An Introduction to
K-Theory for C^*-Algebras

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Preface

About K-theory

K-theory was developed by Atiyah and Hirzebruch in the 1960's based on work of Grothendieck in algebraic geometry. It was introduced as a tool in C^*-algebra theory in the early 1970's through some specific applications described below. Very briefly, K-theory (for C^*-algebras) is a pair of functors, called K_0 and K_1, that to each C^*-algebra A associates two Abelian groups $K_0(A)$ and $K_1(A)$. The group $K_0(A)$ is given an ordering that (in special cases) makes it an ordered Abelian group. There are powerful machines, some of which are described in this book, making it possible to calculate the K-theory of a great many C^*-algebras. K-theory contains much information about the individual C^*-algebras — one can learn about the structure of a given C^*-algebra by knowing its K-theory, and one can distinguish two C^*-algebras from each other by distinguishing their K-theory. For certain classes of C^*-algebras, K-theory is actually a complete invariant. K-theory is also a natural home for index theory.

Two applications demonstrated the importance of K-theory to C^*-algebras. George Elliott showed in the early 1970's (in a work published in 1976, [18]) that AF-algebras (the so-called "approximately finite-dimensional" C^*-algebras; see Chapter 7 for a precise definition) are classified by their ordered K_0-groups. (The K_1-group of an AF-algebra is always zero.) As a consequence, all information about an AF-algebra is contained in its ordered K_0-group. This result indicated the possibility of classifying a more general class of C^*-algebras by their K-theory.

Another important application of K-theory to C^*-algebras was Pimsner and Voiculescu's proof in 1982, [32], of the fact that $C^*_{\text{red}}(F_2)$, the reduced C^*-algebra of the free group of two generators, has no projections other than 0 and 1. Kadison had, at a time when it was not known that there exists a simple unital C^*-algebra with no projections other than 0 and 1, conjectured that $C^*_{\text{red}}(F_2)$ would be such an example. It was shown by Powers in 1975 in [33] that $C^*_{\text{red}}(F_2)$ actually is a simple C^*-algebra, and, as mentioned, Pimsner and Voiculescu then showed that this C^*-algebra has no non-trivial projections by calculating its K-theory. Blackadar had a couple of years before that found another example of a simple unital projectionless C^*-algebra, [2].

A landmark for the use of K-theory in C^*-algebra theory, and for the use of K-theory for C^*-algebras in topology, was Brown, Douglas, and Fillmore's development in the 1970's of K-homology (a dual theory to K-theory) via extensions of C^*-algebras, [8] and [9]. This theory was generalized by Kasparov in his KK-theory that encompasses K-theory and K-homology, [27].

Today K-theory is an active research area, and a much used tool for the study of C^*-algebras. One current line of research concentrates on generalizing Elliott's classification theorem for AF-algebras to a much broader class of C^*-algebras, [19]. Another active branch of research seeks to prove the conjecture by Baum and Connes on the K-theory of the C^*-algebra $C^*_{\text{red}}(G)$ of an arbitrary group G in a way that generalizes Pimsner and Voiculescu's result about $C^*_{\text{red}}(F_2)$. Connes has in his book "Noncommutative Geometry," [11], described how K-theory is useful in the understanding of a big mathematical landscape that contains geometry, physics, C^*-algebras, and algebraic topology among many other subjects!

Besides actually being useful — in the mathematical sense of the word — K-theory is fun to study because of the way it mixes ideas from the different branches of mathematics where it has its roots.

The aim of this book is not to present all the new mathematics that involves K-theory for C^*-algebras, but to give an elementary and, we hope, easy-to-read introduction to the subject.

About the book

This book saw its first light as a set of handwritten lecture notes to a graduate course on K-theory for C^*-algebras at Odense University, Denmark, in the spring of 1995, given by the first named author and with the two other authors among the participants. The handwritten notes were TEX'ed and rewritten in the fall of 1995 as a joint project among the three authors. The K-theory course has since then been given once more in Odense (in the fall of 1997) and once again in Copenhagen (in the fall of 1998). Besides, a number of students have taken a reading course based on these notes. We have in this way received substantial feedback from the many students who have been subjected to the notes and, as a result, the notes have continuously been improved. We started writing this book in the winter 1998/99 in order to make the work that over the years has been put into the notes available to a larger group of readers.

The book is intended as a text for a one-semester first or second year graduate course, or as a text for a reading course for students at that level. As such, the text does not take the reader very far into the vast world of K-theory. The most basic properties of K_0 and K_1 are covered, Bott periodicity is proved and the six-term exact sequence is derived. There is a chapter on inductive limits and continuity of K-theory, and Elliott's classification of AF-algebras is proved. In the last chapter of the book, the theory developed is used to show that each pair of countable Abelian groups arises as the K-groups of a C^*-algebra.

An effort has been made to make the text as self-contained as possible. Chapter 1 contains an overview, mostly without proofs, of what the reader should know, or should learn, about C^*-algebras. The theory in the book is illustrated with examples that can be found in the exercises and in the text.

The subject of this book is treated in several other textbooks, most notably in Bruce Blackadar's book, [3]. Other books treating K-theory for operator algebras include Niels Erik Wegge-Olsen's detailed treatment, [38], Gerard Murphy's book, [28], and the recent books by Ken Davidson, [15], and by Peter Fillmore, [20]. Our treatment is indebted to these books, in particular to Bruce Blackadar's book.

We thank Hans Jørgen Munkholm for sharing with us his point of view on K-theory. We thank also George Elliott for many valuable comments. Last, but not least, we thank those who have read and commented on (earlier versions of) this text. Thanks are especially due to Piotr Dzierzynski, Jacob Hjelmborg, Johan Kustermans, Franz Lehner, Jesper Mygind, Agata Przybyszewska, Rolf Dyre Svegstrup, and Steen Thorbjørnsen.

Sections, examples, and paragraphs in the book marked with an asterisk $*$ contain digressions which the reader can omit or postpone without losing the logic of the overall exposition. Two possible shorter routes through the books are

1) Chapters 1–4 and Chapters 8–12, or
2) Chapters 1–7.

Chapter 7 and Chapter 13 can be omitted or postponed.

The book has a home page:

http://www.math.ku.dk/~rordam/K-theory.html

that contains a list of corrections to the book. Readers are strongly encouraged to report on any mistakes they may have found in the book (see the homepage for address information). We also welcome suggestions of how to make the book better.

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Table of K-groups

C^*-algebra A	$K_0(A)$	$K_1(A)$	References
$\mathbb{C}$, $M_n(\mathbb{C})$	$\mathbb{Z}$	0	Example 3.3.2, Example 8.1.8.
$\mathcal{K}$	$\mathbb{Z}$	0	Corollary 6.4.2, Example 8.2.9.
$B(H)$	0	0	Example 3.3.3, Example 8.1.8.
$\mathcal{Q}(H)$	0	$\mathbb{Z}$	Example 9.4.3, Exercise 12.4.
$C([0,1])$, $C(\mathbb{D})$	$\mathbb{Z}$	0	Exercise 3.3, Exercise 8.2.
$C(X)$, X contractible	$\mathbb{Z}$	0	Example 3.3.6, Exercise 8.2.
$C_0((0,1])$	0	0	Exercise 4.2, Exercise 8.4.
$C_0((0,1))$, $C_0(\mathbb{R})$	0	$\mathbb{Z}$	Example 11.3.2.
$C(S^1)$, $C(\mathbb{T})$	$\mathbb{Z}$	$\mathbb{Z}$	Example 11.3.3.
$C(S^n)$, n even	$\mathbb{Z} \oplus \mathbb{Z}$	0	Example 11.3.3.
$C(S^n)$, n odd	$\mathbb{Z}$	$\mathbb{Z}$	Example 11.3.3.
$C_0(\mathbb{R}^n)$, n even	$\mathbb{Z}$	0	Example 11.3.2.
$C_0(\mathbb{R}^n)$, n odd	0	$\mathbb{Z}$	Example 11.3.2.
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$	Example 11.3.4.
$C(\mathbb{D}/\sim_n)$, Moore space	$\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$	0	Exercise 12.2.
$C(Z_n)$, n-clover	$\mathbb{Z}$	$\mathbb{Z}^n$	Exercise 12.3.

C^*-algebra A	**$K_0(A)$**	**$K_1(A)$**	**References**
A_θ, irrational rotation algebra	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	Exercise 5.8.
$\mathcal{O}_n$, Cuntz algebra	$\mathbb{Z}/(n-1)\mathbb{Z}$	0	Exercise 4.5, Exercise 8.10.
$\mathcal{T}$, Toeplitz algebra	$\mathbb{Z}$	0	Example 9.4.4, Exercise 12.3.
$\mathcal{T}_0$, reduced Toeplitz algebra	0	0	Example 9.4.4, Exercise 12.4.
D_n, dimension drop algebra	0	$\mathbb{Z}/n\mathbb{Z}$	Paragraph 13.1.1.
$M_n(B)$	$K_0(B)$	$K_1(B)$	Proposition 4.3.8, Proposition 8.2.8.
$\mathcal{K}B$, stabilization	$K_0(B)$	$K_1(B)$	Proposition 6.4.1, Proposition 8.2.8.
$\widetilde{B}$, unitization	$K_0(B) \oplus \mathbb{Z}$	$K_1(B)$	Example 4.3.5, Equation 8.4.
$B_1 \oplus B_2$	$K_0(B_1) \oplus K_0(B_2)$	$K_1(B_1) \oplus K_1(B_2)$	Proposition 4.3.4, Proposition 8.2.6.
SB, suspension	$K_1(B)$	$K_0(B)$	Theorem 10.1.3, Corollary 11.3.1.
CB, cone	0	0	Example 4.1.5, Exercise 8.3.
$\mathbb{T}B$	$K_0(B) \oplus K_1(B)$	$K_0(B) \oplus K_1(B)$	Example 11.3.4.
AF-algebra	Dimension group	0	Proposition 7.2.8, Exercise 8.7.
UHF-algebra, type n	$Q(n)$	0	Theorem 7.4.5, Proposition 7.2.8.
II_1-factor	$\mathbb{R}$	0	Exercise 3.12, Exercise 8.11.
Inductive limit of dimension drop algebras	arbitrary countable	arbitrary countable	Theorem 13.2.4.

List of symbols

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