Tensor products of C^*-algebras with the ideal property

Cornel Pasnicu and Mikael Rørdam

Abstract

Combining a construction of Dadarlat of a unital, simple, non-exact C^*-algebra C of real rank zero and stable rank one, which is shape equivalent to a UHF-algebra, with results of Kirchberg and a result obtained by Dadarlat and the first named author, we show that $B(H) \otimes C$ contains an ideal that is not generated by its projections. We also find a unital, *separable* sub-C^*-algebra A of $B(H)$ such that A is of real rank zero and $A \otimes C$ has an ideal that is not generated by its projections.

1 Preliminaries

Definition 1.1 *A C^*-algebra A is said to have* the ideal property *if each closed two-sided ideal in A is generated by its projections.*

All simple, unital C^*-algebras have the ideal property and so do all C^*-algebras of real rank zero (the set of invertible self-adjoint elements is norm dense in the set of self-adjoint elements). Actually one purpose of considering the class of C^*-algebras with the ideal property is to find a common frame for theorems that hold for simple, unital C^*-algebras and for C^*-algebras of real rank zero. This is particularly pertinent within Elliott's classification program (see [5]).

If A is a C^*-algebra with the ideal property and with the cancellation property, then $(K_0(A), K_0(A)^+)$ is an ordered Abelian group and its ideal lattice is in a canonical way order isomorphic to the ideal lattice of A. Recall that a C^*-algebra A is said to have the *cancellation property* if the canonical map from the set of Murray–von Neumann equivalence classes of projections in the stabilized C^*-algebra $A \otimes \mathcal{K}$ to $K_0(A)$ is injective. All C^*-algebras of stable rank one (the invertible elements in the algebra or its unitization are dense) have the cancellation property, cf. [10]. Recall also that an ideal in an ordered Abelian group (G, G^+) is a subgroup H of G which satisfies $H = (H \cap G^+) - (H \cap G^+)$

and which has the hereditary property: If $x \in G$ and $y \in H$ satisfy $0 \leq x \leq y$, then x belongs to H.

The present paper is concerned with the question if the ideal property is preserved under forming minimal tensor products. The answer to this question is "yes" for all C^*-algebras that will be considered in Elliott's classification program due to the following theorem of Kirchberg:

Theorem 1.2 (Kirchberg, [7, Proposition 2.13]) *Let A and B be C^*-algebras of which at least one is exact. Then each closed two-sided ideal K of the minimal tensor product $A \otimes B$ is generated by the family of rectangular ideals $\{I_\alpha \otimes J_\alpha\}_{\alpha \in \mathbb{I}}$ contained in K.*

The symbol $\otimes$ will in this paper always mean the minimal tensor product, and all ideals are closed and two-sided.

As a corollary to Kirchberg's theorem we obtain:

Corollary 1.3 *Let A and B be C^*-algebras with the ideal property and suppose that at least one of A and B is exact. It follows that the minimal tensor product $A \otimes B$ has the ideal property.*

Proof: By Kirchberg's theorem above it suffices to show that whenever I and J are closed two-sided ideals in A and B, respectively, then $I \otimes J$ is generated by projections. By hypothesis, I and J are generated as ideals by sets of projections $\{p_\alpha\}_{\alpha \in \mathbb{A}}$, respectively, $\{q_\beta\}_{\beta \in \mathbb{B}}$. Hence $I \otimes J$ is the ideal generated by the set $\{p_\alpha \otimes q_\beta\}_{(\alpha,\beta) \in \mathbb{A} \times \mathbb{B}}$. □

Observe that each ideal in $A \otimes B$ from Corollary 1.3 actually is generated by elementary tensor projections $p \otimes q$, where $p \in A$ and $q \in B$ are projections.

2 The non-exact case

Dadarlat gave in [4] an example of a non-exact, unital, simple C^*-algebra C of stable rank one and real rank zero which is shape equivalent to an AF-algebra. For the convenience of the reader, and because we need a specific detail from the construction, we reproduce it here. We take the liberty of restricting ourselves to a specific case of Dadarlat's construction.

2.1 (Dadarlat's construction of a simple non-exact C^*-algebra, [4]) Put

$$D = \big(C^*(F_2) \otimes C_0((0,1])\big)^{\sim},$$

where $C^*(F_2)$ is the full C^*-algebra of F_2, the free group with two generators. The C^*-algebra $C^*(F_2)$ is non-exact, cf. S. Wassermann, [11], and it has a countable separating family of finite-dimensional representations by Choi, [2]. It follows that D is non-exact and that D has a countable separating family $\{\pi_n\}_{n=1}^\infty$ of finite-dimensional representations. By repeating each π_n infinitely many times, if necessary, we can assume that $\{\pi_n\}_{n=1}^\infty$ is *strongly separating*, i.e., for each non-zero a in A there are infinitely many natural numbers n such that $\pi_n(a) \neq 0$.

Let d_n be the dimension of the representation π_n and define inductively a sequence $\{k_n\}_{n=1}^\infty$ of integers by $k_1 = 1$ and $k_{n+1} = k_n(d_n + 1)$. We may view the map π_n as a unital *-homomorphism from D to $M_{d_n}(\mathbb{C})$. Composing π_n with the canonical unital *-homomorphism $M_{d_n}(\mathbb{C}) \to M_{d_n}(D)$ yields a unital *-homomorphism $D \to M_{d_n}(D)$. For each n, let $\rho_n\colon M_{k_n}(D_n) \to M_{d_n k_n}(D)$ be the unital *-homomorphism induced by this *-homomorphism.

Let α and β be the canonical unital *-homomorphisms in the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C^*(F_2) \otimes C_0((0,1]) \arrow[r] & D \arrow[r, shift left, "\alpha"] & \mathbb{C} \arrow[l, shift left, "\beta"] \arrow[r] & 0, \end{tikzcd}$$

and let $\alpha_n\colon M_{k_n}(D) \to M_{k_n}(\mathbb{C})$ and $\beta_n\colon M_{k_n}(\mathbb{C}) \to M_{k_n}(D)$ be the unital *-homomorphisms induced by α and β. Observe that

$$\beta \circ \alpha \sim_h \mathrm{id}_D, \qquad \beta_n \circ \alpha_n \sim_h \mathrm{id}_{M_{k_n}(D)}, \tag{2.1}$$

because $C^*(F_2) \otimes C_0((0,1])$ is homotopy equivalent to zero (and where $\sim_h$ denotes homotopy equivalence of *-homomorphisms).

Consider the diagram

$$\begin{tikzcd} D \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha_1"'] & D \arrow[r, "\varphi_1"] & M_{k_2}(D) \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha_2"'] & M_{k_2}(D) \arrow[r, "\varphi_2"] & M_{k_3}(D) \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha_3"'] & \cdots \arrow[r] & C \\ \mathbb{C} \arrow[r, "\mathrm{id}"'] & \mathbb{C} \arrow[r, "\lambda_1"'] \arrow[u, "\beta_1"] & M_{k_2}(\mathbb{C}) \arrow[r, "\mathrm{id}"'] & M_{k_2}(\mathbb{C}) \arrow[r, "\lambda_2"'] \arrow[u, "\beta_2"] & M_{k_3}(\mathbb{C}) \arrow[r, "\mathrm{id}"'] & \cdots \arrow[r] & B, \end{tikzcd} \tag{2.2}$$

where

$$\varphi_n(x) = \begin{pmatrix} x & 0 \\ 0 & \rho_n(x) \end{pmatrix}, \qquad \lambda_n(y) = \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y \end{pmatrix}.$$

The C^*-algebras C and B are the inductive limits of the first and the second row in (2.2), respectively, and so B is a UHF-algebra. Squares number one, three, five, seven, etc. in (2.2) commute up to homotopy by (2.1). The remaining squares in (2.2) commute up to unitary equivalence, and hence also up to homotopy. It follows that (2.2) is a shape equivalence between C and B. Moreover, C is non-exact (having a non-exact sub-C^*-algebra D), and C is clearly separable.

It is proved in [4, Proposition 9] that C is simple (because B is simple) and in [4, Proposition 5] it is shown that C is of stable rank one and of real rank zero. $\square$

It is crucial in (2.2) that the connecting maps (in the first row) are injective. We can therefore conclude that the sequence

$$A \otimes D \xrightarrow{\mathrm{id} \otimes \varphi_1} A \otimes M_{k_2}(D) \xrightarrow{\mathrm{id} \otimes \varphi_2} A \otimes M_{k_3}(D) \xrightarrow{\mathrm{id} \otimes \varphi_3} \cdots \tag{2.3}$$

has inductive limit (isomorphic to) $A \otimes C$ for any C^*-algebra A. In conclusion, upon tensoring (2.2) by an arbitrary C^*-algebra A (and after making a contraction of the diagram) we obtain the following shape equivalence (each triangle commutes up to homotopy):

$$\begin{tikzcd}
A \otimes D \arrow[rr, "\mathrm{id} \otimes \varphi_1"] \arrow[dr] & & A \otimes M_{k_2}(D) \arrow[rr, "\mathrm{id} \otimes \varphi_2"] \arrow[dr] & & A \otimes M_{k_3}(D) \arrow[r] & \cdots \arrow[r] & A \otimes C \\
& A \otimes \mathbb{C} \arrow[rr, "\mathrm{id} \otimes \lambda_1"'] \arrow[ur] & & A \otimes M_{k_2}(\mathbb{C}) \arrow[rr, "\mathrm{id} \otimes \lambda_2"'] \arrow[ur] & & \cdots \quad \cdots \arrow[r] & A \otimes B.
\end{tikzcd}$$

By a result of Dadarlat, [3], and the first named author, see [9, Proposition 4.13] and its proof, there is a natural order preserving bijective correspondence between the ideals generated by projections in $A \otimes C$ and in $A \otimes B$. Combining this result with Dadarlat's construction in Paragraph 2.1 we obtain:

Proposition 2.2 *Let B be the UHF-algebra and let C be Dadarlat's simple, non-exact C^*-algebra of stable rank one and real rank zero from Paragraph 2.1. Let A be any C^*-algebra. It follows that the ordered sets of ideals generated by projections in the two C^*-algebras $A \otimes C$ and $A \otimes B$ are isomorphic.*

From Dadarlat's result [4, Proposition 9] one gets to each simple, unital, infinite-dimensional AF-algebra B a simple, unital, quasidiagonal, separable, non-exact C^*-algebra C of real rank zero and stable rank one such that B and C are shape equivalent. To conclude that $A \otimes B$ and $A \otimes C$ are shape equivalent we used that the connecting maps in a specific shape equivalence between B and C are injective (in general, the connecting maps need not be injective). We do not know if this extra piece of information is also necessary: Suppose that B and C are shape equivalent C^*-algebras and that A is a general (possibly non-exact) C^*-algebra. Does it follow that $A \otimes B$ and $A \otimes C$ are shape equivalent? The difficulty lies in the fact, communicated to us by Kirchberg, that the minimal tensor product with a non-exact C^*-algebra does not preserve inductive limits (with non-injective connecting maps).

As shown by Kirchberg, [6, Theorem 1.1], a C^*-algebra A is exact if and only if for a separable infinite dimensional Hilbert space H, the sequence

$$0 \longrightarrow \mathcal{K} \otimes A \xrightarrow{\iota \otimes \mathrm{id}_A} B(H) \otimes A \xrightarrow{\pi \otimes \mathrm{id}_A} (B(H)/\mathcal{K}) \otimes A \longrightarrow 0$$

is exact, i.e., that $\mathrm{Ker}(\pi \otimes \mathrm{id}_A) = \mathcal{K} \otimes A$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on H. (One always has $\mathcal{K} \otimes A \subseteq \mathrm{Ker}(\pi \otimes \mathrm{id}_A)$. The kernel $\mathrm{Ker}(\pi \otimes \mathrm{id})$ is in [6] called the *Fubini* algebra of $\mathcal{K}$ and A.) Hence if C is a simple C^*-algebra, then C is non-exact if and only if the minimal tensor product $B(H) \otimes C$ contains other ideals than the three trivial ones: 0, $\mathcal{K} \otimes C$, and $B(H) \otimes C$. Combining this with Proposition 2.2 yields:

Theorem 2.3 *Let C be Dadarlat's C^*-algebra from Paragraph 2.1.*

Then C and $B(H)$ are C^*-algebras of real rank zero. In particular, C and $B(H)$ have the ideal property. However, the minimal tensor product $B(H) \otimes C$ does not have the ideal property.

More specifically, $B(H) \otimes C$ has other ideals than the three trivial ones: 0, $\mathcal{K} \otimes C$, and $B(H) \otimes C$, but each projection in $B(H) \otimes C$ either belongs to $\mathcal{K} \otimes C$ or is full (i.e., generates $B(H) \otimes C$ as an ideal).

Proof: Since B from Proposition 2.2 is a UHF-algebra (and therefore is simple and exact), the only ideals of $B(H) \otimes B$ are 0, $\mathcal{K} \otimes B$, and $B(H) \otimes B$. Each of these three ideals is generated by projections. It follows from Proposition 2.2 that $B(H) \otimes C$ has precisely three ideals that are generated by projections; and these three ideals are 0, $\mathcal{K} \otimes C$, and $B(H) \otimes C$ (because they are generated by projections). Since C is non-exact, Kirchberg's characterization of exact C^*-algebras mentioned above shows that $B(H) \otimes C$ has an ideal I other than its three ideals that are generated by projections.

To prove the last claim, let p be a non-zero projection in $B(H) \otimes C$. The ideal generated by p must be either $\mathcal{K} \otimes C$ or $B(H) \otimes C$. In the first case p belongs to $\mathcal{K} \otimes C$, and in the second case p is full in $B(H) \otimes C$. $\square$

Examples of two C^*-algebras each of real rank zero whose minimal tensor product is a C^*-algebra not of real rank zero are known, see [8]. (Actually, one such example is $B(H) \otimes C$ where C is any unital, exact C^*-algebra of real rank zero with non-zero K_1-group.) Theorem 2.3 shows that the examples of [8] can be improved in the direction that the tensor product does not even have the ideal property.

Question 2.4 *Does there exist a simple unital C^*-algebra C such that $B(H) \otimes C$ contains projections that neither belong to $\mathcal{K} \otimes C$ nor are full?*

A counterexample C to Question 2.4 must be non-exact; and Theorem 2.3 shows that not all non-exact simple C^*-algebras are counterexamples. Besides of this, we have no evidence to support a positive answer to Question 2.4.

We conclude this paper by showing that one can replace the non-separable C^*-algebra $B(H)$ in Theorem 2.3 by a separable one. First we need a lemma which is an elaboration of a technique by Blackadar in [1].

Lemma 2.5 *Each separable sub-C^*-algebra of $B(H)$ is contained in a unital, separable sub-C^*-algebra A of $B(H)$ which satisfies: $\mathcal{K} \subseteq A$, A is of real rank zero, and $A/\mathcal{K}$ is simple.*

Proof: Let A_0 be a separable sub-C^*-algebra of $B(H)$. Upon replacing A_0 with $A_0 + \mathcal{K}$ we may assume that A_0 contains $\mathcal{K}$. We find inductively an increasing sequence $\{A_n\}_{n=0}^{\infty}$ of separable sub-C^*-algebras of $B(H)$, countable subsets X_n, Y_n of A_n (for each $n \geq 0$), and countable subsets X'_{n+1}, Y'_{n+1} of A_{n+1} (for each $n \geq 0$) satisfying:

(i) X_n is a dense subset of the set of self-adjoint elements in A_n,

(ii) Y_n is a dense subset of the set $\{a \in A_n : \|a + \mathcal{K}\| = 1\}$,

(iii) for each a in X_n there is an invertible self-adjoint element b in X'_{n+1} with $\|a - b\| \leq 1/(n+1)$,

(iv) for each a in Y_n there are elements b, c in Y'_{n+1} with $1 = bac$, $\|b\| \leq 2$, and $\|c\| \leq 2$.

The construction goes as follows. To begin, take, as we can, any countable subsets X_0 and Y_0 of A_0 satisfying (i) and (ii).

Assume that $n \geq 0$ and that A_n, X_n, and Y_n have been found. Since $B(H)$ is of real rank zero we can for each a in X_n find a self-adjoint invertible element b in $B(H)$ such that $\|a - b\| \leq 1/(n+1)$. Let X'_{n+1} be the (countable) set of these elements b (one element

b for each a in X_n). Similarly, if a is an element in Y_n, then by a standard Hilbert space argument there are elements b, c in $B(H)$ with $1 = bac$, $\|b\| \le 2$, and $\|c\| \le 2$. Let Y'_{n+1} be the set of these elements b, c (one pair of elements b, c for each a in Y_n).

Let A_{n+1} be the separable C^*-algebra generated by $A_n \cup X'_{n+1} \cup Y'_{n+1}$. Finally, take any countable subsets X_{n+1} and Y_{n+1} of A_{n+1} satisfying (i) and (ii).

Now, let A be the closure of $\bigcup_{n=1}^{\infty} A_n$. Then A is separable, $\mathcal{K} \subseteq A$, and $A_0 \subseteq A$.

To see that A is of real rank zero, take a self-adjoint element a in A and let $\varepsilon > 0$. Find a natural number n (that we may assume is greater than $3/\varepsilon$) and a self-adjoint element a' in A_n with $\|a - a'\| \le \varepsilon/3$. Find next an element a'' in X_n with $\|a' - a''\| \le \varepsilon/3$, and take an invertible self-adjoint element b in X'_{n+1} with $\|a'' - b\| \le 1/(n+1) \le \varepsilon/3$. Then $\|a - b\| \le \varepsilon$ and b is an invertible self-adjoint element in A.

We proceed to prove that $A/\mathcal{K}$ is simple. It is for this purpose enough to show that for each element x in $A/\mathcal{K}$ of norm one there are elements y, z in $A/\mathcal{K}$ such that yxz is invertible. Find a in A with $a + \mathcal{K} = x$. Find next a natural number n and an element a' in A_n with $\|a - a'\| < 1/8$ and $\|a' + \mathcal{K}\| = 1$. Find then an element a'' in Y_n with $\|a' - a''\| < 1/8$ and take elements b, c in Y'_{n+1} with $ba''c = 1$, $\|b\| \le 2$, and $\|c\| \le 2$. Then,

$$\|1 - bac\| = \|b(a'' - a)c\| \le 4\|a - a''\| < 1,$$

and so bac is invertible, whence also $(b + \mathcal{K})x(c + \mathcal{K}) = bac + \mathcal{K}$ is invertible. $\square$

Theorem 2.6 *Let C be Dadarlat's simple, real rank zero, non-exact C^*-algebra from Paragraph 2.1.*

There is a separable, unital sub-C^*-algebra A of $B(H)$ such that A is of real rank zero (and hence has the ideal property), $\mathcal{K} \subseteq A$, $A/\mathcal{K}$ is simple, and such that $A \otimes C$ does not have the ideal property.

More specifically, $A \otimes C$ contains an ideal I satisfying $\mathcal{K} \otimes C \subsetneq I \subsetneq A \otimes C$ and each projection in I belongs to $\mathcal{K} \otimes C$. In particular, I is not generated by its projections.

Proof: Since C is non-exact, $B(H) \otimes C$ contains a proper ideal J that properly contains $\mathcal{K} \otimes C$. Take a positive element y in J with $\|y + \mathcal{K} \otimes C\| = 1$. Find next an element z in the algebraic tensor product $B(H) \otimes_{\mathrm{alg}} C$ close enough to $y^{1/2}$ so that $\|y - z^*z\| < 1/2$. Write

$$x = z^*z = \sum_{j=1}^{n} b_j \otimes c_j, \qquad b_j \in B(H), \;\; c_j \in C.$$

Let A_0 be the separable sub-C^*-algebra of $B(H)$ generated by $\{b_1, b_2, \ldots, b_n\}$. Then x belongs to $A_0 \otimes C$. Let $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ be given by $f(t) = \max\{0, t - 1/2\}$. Since $\|x - y\| < 1/2$

we find that $x+J$ is a positive element in $(B(H)\otimes C)/J$ of norm $< 1/2$. It follows that $f(x)+J = f(x+J) = 0$ and hence that $f(x)$ belongs to J. On the other hand,

$$\|f(x)-y\| \leq \|f(x)-x\| + \|x-y\| < 1 = \|y+\mathcal{K}\otimes C\|, \tag{2.4}$$

and so $f(x)$ does not belong to $\mathcal{K}\otimes C$.

Use Lemma 2.5 to find a unital, separable sub-C^*-algebra A of $B(H)$ containing A_0 and $\mathcal{K}$, such that A is of real rank zero and $A/\mathcal{K}$ is simple. Put $I = J\cap(A\otimes C)$. Since J is a proper ideal and therefore does not contain the unit (of $B(H)\otimes C$ and of $A\otimes C$), we see that I is a proper ideal of $A\otimes C$. Clearly $\mathcal{K}\otimes C$ is contained in I. To see that $I\neq\mathcal{K}\otimes C$ observe that $f(x)$ belongs to I and not to $\mathcal{K}\otimes C$.

Since all projections in J belong to $\mathcal{K}\otimes C$, the same holds for all projections in I. Therefore I is not generated by its projections. $\square$

References

[1] B. Blackadar, *Weak expectations and nuclear C^*-algebras*, Indiana Univ. J. Math. **27** (1978), 1021–1026.

[2] M.-D. Choi, *The full C^*-algebra of the free group on two generators*, Pacific J. Math. **87** (1980), 41–48.

[3] M. Dădărlat, private communication to Cornel Pasnicu.

[4] ———, *Nonnuclear subalgebras of AF-algebras*, to appear in Amer. J. Math.

[5] G. A. Elliott, *The classification problem for amenable C^*-algebras*, Proceedings of the International Congress of Mathematicians, vol. 1,2, Birkhäuser, Basel, 1995, pp. 922–932.

[6] E. Kirchberg, *The Fubini Theorem for exact C^*-algebras*, J. Operator Theory **10** (1983), 3–8.

[7] ———, *The classification of Purely Infinite C^*-algebras using Kasparov's Theory*, to appear in the Fields Institute Communication series, 2000.

[8] K. Kodaka and H. Osaka, *Real rank of tensor products of C^*-algebras*, Proc. Amer. Math. Soc. **123** (1995), no. 7, 2213–2215.

[9] C. Pasnicu, *On the AH algebras with the ideal property*, to appear in J. Operator Theory, 2000.

[10] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), no. (3), 301–333.

[11] S. Wasserman, *Exact C^*-algebras and related topics*, Lecture Notes Series, vol. 19, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, Korea, 1994.

Department of Mathematics, University of Puerto Rico, P.O. Box 23355, San Juan, Puerto Rico 00931, USA

E-mail address: `cpasnic@upracd.upr.clu.edu`

Department of Mathematics, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark

E-mail address: `rordam@math.ku.dk`
Internet home page: `www.math.ku.dk/~rordam/`