n is orthogonal to $f_{1/(n-1)}(a)$, we see that the infinite sums $B = \sum b_n$, $C = \sum c_n$, $V = \sum x_n$ and $W = \sum y_n$ are strictly convergent, and therefore belong to $M(A)$. Since $b_i \perp c_j$ for all $i, j \in \mathbb{N}$, B and C are orthogonal. Also, since all b_n are orthogonal to $f_\varepsilon(a)$, so is B. By orthogonality of the sequence $(b_n)_{n=1}^\infty$ we see that $VV^* = \sum x_n x_n^* = \sum g_n(a) = 1$, and, similarly, $WW^* = 1$. It follows that $Z_1 = V^*V$ and $Z_2 = W^*W$ are projections in $M(A)$ that are equivalent to 1. Moreover, Z_1 lies in the hereditary subalgebra of $M(A)$ generated by B, and Z_2 lies in the hereditary subalgebra of $M(A)$ generated by C. Hence $Z_1 \perp Z_2$ and $Z_1 \perp f_\varepsilon(a)$. We have thus shown that the unit 1 in $M(A)$ is properly infinite.

Now, Z_1, being equivalent to 1, is properly infinite and full in $M(A)$. Lemma 2.7 then provides a projection G such that $1 - Z_1 \leq G \leq 1$, $G \sim 1$ and $1 - G \gtrsim 1$. Since $Z_1 \perp f_\varepsilon(a)$ we obtain that $1 - G \perp f_\varepsilon(a)$.

□

Proof of Theorem 2.1:

(a) $\Rightarrow$ (c): It suffices to show that (a) implies property (b) of Proposition 2.2. Assume A is stable. Then A is isomorphic to the C^*-algebra $B = \overline{\bigcup_{n=1}^\infty M_n(A)}$ where $M_n(A)$ is embedded in the upper left hand corner of $M_{n+1}(A)$.

Let $a \in B^+$ and $\varepsilon > 0$ be given. Find $n \in \mathbb{N}$ and $c \in M_n(A)^+$ such that $\|a - c\| < \varepsilon$. Let $x \in M_{2n}(A)$ be defined by

$$x = \begin{pmatrix} 0 & c^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}.$$

Then $x^*x \perp xx^*$ and $\|a - xx^*\| < \varepsilon$, and we are done.

(c) $\Rightarrow$ (e): Let $a \in A$ be strictly positive with $\|a\| = 1$. We construct inductively a sequence of mutually orthogonal projections $E_1, E_2, \dots \in M(A)$ so that

$$E_1 \sim E_2 \sim \cdots \sim E_n \sim 1, \quad 1 - (E_1 + E_2 + \cdots + E_n) \gtrsim 1, \quad \|(1 - (E_1 + E_2 + \cdots + E_n))a\| < 1/n$$

holds for each $n \in \mathbb{N}$. The infinite sum $\sum E_n$ will then converge strictly to 1, and the proof will be completed. We shall in the following use the fact that $\|(1 - E)a\| \le \varepsilon$ if $1 - E \perp f_\varepsilon(a)$.

The existence of E_1 follows from Lemma 2.6. Suppose $n \ge 1$ and that $E_1, E_2, \dots, E_n$ have been found. Set $E = E_1 + E_2 + \cdots + E_n$. Then $1 - E \gtrsim 1$ and $\|(1 - E)a\| < 1/n$. We must find a multiplier projection E_{n+1} such that

$$E_{n+1} \perp E, \quad E_{n+1} \sim 1, \quad 1 - (E + E_{n+1}) \gtrsim 1, \quad \|(1 - E - E_{n+1})a\| < \tfrac{1}{n+1}.$$

Since $1 - E \gtrsim 1$ there is a projection $F \in M(A)$ such that $1 - E \ge F$ and $F \sim 1$. The latter implies that the C^*-algebra FAF is isomorphic to A.

We assert that FaF is a strictly positive element of FAF. Assume to the contrary that $\varphi(FaF) = 0$ for a non-zero positive functional φ on FAF. Let $\tilde{\varphi}$ be the positive functional on A defined by $\tilde{\varphi}(x) = \varphi(FxF)$ for $x \in A$. Since $\tilde{\varphi}(a) = 0$ and $a \in A$ is strictly positive it follows that $\tilde{\varphi} = 0$. But then $\varphi = 0$.

Notice that $M(FAF) = FM(A)F$. Lemma 2.6 (ii) provides for $\varepsilon > 0$, chosen such that $\varepsilon + \varepsilon^{\frac{1}{2}} < 1/(n+1)$, a projection $G \in FM(A)F$ satisfying

$$G \sim F \sim 1, \qquad F - G \gtrsim F \sim 1, \qquad \|FaF - G(FaF)\| \; (= \|(F - G)aF\|) \; \le \varepsilon.$$

We proceed to show that $\|(F - G)a\| < 1/(n+1)$. Since $\|a\| = 1$ we obtain that

$$\begin{aligned} \|(F - G)a(1 - F)\| &\le \|(F - G)a(F - G)\|^{\frac{1}{2}} \|(1 - F)a(1 - F)\|^{\frac{1}{2}} \\ &\le \|(F - G)a(F - G)\|^{\frac{1}{2}} \\ &\le \|(F - G)aF\|^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}}. \end{aligned}$$

The first estimate follows from the inequality $\|paq\|^2 \le \|pap\|\|qaq\|$, which holds when a is positive and p, q are projections. Hence

$$\|(F - G)a\| \le \|(F - G)aF\| + \|(F - G)a(1 - F)\| < \varepsilon + \varepsilon^{\frac{1}{2}} < \frac{1}{n+1}.$$

Lemma 2.7 provides a projection $E_{n+1} \in M(A)$ such that

$$1 - E - (F - G) \leq E_{n+1} \leq 1 - E, \qquad E_{n+1} \sim 1, \qquad 1 - E - E_{n+1} \gtrsim 1.$$

Since $\|(1 - E - E_{n+1})a\| \leq \|(F - G)a\| < 1/(n+1)$, the projection E_{n+1} is as wanted.

(e) $\Rightarrow$ (a): Set $P_n = \sum_{j=1}^{n} E_j$, and let A_n be the hereditary C^*-subalgebra $P_n A P_n$ of A. Since the projections E_j are mutually equivalent, there are partial isometries $V_1, V_2, V_3, \dots$ in $M(A)$ such that $V_1 = E_1$, $V_j^* V_j = E_1$ and $V_j V_j^* = E_j$ for all $j \geq 2$. Let M_n denote the C^*-algebra of n by n matrices over $\mathbb{C}$, and let $\{e_{ij}\}$ be the standard system of matrix units for M_n. Define an isomorphism $\sigma_n \colon A_n \to M_n \otimes A_1$ by

$$\sigma_n(b) = \sum_{1 \leq i,j \leq n} e_{ij} \otimes V_i^* b V_j, \qquad b \in A_n.$$

For each $n \in \mathbb{N}$ let $\iota_{A_n} \colon A_n \to A_{n+1}$ be the inclusion map, let $\psi_n \colon M_n \to M_{n+1}$ be the embedding into the upper left-hand corner, and define $\varphi_n \colon M_n \otimes A_1 \to M_{n+1} \otimes A_1$ by $\varphi_n = \psi_n \otimes \mathrm{id}_{A_1}$. From the construction of σ_n we see that $\varphi_n \circ \sigma_n = \sigma_{n+1} \circ \iota_{A_1}$.

The inductive limit C^*-algebra $\lim_{n\to\infty}(A_n, \iota_{A_n})$ is equal to $\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{\bigcup_{n=1}^{\infty} P_n A P_n}$. Since $(P_n)_{n=1}^{\infty}$ converges strictly to 1 this union is A. All in all we obtain the following commuting diagram:

$$\begin{array}{ccccccccccccc}
A_1 & \overset{\iota_{A_1}}{\hookrightarrow} & A_2 & \hookrightarrow & \cdots & \hookrightarrow & A_n & \overset{\iota_{A_n}}{\hookrightarrow} & A_{n+1} & \hookrightarrow & \cdots & \to & A \\
\| & & \sigma_2 \downarrow \cong & & & & \sigma_n \downarrow \cong & & \sigma_{n+1} \downarrow \cong & & & & \\
A_1 & \xrightarrow{\varphi_1} & M_2 \otimes A_1 & \longrightarrow & \cdots & \longrightarrow & M_n \otimes A_1 & \xrightarrow{\varphi_n} & M_{n+1} \otimes A_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{K} \otimes A_1
\end{array}$$

This intertwining yields an isomorphism $A \to \mathcal{K} \otimes A_1$, and therefore A is stable. $\square$

In the abstract we claimed that a σ-unital C^*-algebra A is stable if and only if for each positive element $a \in A$ and each $\varepsilon > 0$ there exists a positive element $b \in A$ such that $\|ab\| < \varepsilon$ and $x^*x = a$, $xx^* = b$ for some x in A. To see this assume first that A is stable, and let $a \in A^+$ and $\varepsilon > 0$ be given. Find $a_0 \in F(A)$ with $\|a - a_0\|(\|a\| + \|a_0\|) < \varepsilon$ and use property (d) of Proposition 2.2 to find a unitary $u \in \tilde{A}$ such that $ua_0u^* \perp a_0$. Set $x = ua^{1/2}$, and set $b = xx^* = uau^*$. Then $a = x^*x$, and

$$\|ab\| \leq \|a_0ua_0u^*\| + \|ab - a_0ua_0u^*\| \leq 0 + \|ab - aua_0u^*\| + \|aua_0u^* - a_0ua_0u^*\| < \varepsilon.$$

Conversely, assume that the property in the abstract holds, let $a \in F(A)$ and let $\varepsilon > 0$ be given. Then there exist $x \in A$ and $c \in A^+$ such that $x^*x = a$, $xx^* = c$, and $\|ac\| < \varepsilon$. Setting $b = c$ we see that property (b) of Proposition 2.2 holds, and therefore A is stable by Theorem 2.1 and Proposition 2.2.

3 Characterization of stability in terms of projections

The characterization theorem (Theorem 2.1) has a simpler form — and its proof is more direct — for C^*-algebras that admit a countable approximate unit consisting of projections. Theorem 3.3 below is a reformulation of Theorem 2.1 in the case where the C^*-algebra admits a countable approximate unit consisting of projections, and we give a self contained proof of this reformulated theorem.

Lemma 3.1 *Let A be a C^*-algebra and let $(p_n)_{n=1}^\infty$ be an approximate unit for A consisting of projections. For every projection $q \in A$ there exist a sequence of projections $(\tilde{p}_n)_{n=1}^\infty$ in A such that $\tilde{p}_n \geq q$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\tilde{p}_n - p_n\| = 0$.*

Proof: Since $(p_n)_{n=1}^\infty$ is an approximate unit for A we get that $\|p_nqp_n - q\| \to 0$ as $n \to \infty$, and hence $\|(p_nqp_n)^2 - p_nqp_n\| \to 0$ as $n \to \infty$. By a continuous function calculus argument there is a sequence of projections $(\hat{p}_n)_{n=1}^\infty$ such that $\hat{p}_n \in p_nAp_n$ and $\|p_nqp_n - \hat{p}_n\| \to 0$ as $n \to \infty$. Hence $\|\hat{p}_n - q\| \to 0$ as $n \to \infty$ by the triangle inequality.

Find $n_0 \in \mathbb{N}$ such that $\|\hat{p}_n - q\| < 1$ for all $n \geq n_0$. For every $n \geq n_0$ there are unitaries $u_n \in \tilde{A}$ such that $\hat{p}_n = u_nqu_n^*$ and $\|u_n - 1\| \to 0$ as $n \to \infty$. Set $\tilde{p}_n = u_n^*p_nu_n$ for $n \geq n_0$, and set $\tilde{p}_n = q$ otherwise. Then $\tilde{p}_n \in A$, $\tilde{p}_n \geq q$ and $\|\tilde{p}_n - p_n\| \to 0$ as $n \to \infty$. □

For a C^*-algebra A denote by $P(A)$ the set of projections of A.

Lemma 3.2 *Let A be a C^*-algebra and let $(p_n)_{n=1}^\infty$ be an approximate unit for A consisting of projections. The following statements are equivalent:*

(i) For all $p, q \in P(A)$ there exists $r \in P(A)$ such that $p \perp r$ and $q \sim r$.

(ii) For all $p \in P(A)$ there exists $q \in P(A)$ such that $p \perp q$ and $p \sim q$.

Proof: (i) ⇒ (ii) is trivial.

(ii) $\Rightarrow$ (i). Let $p, q \in P(A)$. By Lemma 3.1 there is a sequence $(\tilde{p}_n)_{n=1}^{\infty}$ of projections satisfying

$$p \leq \tilde{p}_n, \qquad \lim_{n\to\infty} \|\tilde{p}_n - p_n\| = 0.$$

Since $\|p_n q p_n - q\| \to 0$ as $n \to \infty$ we obtain that $\|\tilde{p}_n q \tilde{p}_n - q\| < 1$ for some n, and this implies that q is equivalent to a subprojection of $\tilde{p}_n$. By assumption there is a projection p' with $p' \sim \tilde{p}_n$ and $p' \perp \tilde{p}_n$. Hence $p' \gtrsim q$ and since $p \leq \tilde{p}$ it follows that $p' \perp p$, and we may therefore take r to be a subprojection of p' with $r \sim q$. $\square$

Theorem 3.3 *Let A be a C^*-algebra which admits a countable approximate unit consisting of projections. Then A is stable if and only if for each projection $p \in A$ there is a projection $q \in A$ such that $p \sim q$ and $p \perp q$.*

Proof: The "if-part". Note that A possess property (i) of Lemma 3.2. Let $(p_n)_{n=1}^{\infty}$ be an approximate unit for A consisting of projections. We first construct a system of projections $\{q_n^i\}$, where $n \in \mathbb{N}$ and $1 \leq i \leq n$ such that

$$\begin{array}{cccccccc}
q_1^1 & \leq q_2^1 & \leq q_3^1 & \leq q_4^1 & \leq & \cdots \\
 & q_2^2 & \leq q_3^2 & \leq q_4^2 & \leq & \cdots \\
 & & q_3^3 & \leq q_4^3 & \leq & \cdots \\
 & & & q_4^4 & \leq & \cdots \\
 & & & & & \ddots
\end{array}$$

where the projections in each column are mutually orthogonal and equivalent, $q_{n+1}^i - q_n^i \sim q_{n+1}^j - q_n^j$ for $n \in \mathbb{N}$ and $i, j = 1, \dots, n$, and such that the sequence $(q_n^1 + q_n^2 + \cdots + q_n^n)_{n=1}^{\infty}$ is an approximate unit for A.

In the initial step of the construction we set $q_1^1 = p_1$.

Assume now that $n \geq 1$, and that we have constructed $q_m^1, q_m^2, \dots, q_m^m$ with the desired properties, and such that

$$\|p_m(q_m^1 + q_m^2 + \cdots + q_m^m)p_m - p_m\| \leq \tfrac{1}{m},$$

for all $m \le n$. (Observe that if this inequality holds for all $m \in \mathbb{N}$, then $(q_n^1+q_n^2+\cdots+q_n^n)_{n=1}^\infty$ is an approximate unit for A.)

By Lemma 3.1 there is a projection $\tilde{p} \in A$ such that

$$\tilde{p} \ge q_n^1 + q_n^2 + \cdots + q_n^n, \qquad \|p_{n+1}\tilde{p}p_{n+1} - p_{n+1}\| \le \tfrac{1}{n+1}.$$

Set

$$r^1 = \tilde{p} - (q_n^1 + q_n^2 + \cdots + q_n^n), \qquad q_{n+1}^1 = q_n^1 + r^1.$$

Successive applications of Lemma 3.2 yield projections $r^2, r^3, \ldots, r^n$ in A satisfying

$$r^i \sim r^1, \qquad r^i \perp (q_n^1 + q_n^2 + \cdots + q_n^n) + (r^1 + r^2 + \cdots + r^{i-1}).$$

Set $q_{n+1}^i = q_n^i + r^i$ for $2 \le i \le n$. One more application of Lemma 3.2 produces a projection q_{n+1}^{n+1} which is equivalent to q_{n+1}^1 and orthogonal to $q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^n$.

Since $\tilde{p} \le q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^{n+1}$, we get

$$\|p_{n+1}(q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^{n+1})p_{n+1} - p_{n+1}\| \le \|p_{n+1}\tilde{p}p_{n+1} - p_{n+1}\| \le \tfrac{1}{n+1},$$

and this completes the construction of the system $\{q_n^i\}$.

By the properties of the system $\{q_n^i\}$ there exist partial isometries $\{v_n^i\}$ in A such that

$$(v_n^i)^* v_n^i = q_n^1, \qquad v_n^i (v_n^i)^* = q_n^i, \qquad v_n^i q_{n-1}^1 = v_{n-1}^i.$$

For each $n \in \mathbb{N}$ set $A_n = (q_n^1 + q_n^2 + \cdots + q_n^n)A(q_n^1 + q_n^2 + \cdots + q_n^n)$, $B_n = q_n^1 A q_n^1$ and let $\iota_{A_n} \colon A_n \to A_{n+1}$ be the inclusion map. Let $\{e_{ij}\}$ denote the matrix units in the C^*-algebra of $n \times n$ matrices, and define an isomorphism $\sigma_n \colon A_n \to M_n \otimes B_n$ by

$$\sigma_n(a) = \sum_{1 \le i,j \le n} e_{ij} \otimes (v_n^i)^* a v_n^j.$$

Let $\psi_n \colon M_n \to M_{n+1}$ be the embedding into the upper left-hand corner, let $\iota_{B_n} \colon B_n \to B_{n+1}$ be the inclusion map, and define $\varphi_n \colon M_n \otimes B_n \to M_{n+1} \otimes B_{n+1}$ by $\varphi_n = \psi_n \otimes \iota_{B_n}$. By the choice of the partial isometries (v_n^i) we have $\varphi_n \circ \sigma_n = \sigma_{n+1} \circ \iota_{A_n}$.

The inductive limit C^*-algebra $\lim_{n\to\infty}(A_n, \iota_{A_n})$ equals $\overline{\bigcup_{n=1}^{\infty} A_n}$, and this union is A, because $(q_n^1 + q_n^2 + \cdots + q_n^n)_{n=1}^{\infty}$ is an approximate unit for A. The inductive limit C^*-algebra $\lim_{n\to\infty}(M_n \otimes B_n, \varphi_n)$ is isomorphic to $\mathcal{K} \otimes B$, where $B = \lim_{n\to\infty} B_n$. We thus obtain the following commuting diagram:

$$\begin{array}{ccccccccccccc}
A_1 & \overset{\iota_{A_1}}{\hookrightarrow} & A_2 & \hookrightarrow & \cdots & \hookrightarrow & A_n & \overset{\iota_{A_n}}{\hookrightarrow} & A_{n+1} & \hookrightarrow & \cdots & \rightarrow & A \\
\| & & \sigma_2 \downarrow \cong & & & & \sigma_n \downarrow \cong & & \sigma_{n+1} \downarrow \cong & & & & \\
B_1 & \xrightarrow{\varphi_1} & M_2 \otimes B_2 & \longrightarrow & \cdots & \longrightarrow & M_n \otimes B_n & \xrightarrow{\varphi_n} & M_{n+1} \otimes B_{n+1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K} \otimes B
\end{array}$$

The intertwining induces an isomorphism between A and $\mathcal{K} \otimes B$, and A is therefore stable.

The "only if-part". Since A is stable, A is isomorphic to $\overline{\bigcup_{n=1}^{\infty} M_n(A)}$ $(= D)$. Let $p \in P(D)$. There is $n \in \mathbb{N}$ and $p' \in P(M_n(A))$ such that $\|p' - p\| < 1$. Hence $p = up'u^*$ for some unitary u in $\tilde{D}$. There is a projection q' in $M_{2n}(A)$ with $q' \sim p'$ and $q' \perp p'$. It follows that $(q =)\ uq'u^*$ is orthogonal and equivalent to p. □

4 Some applications of the characterization theorem

In this section we present some corollaries to Theorem 2.1.

Corollary 4.1 *If A is the inductive limit of a sequence of stable σ-unital C^*-algebras, then A is stable.*

Proof: By assumption A is the inductive limit of a sequence

$$A_1 \to A_2 \to A_3 \to \cdots$$

of stable σ-unital C^*-algebras A_n. Let $\mu_n \colon A_n \to A$ be the associated homomorphisms. Notice that A is σ-unital because each A_n is σ-unital and the sequence is countable. We show that A satisfies property (b) of Proposition 2.2. Let $a \in F(A)$ and $\varepsilon > 0$ be given. Find $n \in \mathbb{N}$ and $b_0 \in F(A_n)$ such that $\|a - \mu_n(b_0)\| < \varepsilon$. Since A_n is stable there is by Theorem 2.1 an element c_0 in A_n^+ such that $b_0 \perp c_0$ and $b_0 \sim c_0$. Set $b = \mu_n(b_0)$ and $c = \mu_n(c_0)$. Then $b \perp c$, $b \sim c$, and $\|a - b\| < \varepsilon$ as desired. □

Lemma 4.2 *For each $\varepsilon > 0$ and for each $K < \infty$ there is a $\delta > 0$ so that the following holds: For every C^*-algebra A, and for every set of positive elements a_1, a_2, b_1, b_2 in A, if*

$$\|a_i\| \leq K, \quad \|b_j\| \leq K, \quad \|(a_1 + a_2)(b_1 + b_2)\| \leq \delta,$$

then $\|a_i^{1/2} b_j\| \leq \varepsilon$ and $\|a_i b_j\| \leq \varepsilon K^{1/2}$

Proof: Choose $\delta_1 > 0$ such that $\delta_1^{1/2}(2K^3)^{1/4} < \varepsilon$. By Lemma 2.4 we can find $\delta > 0$ such that $\|(a_1 + a_2)(b_1 + b_2)\| \leq \delta$, $\|a_i\| \leq K$, and $\|b_j\| \leq K$ implies $\|(a_1 + a_2)^{1/2}(b_1 + b_2)\| \leq \delta_1$.

Notice that

$$0 \leq (a_1 + a_2)^{1/2} b_j (a_1 + a_2)^{1/2} \leq (a_1 + a_2)^{1/2}(b_1 + b_2)(a_1 + a_2)^{1/2} \leq \delta_1 (2K)^{1/2} \cdot 1.$$

Set $x_j = (a_1 + a_2)^{1/2} b_j^{1/2}$. Then $\|x_j\|^2 \leq \delta_1(2K)^{1/2}$. Now,

$$0 \leq b_j^{1/2} a_i b_j^{1/2} \leq b_j^{1/2}(a_1 + a_2) b_j^{1/2} = x_j^* x_j.$$

This shows that $\|a_i^{1/2} b_j^{1/2}\|^2 = \|b_j^{1/2} a_i b_j^{1/2}\| \leq \|x_j^* x_j\| \leq \delta_1 (2K)^{1/2}$. Hence $\|a_i^{1/2} b_j\|^2 \leq \delta_1 (2K)^{1/2} K \leq \varepsilon^2$, and $\|a_i b_j\| \leq \varepsilon K^{1/2}$. □

Corollary 4.3 *Let A be a stable separable C^*-algebra. For each positive $a \in A$ of norm at most 1, the hereditary C^*-subalgebra $\overline{(1-a)A(1-a)}$ of A is stable.*

Proof: Set $B = \overline{(1-a)A(1-a)}$. Notice that B is σ-unital because A and hence B are separable. We show that B satisfies property (b) of Proposition 2.2. Let $b \in F(B)$ be given. Since A is stable, and using that $F(A)$ is dense in A^+ it follows from Proposition 2.2 (d) that there is a sequence $(u_n)_{n=1}^{\infty}$ of unitaries in $\tilde{A}$ such that $\|u_n(a+b)u_n^*(a+b)\|$ tends to zero. By Lemma 4.2 this implies that $\|u_n b^{1/2} u_n^* a\|$ and $\|u_n b u_n^* b\|$ tend to zero for large n.

Put $x_n = (1-a)u_n b^{1/2} \in B$ and $y_n = u_n b^{1/2} \in A$. Then

$$\|x_n - y_n\| = \|(x_n - y_n)^*\| = \|b^{1/2} u_n^* a\| = \|u_n b^{1/2} u_n^* a\| \to 0.$$

Since $y_n^* y_n = b$, and since $(y_n y_n^*)(y_n^* y_n) = u_n b u_n^* b$ tends to zero, we obtain that

$$\lim_{n\to\infty} \|x_n^* x_n - b\| = 0, \qquad \lim_{n\to\infty} \|(x_n x_n^*)(x_n^* x_n)\| = 0.$$

This shows that property (b) of Proposition 2.2 holds for B.

$\square$

We shall in the next two results consider inclusions $B \subseteq A$ of C^*-algebras with the property that B contains an approximate unit which is also an approximate unit for A. Notice that for such an inclusion $B \subseteq A$ necessarily *every* approximate unit for B is an approximate unit for A. This condition is again equivalent to the property that for each $a \in A$ and each $\varepsilon > 0$ there exists $e \in B$ with $0 \le e \le 1$ such that $\|a - ae\| < \varepsilon$. Since $F(B)$ is dense in B^+, we can in this case also find $e \in F(B)$ with $\|a - ae\| < \varepsilon$.

Proposition 4.4 *Let A be a σ-unital C^*-algebra and suppose $B \subseteq A$ is a C^*- subalgebra containing an approximate unit that is also an approximate unit for A. If B is stable, then so is A.*

Proof: Let $a \in F(A)$ and $\varepsilon > 0$ be given. By assumption there exists $e \in F(B)$ such that $2\|a\|\|a - ae\| < \varepsilon$. By Theorem 2.1 and Proposition 2.2 there is a unitary $w \in \tilde{B}$ such that $e \perp wew^*$. Now $a \sim waw^*$, and we have the following estimate:

$$\begin{aligned} \|awaw^*\| &\le \|awaw^* - aewaw^*\| + \|aewaw^* - aeweaw^*\| + \|aeweaw^*\| \\ &\le (\|a - ae\| + \|a - ea\|)\|a\| + 0 \; < \; \varepsilon. \end{aligned}$$

By Theorem 2.1 and Proposition 2.2 this shows that A is stable. $\square$

In Proposition 4.4 one cannot conclude that B is stable if it is known that A is stable. For example, let $B = c_0(\mathbb{N})$ be embedded as the diagonal in $A = \mathcal{K}$.

Corollary 4.5 *Let A be a σ-unital C^*-algebra, let G be a discrete group, and let $\alpha \colon G \to$* $\mathrm{Aut}(A)$ *be an action of G on A. If A is stable then the crossed product $A \rtimes_\alpha G$ is stable.*

Proof: By Proposition 4.4 it suffices to show that A contains an approximate unit for the crossed product $A \rtimes_\alpha G$. Let $(e_n)_{n=1}^\infty$ be any (bounded) approximate unit for A. The set of elements $x \in A \rtimes_\alpha G$ for which $\|x - e_n x\|$ tends to zero is a norm-closed linear subspace of $A \rtimes_\alpha G$. Since the subalgebra of all finite sums $\sum_{\gamma \in G} a_\gamma u_\gamma$, with coefficients $a_\gamma \in A$ and unitaries u_γ that implement the action of G, is dense in $A \rtimes_\alpha G$, it suffices to show that $\|a_\gamma u_\gamma - e_n a_\gamma u_\gamma\|$ tends to zero for large n. This, however, is trivially the case because $\|a_\gamma - e_n a_\gamma\|$ tends to zero for all a_γ in A. □

It may happen that $A \rtimes_\alpha G$ is stable without A being stable. For example the compacts $\mathcal{K}$ is isomorphic to $c_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ where α acts by left translation.

5 Related remarks

As mentioned in the introduction, an AF-algebra is stable if and only if it does not admit a bounded densely defined trace. This criterion is stronger and more useful than our characterization theorem (Theorem 2.1). For example, it is easy to see that if $0 \to I \to A \to B \to 0$ is an extension of C^*-algebras, and if I and B do not admit any bounded, densely defined trace, then A does not admit a bounded, densely defined trace.

The proposition below contains a related (partial) characterization of stability that holds more generally. It is a compelling question if the statement would remain true without condition (iii).

Proposition 5.1 *Let A be a σ-unital C^*-algebra, and suppose that*

(i) A admits no bounded, densely defined trace,

(ii) no non-zero quotient of A is unital, and

(iii) for every full hereditary subalgebra B of A, such that B does not admit any bounded, densely defined trace, and for every $a \in F(A)$ there exists $b \in B^+$ with $a \sim b$.

It follows that A is stable.

Conversely, if A is stable, then (i) and (ii) hold.

Proof: Let $a \in F(A)$ be given, and let B be the hereditary subalgebra of A consisting of all elements in A that are orthogonal to a. Let $e \in F(A)$ be such that $ae = a$. If B were contained in a proper ideal I in A, then $e + I$ would be a unit for A/I, thus contradicting assumption (ii). Hence B is full.

The domain of every densely defined trace on A contains the Pedersen ideal of A, and the Pedersen ideal contains $F(A)$. Thus $\tau(e) < \infty$ for every densely defined trace τ on A. Assume that the restriction of τ to B were bounded. Then for every $x \in A^+$,

$$\tau(x) = \tau(e^{1/2}xe^{1/2}) + \tau((1-e)^{1/2}x(1-e)^{1/2}) \leq \|x\|\tau(e) + \|\tau|_B\| \cdot \|x\|,$$

which shows that τ is bounded, in contradiction with assumption (i).

It follows that B satisfies the conditions of (iii), and B therefore contains an element b which is equivalent to a. Hence (c) of Theorem 2.1 holds, and A must be stable.

The last statement is trivial. □

We do not know of any C^*-algebras that do not satisfy condition (iii) of Proposition 5.1. If it turns out that there exist stably finite C^*-algebras without traces (i.e. that quasi-traces need not be traces), then one should sharpen (i) to exclude the existence of bounded densely defined quasi-traces. (Recall that Uffe Haagerup has proved that quasi-traces on exact C^*-algebras are traces, [5].)

Condition (iii) is easily seen to be satisfied for all AF-algebras, and it follows from [2] that every exact approximately divisible simple C^*-algebra satisfies property (iii).

Condition (ii) follows from condition (i) for every exact C^*-algebra A with the strong finiteness property that every quotient of A is stably finite. Indeed, if A/I were unital for some proper ideal I, then A/I would admit a bounded trace, being exact, stably finite and unital (see [5]). Hence A would admit a bounded trace.

It follows that if A is an exact C^*-algebra such that every quotient of A is stably finite, and such that (iii) holds, then A is stable if and only if A admits no bounded densely defined trace. This can be applied to AF-algebras.

If it were true that every C^*-algebra that satisfies (i) and (ii) of Proposition 5.1 is stable, then it would also follow that every simple C^*-algebra is either stably finite or purely infinite. Indeed, if A is simple and not stably finite, then so is every non-zero hereditary C^*-subalgebra B of A (by Brown's theorem [3]), and therefore every non-unital hereditary C^*-subalgebra B of A would be stable. The claim now follows from the proposition below:

Proposition 5.2 *Let A be a simple C^*-algebra, not of type I, and with the property that if B is a hereditary C^*-subalgebra of A, then either B is unital or B is stable. It follows that A is purely infinite.*

Proof: We must show that every non-zero hereditary C^*-subalgebra B of A contains an infinite projection (cf. [4]). Assume that B is unital. Then, since A is assumed to be not of type I, B is infinite dimensional, and there is an $a \in F(B)$ such that a is non-invertible and 0 is not an isolated point of the spectrum of a. The hereditary subalgebra $\overline{aAa}$ of B is then non-unital. Upon replacing B with $\overline{aAa}$, we may assume that B is non-unital, and thus, by assumption, stable.

It follows from [1, Thm. 1.2] that either B admits a dimension function defined on its Pedersen ideal, or B contains an infinite projection. We proceed to show that B does not admit a dimension function.

Suppose, to the contrary, that φ is a dimension function defined on the Pedersen ideal of B. Choose (arguing as above) $a, e \in F(B)$ such that $ae = a = ea$, $\|e\| = 1$, and such that the hereditary C^*-subalgebra $(D =)$ $\overline{aAa}$ is non-unital (and non-zero). Then D is stable by our assumption. On the other hand, $ex = xe = x$, whence $\varphi(x) \leq \varphi(e) < \infty$, for every $x \in D$. Hence φ is bounded. One easily deduces from item (c) of Theorem 2.1, that no stable C^*-algebra admits a bounded (non-zero) dimension function.

□

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