Non-simple purely infinite C^*-algebras

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Abstract

A C^*-algebra A is defined to be *purely infinite* if there are no characters on A, and if for every pair of positive elements a, b in A, such that b lies in the closed two-sided ideal generated by a, there exists a sequence $\{r_n\}$ in A such that $r_n^* a r_n \to b$. This definition agrees with the usual definition by J. Cuntz when A is simple.

It is shown that the property of being purely infinite is preserved under extensions, Morita equivalence, inductive limits, and it passes to quotients, and to hereditary sub-C^*-algebras. It is shown that $A \otimes \mathcal{O}_\infty$ is purely infinite for every C^*-algebra A. Purely infinite C^*-algebras admit no traces, and, conversely, an approximately divisible exact C^*-algebra is purely infinite if it admits no non-zero trace.

1 Introduction

Joachim Cuntz introduced in [7] what is now called the Cuntz algebra $\mathcal{O}_n$ (the universal C^*-algebra generated by n isometries whose range projections add up to the unit), and he showed that these C^*-algebras have the property that for every non-zero x in $\mathcal{O}_n$ there exist a, b in $\mathcal{O}_n$ such that $1 = axb$. Later, in [9], he showed that this property is equivalent to the property that every non-zero hereditary sub-C^*-algebra of the given C^*-algebra contains an infinite projection, provided that the C^*-algebra is simple. He named this property "purely infiniteness".

Being simple and purely infinite has been reformulated in several other ways since then. S. Zhang showed that a simple C^*-algebra is purely infinite if, and only if, it is of real rank zero and every non-zero projection is properly infinite ([29], see also [21] for other equivalent conditions).

The list of examples of simple purely infinite C^*-algebras increased over the years. To this list belong the Cuntz-Krieger algebras $\mathcal{O}_A$ corresponding to irreducible shifts of finite type ([10]). It was shown in [5] that simple, unital C^*-algebras with a certain nice property (approximate divisibility) are either stably finite or purely infinite. It remains an important open problem to decide if there exists a simple unital C^*-algebra which is neither stably finite nor purely infinite.

In 1989, George Elliott showed that a certain class of C^*-algebras (inductive limits of circle algebras of real rank zero) are classified up to *-isomorphism by their K-theory, [11]. Elliott's paper started a comprehensive research in what is now called the classification program of Elliott. The following results have been obtained for purely infinite simple C^*-algebras:

- If A and B are purely infinite, simple, nuclear, separable C^*-algebras, then A and B are stably isomorphic if and only if they are KK-equivalent. If, moreover, the UCT holds for A and B, and they are unital, then A is isomorphic to B if and only if there are isomorphisms $\alpha_0 \colon K_0(A) \to K_0(B)$ and $\alpha_1 \colon K_1(A) \to K_1(B)$ with $\alpha_0([1_A]) = [1_B]$. (See [16], [26], [15]).

- For every pair of countable Abelian groups G_0 and G_1 and for every g_0 in G_0 there exists a purely infinite, simple, nuclear, separable, unital C^*-algebra A satisfying the UCT such that there are isomorphisms $\alpha_0 \colon G_0 \to K_0(A)$ and $\alpha_1 \colon G_1 \to K_1(A)$ with $\alpha_0(g_0) = [1_A]$. (See [28, Theorem 3.6]).

- If A is a simple, separable, nuclear, unital C^*-algebra, then A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite. (See [16], [18], [15]).

The purpose of this paper is to provide a possible frame for an extension of these results to non-simple C^*-algebras. The right definition of being purely infinite in the non-simple case should at least lead to the following:

(a) a purely infinite C^*-algebra admits no non-zero trace,

(b) $A \otimes \mathcal{O}_\infty$ is purely infinite for every C^*-algebra A.

A trace or a quasi-trace on a C^*-algebra is here understood to be a function whose domain is an algebraic, not necessarily dense, ideal in the C^*-algebra. Our definition (as described in the abstract — see also Section 4) meets these requirements. We know of no example of a traceless C^*-algebra which is not purely infinite, and it may be realistic to hope that a C^*-algebra A is purely infinite if its ultra power A_ω, corresponding to a free ultra filter ω on $\mathbb{N}$, admits no non-zero lower semi-continuous quasi-traces. We show a partial result in this direction that an exact approximately divisible C^*-algebra indeed is purely infinite if it admits no non-zero trace. It could be true that A is isomorphic to $A \otimes \mathcal{O}_\infty$ if A is a purely infinite, separable, nuclear C^*-algebra. In a forthcoming paper we show that if A is a purely infinite, separable, nuclear C^*-algebra of real rank zero, then A is isomorphic to $A \otimes \mathcal{O}_\infty$.

It has been suggested in [1], [20], and [19] to call a non-simple C^*-algebra purely infinite if all its non-zero hereditary sub-C^*-algebras contain an infinite projection. With this definition, (a) and (b) above do not hold. Indeed, $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ does not contain any non-zero projections, let alone infinite projections, and this C^*-algebra should be purely infinite if (b) is true. It is easy to construct examples of C^*-algebras that are purely infinite in the sense of [1], [20], and [19] and which admit a non-zero trace thus violating (a) (see Example 4.6).

We show that a C^*-algebra A is purely infinite (in the sense of this paper) if every non-zero hereditary sub-C^*-algebra of *every quotient* of A contains an infinite projections, cf. Proposition 4.7.

It would be desirable to have classification results for non-simple purely infinite C^*-algebra analogous to those for simple purely infinite C^*-algebras. A number of partial results in this direction have been obtained — see for example J. Mortensen's paper [24].

Our definition of pure infiniteness is based on the comparison theory for positive elements defined by J. Cuntz. This theory is reviewed and further developed in Section 2. Section 3 contains a discussion of finite, infinite, respectively, properly infinite positive elements in a C^*-algebra, being generalizations of the similar concepts for projections in a C^*-algebra. Pure infiniteness is defined in Section 4, and this section contains results on inductive limits, extensions, and Morita equivalence of purely infinite C^*-algebras. In Section 5 the relation between absence of traces and being purely infinite is treated, and this section also contains some tensor product results.

2 Preliminaries

With the purpose of constructing dimension functions and traces on C^*-algebra, Joachim Cuntz introduced in [8] a notion of comparison of (positive) elements in a C^*-algebra, that generalizes the comparison of projections in von Neumann algebras and in C^*-algebra defined by Murray and von Neumann. We shall make extensive use of Cuntz' ideas, and we will therefore spend some time explaining the main points (as developed in [8], [27] and [2]).

Denote the set of positive element in a C^*-algebra A by A^+.

Definition 2.1 (Cuntz) *Let A be a C^*-algebra, and let a, b be positive elements in A. Write $a \precsim b$ if there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in A with $x_k^* b x_k \to a$.*

More generally, if a is a positive element in $M_n(A)$ and if b is a positive element in $M_m(A)$, then write $a \precsim b$ if there exists a sequence of rectangular matrices $\{x_k\}_{k=1}^{\infty}$ in $M_{m,n}(A)$ with $x_k^* b x_k \to a$ (using obvious matrix multiplication).

Write $a \approx b$ if $a \precsim b$ and $b \precsim a$.

The relation $\precsim$ is clearly transitive and reflexive, and $\approx$ is an equivalence relation.

Lemma 2.2 *Let A be a C^*-algebra*

(i) *For every $a \in A^+$ and for every continuous function $\varphi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, we have $\varphi(a) \precsim a$.*

(ii) *$a \approx a^2$ for every $a \in A^+$.*

(iii) *If B is a hereditary sub-C^*-algebra of A, and if a, b are positive elements in B such that $a \precsim b$ relatively to A, then $a \precsim b$ relatively to B.*

Proof: (i). We can find a sequence of continuous functions $\psi_k \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_k(0) = 0$, and such that $t\psi_k(t)$ converges uniformly to $\varphi(t)$ on the spectrum of a. Put $x_k = \psi_k(a)^{1/2}$. Then $x_k^* a x_k \to \varphi(a)$.

(ii) follows from (i).

(iii). If $a \precsim b$ in A, then $a^2 \precsim b^{1/2}$ by (ii). Find x_k in A with $x_k^* b^2 x_k \to a^{1/2}$. Put $y_k = b^{1/2} x_k a^{1/4}$. Then y_k belongs to B and $y_k^* b y_k \to a$, thus showing that $a \precsim b$ in B. □

We let $\mathcal{M}(A)$ denote the *multiplier algebra* of A. The lemma above shows that if a, b are positive elements in A and if $a \precsim b$ in $\mathcal{M}(A)$, then $a \precsim b$ in A.

If p, q are projections in a C^*-algebra A, then $p \precsim q$ (in the sense of Definition 2.1) if and only if p is equivalent to a sub-projection of q, thus agreeing with the usual definition of comparison of two projection (cf. Proposition 2.7 below). Observe that $p \approx q$ in the sense of Definition 2.1 does not imply that p and q are Murray-von Neumann equivalent — unless A is finite. Another notion of equivalence of positive elements, that clearly extend Murray-von Neumann equivalence of projections, is given as follows:

Definition 2.3 *If a, b are positive elements in a C^*-algebra A, then write $a \sim b$ if there exists x in A such that $x^*x = a$ and $xx^* = b$.*

Similarly, if a in $M_n(A)$ and b in $M_m(A)$ are positive, then $a \sim b$ if there exists x in $M_{m,n}(A)$ such that $x^*x = a$ and $xx^* = b$.

It was shown by G. K. Pedersen in [25] that $\sim$ is an equivalence relation.

Notice also that $a \sim b$ implies that $a \approx b$. Indeed, if $x^*x = a$ and $xx^* = b$, then $x^*bx = a^2$ and so $a \precsim a^2 \precsim b$, and vice versa.

For a in $M_n(A)$ and b in $M_m(A)$ let $a \oplus b$ denote the element

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A).$$

Let $a \otimes 1_k$ denote the element $a \oplus a \oplus \cdots \oplus a$ (with k summands).

Let $a \in A^+$. Then $\overline{aAa}$ is the hereditary sub-C^*-algebra of A generated by a, and $\overline{AaA}$ denotes the closed two-sided ideal in A generated by a, where AaA implicitly is understood to be the *linear span* of the set of elements of the form xay, where x, y lie in A.

Consider again a positive element a in A, and let $\varepsilon > 0$ be given. Let $(a - \varepsilon)_+$ be the positive part of the self-adjoint element $a - \varepsilon \cdot 1$, where 1 is the unit of $\widetilde{A}$, the unitazation of A. Equivalently, $(a - \varepsilon)_+ = \varphi(a)$, where $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\varphi(t) = \max\{t - \varepsilon, 0\}$.

The lemma below is well-known. We include a brief sketch of its proof.

Lemma 2.4 *Let a, b be positive elements in a C^*-algebra A. If $a \sim b$, then there is an isomorphism $\varphi\colon \overline{aAa} \to \overline{bAb}$ with $\varphi(a) = b$.*

Proof: We have $a = x^*x$ and $b = xx^*$ for some x in A. Let $x = v|x|$ be the polar decomposition for x, where v is a partial isometry in the enveloping von Neumann algebra A^{**}. Then $c \mapsto vcv^*$ defines an isomorphism from $\overline{aAa}$ to $\overline{bAb}$ that maps a onto b. □

The next lemma contains information about continuity properties of the relation $\precsim$.

Lemma 2.5 *Let A be a C^*-algebra, and let a be a positive element in A. Then*

(i) $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$ *for all* $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$,

(ii) *if* $\varepsilon > 0$ *and* b *is a positive element in* A *with* $\|a - b\| < \varepsilon$, *then* $(a - \varepsilon)_+ = x^*bx$ *for some* $x \in A$, *and in particular,* $(a - \varepsilon)_+ \precsim b$,

(iii) *if* $\{b_n\}$ *is a sequence of positive elements in* A *so that* $b_n \to a$, *and if* $0 \le \varepsilon_1 < \varepsilon_2$, *then* $(a - \varepsilon_2)_+ \precsim (b_n - \varepsilon_1)_+$ *and* $(b_n - \varepsilon_2)_+ \precsim (a - \varepsilon_1)_+$ *for all sufficiently large* n.

Proof: (i) is straightforward, and (ii) is contained in [27, Proposition 2.2].

(iii). If $b_n \to a$, then $\|(b_n - \varepsilon_1)_+ - (a - \varepsilon_1)_+\| < \varepsilon_2 - \varepsilon_1$ when n is large enough. For such n we have $(a - \varepsilon_2)_+ \precsim (b_n - \varepsilon_1)_+$ and $(b_n - \varepsilon_2)_+ \precsim (a - \varepsilon_1)_+$ by (i) and (ii). □

Proposition 2.6 ([27, Proposition 2.4]) *Let A be a C^*-algebra, and let a, b be positive elements in A. The following are equivalent:*

(i) $a \precsim b$,

(ii) *for all* $\varepsilon > 0$, $(a-\varepsilon)_+ \precsim b$,

(iii) *for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $(a-\varepsilon)_+ \precsim (b-\delta)_+$,

(iv) *for all* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* x *in* A *such that* $(a-\varepsilon)_+ = x^*(b-\delta)_+ x$.

It is a consequence of this proposition that if p is a projection, and if a is a positive element in A with $p \precsim a$, then $p = x^*ax$ for some $x \in A$ — because $(p - \frac{1}{2})_+ = \frac{1}{2}p$.

If $0 \le \varepsilon_1 < \varepsilon_2$ and if a and b are positive elements in A with $(a-\varepsilon_1)_+ \precsim b$, then by Proposition 2.6 (iv) and Lemma 2.5 (i) there exist $\delta > 0$ and x in A with $(a-\varepsilon_2)_+ = x^*(b-\delta)_+ x$.

Recall that an element a in a C^*-algebra A is *strictly positive* if $\rho(a) > 0$ for every non-zero positive linear functional on A. Equivalently, a is strictly positive if and only if $\overline{aAa} = A$.

Proposition 2.7 *Let A be a C^*-algebra.*

(i) *If a is a positive element in A and if b is a positive element in $\overline{aAa}$, then $b \precsim a$.*

(ii) *If a is a strictly positive element in A, then $b \precsim a$ for every positive element b in A.*

(iii) *If a, b are positive elements in A satisfying $b \precsim a$, then for each $\varepsilon > 0$ there exists a positive element a_0 in $\overline{aAa}$ with $(b-\varepsilon)_+ \sim a_0$.*

(iv) *If $b = \sum_{j=1}^k x_j^* a x_j$ for some $x_1, x_2, \dots, x_k$ in A, then $b \precsim a \otimes 1_k$.*

(v) *If a is a positive element in A, and if b is a positive element in $\overline{AaA}$, then for each $\varepsilon > 0$ there exists k in $\mathbb{N}$ such that $(b-\varepsilon)_+ \precsim a \otimes 1_k$.*

Proof: (i). Find a sequence x_n in A with $ax_na \to b^{1/2}$, and put $y_n = x_n a x_n^*$. Then each y_n is positive and $ay_na \to b$. As shown in [27, Lemma 2.3], if $c \le d$, then $c \precsim d$. Since $ay_na \le \|y_n\| a^2 \precsim a$, we get $ay_na \precsim a$ for all n. Using Lemma 2.5, this shows that $(b-\varepsilon)_+ \precsim a$ for all $\varepsilon > 0$, and by Proposition 2.6, we conclude that $b \precsim a$.

(ii) is an immediate consequence of (i).

(iii). Use Proposition 2.6 to find y in A with $y^*ay = (b-\varepsilon)_+$, and put $x = a^{1/2}y$. Then $(b-\varepsilon)_+ = x^*x$, and xx^* belongs to $\overline{aAa}$.

(iv). Let x in $M_{k,1}(A)$ be the column matrix with entries $x_1, x_2, \dots, x_k$. Then $b = x^*(a \otimes 1_k)x$.

(v). If b is a positive element in $\overline{AaA}$ and if $\varepsilon > 0$, then there exist $x_1, x_2, \dots, x_k$ in A such that $\|b - \sum_{j=1}^k x_j^* a x_j\| < \varepsilon$. From (iv) and Lemma 2.5 we get

$$(b-\varepsilon)_+ \;\precsim\; \sum_{j=1}^k x_j^* a x_j \;\precsim\; a \otimes 1_k.$$

□

Lemma 2.8 *Let A be a C^*-algebra.*

(i) *If $a \in A^+$ and if p is a projection in $\mathcal{M}(A)$, then $a \precsim pap + (1-p)a(1-p)$.*

(ii) *If a, b are positive elements in A, then $a + b \precsim a \oplus b$.*

(iii) *If a, b are positive and mutually orthogonal elements in A, then $a + b \sim a \oplus b$.*

Proof: (i). Let s be the symmetry $p - (1-p)$ in $\mathcal{M}(A)$. Then

$$a \;\le\; a + sas \;=\; 2(pap + (1-p)a(1-p)) \;\precsim\; pap + (1-p)a(1-p),$$

and this shows (i) — recalling that $b \le c$ implies $b \precsim c$, cf. Proposition 2.7.

(ii). Put $x = (a^{1/2}\; b^{1/2}) \in M_{1,2}(A)$. Then, by (i),

$$a + b \;=\; xx^* \;\sim\; x^*x \;=\; \begin{pmatrix} a & b^{1/2}a^{1/2} \\ a^{1/2}b^{1/2} & b \end{pmatrix} \;\precsim\; a \oplus b.$$

(iii). If a and b are orthogonal, then $x^*x = a \oplus b$, when x is as in the proof of (ii) above. □

Lemma 2.9 *Let a_1, a_2, b_1, b_2 be positive elements in a C^*-algebra A with $a_j \precsim b_j$. Then $a_1 \oplus a_2 \precsim b_1 \oplus b_2$. If also $a_1 \perp a_2$ and $b_1 \perp b_2$, then $a_1 + a_2 \precsim b_1 + b_2$.*

Proof: If $x_n^* b_1 x_n \to a_1$ and $y_n^* b_2 y_n \to a_2$, then

$$(x_n \oplus y_n)^*(b_1 \oplus b_2)(x_n \oplus y_n) \to a_1 \oplus a_2.$$

The second statement, which is contained in [8, Proposition 1.1], can be derived from the first statement and from Lemma 2.8 (iii). □

3 Finite, infinite and properly infinite positive elements

Recall that a projection p in a C^*-algebra A is said to be *infinite* if it is equivalent to a proper sub-projection of itself, and p is *finite* otherwise. If p has mutually orthogonal sub-projections q_1 and q_2 such that $p \sim q_1 \sim q_2$, then p is said to be *properly infinite*. Equivalently, p is properly infinite if p is non-zero and $p \oplus p \precsim p$. A unital C^*-algebra A is said to be finite, infinite, respectively, properly infinite, if the unit of A has this property.

Lemma 3.1 *Let p be a projection in a C^*-algebra A. Then p is infinite if and only if there exists a non-zero positive element a in A such that $p \oplus a \precsim p$.*

Proof: If p is infinite, then p is equivalent to a proper sub-projection p_0 of p. Put $q = p - p_0 \neq 0$. Then $p \oplus q \sim p_0 + q = p$, and so in particular, $p \oplus q \precsim p$.

Assume that there exists a non-zero positive element a with $p \oplus a \precsim p$. Choose ε in $(0,1)$ such that $(a-\varepsilon)_+ \neq 0$, and put $b = (a-\varepsilon)_+$. Since $((p \oplus a) - \varepsilon)_+ = (1-\varepsilon)p \oplus b$, it follows from Proposition 2.6 that there exists $r = (r_1\ r_2)$ in $M_{1,2}(A)$ with $p \oplus b = r^*pr$. This entails that

$$\begin{pmatrix} p & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} r_1^* \\ r_2^* \end{pmatrix} p \begin{pmatrix} r_1 & r_2 \end{pmatrix} = \begin{pmatrix} r_1^*pr_1 & r_1^*pr_2 \\ r_2^*pr_1 & r_2^*pr_2 \end{pmatrix}.$$

Put $v = pr_1$. Then $v^*v = p$, and $(p_0 =)\, vv^* \le p$. Assume that $p_0 = p$. Then

$$b = r_2^*pr_2 = r_2^*vv^*r_2 = r_2^*pr_1v^*r_2 = 0,$$

a contradiction. Hence p_0 is a proper sub-projection of p, and p is infinite. $\square$

Definition 3.2 *A positive element a in a C^*-algebra A is called* infinite *if there exists a non-zero positive element b in A such that $a \oplus b \precsim a$. If a is not infinite, then we say that a is* finite. *If a is non-zero and if $a \oplus a \precsim a$, then a is said to be* properly infinite.

Lemma 3.1 above shows that being finite and infinite in the sense of Definition 3.2 extends the usual definition of finite and infinite projections, and also, a properly infinite projection is clearly properly infinite in the sense of Definition 3.2.

Every properly infinite positive element is infinite. If I is a closed two-sided ideal in A and if a is a properly infinite element in A, then $a + I$ is either zero or properly infinite in A/I (because the relation $a \oplus a \precsim a$ is preserved under *-homomorphisms).

Proposition 3.3 *Let A be a C^*-algebra, and let a be a non-zero positive element in A. The following conditions are equivalent:*

(i) *a is properly infinite,*

(ii) *$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a$ for every $\varepsilon > 0$,*

(iii) *for every $\varepsilon > 0$ there exist mutually orthogonal positive elements a_1, a_2 in $\overline{aAa}$ such that $(a-\varepsilon)_+ \precsim a_j$,*

(iv) *there are sequences $\{x_n\}$ and $\{y_n\}$ in $\overline{aAa}$ with $x_n^*x_n \to a$, $y_n^*y_n \to a$, and $x_n^*y_n \to 0$,*

(v) *for every $\varepsilon > 0$ there are x, y in $\overline{aAa}$ with $x^*x = y^*y = (a-\varepsilon)_+$, and $xx^* \perp yy^*$.*

Moreover, (i) – (v) *are implied by*

(vi) *$\mathcal{M}(\overline{aAa})$ is properly infinite.*

Proof: We show

$$\text{(i)} \Rightarrow \text{(v)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)}, \qquad \text{(v)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}, \qquad \text{(vi)} \Rightarrow \text{(iv)}.$$

(i) $\Rightarrow$ (v). Assume $a \oplus a \precsim a$. Let $\varepsilon > 0$. There is, by Proposition 2.6, an element $r = (s\ t)$ in $M_{1,2}(A)$ with $(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ = r^*ar$, or, equivalently, $s^*as = t^*at = (a-\varepsilon)_+$ and $s^*at = 0$. Hence condition (v) holds with $x = a^{1/2}s$ and $y = a^{1/2}t$.

(v) $\Rightarrow$ (iv). Take x_n and y_n as in (v) corresponding to $\varepsilon = 1/n$.

(iv) $\Rightarrow$ (i). Let $\{x_n\}$ and $\{y_n\}$ be given as in (iv). In particular, each x_n and y_n lie in $\overline{a^{1/2}A}$, and we can therefore find s_n and t_n in A with $\|a^{1/2}s_n - x_n\| \to 0$ and $\|a^{1/2}t_n - y_n\| \to 0$. Then $s_n^*as_n \to a$, $t_n^*at_n \to a$, and $s_n^*at_n \to 0$. Put $z_n = (s_n\ t_n)$ in $M_{1,2}(A)$. Then $z_n^*az_n \to a \oplus a$, showing that $a \oplus a \precsim a$.

(v) $\Rightarrow$ (iii). Find x, y in $\overline{aAa}$ with $x^*x = y^*y = (a-\varepsilon)_+$, and $xx^* \perp yy^*$. Put $a_1 = xx^*$ and $a_2 = yy^*$. Then a_1 and a_2 belong to $\overline{aAa}$, $a_1 \perp a_2$, and $(a-\varepsilon)_+ \sim a_j$, so (iii) holds.

(iii) $\Rightarrow$ (ii). Assume a_1, a_2 are mutually orthogonal positive elements in $\overline{aAa}$ with $(a-\varepsilon)_+ \precsim a_j$ for some $\varepsilon > 0$. Then

$$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \ \precsim\ a_1 \oplus a_2 \ \sim\ a_1 + a_2 \ \precsim\ a,$$

by Proposition 2.7 (i) and Lemma 2.8 (iii).

(ii) $\Rightarrow$ (i). If $(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim a$ for every $\varepsilon > 0$, then $a \oplus a \precsim a$ by Proposition 2.6.

(vi) $\Rightarrow$ (iv). Let s_1, s_2 be isometries in $\mathcal{M}(\overline{aAa})$ with orthogonal range projections, and put $x = s_1 a^{1/2}$, $y = s_2 a^{1/2}$. Then x, y lie in $\overline{aAa}$, $x^*x = y^*y = a$, and $x^*y = 0$. Hence (iv) holds. $\square$

Question 3.4 *Is $\mathcal{M}(\overline{aAa})$ properly infinite if a is properly infinite?*

Proposition 3.5 *Let a be a properly infinite positive element in a C^*-algebra A. Then:*

(i) *$a \otimes 1_k \precsim a$ for every k in $\mathbb{N}$.*

(ii) *$b \precsim a$ for every b in $\overline{AaA}$.*

Proof: (i). If a is properly infinite and if $a \otimes 1_{k-1} \precsim a$ for some k in $\mathbb{N}$, then

$$a \otimes 1_k \;\precsim\; a \oplus (a \otimes 1_{k-1}) \;\precsim\; a \oplus a \;\precsim\; a,$$

by Lemma 2.9. (i) therefore follows by induction on k.

(ii). Combining (i) with Proposition 2.7 (v) yields $(b - \varepsilon)_+ \precsim a$ for every b in $\overline{AaA}$ and for every $\varepsilon > 0$. By Proposition 2.6, this implies $b \precsim a$. $\square$

Definition 3.6 *A positive element a in a C^*-algebra A is called* stable *if $\overline{aAa}$ is a stable C^*-algebra.*

Proposition 3.7 *Every stable positive element is properly infinite.*

Proof: The multiplier algebra of a stable C^*-algebra is properly infinite (because it contains the bounded operators on an infinite-dimensional Hilbert space as a unital sub-C^*-algebra). The proposition therefore follows from Proposition 3.3 (vi). $\square$

We see from Proposition 3.7 that the C^*-algebra $\mathcal{K}$ of all compact operators on a separable infinite-dimensional Hilbert space contains properly infinite elements. This may conflict with the intuition that $\mathcal{K}$ is a finite C^*-algebra. The finiteness of $\mathcal{K}$ can be recovered from the fact that the *Pedersen ideal* of $\mathcal{K}$ contains no infinite elements (and no properly infinite elements).

Lemma 3.8 *Let a be a properly infinite positive element in a C^*-algebra A, and let b be a positive element in $\overline{AaA}$ satisfying $a \precsim b$. Then b is properly infinite.*

Proof: If b is a positive element in $\overline{AaA}$, then $b \precsim a$ by Proposition 3.5. If, in addition, $a \precsim b$, then $a \approx b$, and this clearly entails that b is properly infinite. $\square$

Lemma 3.9 *Let a, b be properly infinite positive elements in a C^*-algebra A. Then $a \oplus b$ is a properly infinite element in $M_2(A)$, and if $a \perp b$, then $a+b$ is a properly infinite element in A.*

Proof: The first claim follows from Lemma 2.9, and the second claim follows from the first and the fact that $a + b \sim a \oplus b$ when a and b are orthogonal (cf. Lemma 2.8 (iii)). $\square$

Question 3.10 *Is the sum of two (not necessarily orthogonal) positive properly infinite elements always properly infinite?*

Definition 3.11 *Let A be a C^*-algebra, and let a be a positive element in A. Put*

$$J(a) = \{x \in A : a \oplus |x| \precsim a\}.$$

Lemma 3.12 *Let A be a C^*-algebra, let a be a positive element in A, and let $J(a)$ be as in Definition 3.11 above. Then:*

(i) *$J(a)$ is a closed two-sided ideal in A.*

(ii) *$a \oplus |x| \precsim a$ for every k in $\mathbb{N}$ and for every x in $M_k(J(a))$.*

(iii) *$J(a)$ is contained in $\overline{AaA}$.*

(iv) *a is finite if and only if $J(a) = 0$, a is infinite if and only if $J(a) \neq 0$, and a is properly infinite if and only if $J(a) = \overline{AaA} \neq 0$.*

Proof: (i). We show first that

$$|xy| \precsim |x|, \tag{3.1}$$
$$|yx| \precsim |x|, \tag{3.2}$$
$$|x+y| \precsim |x| \oplus |y|. \tag{3.3}$$

for every pair of elements x, y in A. Using $c \approx c^2$, cf. Lemma 2.2, we calculate

$$\begin{aligned} |yx| &\approx x^*y^*yx \le \|y\|^2 x^*x \precsim |x|^2 \precsim |x|, \\ |xy| &\approx y^*x^*xy \sim xyy^*x^* \le \|y\|^2 xx^* \sim \|y\|^2 x^*x \precsim |x|^2 \precsim |x|. \end{aligned}$$

This proves (3.1) and (3.2). Since $|x+y|$ belongs to the hereditary sub-C^*-algebra of A generated by $|x|+|y|$, (3.3) follows from Proposition 2.7 (i).

It follows from (3.1), (3.2), and (3.3) that $J(a)$ is a two-sided ideal in A. The set of positive elements b in $M_2(A)$ for which $b \precsim a$ is norm-closed, and the map $x \mapsto a \oplus |x|$ is continuous. Hence $J(a)$ is norm-closed.

(ii). Assume that x belongs to $M_k(J(a))$ for some k in $\mathbb{N}$. Let $b_1, b_2, \dots, b_k$ be the diagonal entries of $|x|$. Then each b_j lies in $J(a)$, and $|x| \precsim b_1 \oplus b_2 \oplus \cdots \oplus b_k$ by Lemma 2.8. Hence

$$a \oplus |x| \;\precsim\; a \oplus b_1 \oplus b_2 \oplus \cdots \oplus b_k \;\precsim\; a \oplus b_2 \oplus b_3 \oplus \cdots \oplus b_k \;\precsim\; \cdots \;\precsim\; a.$$

(iii). If $a \oplus |x| \precsim a$, then $|x| \precsim a$. Hence $|x|$ and x belong to $\overline{AaA}$.

(iv). By definition, a is properly infinite if and only if $a \in J(a)$ and a is non-zero. The former is clearly equivalent to $J(a)$ being equal to $\overline{AaA}$. The claims about a being finite, respectively, infinite, follow immediately from the definition of a being finite, respectively, infinite. □

Lemma 3.13 *The element $a + J(a)$ in $A/J(a)$ is finite for every C^*-algebra A and for every positive element a in A.*

Proof: Denote by π the quotient mapping $A \to A/J(a)$. Assume, to reach a contradiction, that $\pi(a)$ is infinite. Then $\pi(a) \oplus c \precsim \pi(a)$ for some non-zero positive element c in $A/J(a)$. Let $\varepsilon > 0$. Find s in $M_{1,2}(A/J(a))$ with

$$s^*\pi(a)s = (\pi(a) - \varepsilon)_+ \oplus (c - \varepsilon)_+ = (\pi(a) \oplus c - \varepsilon)_+.$$

Lift c to a positive element b in A, lift s to an element r in $M_{1,2}(A)$, and put

$$x = (a - \varepsilon)_+ \oplus (b - \varepsilon)_+ - r^*ar.$$

Then x belongs to $M_2(J(a))$, and so, by Lemma 2.8 (ii),

$$(a - \varepsilon)_+ \oplus (b - \varepsilon)_+ \;=\; r^*ar + x \;\leq\; r^*ar + |x| \;\precsim\; r^*ar \oplus |x| \;\precsim\; a \oplus |x| \;\precsim\; a.$$

This holds for all $\varepsilon > 0$ and therefore $a \oplus b \precsim a$. Hence b lies in $J(a)$ in contradiction with the assumption that $c = \pi(b) \neq 0$. □

Proposition 3.14 *A non-zero positive element a in a C^*-algebra A is properly infinite if and only if the element $a+I$ in A/I is either zero or infinite for every closed two-sided ideal I in A.*

Proof: The relation $a \oplus a \precsim a$ passes to quotients, and so, if a is properly infinite, then $a+I$ is either zero or properly infinite for every closed two-sided ideal I in A.

Conversely, if a is non-zero and not properly infinite, then $J(a)$ is a proper ideal in $\overline{AaA}$ by Lemma 3.12, and $a+J(a)$ is then a non-zero finite element in $A/J(a)$ by Lemma 3.13. □

The corollaries below are contained in Proposition 3.14. Since these results may be of independent interest we add an alternative proof (of Corollary 3.16) that does not use the theory of comparison of positive elements.

Corollary 3.15 *A non-zero projection p in a C^*-algebra A is properly infinite if and only if the element $p+I$ in A/I is either zero or infinite for every closed two-sided ideal I in A.*

Observing that a projection p in A is properly infinite if and only if the corner algebra pAp contains a unital copy of the Cuntz algebra $\mathcal{O}_\infty$, we get the following reformulation of Corollary 3.15:

Corollary 3.16 *A unital C^*-algebra A contains a unital copy of $\mathcal{O}_\infty$ if and only if every non-zero quotient of A contains a non-unitary isometry.*

Proof: Denote by $\mathcal{P}$ the set of projections in A which are of the form $1-ss^*$ for some isometry s in A. Let J be the closed two-sided ideal in A generated by $\mathcal{P}$.

We claim that the set

$$X = \{x^*px \mid x \in A, \;\; p \in \mathcal{P}\}$$

is dense in J^+. A standard argument shows that the additive span of X is dense in J^+, and we need therefore only show that $X+X$ is contained in X. Let p,q be in $\mathcal{P}$ and let x,y be in A. Let s and t be isometries in A with $p=1-ss^*$ and $q=1-tt^*$. Put $r=1-(st)(st)^*$ and put $z=px+sqy$. Then r belongs to $\mathcal{P}$, $r=p+sqs^*$, and $z^*rz=x^*px+y^*qy$, showing that x^*px+y^*qy belongs to X.

We proceed to show that A/J contains no non-unitary isometries (if $J \neq A$). Assume, to reach a contradiction, that v is a non-unitary isometry in A/J. Lift v to a contraction a in A. Then $1-a^*a$ is positive and belongs to J, but $1-aa^*$ does not belong to J. Since

$1-a^*a$ lies in the closure of X we can find p in $\mathcal{P}$ and x in A such that $\|(1-a^*a)-x^*px\| < 1$. Let s in A be an isometry with $p = 1 - ss^*$, and put $b = sa + px$. Then

$$\|1 - b^*b\| = \|(1 - a^*a) - x^*px\| < 1,$$

and so b^*b is invertible. Put $u = b(b^*b)^{-1/2}$. Then $u^*u = 1$, and so $1 - uu^*$ belongs to J. That implies that the elements $u + J$ and $b + J$ in A/J are invertible. But $b + J = (s + J)v$ is a product of a unitary and a non-unitary isometry, and hence non-invertible.

We conclude that $J = A$. Then 1 lies in the closure of X and we can find p in $\mathcal{P}$ and x in A such that $\|1 - x^*px\| < 1$. Let s be an isometry in A with $p = 1 - ss^*$. Put $t = px(x^*px)^{-1/2}$. Then t is an isometry with $tt^* \le p$. Hence s, t are isometries with orthogonal range projections, and the C^*-algebra generated by s and t contains a unital copy of $\mathcal{O}_\infty$. □

Lemma 3.17 *Let A be a C^*-algebra, let a be a positive element in A, and let p be a projection in A satisfying $p \precsim a$.*

(i) *If p is equivalent to a sub-projection p_0 of p, then $p - p_0$ belongs to $J(a)$.*

(ii) *If p is properly infinite, then p belongs to $J(a)$.*

Proof: (i). If $p \precsim a$, then $p = r^*ar$ for some r in A by Proposition 2.6. It follows that p is equivalent to the projection $p' = a^{1/2}rr^*a^{1/2}$ in $\overline{aAa}$, and we may for this reason assume that p lies in $\overline{aAa}$.

The two projections $q = (1 - p) + p_0$ and $1 = (1 - p_0) + p$ are equivalent in $\mathcal{M}(\overline{aAa})$. Let v in $\mathcal{M}(\overline{aAa})$ be such that $v^*v = 1$ and $vv^* = q$. Then

$$a \oplus (p - p_0) \;\approx\; vav^* \oplus (p - p_0) \;\sim\; vav^* + (p - p_0) \;\precsim\; a,$$

where the last claim follows from Proposition 2.7 (i).

(ii). If p is properly infinite, then there exist mutually orthogonal sub-projections p_1, p_2 of p, which are equivalent to p. It follows from (i) that $p - p_2$ lies in $J(a)$. This shows that p lies in $J(a)$ being equivalent to p_1 which is less than $p - p_2$. □

It follows from Lemma 3.17 that if A is a C^*-algebra, p is a projection in A, and a is a positive element in A such that $p \precsim a$, then a is infinite if p is infinite. In other words, a positive element is infinite if it dominates an infinite projection. The examples below show that the assumption that p is a projection cannot be omitted:

Example 3.18 We find examples of positive elements a, b in a C^*-algebra such that $b \precsim a$, a is finite, and b is infinite — even properly infinite.

As the first example, let A be the C^*-algebra $\widetilde{\mathcal{K}}$, the unitazation of the compact operators on an infinite dimensional separable Hilbert space. Let b be a strictly positive element in $\mathcal{K}$, and let a be the unit of $\widetilde{\mathcal{K}}$. Then $b \precsim a$, a is finite, being a finite projection, and b is properly infinite by Proposition 3.7.

In the next example, b is even more infinite, in that $(b-\varepsilon)_+$ is properly infinite whenever $0 \leq \varepsilon < \|b\|$ (and b is non-zero). Let here A be the unitazation of $\mathcal{O}_2 \otimes C_0(\mathbb{R})$. Let a be the unit of A, and let b be any non-zero positive element in $\mathcal{O}_2 \otimes C_0(\mathbb{R})$. Then $b \precsim a$, and a is finite being a minimal projection in A, cf. Lemma 3.1. Also, $(b-\varepsilon)_+$ is a non-zero positive element in $\mathcal{O}_2 \otimes C_0(\mathbb{R})$ whenever $0 \leq \varepsilon < \|b\|$, and so $(b-\varepsilon)_+$ is properly infinite, cf. Proposition 4.5 below.

4 Purely infinite C^*-algebra

Joachim Cuntz considered in [9] the property of a *simple* C^*-algebra that each of its non-zero hereditary sub-C^*-algebras contain an infinite projection, and he called simple C^*-algebras with this property *purely infinite.* We extend his definition to cover non-simple C^*-algebras as follows:

Definition 4.1 *A C^*-algebra A is said to be* purely infinite *if there are no characters on A and if for every pair of positive elements a, b in A, such that a lies in the closed two-sided ideal generated by b, there exists a sequence $\{r_j\}_{j=1}^\infty$ in A with $r_j^* b r_j \to a$.*

The latter condition can, with the terminology of Section 2, be rephrased as: For every pair of positive elements a, b in A, $a \precsim b$ if and only if a belongs to $\overline{AbA}$.

This definition agrees with Cuntz' definition in the simple case as shown in Proposition 5.4 — a fact which is also contained in [21]. Our definition entails that the zero C^*-algebra is purely infinite. Whether one should exclude the zero C^*-algebra from the list of purely infinite C^*-algebras is a matter of taste, but it is actually convenient to include it as a purely infinite C^*-algebra.

Lemma 4.2 *If A is a C^*-algebra in which every non-zero positive element is properly infinite, then A is purely infinite.*

Proof: To see that A has no characters it suffices to show that no non-zero quotients of A are Abelian. A properly infinite element in A is under a quotient homomorphism mapped to a properly infinite element or to zero. But an Abelian C^*-algebra contains no properly infinite elements (this is an easy consequence of Proposition 3.3). This shows that no non-zero quotient of A is Abelian.

Proposition 3.5 (ii) shows that $b \precsim a$ whenever b lies in $\overline{AaA}$ and a is properly infinite. It follows that A is purely infinite. $\square$

Proposition 4.3 *Every ideal in a purely infinite C^*-algebra, and every quotient of a purely infinite C^*-algebra are again purely infinite.*

Proof: Let A be a purely infinite C^*-algebra and let I be an ideal in A. Notice that A/I cannot have any characters, because A does not have characters, being purely infinite. Also, I cannot have characters. Indeed, if I admitted a character, then there would be a closed two-sided ideal J in I such that $I/J \cong \mathbb{C}$. Then I/J would a closed two-sided *unital* ideal in A/J and hence a direct summand of A/J. That would imply that A/J has a character, but we know that no such exists.

Let a, b be positive elements in A/I such that b lies in $\overline{(A/I)a(A/I)}$. Lift a to a positive element c in A. The quotient mapping $\pi\colon A \to A/I$ maps $\overline{AcA}$ onto $\overline{(A/I)a(A/I)}$, and we can therefore lift b to a positive element d in $\overline{AcA}$. Since A is purely infinite, $d \precsim c$. The *-homomorphism π preserves this relation, thus yielding $b \precsim a$ in A/I as desired.

If a, b are positive elements in I such that b lies in $\overline{IaI}$, then b lies in $\overline{AaA}$, which implies that $b \precsim a$ relatively to A. But then $b \precsim a$ relatively to I, cf. Lemma 2.2 (iii). $\square$

Proposition 4.4 *No C^*-algebra of type I is purely infinite.*

Proof: Notice first that *simple* type I C^*-algebras, i.e., C^*-algebras isomorphic to $\mathcal{K}(H)$, the compact operators on a Hilbert space H, are not purely infinite. Indeed, if $\dim(H) \geq 2$, and if e and f are projections in $\mathcal{K}(H)$ of dimensions 1, respectively, 2, then $f \not\precsim e$ although f lies in the ideal generated by e. If $\dim(H) = 1$, then $\mathcal{K}(H)$ has a character.

If A is a non-simple C^*-algebra not of type I, then there are ideals I and J in A with $I \subseteq J \subseteq A$ such that J/I is simple and of type I. Therefore J/I is not purely infinite, and so A cannot be purely infinite by Proposition 4.3. $\square$

Proposition 4.5 *Let A be a unital, simple, separable, purely infinite, nuclear C^*-algebra, and let B be any C^*-algebra. Then $A \otimes B$ is purely infinite. In particular, $\mathcal{O}_\infty \otimes B$ and $\mathcal{O}_2 \otimes B$ are purely infinite for every C^*-algebra B.*

Proof: It is a consequence of [16] (see also [18] and [15]) that

$$A \cong A \otimes (\otimes_{n=1}^{\infty} \mathcal{O}_\infty).$$

In particular there exists a sequence of unital *-homomorphisms $\varphi_n \colon \mathcal{O}_\infty \to A \otimes \widetilde{B}$ satisfying $\varphi_n(a)b - b\varphi_n(a) \to 0$ for every a in $\mathcal{O}_\infty$ and every b in $A \otimes B$. Let now c be a positive element in $A \otimes B$, let s_1, s_2 be isometries in $\mathcal{O}_\infty$ with orthogonal range projections, and set $x_n = c^{1/4}\varphi_n(s_1)c^{1/4}$ and $y_n = c^{1/4}\varphi_n(s_2)c^{1/4}$. Then $x_n^* x_n \to c$, $y_n^* y_n \to c$, and $x_n^* y_n \to 0$, and so c is properly infinite (or zero) by Proposition 3.3. This shows that $A \otimes B$ is purely infinite.

The last claims follows from the facts that $\mathcal{O}_\infty$ and $\mathcal{O}_2$ are simple, separable, purely infinite and nuclear, cf. [7]. $\square$

Example 4.6 The property that every non-zero hereditary sub-C^*-algebra contains an infinite projection is, for non-simple C^*-algebras, neither weaker nor stronger than being purely infinite (in the sense of Definition 4.1).

The C^*-algebra A obtained by adjoining a unit to $\mathcal{O}_2 \otimes \mathcal{K}$ has the property that every non-zero hereditary sub-C^*-algebra of A contains an infinite projection, but it is not purely infinite, having a character. For another such example — that does not have characters — let, for instance, B be a UHF-algebra and consider $A \otimes B$. Now, $A \otimes B$ still has the property that all its non-zero hereditary sub-C^*-algebras contain infinite projections, but $A \otimes B$ is not purely infinite. Indeed, $A \otimes B$ has a quotient isomorphic to B, and B is not purely infinite, and therefore $A \otimes B$ cannot be purely infinite, cf. Proposition 4.3.

The C^*-algebra $\mathcal{O}_2 \otimes C_0(\mathbb{R})$ is purely infinite (by Proposition 4.5) and this C^*-algebra contains no non-zero projections.

The following mild — and obvious — strengthening of the property that all hereditary sub-C^*-algebras contain infinite projections, we do get a condition that is stronger than pure infiniteness:

Proposition 4.7 *Let A be a C^*-algebra with the property that every non-zero hereditary sub-C^*-algebra in every quotient of A contains an infinite projection. Then A is purely infinite.*

Proof: It follows from Lemma 4.2 that A is purely infinite if every non-zero positive element a in A is properly infinite. By Proposition 3.14 this is the case if $a + I$ is infinite or zero in A/I for every ideal I in A. Let I be a closed two-sided ideal in A that does not contain a. The (non-zero) hereditary sub-C^*-algebra $\overline{(a+I)A/I(a+I)}$ contains by assumption an infinite projection p, and $p \precsim a+I$ by Proposition 2.7 (i). Lemma 3.17 (i) yields that $J(a+I)$ is non-zero, hence $a+I$ is infinite. □

Question 4.8 *Is it true that a C^*-algebra A is purely infinite if and only if every non-zero hereditary sub-C^*-algebra in every quotient of A contains an infinite positive element?*

We shall in the following establish the converse to Proposition 4.7, that every non-zero positive element in a purely infinite C^*-algebra is properly infinite, and at the same time we shall show that the class of purely infinite C^*-algebras is closed under forming extensions. This, together with Proposition 4.3, shows that the "only if" part of Question 4.8 is true.

Proposition 4.9 (Loring [22, Theorem 10.2.1], [23, Theorem 3.5]) *The C^*-algebra $M_n(C_0((0,1]))$ is* projective *for each $n \in \mathbb{N}$. That is, for every short-exact sequence*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

of C^*-algebra, and for every *-homomorphism $\varphi\colon M_n(C_0((0,1])) \to B$ there exists a *-homomorphism $\psi\colon M_n(C_0((0,1])) \to A$ so that $\varphi = \pi \circ \psi$.

Proposition 4.10 (Glimm) *Let A be a C^*-algebra not of type I. Then A contains a sub-C^*-algebra isomorphic to the C^*-algebra $M_n(C_0((0,1]))$ for each $n \in \mathbb{N}$.*

Proof: Glimm's classical theorem says that there exists a sub-C^*-algebra B of A and a closed two-sided ideal I in B such that the CAR-algebra embeds into B/I. From this we get an embedding $\psi_1\colon M_n(\mathbb{C}) \to B/I$ such that the CAR-algebra embeds unitally into the relative commutant $B/I \cap \psi_1(M_n(\mathbb{C}))'$. Since the C^*-algebra $C_0((0,1])$ can be embedded into the CAR-algebra, there is an embedding ψ_2 of $C_0((0,1])$ into $B/I \cap \psi_1(M_n(\mathbb{C}))'$. The *-homomorphism

$$\psi\colon M_n(\mathbb{C}) \otimes C_0((0,1]) \to B$$

given by $\psi(a \otimes f) = \psi_1(a)\psi_2(f)$ is an embedding of $M_n(C_0((0,1]))$ into B. Use now the result of Loring (Proposition 4.9) to lift this embedding to an embedding of $M_n(C_0((0,1]))$ into B. □

Lemma 4.11 *Let A be a purely infinite C^*-algebra.*

(i) *Let a be a positive element in A and assume that there are mutually orthogonal positive elements a_1, a_2 in A such that $a \precsim a_j$. Then a is properly infinite.*

(ii) *Let a be a properly infinite element in A. Then every positive element in $\overline{AaA}$ is properly infinite.*

(iii) *Let $\psi\colon M_2(C_0((0,1])) \to A$ be a *-homomorphism. Then every element in the ideal in A generated by the image of ψ is properly infinite.*

Proof: (i). Let $\varepsilon > 0$. By Proposition 2.6 there exist elements r_1, r_2 in A such that $(a-\varepsilon)_+ = r_j^* a_j r_j$. Put $b_j = a_j^{1/2} r_j r_j^* a_j^{1/2}$. Then $b_1 \perp b_2$, and $(a-\varepsilon)_+ \sim b_1 \sim b_2$. Now, $b_1 + b_2$ lies in $\overline{Ab_1A}$, and because A is purely infinite, we get

$$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \ \sim \ b_1 \oplus b_2 \ \sim \ b_1 + b_2 \ \precsim \ b_1 \ \sim \ (a-\varepsilon)_+ \ \precsim \ a.$$

Since $\varepsilon > 0$ was arbitrary, it follows from Proposition 3.3 that a is properly infinite.

(ii). Let b be a positive element in $\overline{AaA}$. Then $b \precsim a$ by Proposition 3.5. Let $\varepsilon > 0$, and find $\delta > 0$ such that $(b-\varepsilon)_+ \precsim (a-\delta)_+$, cf. Proposition 2.6. By Proposition 3.3 we can find mutually orthogonal positive elements a_1, a_2 in $\overline{aAa}$ with $(a-\delta)_+ \precsim a_j$. We then have $(b-\varepsilon)_+ \precsim a_j$, and by (i) this entails that $(b-\varepsilon)_+$ is properly infinite. It follows that $(b-\varepsilon)_+ \oplus (b-\varepsilon)_+ \precsim b$ for every $\varepsilon > 0$, and therefore b is properly infinite by Proposition 3.3.

(iii). Let d be the canonical generator of $C_0((0,1])$, and let d_1, d_2 in $M_2(C_0((0,1]))$ be the two diagonal elements that have d in one corner and zero in the other corner. Then $\psi(d_1)$ is properly infinite in A by (i). Since d_1 is full in $M_2(C_0((0,1]))$, the ideal in A generated by the image of ψ is equal to $\overline{A\psi(d_1)A}$, and every element in this ideal is properly infinite by (ii). $\square$

Lemma 4.12 *Let A be a C^*-algebra, let I be a closed two-sided ideal in A, and let $\pi\colon A \to A/I$ be the quotient mapping. Let a, b be positive elements in A, let $\varepsilon \geq 0$ be given, and suppose that $\pi((a-\varepsilon)_+) \precsim \pi(b)$. Then for each $\delta > \varepsilon$ there exists a positive element c in I with $(a-\delta)_+ \precsim b \oplus c$.*

Proof: Use Proposition 2.6 (and the remarks below it) to find y in A/I with $\pi((a-\delta)_+) = y^*\pi(b)y$. Lift y to an element x in A. Then $\pi(x^*bx - (a-\delta)_+) = 0$. It follows that

$c = |x^*bx - (a-\delta)_+|$ is a positive element in I, and

$$(a-\delta)_+ \;\leq\; x^*bx + c \;\precsim\; x^*bx \oplus c \;\precsim\; b \oplus c.$$

□

Lemma 4.13 *Let a be a properly infinite positive element in a C^*-algebra A. Let $n \in \mathbb{N}$ be given. Let d be the canonical generator of the C^*-algebra $C_0((0,1])$, and let d_j be the diagonal element in $M_n(C_0((0,1]))$ with d in the (j,j)th position and zeros elsewhere.*

Then for each $\varepsilon > 0$ there exists a *-homomorphism $\varphi \colon M_n(C_0((0,1])) \to \overline{aAa}$ with $(a-\varepsilon)_+ \approx \varphi(d_1)$.

Proof: By [22, p. 6], $M_n(C_0((0,1]))$ is the universal C^*-algebra generated by elements $\{w_i\}_{i=2}^n$ satisfying $\|w_i\| \leq 1$, $w_iw_j = 0$, $w_i^*w_j = \delta_{ij}w_2^*w_2$. Moreover, $d_1 = w_i^*w_i$.

As in the proof of "(i) $\Rightarrow$ (v)" in Proposition 3.3, and using that $a \otimes 1_n \precsim a$ when a is properly infinite, we find elements $x_1, x_2, \ldots, x_n$ in $\overline{aAa}$ such that $x_j^*x_j = (a-\varepsilon)_+$ and $x_i^*x_j = 0$ when $i \neq j$. Put $z_i = x_ix_1^*/\|x_ix_1^*\|$ for $i = 2, 3\ldots, n$. Then z_i belongs to $\overline{aAa}$, and for all i,j,

$$\|z_i\| \leq 1, \qquad z_iz_j = 0, \qquad z_i^*z_j = \delta_{ij}z_2^*z_2.$$

There is a *-homomorphism $\varphi \colon M_n(C_0((0,1])) \to \overline{aAa}$ with $\varphi(w_i) = z_i$. Now,

$$\varphi(d_1) \;=\; z_i^*z_i \;=\; x_1(a-\varepsilon)_+x_1^*/\|x_ix_1^*\|^2 \;\approx\; x_1(a-\varepsilon)_+x_1^*,$$

and

$$(a-\varepsilon)_+ \;\approx\; (a-\varepsilon)_+^3 \;=\; x_1^*x_1(a-\varepsilon)_+x_1^*x_1 \;\precsim\; x_1(a-\varepsilon)_+x_1^* \;\precsim\; (a-\varepsilon)_+.$$

This shows that $\varphi(d_1) \approx (a-\varepsilon)_+$. □

Lemma 4.14 *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\;\pi\;} B \longrightarrow 0 \tag{4.1}$$

be a short-exact sequence of C^*-algebras, and assume that all non-zero positive elements in B are properly infinite. Then for each positive non-zero element a in A and for each $\varepsilon > 0$ there exist mutually orthogonal, positive elements a_1, a_2, a_3 in $\overline{aAa}$, and a positive element c in $\overline{aIa}$ with $a_1 \sim a_2 \sim a_3$, and $(a-\varepsilon)_+ \precsim a_j \oplus c$.

Proof: Upon replacing (4.1) with

$$0 \longrightarrow \overline{aIa} \longrightarrow \overline{aAa} \longrightarrow \overline{\pi(a)B\pi(a)} \longrightarrow 0\,, \tag{4.2}$$

we can assume that a is a strictly positive element in A, and this will ensure that the elements a_1, a_2, a_3, c, found below, all belong to $\overline{aAa}$.

Since $\pi(a)$ is properly infinite we can use Lemma 4.13 to find a *-homomorphism $\varphi\colon M_3(C_0((0,1])) \to \overline{\pi(a)B\pi(a)}$ so that $(\pi(a)-\varepsilon/2)_+ \precsim \varphi(d_j)$ for $j = 1,2,3$ (with the notation of Lemma 4.13). It follows from Loring's lifting result, Proposition 4.9, that φ lifts to a *-homomorphism $\psi\colon M_3(C_0((0,1])) \to \overline{aAa}$. Put $a_j = \psi(d_j)$. Then $a_1 \sim a_2 \sim a_3$, $\pi((a-\varepsilon/2)_+) \precsim \pi(a_1)$, and there is a positive element c in I with $(a-\varepsilon)_+ \precsim a_1 \oplus c$ by Lemma 4.12. □

Lemma 4.15 *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

be a short-exact sequence of C^*-algebra, and assume that all non-zero positive elements in I and B are properly infinite. Then all non-zero positive elements in A are properly infinite.

Proof: Let a be an arbitrary non-zero positive element in A, and let $\varepsilon > 0$ be given. By Lemma 4.14 we can find mutually orthogonal, positive elements a_1, a_2, a_3 in A, and a positive element c in I with

$$c \precsim a, \qquad a_j \precsim a, \qquad a_1 \sim a_2 \sim a_3, \qquad (a-\varepsilon/2)_+ \precsim a_j \oplus c.$$

By Proposition 2.6, and the remarks below it, there is a $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (a_j-\delta)_+ \oplus (c-\delta)_+$.

Let I_0 be the hereditary sub-C^*-algebra of I consisting of those elements in $\overline{aIa}$ that are orthogonal to $(a_1-\delta)_+ + (a_2-\delta)_+$. Then I_0 is full in $\overline{aIa}$. Indeed, I_0 contains $\overline{a_3Ia_3}$, and the ideal generated by I_0 will therefore contain $\overline{a_1Ia_1}$ and $\overline{a_2Ia_2}$, cf. Lemma 2.4 and its proof. We can therefore find a (non-zero) positive element d in I_0 so that $(c-\delta/2)_+$ belongs to $\overline{IdI}$. By hypothesis, d is properly infinite, and so $(c-\delta/2)_+ \precsim d$ by Proposition 3.5. Hence $(c-\delta)_+ \precsim (d-\eta)_+$ for some $\eta > 0$ by Proposition 2.6. By Proposition 3.3 we can find mutually orthogonal positive elements d_1, d_2 in $\overline{dId}$ with $(d-\eta)_+ \precsim d_j$.

Put $b_j = (a_j - \delta)_+ + d_j$. Then b_1, b_2 are orthogonal elements in $\overline{aAa}$, and, because d_j is orthogonal to $(a_j - \delta)_+$, we get

$$(a - \varepsilon)_+ \precsim (a_j - \delta)_+ \oplus (c - \delta)_+ \precsim (a_j - \delta)_+ \oplus d_j \precsim b_j.$$

This shows that a is properly infinite, cf. Proposition 3.5. □

Theorem 4.16 *A C^*-algebra A is purely infinite if and only if every non-zero positive element in A is properly infinite.*

Proof: The "if"-part is contained in Lemma 4.2.

Assume that A is purely infinite. Let $\mathcal{I}$ be the collection of all closed two-sided ideals I in A for which all non-zero positive elements in I are properly infinite. We must show that A belongs to $\mathcal{I}$. The set $\mathcal{I}$ is upwards directed: if I and J lie in $\mathcal{I}$, then so does $I + J$. To see this, consider the short-exact sequence

$$0 \to I \to I + J \to (I + J)/I \to 0.$$

By assumption, all non-zero positive elements of I and J are properly infinite, and $(I+J)/I$ is isomorphic to $J/(I \cap J)$. Being properly infinite passes to quotients, so all non-zero positive elements of $(I + J)/I$ are properly infinite. It therefore follows from Lemma 4.14 that $I + J$ lies in $\mathcal{I}$.

Let I_0 be the closure of the union of all ideals in $\mathcal{I}$. We show that I_0 lies in $\mathcal{I}$. Let a be a non-zero positive element in I_0 and let $\varepsilon > 0$. Then $(a - \varepsilon)_+$ lies in the Pedersen ideal of I_0, and hence in some I in $\mathcal{I}$. This entails that $(a - \varepsilon)_+$ is either properly infinite or zero, and in both cases $(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \precsim (a - \varepsilon)_+ \precsim a$. This shows that a is properly infinite in I_0, cf. Proposition 3.3.

We claim that $I_0 = A$ (and this will complete the proof). Assume, to reach a contradiction, that $I_0 \neq A$. It follows from Proposition 4.3 that the quotient A/I_0 is purely infinite. Use Glimm's lemma (Proposition 4.10) combined with Proposition 4.4 to find an injective *-homomorphism $\varphi\colon M_2(C_0((0,1])) \to A/I_0$. The closed two-sided ideal, J, of A/I_0 generated by the image of φ has the property that all its non-zero positive elements are properly infinite by Lemma 4.11 (iii). Letting I_1 be the pre-image of J under the quotient mapping $A \to A/I_0$, we obtain a short-exact sequence $0 \to I_0 \to I_1 \to J \to 0$. All non-zero positive elements in I_0 and in J are properly infinite. Apply Lemma 4.15 to conclude that I_1 belongs to $\mathcal{I}$. Hence $I_0 = I_1$ which is a contradiction because J is non-zero. □

One can easily prove the proposition below without Theorem 4.16 — one needs only show that a hereditary sub-C^*-algebra of a purely infinite C^*-algebra does not admit a character — but the proof becomes pleasantly trivial *with* Theorem 4.16:

Proposition 4.17 *Every non-zero hereditary sub-C^*-algebra of a purely infinite C^*-algebra is again purely infinite.*

Proof: Let B be a non-zero hereditary sub-C^*-algebra of a purely infinite C^*-algebra A. Then $b \oplus b \precsim b$ relatively to A for every (non-zero) positive element b in B by Theorem 4.16. By Lemma 2.2, $b \oplus b \precsim b$ relatively to B, and this shows that B is purely infinite. □

Proposition 4.18 *Let A be the inductive limit of a directed system of purely infinite C^*-algebras $\{A_i\}_{i \in \mathbb{I}}$ with arbitrary connecting *-homomorphisms $\varphi_{i,j} \colon A_i \to A_j$ for $i \le j$. Then A is purely infinite.*

Proof: Denote the inductive limit map $A_i \to A$ by ψ_i. Let a be a non-zero positive element in A and let $\varepsilon > 0$. It follows from Lemma 2.5 (iii) that there exist i in $\mathbb{I}$ and a positive element b in A_i such that $(a-\varepsilon)_+ \precsim \psi_i((b-\varepsilon/2)_+) \precsim a$. Since A_i is purely infinite, $(b-\varepsilon/2)_+$ is properly infinite or zero by Theorem 4.16, and in either case we have

$$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \;\precsim\; \psi_i((b-\varepsilon/2)_+) \oplus \psi_i((b-\varepsilon/2)_+) \;\precsim\; \psi_i((b-\varepsilon/2)_+) \;\precsim\; a.$$

Hence a is properly infinite, cf. Proposition 3.3. □

Theorem 4.19 *Given a short exact sequence*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of C^*-algebras. Then A is purely infinite if and only if both I and B are purely infinite.

Proof: The "only if" part follows from Propositions 4.3 and 4.17. The "if" part follows from Lemma 4.15 and Theorem 4.16. □

To a family, $\{A_i\}_{i \in \mathbb{I}}$, of C^*-algebras one associates the product C^*-algebra, $\prod_{i \in \mathbb{I}} A_i$, and the sum C^*-algebra, $\sum_{i \in \mathbb{I}} A_i$, being the set of all tuples $\{a_i\}_{i \in \mathbb{I}}$, with a_i in A_i, where $\{\|a_i\|\}$ is bounded, respectively, where $\|a_i\|$ tends to zero outside finite subsets of I.

To a C^*-algebra A let $\ell_\infty(A)$ be the C^*-algebra of all bounded sequences $\{a_n\}_{n\in\mathbb{N}}$, where each a_n lies in A. If ω is a filter on $\mathbb{N}$, then let $c_\omega(A)$ be the ideal in $\ell_\infty(A)$ consisting of those sequences for which $\lim_\omega \|a_n\| = 0$, and put $A_\omega = \ell_\infty(A)/c_\omega(A)$. The C^*-algebra A_ω is called an *ultra power* of A.

Proposition 4.20 *If $\{A_i\}_{i\in\mathbb{I}}$ is a family of purely infinite C^*-algebras, then $\prod_{i\in\mathbb{I}} A_i$ and $\sum_{i\in\mathbb{I}} A_i$ are purely infinite C^*-algebras.*

If A is a purely infinite C^*-algebra, then $\ell_\infty(A)$ is purely infinite, and A_ω is purely infinite for every filter ω on $\mathbb{N}$.

Proof: If A is a purely infinite C^*-algebra, then each non-zero positive element a in A is properly infinite by Theorem 4.16. For each $\varepsilon > 0$ there exist elements x, y in $\overline{(a-\varepsilon/2)_+ A (a-\varepsilon/2)_+}$ such that $x^*x = y^*y = (a-\varepsilon)_+$ and $x^*y = 0$, cf. Proposition 3.3 and Lemma 2.5 (i). Let $\varphi_\varepsilon \colon \mathbb{R}^+ \to [0,1]$ be the continuous function given by

$$\varphi_\varepsilon(t) = \begin{cases} 0, & t \le \varepsilon, \\ \varepsilon^{-1}(t-\varepsilon), & \varepsilon \le t \le 2\varepsilon, \\ 1, & t \ge 2\varepsilon. \end{cases} \tag{4.3}$$

Set $e = \varphi_{\varepsilon/4}(a)$, and notice that $ex = xe = x$ and $ey = ye = y$.

Assume that all A_i are purely infinite. Let $a = \{a_i\}_{i\in\mathbb{I}}$ be a positive non-zero element in $\prod_{i\in\mathbb{I}} A_i$, and let $\varepsilon > 0$ be given. Set $e_i = \varphi_{\varepsilon/4}(a_i)$. Find x_i, y_i in A_i such that

$$x_i^*x_i = y_i^*y_i = (a-\varepsilon)_+, \qquad x_i^*y_i = 0, \qquad e_ix_i = x_ie_i = x_i, \qquad e_iy_i = y_ie_i = y_i.$$

Notice that $\|x_i\| = \|y_i\| = \|(a_i-\varepsilon)_+\|^{1/2}$. This shows that $x = \{x_i\}$ and $y = \{y_i\}$ both lie in $\prod_{i\in\mathbb{I}} A_i$. Put $e = \varphi_{\varepsilon/4}(a)$, and observe that $e = \{e_i\}$, $ex = xe = x$, and $ey = ye = y$. Therefore x and y belong to the hereditary sub-C^*-algebra of $\prod_{i\in\mathbb{I}} A_i$ generated by a. Also, $x^*x = y^*y = (a-\varepsilon)_+$, and $x^*y = 0$. It follows from Proposition 3.3 that a is properly infinite. Hence $\prod_{i\in\mathbb{I}} A_i$ is purely infinite.

The C^*-algebra $\sum_{i\in\mathbb{I}} A_i$ is purely infinite because it is a closed two-sided ideal in $\prod_{i\in\mathbb{I}} A_i$.

The C^*-algebra $\ell_\infty(A)$ is equal to $\prod_{i\in\mathbb{N}} A$, and is therefore purely infinite. Finally, A_ω is purely infinite being a quotient of the purely infinite C^*-algebra $\ell_\infty(A)$. □

Proposition 4.18 can be improved as follows:

Proposition 4.21 *Let*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \tag{4.4}$$

be a sequence of C^*-algebras, and let A be its inductive limit. Then A is purely infinite if and only if for every n in $\mathbb{N}$, for every positive element a in A_n, and for every $\varepsilon > 0$ there are $m > n$ and x, y in $\overline{\varphi_{n,m}(a) A_m \varphi_{n,m}(a)}$ such that

$$\|x^*x - \varphi_{n,m}(a)\| < \varepsilon, \qquad \|y^*y - \varphi_{n,m}(a)\| < \varepsilon, \qquad \|x^*y\| < \varepsilon, \tag{4.5}$$

where $\varphi_{n,m} \colon A_n \to A_m$ are the connecting map from the sequence (4.4).

Proof: Denote the inductive limit map $A_n \to A$ by ψ_n. We begin by showing the "only if" direction. We show first that every positive non-zero element a in $\bigcup_{n=1}^\infty \psi_n(A_n)$ is properly infinite. Let such an element a be given, and let $\varepsilon > 0$. Find n in $\mathbb{N}$ and a positive element b in A_n such that $\psi_n(b) = a$. Find next $m \geq n$ and x, y in $\overline{\varphi_{n,m}(b) A_m \varphi_{n,m}(b)}$ with

$$\|x^*x - \varphi_{n,m}(b)\| < \varepsilon, \qquad \|y^*y - \varphi_{n,m}(b)\| < \varepsilon, \qquad \|x^*y\| < \varepsilon.$$

Put $s = \psi_m(x)$ and $t = \psi_m(y)$. Then s, t lie in $\overline{aAa}$, and $\|s^*s - a\| \leq \varepsilon$, $\|t^*t - a\| \leq \varepsilon$, and $\|s^*t\| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that a is properly infinite, cf. Proposition 3.3.

Consider now an arbitrary positive non-zero element a in A, and let $\varepsilon > 0$. By Lemma 2.5 we can find a positive element b in $\bigcup_{n=1}^\infty \psi_n(A_n)$ with $(a - \varepsilon)_+ \precsim (b - \varepsilon/2)_+ \precsim a$. The element $(b - \varepsilon/2)_+$ also lies in $\bigcup_{n=1}^\infty \psi_n(A_n)$ and is therefore properly infinite or zero. Hence

$$(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \ \precsim\ (b - \varepsilon/2)_+ \oplus (b - \varepsilon/2)_+ \ \precsim\ (b - \varepsilon/2)_+ \ \precsim\ a.$$

This shows that a is properly infinite.

To prove the "if" part, assume that A is purely infinite, and let n in $\mathbb{N}$, a positive element a in A_n, and $\varepsilon > 0$ be given. Then, by Proposition 3.3, there exist s, t in $\overline{\psi_n(a) A \psi_n(a)}$ so that $\|s^*s - \psi_n(a)\| < \varepsilon/3$, $\|t^*t - \psi_n(a)\| < \varepsilon/3$, and $\|s^*t\| < \varepsilon/3$. Find $k \geq n$ and u, v in $\overline{\varphi_{n,k}(a) A_k \varphi_{n,k}(a)}$ with $\psi_k(u)$ and $\psi_k(v)$ close enough to s and t to ensure that

$$\|\psi_k(u)^*\psi_k(u) - \psi_n(a)\| < 2\varepsilon/3, \quad \|\psi_k(v)^*\psi_k(v) - \psi_n(a)\| < 2\varepsilon/3, \quad \|\psi_k(u)^*\psi_k(v)\| < 2\varepsilon/3.$$

Then, by choosing $m \geq k$ large enough, and letting $x = \varphi_{k,m}(u)$ and $y = \varphi_{k,m}(v)$, (4.5) is satisfied. □

Corollary 4.22 *Let A be a purely infinite C^*-algebra, and let X be a countable subset of A. Then there is a separable, purely infinite sub-C^*-algebra B of A that contains X.*

Proof: We construct inductively an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of countable subsets of A, and an increasing sequence of separable sub-C^*-algebras $\{B_n\}_{n=1}^{\infty}$ of A as follows. Let $X_1 = X$, let B_n be the sub-C^*-algebra generated by X_n, and choose for each n a countable dense subset Y_n of B_n^+. We describe now how one obtains X_{n+1} from X_n and Y_n.

Since A is purely infinite we can for each a in Y_n find elements $x(a), y(a)$ in $\overline{aAa}$ such that

$$\|x(a)^*x(a) - a\| < 1/n, \qquad \|y(a)^*y(a) - a\| < 1/n, \qquad \|x(a)^*y(a)\| < 1/n,$$

cf. Proposition 3.3. Put

$$X_{n+1} = \{x(a) \mid a \in Y_n\} \cup \{y(a) \mid a \in Y_n\} \cup X_n.$$

Let B be the closure of the union of the C^*-algebras B_n. Then B is isomorphic to the inductive limit of the sequence $B_1 \to B_2 \to B_3 \to \cdots$, where the connecting maps are inclusion maps, and B is a separable sub-C^*-algebra of A that contains X. From the construction of $\{B_n\}_{n=1}^{\infty}$ we see that for every $n \in \mathbb{N}$ and for every (non-zero) positive element a in B_n there exist elements x, y in B_{n+1} such that $\|x^*x - a\| < 1/n$, $\|y^*y - a\| < 1/n$, and $\|x^*y\| < 1/n$. It follows from Proposition 4.21 that B is purely infinite. $\square$

Theorem 4.23 *If A and B are stably isomorphic C^*-algebra, and if A is purely infinite, then B is purely infinite.*

Proof: We show first that if A is purely infinite, then $M_n(A)$ is purely infinite for every n in $\mathbb{N}$.

Let a be a positive element in $M_n(A)$, and let $\varepsilon > 0$. Let b in A^+ be the sum of the diagonal entries in a. Then, viewing A as a hereditary sub-C^*-algebra of $M_n(A)$, a lies in the ideal generated by b. Since b is properly infinite, $a \precsim b$ by Proposition 3.7 (iii). By Proposition 2.7 (ii) we can find an element b_0 in the hereditary sub-C^*-algebra $\overline{bM_n(A)b}$ such that $(a - \varepsilon)_+ \sim b_0$, and b_0 lies in A because b lies in A. Being properly infinite is preserved under the equivalence $\approx$, and therefore also under the equivalence $\sim$. Hence

$(a-\varepsilon)_+$ is properly infinite for every $\varepsilon > 0$. This shows that a is properly infinite, cf. Proposition 3.3.

Use Proposition 4.18 to conclude that $A \otimes \mathcal{K}$ is purely infinite when A is purely infinite. Hence $B \otimes \mathcal{K}$ is purely infinite, and B is (isomorphic to) a hereditary sub-C^*-algebra of $B \otimes \mathcal{K}$. Therefore B is purely infinite by Proposition 4.17. $\square$

The next result is a generalization of S. Zhang's dichotomy, [30], that a σ-unital purely infinite simple C^*-algebra is either unital or stable.

Theorem 4.24 *Let A be a σ-unital purely infinite C^*-algebra. Then A is stable if and only if no non-zero quotient of A is unital.*

Proof: If A is stable, then every quotient of A is stable and hence no quotient of A can be unital. Suppose conversely that no non-zero quotient of A is unital. It then follows from [14, Proposition 5.1] that A is stable if for every full hereditary sub-C^*-algebra B of A and for every positive element a in A, for which there exists a positive element e in A with $ea = a$, there is a positive element b in B such that $b \sim a$.

Let φ_ε be given as in (4.3). Put $e_1 = \varphi_{1/4}(e)$ and $e_2 = \varphi_{1/2}(e)$. Then $e_2 a = a$ and e_1 lies in the Pedersen ideal of A, and hence in the algebraic ideal generated by B. In particular, e_1 lies in $\overline{Ab_1A}$ for some positive element b_1 in B. Hence $e_1 \precsim b_1$ because b_1 is properly infinite, and consequently $(e_1 - \frac{1}{2})_+ = y^* b_1 y$ for some y in A. Next, $e_2 = w^*(e_1 - \frac{1}{2})_+ w$ for some w in A, and this shows that $e_2 = z^* b_1 z$ for some z in A. Put $x = b_1^{1/2} z a^{1/2}$ and put $b = xx^*$. Then b lies in B, $x^*x = a$, and so $b \sim a$ as desired. $\square$

5 Approximate divisibility and traces

We shall here investigate the relations between having no (quasi-)traces and being purely infinite.

By a *trace* on a C^*-algebra A we shall here understand a trace whose domain I is a (not necessarily closed or dense) two-sided ideal in A. A trace τ with domain I extends uniquely to a trace on $M_n(I)$ for every n in $\mathbb{N}$.

A *dimension function* on A is a function $d\colon \bigcup_{n=1}^{\infty} M_n(I)^+ \to \mathbb{R}^+$, where I as above is an algebraic ideal in A, satisfying $d(a \oplus b) = d(a) + d(b)$, and $d(a) \leq d(b)$ when $a \precsim b$. The ideal I is called the domain of d. A dimension function is *lower semi-continuous* if and only if

$$d(a) = \lim_{\varepsilon \to 0^+} d((a-\varepsilon)_+), \qquad a \in \bigcup_{n=1}^{\infty} M_n(I)^+. \tag{5.1}$$

If d is a dimension function on A with domain I, then

$$\bar{d}(a) = \lim_{\varepsilon \to 0^+} d((a-\varepsilon)_+), \qquad a \in \bigcup_{n=1}^{\infty} M_n(I)^+. \tag{5.2}$$

defines a lower semi-continuous dimension function with domain I, cf. [4].

Let τ be a trace on A with domain I. For each positive a in $M_n(I)$ set

$$d_\tau(a) = \lim_{\varepsilon \to 0^+} \tau(\varphi_\varepsilon(a)),$$

where φ_ε is as defined in (4.3). Then d_τ is a lower semi-continuous dimension function (or *rank function*) with domain I, cf. [8, Proposition 2.1]. If τ is non-zero, then d_τ is non-zero.

Conversely, as shown in [4], if d is a lower semi-continuous dimension function, then $d = d_\tau$ for some *quasi-trace* τ given by

$$\tau(a) = \int_0^{\|a\|} d((a-t)_+)\, dt, \qquad a \in A^+.$$

We refer to a theorem of Uffe Haagerup in [12] (see also [13]) that every quasi-trace on a unital *exact* C^*-algebra is, in fact, a trace. Following the methods of [4] one should be able to extend quasi-traces on non-unital exact C^*-algebras to certain unital exact C^*-algebras and in this way obtain that every quasi-trace on every exact (unital and non-unital) C^*-algebra is a trace. This is done in [17] for simple stably projectionless C^*-algebras.

Proposition 5.1 *A purely infinite C^*-algebra admits no non-zero dimension function and no non-zero trace.*

Proof: Assume that A is a purely infinite C^*-algebra and that d is a dimension function on A with domain I. Let a be a positive element in I. Then $a \oplus a \precsim a$, and so $d(a) + d(a) = d(a \oplus a) \leq d(a)$, whence $d(a) = 0$. This shows that A admits no non-zero dimension functions.

If τ were a non-zero trace on A, then it would induce a non-zero dimension function d_τ, but no such exists. □

Question 5.2 *Does there exist a C^*-algebra with no non-zero (quasi-)trace that is not purely infinite?*

It is not known if all simple C^*-algebras fall into the two classes: the stably finite C^*-algebras, and the purely infinite C^*-algebras. Every stably finite C^*-algebra has a quasi-trace, but it is not known if all quasi-traces are traces (for non-exact C^*-algebras). Question 5.2 could be answered in the affirmative by constructing an example of a *non-simple* C^*-algebra without (quasi-)traces that is not purely infinite, thus leaving the question for simple C^*-algebras unanswered. We are more optimistic about a positive answer to the following:

Question 5.3 *Let A be a C^*-algebra such that A_ω has no quasi-traces for some free ultra-filter ω on $\mathbb{N}$. Does it follow that A is purely infinite?*

We can answer Question 5.3 in the affirmative for simple C^*-algebras and for C^*-algebras of real rank zero. These results will be contained in a sequel to this paper. For the converse to Question 5.3 notice that A_ω is purely infinite when A is purely infinite by Proposition 4.20, and that A_ω therefore has no quasi-traces by Proposition 5.1.

Proposition 5.4 *If A is a simple C^*-algebra, which is purely infinite in the sense of Definition 4.1, then every non-zero hereditary sub-C^*-algebra of A contains an infinite projection.*

Proof: It is shown in [3] that $A \otimes \mathcal{K}$ contains an infinite projection if A is a simple C^*-algebra and $A \otimes \mathcal{K}$ admits no non-zero dimension function defined on its Pedersen ideal. Assuming that A is simple and purely infinite, we conclude from this and from Proposition 5.1 that $A \otimes \mathcal{K}$ contains an infinite projection p.

Let B be a non-zero hereditary sub-C^*-algebra of A, and view B as a hereditary sub-C^*-algebra of $A \otimes \mathcal{K}$. Since p, being a projection, lies in the *algebraic ideal* generated by B, there exists a positive element b in B so that p lies in (closed two-sided) ideal generated by b. Now, $p \precsim b$ because b is properly infinite (Theorem 4.16 and Proposition 3.7). Thus $p = x^*bx$ for some x in $A \otimes \mathcal{K}$ (using Proposition 3.3 and that p is a projection). Put $v = b^{1/2}x$ and put $q = vv^*$. Then $p = v^*v$ and q lies in B. Finally, q is an infinite projection, because p is an infinite projection and $q \sim p$. $\square$

We establish below a converse to Proposition 5.1 for *approximate divisible* C^*-algebras. Approximate divisibility for *unital* C^*-algebras was considered in [5]. We extend this property to *non-unital* C^*-algebras as follows, where $\mathcal{M}(A)$ denotes the multiplier algebra of A.

Definition 5.5 *A C^*-algebra A is called* approximately divisible *if for every n in $\mathbb{N}$, for every finite subset F of A, and for every $\varepsilon > 0$ there exists a unital *-homomorphism*

$$\varphi\colon M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to \mathcal{M}(A)$$

so that $\|a\varphi(b) - \varphi(b)a\| \leq \varepsilon\|b\|$ for every a in F and every b in $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$.

If A is approximately divisible, then the *-homomorphisms $\varphi\colon M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to \mathcal{M}(A)$ can always be chosen to be *injective*. Indeed, given n in $\mathbb{N}$, then for each $m > n(n+1)$ there are injective *-homomorphisms from $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ into $M_m(\mathbb{C})$ and into $M_{m+1}(\mathbb{C})$.

Using that $\mathcal{M}(A) \otimes \mathcal{M}(B)$ is contained in $\mathcal{M}(A \otimes B)$ (both tensor products are the minimal tensor products) we get:

Lemma 5.6 *Let A be an approximately divisible C^*-algebra, and let B be any C^*-algebra. Then $A \otimes B$ is approximately divisible.*

The three next results, and their proofs, are similar to the results of [27, Section 5].

Proposition 5.7 *Let A be a C^*-algebra with no non-zero, lower semi-continuous dimension functions. Then for each non-zero positive a in A and for each $\varepsilon > 0$ there exists k_0 in $\mathbb{N}$ such that $((a-\varepsilon)_+ \oplus (a-\varepsilon)_+) \otimes 1_k \precsim a \otimes 1_k$ for every $k \geq k_0$.*

Proof: If A admits no non-zero dimension functions (lower semi-continuous or not), then a variation of the proof given below will show that $(a \oplus a) \otimes 1_k \precsim a \otimes 1_k$ for all sufficiently large k in $\mathbb{N}$, showing that $a \otimes 1_k$ is properly infinite. It is — at least not a priory — clear if A could admit a non-zero dimension function but no non-zero lower semi-continuous dimension functions. In the proof, given below, we allow for the possibility that A does admit non-zero (non lower semi-continuous) dimension functions.

Let a and $\varepsilon > 0$ be given as in the proposition. Let I be the two-sided, not necessarily closed, ideal in A generated by a. Let T be the set of all positive elements in $\bigcup_{n=1}^\infty M_n(I)$, and let S be the set of equivalence classes of elements from T with respect to the equivalence relation $\approx$. Let $\langle b \rangle$ in S denote the equivalence class containing b in T. Define $+$ and $\leq$ on S by $\langle b_1 \rangle + \langle b_2 \rangle = \langle b_1 \oplus b_2 \rangle$, and $\langle b_1 \rangle \leq \langle b_2 \rangle$ if $b_1 \precsim b_2$. Then $(S, +, \leq, \langle a \rangle)$ is an ordered Abelian semi-group with distinguished order unit $\langle a \rangle$. A *state* ρ on $(S, +, \leq, \langle a \rangle)$ is an order preserving semi-group homomorphism $\rho\colon S \to \mathbb{R}$ with $\rho(\langle a \rangle) = 1$. There is a one-to-one connection between states ρ on S and dimension functions d on A with $d(a) = 1$ given by $d(b) = \rho(\langle b \rangle)$. We know that if d is a dimension function on A, then $\bar{d}$ (from (5.2)) is a lower semi-continuous dimension function on A, and by assumption $\bar{d}$ must be zero.

Therefore $d((a-\varepsilon)_+) \le \bar{d}(a) = 0$. We conclude that $\rho(\langle(a-\varepsilon)_+\rangle) = 0$ for every state ρ on S.

Let S_0 be the sub-semigroup of S generated by $\langle a\rangle$ and $\langle(a-\varepsilon)_+\rangle$. It follows from [6, Corollary 2.7] that every state on S_0 extends to a state on S. Hence $\rho(\langle(a-\varepsilon)_+\rangle) = 0$ for every state ρ on S_0. We can now use [6, Lemma 2.8] (with $x = 3\langle(a-\varepsilon)_+\rangle$ and $y = \langle a\rangle$) to find n in $\mathbb{N}$ and $\langle z\rangle$ in S_0 with

$$3n\langle(a-\varepsilon)_+\rangle + \langle z\rangle \le n\langle a\rangle + \langle z\rangle.$$

Write $z = ((a-\varepsilon)_+ \otimes 1_j) \oplus (a \otimes 1_l)$ for appropriate j, l in $\mathbb{N}$, and reorganize the expression above — and rephrase it in T — to obtain

$$((a-\varepsilon)_+ \otimes 1_{3n+j}) \oplus (a \otimes 1_l) \;\precsim\; ((a-\varepsilon)_+ \otimes 1_j) \oplus (a \otimes 1_{n+l}).$$

Iterating this expression m times yields

$$((a-\varepsilon)_+ \otimes 1_{3mn+j}) \oplus (a \otimes 1_l) \;\precsim\; ((a-\varepsilon)_+ \otimes 1_j) \oplus (a \otimes 1_{mn+l}),$$

and using $(a-\varepsilon)_+ \precsim a$, we get

$$(a-\varepsilon)_+ \otimes 1_{3mn+j+l} \;\precsim\; a \otimes 1_{mn+j+l}, \quad m \in \mathbb{N}. \tag{5.3}$$

There exists k_0 so that

$$[k_0, \infty) \;\subseteq\; \bigcup_{m=1}^{\infty} [mn+j+l, (3mn+j+l)/2].$$

If $k \ge k_0$, then $2k \le 3mn+j+l$ and $k \ge mn+j+l$ for some $m \in \mathbb{N}$. In combination with (5.3) this yields $((a-\varepsilon)_+ \oplus (a-\varepsilon)_+) \otimes 1_k \precsim a \otimes 1_k$ for every $k \ge k_0$ as desired. $\square$

Lemma 5.8 *Let A be a unital C^*-algebra, let B be a sub-C^*-algebra of A that contains the unit of A, and such that $A \cap B'$ contains a unital copy of $M_k(\mathbb{C}) \oplus M_{k+1}(\mathbb{C})$. Let a, b be positive elements in some matrix algebras over B and assume that $a \otimes 1_k \precsim b \otimes 1_k$ and $a \otimes 1_{k+1} \precsim b \otimes 1_{k+1}$ relatively to B. Then $a \precsim b$ relatively to A.*

Proof: For each $\varepsilon > 0$ there are $x = (x_{ij})$ in $M_k(B)$ and $y = (y_{ij})$ in $M_{k+1}(B)$ with $(a-\varepsilon)_+ \otimes 1_k = x^*(b \otimes 1_k)x$ and $(a-\varepsilon)_+ \otimes 1_{k+1} = y^*(b \otimes 1_{k+1})y$, cf. Proposition 2.6. Let

$\{e_{ij}\}$ and $\{f_{ij}\}$ be matrix units for the copies of $M_k(\mathbb{C})$ and $M_{k+1}(\mathbb{C})$ in $A \cap B'$. Put

$$r = \sum_{i,j=1}^{k} x_{ij} e_{ij} + \sum_{i,j=1}^{k+1} y_{ij} f_{ij}.$$

Then r belongs to A, and a direct calculation shows that $r^* b r = a$. $\Box$

Theorem 5.9 *Let A be an approximately divisible C^*-algebra with no non-zero lower semi-continuous dimension functions. Then A is purely infinite.*

In particular, if A is an exact, unital, approximately divisible C^*-algebra with no non-zero traces, then A is purely infinite.

Proof: As mentioned above, by the theorems of Haagerup, [12], and of Blackadar and Handelman, [4], if A is exact and unital without traces, then A has no non-zero lower semi-continuous dimension function. It therefore suffices to show that if A is an approximately divisible C^*-algebra without non-zero lower semi-continuous dimension functions, then A is purely infinite.

Let a be a non-zero positive element in A, let $\varepsilon > 0$, and use Proposition 5.7 to find k in $\mathbb{N}$ so that $(a - \varepsilon/4)_+ \otimes 1_l \precsim a \otimes 1_l$ for $l = k, k+1$. Then

$$((a - \varepsilon/3)_+ \oplus (a - \varepsilon/3)_+) \otimes 1_l = x_l^*((a - \delta)_+ \otimes 1_l) x_l, \quad l = k, k+1, \tag{5.4}$$

for some $\delta > 0$, some x_k in $M_{k,2k}(A)$ and x_{k+1} in $M_{k+1,2(k+1)}(A)$, cf. Proposition 2.6. Let F be the finite subset of A consisting of a and of all matrix entries of x_k and x_{k+1}.

Since A is approximately divisible, there is a sequence of unital *-homomorphisms $\varphi_n \colon M_k(\mathbb{C}) \oplus M_{k+1}(\mathbb{C}) \to \mathcal{M}(A)$ satisfying $\|d\varphi_n(b) - \varphi_n(b)d\| \to 0$ for all d in $C^*(F)$ and all b in $M_k(\mathbb{C}) \oplus M_{k+1}(\mathbb{C})$. Let

$$E_n \colon A \to \mathcal{M}(A) \cap \mathrm{Im}(\varphi_n)'$$

be conditional expectations such that $\|E_n(d) - d\| \to 0$ for all d in F. Extending E_n to square and rectangular matrices over A we get from (5.4) that

$$\|((E_n(a) - \varepsilon/3)_+ \oplus (E_n(a) - \varepsilon/3)_+) \otimes 1_l - E_n(x_l^*)((E_n(a) - \delta)_+ \otimes 1_l) E_n(x_l)\| \to 0$$

for $l = k, k+1$. It follows from Lemma 2.5 that

$$((E_n(a) - 2\varepsilon/3)_+ \oplus (E_n(a) - 2\varepsilon/3)_+) \otimes 1_l \precsim (E_n(a) - \delta)_+ \otimes 1_l, \quad l = k, k+1, \tag{5.5}$$

in $\mathcal{M}(A) \cap \mathrm{Im}(\varphi_n)'$ when n is sufficiently large. By Lemma 5.8 and (5.5) we get

$$(E_n(a) - 2\varepsilon/3)_+ \oplus (E_n(a) - 2\varepsilon/3)_+ \precsim (E_n(a) - \delta)_+. \tag{5.6}$$

From Lemma 2.5, by choosing n large enough, we obtain

$$(a - \varepsilon)_+ \precsim (E_n(a) - 2\varepsilon/3)_+, \qquad (E_n(a) - \delta)_+ \precsim a,$$

in $\mathcal{M}(A)$. Combining this with (5.6), we get $(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \precsim a$ in $\mathcal{M}(A)$ and hence in A by Lemma 2.2. Since $\varepsilon > 0$ was arbitrary, this shows that a is properly infinite, cf. Proposition 3.3. $\square$

Lemma 5.10 *Let A and B be two C^*-algebras, where one of them is purely infinite and one of them is exact. It follows that the minimal tensor product $A \otimes B$ does not admit a non-zero lower semi-continuous dimension function.*

Proof: Observe first that if A and B are any pair of C^*-algebras, and if a is a positive, properly infinite element in A and b is a positive non-zero element in B, then $a \otimes b$ is properly infinite in $A \otimes B$. To see this, let $\{x_n\}$ and $\{y_n\}$ be sequences in $\overline{aAa}$ satisfying $x_n^* x_n \to a$, $y_n^* y_n \to a$, and $x_n^* y_n \to 0$, cf. Proposition 3.3. Put $s_n = x_n \otimes b^{1/2}$ and $t_n = y_n \otimes b^{1/2}$ in $A \otimes B$. Then s_n and t_n lie in the hereditary sub-C^*-algebra generated by $a \otimes b$, and

$$s_n^* s_n = x_n^* x_n \otimes b \to a \otimes b, \qquad t_n^* t_n = y_n^* y_n \otimes b \to a \otimes b, \qquad s_n^* t_n = x_n^* y_n \otimes b \to 0.$$

Hence $a \otimes b$ is properly infinite by Proposition 3.3.

Assume that A is purely infinite and that d is a lower semi-continuous dimension function on $A \otimes B$ with domain I_0. Let I be the closure of the algebraic ideal I_0. We shall show that d is zero. By the observation above, for every positive elementary tensor $a \otimes b$ in I_0, we have $d(a \otimes b) = 0$, because $a \otimes b$ is either zero or properly infinite (cf. the proof of Proposition 5.7).

Let M be the set of all x in I_0 for which $d(x^* x) = 0$. Then M is a two-sided ideal in $A \otimes B$. Moreover, M is closed in in I_0. Indeed, let x be an element in I_0 that lies in the closure of M. Find a sequence $\{x_n\}$ in I_0 with $x_n \to x$. Then $x_n^* x_n \to x^* x$, and for each

$\varepsilon > 0$ there exists n in $\mathbb{N}$ with $(x^*x - \varepsilon)_+ \precsim x_n^* x_n$. Hence $d((x^*x - \varepsilon)_+) \leq d(x_n^* x_n) = 0$. Because d is lower semi-continuous, this implies that $d(x^*x) = 0$, and so x lies in M.

We show next that $I_0 \cap (J \otimes K) \subseteq M$ for every rectangular (or elementary) ideal $J \otimes K$ in $A \otimes B$ contained in I. The Pedersen ideal of $J \otimes K$ is contained in I_0. We can therefore find approximate units $\{e_i\}$ for J and $\{f_j\}$ for K such that $e_i \otimes f_j$ lie in I_0 for all i and j. If c is a positive element in $(e_i \otimes f_j)(A \otimes B)(e_i \otimes f_j)$, then $c \precsim e_i \otimes f_j$, and hence $d(c) \leq d(e_i \otimes f_j) = 0$. This shows that $(e_i \otimes f_j)(A \otimes B)(e_i \otimes f_j)$ is contained in M for all i, j. The union of these hereditary sub-C^*-algebras is dense in $I_0 \cap (J \otimes K)$, and we know that M is closed in I_0. Hence $I_0 \cap (J \otimes K)$ is contained in M.

Let $\{J_i \otimes K_i\}_{i \in \mathbb{I}}$ be the family of all rectangular ideals in $A \otimes B$ that are contained in I. By the assumption that one of A and B is exact, it follows from [15, Proposition 2.13] that I is generated as an ideal by this family. As shown above, $I_0 \cap (J_i \otimes K_i)$ is contained in M for all i in $\mathbb{I}$. Hence $J_i \otimes K_i$ is contained in $\overline{M}$, the closure of M, for all i; whence $I \subseteq \overline{M}$. Consequently, I_0 is contained in $\overline{M} \cap I_0$. Since $\overline{M} \cap I_0$ is contained in M, as shown above, we conclude that $M = I_0$ and that d is zero. □

The theorem below extends Proposition 4.5:

Theorem 5.11 *Let A be an exact, purely infinite, approximately divisible C^*-algebra. Then the minimal tensor product $A \otimes B$ is a purely infinite C^*-algebra for every C^*-algebra B.*

Proof: This is a immediate consequence of Lemmas 5.6 and 5.10 and of Theorem 5.9. □

Question 5.12 *Is the minimal tensor product $A \otimes B$ purely infinite if one of A and B is purely infinite?*

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