ON THE ORDERED $K_0$-GROUP OF
UNIVERSAL FREE PRODUCT $C^*$-ALGEBRAS

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Abstract. The ordered $K_0$-group of the universal, unital free product $C^*$-algebra $M_k(\mathbb{C}) * M_l(\mathbb{C})$ is calculated in the case where $k$ is prime and not a divisor in $l$. It is shown that the positive cone of $K_0(M_k(\mathbb{C}) * M_l(\mathbb{C}))$ is as small as possible in this case. The article also contains results (full and partial) on the ordered $K_0$-group of more general universal, unital free product $C^*$-algebras.

1. Introduction

Let $A * B$ denote the universal, unital free product $C^*$-algebra of the unital $C^*$-algebras $A$ and $B$. The purpose of this paper is to calculate the ordered $K_0$-group $(K_0(A * B), K_0(A * B)^+)$, at least for some specific $C^*$-algebras $A$ and $B$. Along the same line, we consider the question of what ordered abelian groups $(G, G^+)$ arise as the ordered $K_0$-groups of some $C^*$-algebra with particular focus on the case where $G = \mathbb{Z}$.

The $K_0$-groups — just as abelian groups — of these algebras have been calculated by E. Germain in [G] when $A$ and $B$ are $K$-nuclear. The ordered $K_0$-group of the reduced free product $C^*$-algebra $A * r B$ has been determined in [DR] (under some stronger assumptions on $A$ and $B$), and it has been shown that these $K_0$-groups are weakly unperforated, i.e. $ng > 0$ for some $n \in \mathbb{Z}^+$ and $g \in K_0(A * B)$ implies $g > 0$. One can rephrase this result by the somewhat imprecise statement, that the positive cone $K_0(A * r B)^+$ is as large as one can imagine. The results on the ordered $K_0$-group of the universal, unital, free product $C^*$-algebra referred to above can similarly be rephrased by saying that $K_0(A * B)^+$ is as small as one can imagine.

2. Ordered abelian groups and $K_0$-groups

Recall that an ordered abelian group is a pair $(G, G^+)$ where $G$ is an abelian group, $G^+ \subseteq G$ and

$$G^+ + G^+ \subseteq G^+, \quad G^+ - G^+ = G, \quad G^+ \cap -G^+ = \{0\}.$$ 

An element $u \in G^+$ is called an order unit in $G$ if for every $x \in G$ there exists a $k \in \mathbb{N}$ such that $-ku \leq x \leq ku$. If every $u \in G^+ \setminus \{0\}$ is an order unit, then

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$(G, G^+)$ is said to be simple. A simple ordered group $(G, G^+)$ is said to be weakly unperforated if $kx > 0$ for some $x \in G$ and some $k \in \mathbb{N}$ implies $x > 0$.

If $A$ is a stably finite C*-algebra with an approximate unit consisting of projections then $(K_0(A), K_0(A)^+)$ is an ordered abelian group. If, in addition, $A$ is simple then so is $(K_0(A), K_0(A)^+)$. George Elliott has proved in [E] that every simple, weakly unperforated, countable abelian ordered group is isomorphic to the ordered $K_0$-group of a unital, separable, simple, stably finite, nuclear C*-algebra. (This result fits into a program whereby the ordered $K_0$-group together with other invariants are conjectured — and in specific cases proved — to be a complete invariant for nuclear, simple, unital, separable, stably finite C*-algebras.) The second named author showed in [V] that the ordered $K_0$-group of a nuclear, simple, unital, separable, stably finite C*-algebra need not be weakly unperforated. This raises the following:

**Question 2.1.**

a) Which simple, countable ordered groups are isomorphic to the ordered $K_0$-group of some stably finite C*-algebra?

b) Which such groups are isomorphic to the ordered $K_0$-group of some simple, stably finite, unital C*-algebra?

c) Which such groups are isomorphic to the ordered $K_0$-group of some simple, stably finite, unital and nuclear C*-algebra?

We know of no examples of countable ordered groups (simple or not) that do not arise as the $K_0$-group of a C*-algebra. The example below may illustrate the complexity of ordered groups with perforation.

**Example 2.2.** Let $n \geq 2$ be an integer, and let $\Gamma_0, \ldots, \Gamma_{n-1}$ be arbitrary subsets of $\mathbb{Z}$. Set $G = \mathbb{Z} \oplus \mathbb{Z}$ and

$$G^+ = \{(0,0)\} \cup \{(n+j,x) : 0 \leq j < n, x \in \Gamma_j\} \cup \{(k,x) : k \geq 2n, x \in \mathbb{Z}\}.$$ 

Then $(G, G^+)$ is a simple ordered group (which is not weakly unperforated).

In the following we shall primarily study the case where $G = \mathbb{Z}$. In that case $G^+$ is either a subset of $\mathbb{Z}^+$ (= \{0, 1, 2, \ldots\}) or a subset of $-\mathbb{Z}^+$. We may assume the former by replacing $G^+$ with $-G^+$. If $S \subseteq \mathbb{Z}^+$, then $(\mathbb{Z}, S)$ is an ordered group if and only if $S$ is a subsemigroup of $\mathbb{Z}^+$, $0 \in S$ and $S - S = \mathbb{Z}$.

Let $n_1, \ldots, n_r \in \mathbb{Z}^+$ and set

$$S = \langle n_1, \ldots, n_r \rangle = \{k_1n_1 + \cdots + k_rn_r : k_1, \ldots, k_r \in \mathbb{Z}^+\}.$$ 

Then $S$ is a subsemigroup of $\mathbb{Z}^+$ and $0 \in S$. Moreover $S - S = \mathbb{Z}$ if and only if $\gcd(n_1, n_2, \ldots, n_r) = 1$. 

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Lemma 2.3. Let integers $k, l \geq 2$ with $gcd(k, l) = 1$ be given. Set $N = kl - k - l$. Then $N \notin \langle k, l \rangle$, but $\{N + 1, N + 2, N + 3, \ldots \} \subseteq \langle k, l \rangle$.

Moreover, if $S$ is a subsemigroup of $\mathbb{Z}^+$, then $S = \langle k, l \rangle$ if (and only if) $k, l \in S$ and $N \notin S$.

Proof. Each $m \in \mathbb{Z}$ can in a unique way be written as $m = ak + bl$ where $a, b \in \mathbb{Z}$ and $0 \leq a < l$. Clearly, $m \in \langle k, l \rangle$ if and only if $b \geq 0$ in this decomposition. Since $N = (l - 1)k - l$ it follows that $N \notin \langle k, l \rangle$. Assume that $m > N$. Then $bl = m - ak > N - (l - 1)k = (-1)l$, whence $b \geq 0$ and $m \in \langle k, l \rangle$.

Assume now that $S$ is a subsemigroup of $\mathbb{Z}^+$, that $k, l \in S$, and that $N \notin S$. Then $\langle k, l \rangle \subseteq S$. Suppose $\langle k, l \rangle \neq S$, and choose $m \in S \setminus \langle k, l \rangle$. Then, by the argument above, we can write $m = ak + bl$ where $a, b \in \mathbb{Z}$, $0 \leq a < l$, and $b < 0$. Hence

$$N - m = ((l - 1) - a)k + ((-b) - 1)l \in \langle k, l \rangle \subseteq S.$$ 

But then $N = m + (N - m) \in S$, in contradiction with our assumptions. □

Proposition 2.4. Let $S$ be a subsemigroup of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$. Then

i) there exists $N \in \mathbb{Z}^+$ such that $\{N + 1, N + 2, N + 3, \ldots \} \subseteq S$,

ii) there exists a finite set $n_1, \ldots, n_r \in \mathbb{Z}^+$ such that $S = \langle n_1, \ldots, n_r \rangle$.

Proof. i). By assumption $1 \in S - S$ which entails that $k, k + 1 \in S$ for some $k$. Set $N = k(k + 1) - 2k - 1$. Then by Lemma 2.3,

$$\{N + 1, N + 2, N + 3, \ldots \} \subseteq \langle k, k + 1 \rangle \subseteq S.$$ 

ii). Let $k$ and $N$ be as above, and let $n_1, n_2, \ldots, n_r$ be the elements of the set $S \cap \{1, 2, \ldots, N\}$. Then clearly $\langle n_1, \ldots, n_r \rangle \subseteq S$, and $S \cap \{1, 2, \ldots, N\} \subseteq \langle n_1, \ldots, n_r \rangle$. Since $k, k + 1$ belong to $\{n_1, \ldots, n_r\}$ it follows from i) that

$$\{N + 1, N + 2, N + 3, \ldots \} \subseteq \langle k, k + 1 \rangle \subseteq \langle n_1, \ldots, n_r \rangle,$$

and therefore $S = \langle n_1, \ldots, n_r \rangle$ as desired. □

Proposition 2.5. Let $S$ be a subsemigroup of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$. Then $S$ has a (unique) smallest generating set $(n_1, \ldots, n_r)$, i.e. $S = \langle n_1, \ldots, n_r \rangle$, and if $S = \langle m_1, \ldots, m_s \rangle$ then $\{n_1, \ldots, n_r\} \subseteq \{m_1, \ldots, m_s\}$.

Moreover, if $n_1, \ldots, n_r$ are listed increasingly with respect to the usual order on $\mathbb{Z}$, then

$$n_1 = \min S \setminus \{0\}, \quad n_j = \min S \setminus \langle n_1, \ldots, n_{j-1} \rangle, \quad 2 \leq j \leq r.$$  

Proof. Let $S$ be given, and let $n_1, n_2, \ldots$ be the finite or infinite set given recursively by (2.1). (It will follow that this set is actually finite.) By Proposition 2.4 there
exists a finite set \((m_1, \ldots, m_s) \in \mathbb{Z}^+\) such that \(S = \langle m_1, \ldots, m_s \rangle\). We may assume that \(m_1 \leq m_2 \leq \cdots \leq m_s\), and for the purpose of proving the given statement, we may refine this set in such a way that \(m_i \notin \langle m_1, \ldots, m_{i-1} \rangle\) for all \(i = 2, 3, \ldots, s\). To prove the proposition, it suffices to show that \(\{n_1, n_2, \ldots, m_1, m_2, \ldots, m_s\}\).

Notice first that \(m_1 = \min S \setminus \{0\} = n_1\). Assume that \(m_i = n_1, \ldots, m_{i-1} = n_{i-1}\), where \(2 \leq i \leq s\). Let \(t\) be an element in \(S \setminus \langle m_1, \ldots, m_{i-1} \rangle\). Then
\[
t = k_1 m_1 + \cdots + k_{i-1} m_{i-1} + k_i m_i + \cdots + k_s m_s \geq k_i m_i + \cdots + k_s m_s,
\]
for some \(k_1, \ldots, k_s \in \mathbb{Z}^+\) where at least one of \(k_i, k_{i+1}, \ldots, k_s\) is nonzero. Hence \(t \geq m_i\) and so
\[
m_i = \min S \setminus \langle m_1, \ldots, m_{i-1} \rangle = \min S \setminus \langle n_1, \ldots, n_{j-1} \rangle = n_i.
\]
It follows by induction that \(\{n_1, n_2, \ldots, m_1, m_2, \ldots, m_s\}\) as desired. \(\square\)

Notice that there are countably infinitely many orderings on \(\mathbb{Z}\) by the two previous propositions. The proposition below shows (among other things) that only one of these orderings is weakly unperforated.

Recall that an ordered group \((G, G^+)\) is said to have the Riesz Interpolation Property if whenever \(x_1, x_2, y_1, y_2 \in G\) satisfy \(x_i \leq y_j\), for \(i, j = 1, 2\), there exists an element \(z \in G\) such that \(x_i \leq z \leq y_j\) for \(i, j = 1, 2\).

**Proposition 2.6.** Let \(S\) a subsemigroup of \(\mathbb{Z}^+\) which satisfies \(0 \in S\) and \(S - S = \mathbb{Z}\). Then the following three conditions are equivalent:

i) \(S = \mathbb{Z}^+\),

ii) \((\mathbb{Z}, S)\) is weakly unperforated,

iii) \((\mathbb{Z}, S)\) has the Riesz Interpolation Property.

**Proof.** The implications i) \(\Rightarrow\) ii) and i) \(\Rightarrow\) iii) are trivial.

ii) \(\Rightarrow\) i). Since \(S - S = \mathbb{Z}\) there is \(k \in S\) with \(k > 0\). Accordingly, \(k \cdot 1 \in S\), and so if \((\mathbb{Z}, S)\) is weakly unperforated, then \(1 \in S\). This entails that \(S = \mathbb{Z}^+\).

iii) \(\Rightarrow\) i). Suppose that \(S \neq \mathbb{Z}^+\). We show that \((\mathbb{Z}, S)\) does not have the Riesz decomposition property, and hence not the Riesz interpolation property. To do so we must find \(a, b, c \geq 0\) such that \(a + b \geq c\) and so that there is no pair \(a_1, b_1 \in \mathbb{Z}\) with \(0 \leq a_1 \leq a, 0 \leq b_1 \leq b\), and \(c = a_1 + b_1\). (All inequalities are with respect to the ordering on \(\mathbb{Z}\) given by \(S\).

Let \(n\) be the smallest (with respect to the usual order) number in \(S \setminus \{0\}\). Then \(n \neq 1\), and therefore \(S\) is not contained in the set \(\{kn \mid k \in \mathbb{Z}^+\}\). Let \(m\) be the smallest (with respect to the usual order) number in \(S \setminus \{kn \mid k \in \mathbb{Z}^+\}\). Let \(k\) be the smallest (with respect to the usual order) integer such that \(kn - m \in \mathbb{S}\), and observe that \(k \in \{2, 3, 4, \ldots\}\). Put \(a = n, b = (k-1)n, \) and \(c = m\). Assume that there were \(a_1, b_1 \in \mathbb{Z}\) with \(0 \leq a_1 \leq a, 0 \leq b_1 \leq b\), and \(c = a_1 + b_1\). Then either \(a_1 = 0\) or \(a_1 = a = n\). In the first case, \(m = c = b_1 \leq b = (k-1)n\), in contradiction with the choice of \(k\). In the other case, \(m - n = c - a_1 = b_1 \in \mathbb{S}\), in contradiction with the choice of \(m\). \(\square\)
3. The ordered $K_0$-groups of free products of matrix algebras

Let $A$ and $B$ be unital $C^*$-algebras, and let $\mathcal{A} = A \ast B$ be the universal free product of $A$ and $B$ (in the category of unital $C^*$-algebras). There are inclusion maps $\iota_A: A \rightarrow \mathcal{A}$ and $\iota_B: B \rightarrow \mathcal{A}$; and for each unital $C^*$-algebra $D$ with unital *-homomorphisms $\phi_A: A \rightarrow D$ and $\phi_B: B \rightarrow D$ there is a unique *-homomorphism $\phi: \mathcal{A} \rightarrow D$ making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_A} & \mathcal{A} & \xrightarrow{\phi} & D \\
\downarrow{\phi_A} & & \downarrow{\phi} & \downarrow{\iota_B} \\
B & \xrightarrow{\phi_B} & D
\end{array}
\]

commutative. Define maps

\[
\lambda: \mathbb{Z} \rightarrow K_0(A) \oplus K_0(B), \quad \mu: K_0(A) \oplus K_0(B) \rightarrow K_0(\mathcal{A})
\]

by

\[
\lambda(k) = (k[1_A]_0, -k[1_B]_0), \quad \mu(g, h) = K_0(\iota_A)(g) + K_0(\iota_B)(h),
\]

and consider the sequence:

\[
(3.1) \quad \mathbb{Z} \xrightarrow{\lambda} K_0(A) \oplus K_0(B) \xrightarrow{\mu} K_0(\mathcal{A}) \rightarrow 0.
\]

Then, obviously, $\text{Im}(\lambda) \subseteq \text{Ker}(\mu)$. E. Germain has proved, in [G], that if $A$ and $B$ are $K$-nuclear, then we have an exact six-term sequence:

\[
\begin{array}{ccc}
K_0(\mathbb{C}) & \xrightarrow{\lambda} & K_0(A) \oplus K_0(B) & \xrightarrow{\mu} & K_0(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(\mathcal{A}) & \xleftarrow{\lambda} & K_1(A) \oplus K_1(B) & \xleftarrow{\lambda} & K_1(\mathbb{C})
\end{array}
\]

In particular (3.1) is exact. Notice that

\[
(3.2) \quad \mu(K_0(A)^+ \oplus K_0(B)^+) \subseteq K_0(\mathcal{A})^+.
\]

**Question 3.1.** Is

\[
K_0(\mathcal{A})^+ = \mu(K_0(A)^+ \oplus K_0(B)^+)
\]

for all ($K$-nuclear) unital $C^*$-algebras $A$ and $B$?
**Question 3.2.** Let $p \in A$ and $q \in B$ be projections, and suppose that $p \neq 0$ and that there exists no subprojection of $q$ which is equivalent to $1_B$, i.e. $1_B \nless q$ in $B$. Does it follow that $\iota_A(p) \nless \iota_B(q)$ in $A$?

In other words, if $p \in A$ and $q \in B$ are projections such that $\iota_A(p) \nless \iota_B(q)$ in $A$, does it then follow that either $p = 0$ or $1 \nless q$?

**Remark 3.3.** Recall that a projections $p$ in a C*-algebra $D$ is called properly infinite if there exist sub-projections $p_1, p_2$ of $p$ such that $p_1 \sim p_2 \sim p$ and $p_1 \perp p_2$.

Assume that $e_1, e_2 \in D$ are projections such that $e_1 \sim e_2$ and $e_1 \perp e_2$. It is not known if $e_1 + e_2$ is properly infinite implies that $e_1$ (and $e_2$) are properly infinite. If the answer to Question 3.2 is affirmative, then there do exist $e_1$ and $e_2$ as above with $e_1 + e_2$ properly infinite and with $e_1$ and $e_2$ not properly infinite as shown below:

Let $A$ be the Cuntz algebra $O_2$, and set $B = M_2(\mathbb{C})$ with $p = 1$, the unit of $O_2$ and $q$ a 1-dimensional projection in $B$. Set $e_1 = \iota_B(q)$ and $e_2 = \iota_B(1 - q)$. Then $e_1 \sim e_2$, $e_1 \perp e_2$, and $e_1 + e_2 = 1$ is properly infinite because the unit of $O_2$ is properly infinite. If $e_1$ is properly infinite in $A$, then there exist subprojections $f_1, f_2 \in A$ of $e_1$ with $f_1 \perp f_2$, and $f_1 \sim f_2 \sim e_1 \sim e_2$. Hence

$$\iota_A(p) = 1 = e_1 + e_2 \sim f_1 + f_2 \leq e_1 = \iota_B(q),$$

in which case we could answer Question 3.2 in the negative.

**Note added in proof:** The existence of non properly infinite (actually finite) projections $e_1, e_2$ in some C*-algebra $D$ with $e_1 \sim e_2$, $e_1 \perp e_2$, and $e_1 + e_2$ properly infinite, has subsequently been found in [R].

For a C*-algebra $D$ let $V(D)$ be the ordered semi-group of Murray-von Neumann equivalence classes of projections in $D \otimes K$. If $p \in D \otimes K$ is a projection, then $[p]$ and $[p]_0$ will denote its classes in $V(D)$ and $K_0(D)$ respectively. Let $\eta: V(D) \rightarrow K_0(D)$ be the canonical map. Then $\eta([p]) = [p]_0$.

**Theorem 3.4.** Let $k, l \geq 2$ be integers with $k$ prime and $l$ not divisible by $k$. There exists a C*-algebra $D$, projections $p, q \in D$, and an element $g \in K_0(D)$ such that

i) $[p]_0 = kg$, \; $[q]_0 = lg$;

ii) $l[p] = k[q]$;

iii) $mg \notin K_0(D)^+$ if $m \notin \langle k, l \rangle$ (the semi-group generated by $k$ and $l$).

It follows in particular from Theorem 3.4 that

$$\{n \in \mathbb{Z} : ng \in K_0(D)^+\} = \langle k, l \rangle.$$

**Proof.** Set $B_k = \mathbb{D}/\sim$, where $\mathbb{D} \subseteq \mathbb{C}$ is the unit disc and $z \sim w$ if $z, w \in \mathbb{T}$ and $z^k = w^k$. Then $H^2(B_k) \cong \mathbb{Z}_k$.

Let $\omega$ be a non-trivial complex line bundle over $B_k$. Then the Euler class of $\omega$ is nonzero and $k\omega \cong \theta_k$, i.e. the $k$-fold direct sum of $\omega$ is a trivial vector bundle.
Let $B_k^{k-1}$ denote the $(k - 1)$-fold Cartesian product of $B_k$ with itself and let $\rho_i : B_k^{k-1} \to B_k$, $1 = i, \ldots, k - 1$ denote the coordinate projections. Once again, let $(B_k^{k-1})^l$ denote the $l$-fold Cartesian product of $B_k^{k-1}$ and let $\pi_j : (B_k^{k-1})^l \to B_k^{k-1}$, $j = 1, \ldots, l$, be the corresponding coordinate projections. Put

$$
\zeta = \rho_1^*(\omega) \otimes \cdots \otimes \rho_{k-1}^*(\omega), \quad \xi = \zeta^x \cong \pi_1^*(\zeta) \otimes \cdots \otimes \pi_l^*(\zeta).
$$

Then $\zeta$ is a line-bundle over $B_k^{k-1}$, and $\xi$ is a vector bundle over $(B_k^{k-1})^l$ of fiber dimension $l$. Successive applications of the isomorphism $k\omega \cong \theta_k \cong k\theta_1$ yields $k\zeta \cong \theta_k$, and this in turns implies that $k\xi \cong \theta_{kl}$.

We show that the Euler class $e((k - 1)\xi)$ is nonzero. Using the product formula for the Euler class we get

$$
e((k - 1)\xi) = e(\xi)^{k-1} = \prod_{j=1}^l \pi_j^*(e(\xi)^{k-1}).$$

From the definition of $\zeta$ it follows that

$$
e(\zeta) = \sum_{i=1}^{k-1} \rho_i^*(e(\omega)),$$

and since $e(\omega)^2 = 0$ we get

$$
e(\xi)^{k-1} = (k - 1)! \prod_{i=1}^{k-1} \rho_i^*(e(\omega)).$$

Let

$$
\mu : \bigotimes_{1}^{(k-1)^l} H^2(B_k; \mathbb{Z}) \to H^2((B_k^{k-1})^l; \mathbb{Z})
$$

be given by

$$
\mu(x_{11} \otimes \cdots \otimes x_{k-1,l}) = \prod_{ij} \pi_j^*(\rho_i^*(x_{ij})).
$$

By the Künneth Theorem for singular cohomology, $\mu$ is injective. Now,

$$
e((k - 1)\xi) = \mu(((k - 1)!e(\omega) \otimes e(\omega) \otimes \cdots \otimes e(\omega))$$

and since $k$ does not divide $((k - 1)!$ it follows that $e((k - 1)\xi)$ is nonzero.

Consider the $C^*$-algebra

$$
D = C(B_k^{(k-1)!}) \otimes \mathcal{K}.
$$

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Let \( p \in D \) be a projection corresponding to a trivial bundle of dimension \( k \), and let \( q \in D \) be a projection which corresponds to the bundle \( \xi \). Then \( k|q| = l|p| \) (these classes both correspond to the trivial bundle of dimension \( kl \)). Choose \( a, b \in \mathbb{Z} \) such that \( 1 = ak + bl \), and put \( g = a[p]_0 + b[q]_0 \in K_0(D) \). Then \( kg = [p]_0 \) and \( lg = [q]_0 \).

We must show that \( mq \notin K_0(D)^+ \) if \( m \notin (k, l) \). By Lemma 2.3 it is enough to show that \( (kl - k - l)g \notin K_0(D)^+ \). Suppose, for a moment, that \( (kl - k - l)g \) is positive. Since

\[
(kl - k - l)g = (k - 1)lg - kg = (k - 1)[q]_0 - [p]_0
\]

it follows that \( (k - 1)[\xi] - [\theta_k] \) is positive in \( K^0(B_k^{(k-1)l}) \), and so, in particular, \( (k - 1)[\xi] - [\theta_1] \) is positive. Consequently, there exists a complex vector bundle \( \vartheta \) and \( d \in \mathbb{N} \) such that \( (k - 1)\xi \oplus \theta_d \cong \vartheta \oplus \theta_{d+1} \). Hence,

\[
e((k - 1)\xi) = e(\vartheta \oplus \theta_1) = e(\vartheta)e(\theta_1) = 0
\]

in contradiction with the choice of \( \xi \). \( \square \)

We shall consider the ordered \( K_0 \)-group of the free product \( \mathcal{A} = M_k(\mathbb{C}) \ast M_l(\mathbb{C}) \). It follows from Germain's theorem (quoted earlier in this section), that \( K_0(\mathcal{A}) = \mathbb{Z} \) when \( k \) and \( l \) are relatively prime. We give below an elementary proof of this fact.

**Proposition 3.5.** Let \( k, l \in \mathbb{N} \) be relatively prime, set \( \mathcal{A} = M_k(\mathbb{C}) \ast M_l(\mathbb{C}) \), let

\[
\phi_1: M_k(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_l(\mathbb{C}), \quad \phi_2: M_l(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_l(\mathbb{C}),
\]

be the natural homomorphisms, and let \( \tau: \mathcal{A} \to M_k(\mathbb{C}) \otimes M_l(\mathbb{C}) \) be the homomorphism induced by \( \phi_1 \) and \( \phi_2 \).

Then \( K_0(\tau) \) and \( K_1(\tau) \) are isomorphisms, and in particular we have that \( K_0(\mathcal{A}) \cong \mathbb{Z} \) and \( K_1(\mathcal{A}) = 0 \).

**Proof.** Let \( \iota_1: M_k(\mathbb{C}) \to \mathcal{A} \) and \( \iota_2: M_l(\mathbb{C}) \to \mathcal{A} \) denote the canonical inclusion mappings. Let \( e_1 \in M_k(\mathbb{C}) \) and \( e_2 \in M_l(\mathbb{C}) \) be one-dimensional projections, and set \( f_j = \iota_j(e_j) \in \mathcal{A} \). Then \( \tau(f_j) = \phi_j(e_j) \), and hence \( \tau(f_1) \) has dimension \( l \) and \( \tau(f_2) \) has dimension \( k \). Since \( k \) and \( l \) are relatively prime, this shows that \( K_0(\tau) \) is onto.

The diagram,

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mu_k} & M_k(\mathbb{C}) \otimes \mathcal{A} \\
\Downarrow{\tau} & & \Downarrow{id \otimes \iota_2} \\
M_k(\mathbb{C}) \otimes M_l(\mathbb{C}) & & 
\end{array}
\]

where \( \mu_k \) is given by \( a \mapsto 1_k \otimes a \), commutes up to homotopy. Indeed, the diagram commutes exactly on the image of \( \iota_2 \), and it commutes up to homotopy on the
image of \( \iota_1 \) (since the two canonical homomorphisms \( M_k(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \) are homotopic). It follows that the diagram commutes exactly at the level of K-groups, and therefore that

\[
\text{Ker}(K_j(\tau)) \subseteq \text{Ker}(K_j(\mu_k)) = \text{Ker}(k \cdot \text{id}_{K_j(A)}).
\]

By interchanging the roles of the first and second factor in \( A \), we get

\[
\text{Ker}(K_j(\tau)) \subseteq \text{Ker}(l \cdot \text{id}_{K_j(A)}).
\]

Since \( k \) and \( l \) are relatively prime,

\[
\text{Ker}(k \cdot \text{id}_{K_j(A)}) \cap \text{Ker}(l \cdot \text{id}_{K_j(A)}) = 0,
\]

and this proves that \( K_j(\tau) \) is injective for \( j = 0, 1 \). \( \square \)

**Theorem 3.6.** Let \( k, l \geq 2 \) be integers with \( k \) prime and \( l \) not divisible by \( k \). Put \( A = M_k(\mathbb{C}) \ast M_l(\mathbb{C}) \). Then

i) \( (K_0(A), K_0(A)^+) \cong (\mathbb{Z}, \langle k, l \rangle) \),

ii) \( K_0(A)^+ = \mu(K_0(M_k(\mathbb{C}))^+ \oplus K_0(M_l(\mathbb{C}))^+) \).

Moreover, if \( p \in M_k(\mathbb{C}) \) and \( q \in M_l(\mathbb{C}) \) are projections with \( p \neq 0 \) and \( q \neq 1 \), then \( p \not\cong q \) in \( A \).

**Proof.** Let \( \tau : A \to M_k(\mathbb{C}) \otimes M_l(\mathbb{C}) \) be as in Proposition 3.5. Identify \( K_0(M_k(\mathbb{C})) \), \( K_0(M_l(\mathbb{C})) \) and \( K_0(M_k(\mathbb{C}) \otimes M_l(\mathbb{C})) \) with \( \mathbb{Z} \) in the natural way. Let \( \mu \) be as in (3.1), and set \( \mu'(x, y) = lx + ky \). Then by Proposition 3.5 the diagram

\[
\begin{array}{ccc}
K_0(M_k(\mathbb{C})) \oplus K_0(M_l(\mathbb{C})) & \xrightarrow{\mu} & K_0(A) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\mu'} & \mathbb{Z}
\end{array}
\]

commutes, and hence

\[
(3.3) \quad (K_0(\tau) \circ \mu)(K_0(M_k(\mathbb{C}))^+ \oplus K_0(M_l(\mathbb{C}))^+) = \mu'(\mathbb{Z}^+ \oplus \mathbb{Z}^+) = \langle k, l \rangle.
\]

Put \( S = K_0(\tau)(K_0(A)^+) \subseteq \mathbb{Z}^+ \). Then, by (3.1) and (3.3), \( \langle k, l \rangle \subseteq S \). We proceed to show that \( S \subseteq \langle k, l \rangle \), and this will prove i) and ii).

Let \( D, p, q \in D \) and \( g \in K_0(D) \) be as in Theorem 3.4. Upon replacing \( D \) with a corner of \( D \otimes \mathcal{K} \), we may assume that \( D \) is unital and that \( l[p] = k[q] = |1_D| \). Let \( e_1 \in M_k(\mathbb{C}) \) and \( e_2 \in M_l(\mathbb{C}) \) be 1-dimensional projections. By the choice of \( p \) and \( q \) there exist unital *-homomorphisms \( \psi_1 : M_k(\mathbb{C}) \to A \) and \( \psi_2 : M_l(\mathbb{C}) \to A \) such
that $\psi_1(e_1) = q$ and $\psi_2(e_2) = p$. By the universal property of $\mathcal{A}$, $\psi_1$ and $\psi_2$ factor through a unital *-homomorphism $\psi: \mathcal{A} \to D$:

![Diagram](image)

At the level of K-theory, we have the following commuting diagram

$$
\begin{array}{c}
S & \xrightarrow{V(\tau)} & V(\mathcal{A}) & \xrightarrow{V(\psi)} & V(D) \\
\downarrow{\cong} & & \downarrow{\eta} & & \downarrow{\eta} \\
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{K}_0(\mathcal{A}) & \xrightarrow{\mathbb{K}_0(\psi)} & \mathbb{K}_0(D).
\end{array}
$$

Let $h \in \mathbb{K}_0(\mathcal{A})$ be the element satisfying $\mathbb{K}_0(\tau)(h) = 1$. Viewing $e_1$ and $e_2$ as elements of $\mathcal{A}$, we have $\psi(e_1) = q, \psi(e_2) = p, \mathbb{K}_0(\tau)([e_1]_0) = l$, and $\mathbb{K}_0(\tau)([e_2]_0) = k$. Hence $lh = [e_1]_0$ and $kh = [e_2]_0$, and so

$$l \cdot \mathbb{K}_0(\psi)(h) = \mathbb{K}_0(\psi)([e_1]_0) = [q]_0 = lg, \quad k \cdot \mathbb{K}_0(\psi)(h) = \mathbb{K}_0(\psi)([e_2]_0) = [p]_0 = kg,$$

which implies that $\mathbb{K}_0(\psi)(h) = g$.

Assume $m \in S$, and find $x \in V(\mathcal{A})$ which is mapped to $m$. Then $mh = \eta(x)$, and therefore

$$mg = \mathbb{K}_0(\psi)(mh) = \mathbb{K}_0(\psi) \circ \eta(x) = \eta \circ V(\psi)(x) \in \mathbb{K}_0(D)^+.$$

By Theorem 3.4 this implies that $m \in \langle k, l \rangle$. We have now shown that $S = \langle k, l \rangle$ and the proof of i) and ii) is complete.

Let $p \in \mathbb{M}_k(\mathbb{C})$ and $q \in \mathbb{M}_l(\mathbb{C})$ be given such that $p \neq 0$ and $q \neq 1$. Then $[p]_0 = x[e_1]_0$ and $[q]_0 = y[e_2]_0$ where $x, y \in \mathbb{Z}$ satisfy $1 \leq x \leq k$ and $0 \leq y < l$. Since

$$\mathbb{K}_0(\tau)([q]_0 - [p]_0) = yk - xl \notin \langle k, l \rangle = S,$$

it follows that $[q]_0 - [p]_0 \notin \mathbb{K}_0(\mathcal{A})^+$, and so $p, q \notin \mathcal{A} \quad \Box$

**Remark 3.7.** It would be interesting to know if

$$\mathbb{K}_0(\mathbb{M}_k(\mathbb{C}) * \mathbb{M}_l(\mathbb{C}))^+ = \mu(\mathbb{K}_0(\mathbb{M}_k(\mathbb{C}))^+ \oplus \mathbb{K}_0(\mathbb{M}_l(\mathbb{C}))^+)$$

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for all integers $k, l \geq 2$. It should be noted that if $\gcd(k, l) = m > 1$, then $K_0(M_k(\mathbb{C}) \ast M_l(\mathbb{C})) \cong \mathbb{Z} \oplus \mathbb{Z}/m$.

Remark 3.8. It would also be interesting to know if one to each subsemigroup $S$ of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$ can find a C*-algebra $A$ with $(K_0(A), K_0(A)^+) \cong (\mathbb{Z}, S)$.

One obvious generalization of Theorem 3.6 is as follows: Let $k_1, k_2, \ldots, k_r \geq 2$ be integers and consider the unital C*-algebra

$$
\mathcal{A} = M_{k_1}(\mathbb{C}) \ast M_{k_2}(\mathbb{C}) \ast \cdots \ast M_{k_r}(\mathbb{C}),
$$

where the free product, $\ast$, is the universal free product in the category of unital C*-algebras. By Germain's theorem (see below (3.1)), $K_0(A) \cong \mathbb{Z}$ if and only if $\gcd(k_i, k_j) = 1$ whenever $i \neq j$.

Assume now that $\gcd(k_i, k_j) = 1$ for all $i \neq j$. Set

$$
(3.4) \quad n_j = \prod_{i \neq j} k_i.
$$

Set $S = \{n_1, n_2, \ldots, n_r\}$. One can show that $(K_0(A), K_0(A)^+) \cong (\mathbb{Z}, S)$ provided that $S$ satisfies the following condition:

$(S)$ For each $m \in \mathbb{Z}^+ \setminus S$ there exist a C*-algebra $B$ and homomorphisms $\beta: S \to V(B)$ and $\gamma: \mathbb{Z} \to K_0(B)$ such that

$$
\begin{array}{ccc}
S & \xrightarrow{\beta} & V(B) \\
\subseteq & \downarrow & \downarrow \eta \\
\mathbb{Z} & \xrightarrow{\gamma} & K_0(B)
\end{array}
$$

commutes, and such that $\gamma(m) \notin K_0(B)^+$. Only rather special semi-groups $S$ arise from such a set $k_1, k_2, \ldots, k_r$. For example, if $S$ has minimal generating set $(n_1, n_2, \ldots, n_r)$, where $r \geq 3$ and $\gcd(n_i, n_j) = 1$ for some $i \neq j$, then $(n_1, n_2, \ldots, n_r)$ cannot be obtained from any set $k_1, k_2, \ldots, k_r$ as in (3.4).

We sketch below another construction which to an arbitrary subsemigroup $S$ of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$ associates a C*-algebra $\mathcal{F}(S)$, whose ordered $K_0$-group is likely to be isomorphic to $(\mathbb{Z}, S)$.

Let $(n_1, n_2, \ldots, n_r)$ be the minimal generating set for $S$ (cf. Proposition 2.5). For each pair of indices $i, j$ with $1 \leq i, j \leq r$, $i \neq j$, set $m^i_j = n_i/\gcd(n_i, n_j)$ and find projections $e^i_j \in \mathcal{K}$ with $\dim(e^i_j) = m^i_j$ and such that the projections

$$e^1_j, e^2_j, \ldots, e^{j-1}_j, e^{j+1}_j, e^{j+2}_j, \ldots, e^r_j$$
are mutually orthogonal for every $j = 1, 2, \ldots, r$.

Put $\mathcal{F}_r = \mathcal{K} * \mathcal{K} * \cdots * \mathcal{K}$ (with $r$ copies of $\mathcal{K}$, and where $*$ denotes universal free product with no amalgamation). Let $\phi'_1, \phi'_2, \ldots, \phi'_r : \mathcal{K} \to \mathcal{F}_r$ be the canonical inclusions. Let $I$ be the closed two-sided ideal in $\mathcal{F}_r$ generated by the set
\[
\{ \phi'_i(e_i^j) - \phi'_j(e_i^j) : 1 \leq i, j \leq r, i \neq j \}.
\]

Set $\mathcal{F}(S) = \mathcal{F}_r/I$, let $\pi : \mathcal{F}_r \to \mathcal{F}(S)$ be the quotient mapping, and set $\phi_j = \pi \circ \phi'_j : \mathcal{K} \to \mathcal{F}(S)$.

Then $\mathcal{F}(S) = C^*(\phi_1(\mathcal{K}), \phi_2(\mathcal{K}), \ldots, \phi_r(\mathcal{K}))$, i.e., $\mathcal{F}(S)$ is generated by $r$ copies of $\mathcal{K}$. Moreover, it can be shown that
\[
(K_0(\mathcal{F}(S)), K_0(\mathcal{F}(S))^+) \cong (\mathbb{Z}, S)
\]
provided that the semi-group $S$ satisfies the condition $(S)$ described above, and provided that
\[
K_0(\mathcal{F}(S)) = K_0(\phi_1(K_0(\mathcal{K}))) + \cdots + K_0(\phi_r(K_0(\mathcal{K}))).
\]

4. Comparison of projections in free products

Combining Theorem 3.6 with Propositions 4.1 and 4.2 below we shall show that one can answer Questions 3.1 and 3.2 in the affirmative for a rather large class of pairs of $C^*$-algebras. To systematize our treatment, let $\mathcal{C}_1$ denote the class of all pairs of K-nuclear $C^*$-algebras $(A, B)$ such that
\[
K_0(A * B)^+ = \mu(K_0(A)^+ \oplus K_0(B)^+),
\]
(cf. (3.1) and (3.2)). Let $\mathcal{C}_2$ denote the class of pairs of $C^*$-algebras $(A, B)$ such that for every pair of projections $p \in A$ and $q \in B$, if $p \neq 0, q \neq 0, 1 \not\prec p$, and $1 \not\prec q$, then $p$ and $q$ are incomparable in $A * B$, i.e., $p \not\succ q$ and $q \not\succ p$ in $A * B$.

If follows from Theorem 3.6 that $(\mathcal{M}_k(\mathbb{C}), \mathcal{M}_l(\mathbb{C})) \in \mathcal{C}_j$ for $j = 1, 2$ when $k$ is prime and $\gcd(k, l) = 1$ (or vice versa).

Recall that a (unital) $C^*$-algebra $A$ is said to be finite if for every pair of projections $p, q \in A$, $p \not\lesssim q \lesssim p$ implies $p = q$. If $A$ and $B$ are finite $C^*$-algebras, then $(A, B) \in \mathcal{C}_2$ if and only if for every pair of projections $p \in A$ and $q \in B$, $p \not\lesssim q$ implies that either $p = 0$ or $q = 1$ (and vice versa).

Proposition 4.1.

i) If $(A, B) \in \mathcal{C}_1$, then $(B, A) \in \mathcal{C}_1$.

ii) If $(A_1, B), (A_2, B) \in \mathcal{C}_1$, then $(A_1 \oplus A_2, B) \in \mathcal{C}_1$.

iii) If $A$ is the inductive limit of a sequence of unital $C^*$-algebras
\[
A_1 \to A_2 \to A_3 \to \cdots
\]
with unital connecting maps, and if $(A_n, B) \in \mathcal{C}_1$ for all $n$, then $(A, B) \in \mathcal{C}_1$.  

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Proposition 4.2.

i) If \((A, B) \in C_2\), then \((B, A) \in C_2\).

ii) If \((A_1, B), (A_2, B) \in C_2\), then \((A_1 \oplus A_2, B) \in C_2\).

iii) If \(A\) is the inductive limit of a sequence of unital \(C^*\)-algebras

\[ A_1 \to A_2 \to A_3 \to \ldots \]

with unital connecting maps, and if \((A_n, B) \in C_2\) for all \(n\), then \((A, B) \in C_2\).

iv) Assume \(A\) and \(B\) are finite \(C^*\)-algebras with \((A, B) \in C_2\). Then \((A_0, B_0) \in C_2\) for all sub-\(C^*\)-algebras \(A_0\) of \(A\) and \(B_0\) of \(B\) with \(1_A \in A_0\) and \(1_B \in B_0\).

We prove Propositions 4.1 and 4.2 simultaneously, and we note that part i) of these propositions is trivial. To prove part ii) we shall need the following:

Lemma 4.3. Assume \((A, B) \in C_1\), let \(x \in K_0(A)\), \(y \in K_0(B)\), and set

\[ g = K_0(\iota_A)(x) + K_0(\iota_B)(y) \in K_0(A \ast B). \]

Then \(g \in K_0(A \ast B)^+\) if and only if \(x + n[1_A]_0 \in K_0(A)^+\) and \(y - n[1_B]_0 \in K_0(B)^+\) for some integer \(n\).

Proof. The “if” part follows from (3.2) and the exactness of (3.1). Assume \(g \in K_0(A \ast B)^+\). Then, by the assumption that \((A, B) \in C_1\), there exist \(x' \in K_0(A)^+\) and \(y' \in K_0(B)^+\) such that \(g = K_0(\iota_A)(x') + K_0(\iota_B)(y')\). Hence

\[ K_0(\iota_A)(x' - x) + K_0(\iota_B)(y' - y) = 0. \]

By exactness of the sequence (3.1) there is an integer \(n\) such that \(x' - x = n[1_A]_0\) and \(y' - y = -n[1_B]_0\). This completes the proof. \(\square\)

Proof of part ii) of Propositions 4.1 and 4.2. From the universal property of the free product, the maps \(A_1 \oplus A_2 \to A_j \to A_j \ast B\) factor through \(*\)-homomorphisms \(\phi_j\): \((A_1 \oplus A_2) \ast B \to A_j \ast B\).

To show that \((A_1 \oplus A_2, B) \in C_1\) it suffices to show that

\[ K_0((A_1 \oplus A_2) \ast B)^+ \subseteq \mu(K_0(A_1 \oplus A_2)^+ \oplus K_0(B)^+). \]

Let \(g \in K_0((A_1 \oplus A_2) \ast B)^+\). By exactness of the sequence (3.1), there exist \(x_1 \in K_0(A_1), x_2 \in K_0(A_2),\) and \(y \in K_0(B)\), such that

\[ g = K_0(\iota_{A_1 \oplus A_2})(x_1, x_2) + K_0(\iota_B)(y). \]

Now,

\[ K_0(\iota_{A_j})(x_j) + K_0(\iota_B)(y) = K_0(\phi_j)(g) \in K_0(A_j \ast B)^+. \]
By Lemma 4.3 this entails that $x_j + n_j[1_{A_j}]_0 \geq 0$ and $y - n_j[1_{B}]_0 \geq 0$ for some integers $n_1$ and $n_2$. Set $n = \text{max}\{n_1, n_2\}$. Then

$$x' = (x_1 + n[1_{A_1}]_0, x_2 + n[1_{A_2}]_0) \in K_0(A_1 \oplus A_2)^+, \quad y' = y - n[1_{B}]_0 \in K_0(B)^+,$$

$$g = K_0(\iota_{A_1, A_2})_0(x') + K_0(\iota_B)(y') \in \mu(K_0(A_1 \oplus A_2)^+ \oplus K_0(B)^+)$$
as desired.

We proceed to show that $(A_1 \oplus A_2, B) \in C_2$ if $(A_1, B), (A_2, B) \in C_2$. Let $p = (p_1, p_2)$ be a projection in $A_1 \oplus A_2$, let $q$ be a projection in $B$, and assume that $p \preceq q$ in $(A_1 \oplus A_2)^*B$. Then $p_j = \phi_j(p) \preceq \phi_j(q) = q$ in $A_j^*B$. Hence either $p_j = 0$ or $1 \preceq q$ for $j = 1, 2$. This entails that either $p = 0$ or $1 \preceq q$. \[\square\]

Part iii) of Propositions 4.1 and 4.2 are easy consequences of the following:

**Lemma 4.4.** Let

$$A_1 \to A_2 \to A_3 \to \cdots$$

be a sequence of unital $C^*$-algebras with unital connecting $^*$-homomorphisms, and let $B$ be a unital $C^*$-algebra. Then there exists an isomorphism $\psi$ making the diagram

$$
\begin{array}{c}
\lambda_m*\text{id}_B \\
A_m* B \\
\uparrow \lambda_m \\
(\lim A_n)* B \\
\Downarrow \approx \\
\psi \\
\lim(A_n* B)
\end{array}
$$

commutative. (The maps $\lambda_m*\text{id}_B$ and $\lambda'_m$ are the natural maps arising from the functoriality of the free products and the inductive limits.)

**Proof.** We have $^*$-homomorphisms $\alpha_n$ and $\alpha$ and the commuting diagram

$$
\begin{array}{c}
A_1 \\
\alpha_1 \\
\downarrow \\
A_1* B \\
\alpha \\
\downarrow \\
(\lim A_n)* B \\
\Rightarrow \\
\alpha \\
\Downarrow \\
\lim(A_n* B)
\end{array}
$$

which yields a $^*$-homomorphism $\psi$:

$$
\begin{array}{c}
\lim A_n \\
\alpha \\
\downarrow \\
(\lim A_n)* B \\
\Rightarrow \\
\psi \\
\leftarrow \\
\Downarrow \\
B \\
\lim(A_n* B)
\end{array}
$$

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Moreover, \(\psi\) makes the diagram in the lemma commutative. (One can check this for example by inspecting elements in \(A_m\) and in \(B\) separately.)

By commutativity of the diagram:

\[
\begin{array}{c}
A_m \ast B \\
\downarrow \lambda_m \ast \text{id}_B \\
\left( \varprojlim A_n \right) \ast B \\
\end{array}
\]

we get a \(*\)-homomorphism \(\varphi\): \(\varprojlim (A_n \ast B) \rightarrow \left( \varprojlim A_n \right) \ast B\), which makes

\[
\begin{array}{c}
\varprojlim (A_n \ast B) \\
\downarrow \varphi \\
\left( \varprojlim A_n \right) \ast B \\
\end{array}
\]

commutative. Finally, \(\psi \circ \varphi\) is the identity on the image of \(\lambda'_m\), and \(\varphi \circ \psi\) is the identity on the image of \(\lambda_m \ast \text{id}_B\) for each \(m\). This shows that \(\psi\) and \(\varphi\) are each others inverses. Hence \(\psi\) is an isomorphism. \(\square\)

**Proof of Proposition 4.2 iv.** Let \(A_0 \subseteq A\) and \(B_0 \subseteq B\) with \(1_A \in A_0\) and \(1_B \in B_0\) be given, and let \(\varphi\): \(A_0 \ast B_0 \rightarrow A \ast B\) be the canonical \(*\)-homomorphism arising from these inclusions. Let \(p \in A_0\) and \(q \in B_0\) be projections, and assume that \(p \preceq q\) in \(A_0 \ast B_0\). Then \(p = \varphi(p) \preceq \varphi(q) = q\) in \(A \ast B\). Since \(B\) is assumed to be finite, and since \((A, B) \in \mathcal{C}_2\), this implies that \(p = 0\) or \(q = 1\). Reverting the roles of \(A\) and \(B\) yields the other case. \(\square\)

If we knew that \((M_k(\mathbb{C}), M_l(\mathbb{C})) \in \mathcal{C}_1\) for all positive integers \(k\) and \(l\), then we could conclude from Proposition 4.1 that \((A, B) \in \mathcal{C}_1\) for all unital AF-algebras \(A\) and \(B\). With the present restrictions on \(k\) and \(l\) in Theorem 3.6 we can still use Proposition 4.1 to reach conclusions about the ordered \(K_0\)-group at least for some non-trivial AF-algebras:

**Corollary 4.5.** Let \(\tilde{\mathcal{K}}\) denote the \(C^*\)-algebra of compact operators on a separable Hilbert space with a unit adjoined. Then \((\tilde{\mathcal{K}}, \tilde{\mathcal{K}}) \in \mathcal{C}_1\), ie.,

\[
K_0(\tilde{\mathcal{K}} \ast \tilde{\mathcal{K}})^+ = \mu(K_0(\tilde{\mathcal{K}})^+ \oplus K_0(\tilde{\mathcal{K}})^+).
\]

**Proof.** Let \(\{p_n\}\) and \(\{q_n\}\) be two increasing, disjoint, sequences of primes, set

\[
A_n = M_{p_n}(\mathbb{C}) \oplus \mathbb{C}, \quad B_n = M_{q_n}(\mathbb{C}) \oplus \mathbb{C}.
\]

Then \(\tilde{\mathcal{K}} \cong \varprojlim A_n \cong \varprojlim B_n\) (with appropriate choices of unital connecting \(*\)-homomorphisms \(A_n \rightarrow A_{n+1}\) and \(B_n \rightarrow B_{n+1}\)). By Proposition 4.1 (ii) and Theorem 3.6.
we have that \((A_n, B_m) \in \mathcal{C}_1\) for all \(n\) and \(m\). Two applications of Proposition 4.1 (iii) yield \((\mathcal{K}, B_m) \in \mathcal{C}_1\) for all \(m\), and \((\mathcal{K}, \mathcal{K}) \in \mathcal{C}_1\). \(\square\)

We have \(K_0(\mathcal{K}) = \mathbb{Z}[1]_0 + \mathbb{Z}[e]_0\), where \(e\) is a 1-dimensional projection in \(\mathcal{K}\). Hence we can identify \((K_0(\mathcal{K}), K_0(\mathcal{K})^+)\) with \((\mathbb{Z} \oplus \mathbb{Z}, G)\), where

\[
G = \{(0, y) : y \geq 0\} \cup \{(x, y) : x \geq 1\}.
\]

By Corollary 4.5 we get that \((K_0(\mathcal{K} \ast \mathcal{K}), K_0(\mathcal{K} \ast \mathcal{K})^+)\) equals \((\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, H)\), where

\[
H = \{(0, y, z) : y \geq 0, z \geq 0\} \cup \{(1, y, z) : y \geq 0, z \geq 0\} \cup \{(x, y, z) : x \geq 2\}.
\]

We call a unital \(C^*\)-algebra \(A\) non-divisible if there for no integer \(n \geq 2\) exists \(g \in K_0(A)\) such that \([1_A]_0 = ng\).

**Corollary 4.6.** Every pair of unital \(C^*\)-algebras, each of which can be unitally embedded into unital, simple, non-divisible AF-algebras, belong to \(\mathcal{C}_2\) (cf. the first paragraph of this section).

**Proof.** Find four mutually disjoint sequences of primes \(\{p_n\}, \{p'_n\}, \{q_n\}\) and \(\{q'_n\}\) such that if

\[
A_n = M_{p_n}(\mathbb{C}) \oplus M_{p'_n}(\mathbb{C}), \quad B_n = M_{q_n}(\mathbb{C}) \oplus M_{q'_n}(\mathbb{C}),
\]

then there exist unital connecting maps \(A_n \to A_{n+1}\) and \(B_n \to B_{n+1}\) which map each non-zero element of \(A_n\), respectively, \(B_n\), to a full element of \(A_{n+1}\), respectively, \(B_{n+1}\). Set \(A = \lim A_n\) and \(B = \lim B_n\). Arguing as in the proof of Corollary 4.5 we see that \((A, B) \in \mathcal{C}_2\).

The AF-algebras \(A\) and \(B\) are unital, simple and infinite-dimensional. The ordered \(K_0\)-groups of a unital, simple and infinite-dimensional AF-algebra has the property that for each non-zero positive element \(g\) and for each set of positive integers \(d_1, d_2, \ldots, d_r\), with greatest common divisor equal to 1, there exist non-zero positive elements \(g_1, g_2, \ldots, g_r\) such that \(g = d_1 g_1 + d_2 g_2 + \cdots + d_r g_r\).

Using this property it can be shown that every unital, simple, non-divisible AF-algebra can be unitally embedded into \(A\) and \(B\) (first at the level of \(K\)-theory, and then, by the classification theorem for AF-algebras, at the level of algebras). Hence any pair \((A', B')\) of \(C^*\)-algebras that can be unitally embedded into unital, non-divisible AF-algebras can be unitally embedded into \(A\) and \(B\). Therefore \((A', B') \in \mathcal{C}_2\) by Proposition 4.2 (iv). \(\square\)

**Remark 4.7.** Whereas the conclusions of Corollary 4.6 may apply to a very large class of unital, separable, exact \(C^*\)-algebras, it does not give us information about infinite \(C^*\)-algebras, cf. Remark 3.3.
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