On sums of finite projections

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Abstract

It is shown that there exists a (unital) finite C^*-algebra A such that $M_2(A)$ is properly infinite (in the sense that $M_2(A)$ contains two isometries with orthogonal range projections). This also shows that there exist finite C^*-algebras without tracial states.

We discuss different notions of finiteness and how these behave with respect to forming sums.

1 Introduction

A well-known — and non-trivial — theorem by Murray and von Neumann says that the sum of two finite orthogonal projections in a von Neumann algebra is again finite. Finiteness is defined in terms of the comparison theory of Murray and von Neumann. This theorem is an ingredient in the proof that any finite von Neumann algebra has a tracial state.

The situation for C^*-algebra is different. It is ambiguous what it should mean for a projection in a C^*-algebra to be finite; and almost whichever way finiteness is being defined, the sum of two finite projections will fail to be finite (in general). The word "finite" for a projection in a C^*-algebra is usually given the same meaning as it has for von Neumann algebras (c.f. Definition 2.1 below). With this meaning, an example of a finite C^*-algebra A such that $M_2(A)$ is not finite was discovered by N. Clarke [4] (see [2, Exercise 6.10.1]). Clarke's construction involves a generalized Toeplitz algebra. In his example, $M_2(A)$ is not properly infinite, and $M_2(A)$ has a (non-faithful) tracial state, and so $M_2(A)$ is not very infinite.

A still unresolved question asks if every unital *simple* C^*-algebras is either finite in the sense that it admits a tracial state, or is purely infinite in the sense of J. Cuntz (c.f. [6]). In particular, can a simple C^*-algebra contain simultaneously a finite and an infinite projection? Infinite projections in a simple C^*-algebra are automatically properly infinite due to a result of J. Cuntz. If one could show the stronger statement, that $e \oplus e$ cannot be properly infinite when e is a finite projection in any C^*-algebra, then one could conclude that simple C^*-algebras cannot contain both finite and infinite projections (see Remark 4.4). It is shown here that this stronger statement is false. This may indicate that perhaps there exists a simple C^*-algebra with finite and infinite projections.

The example of a finite C^*-algebra A with $M_2(A)$ properly infinite is obtained from an example in [18] of a simple non-stable C^*-algebra B with $M_2(B)$ stable, and this example comes from a construction of J. Villadsen [19]. In Section 3 we establish some new relations between stability of a C^*-algebra and the structure of its multiplier algebra. These results build on a characterization of stable C^*-algebras given in [13].

2 A preliminary discussion

We remind the reader of some standard (and some less standard) notations concerning projections in C^*-algebras. Let A be a C^*-algebra, and let $e \in M_n(A)$ and $f \in M_m(A)$ be projections. Set

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in M_{n+m}(A),$$

and denote by $e \otimes 1_k$ the k-fold direct sum $e \oplus e \oplus \cdots \oplus e \in M_{kn}(A)$. Write $e \sim f$ if there exists $v \in M_{m,n}(A)$ such that $e = v^*v$ and $f = vv^*$, and write $e \precsim f$ if $e \sim f_0 \leq f$ for some projection $f_0 \in M_m(A)$.

Definition 2.1 *Let e be a projection in a C^*-algebra A. Then e is*

- tracially finite, *if there exists a separating family of tracial states on the corner C^*-algebra eAe,*
- weakly tracially finite, *if there exists a (not necessarily faithful) tracial state on the corner C^*-algebra eAe,*
- finite, *if for all projections $f \in A$, $e \sim f \leq e$ implies $f = e$,*
- infinite, *if e is not finite,*
- properly infinite, *if e is non-zero and $e \oplus e \precsim e$.*

If A is unital, then A is said to be tracially finite, weakly tracially finite, finite, infinite, *respectively,* properly infinite *if the unit 1_A of A has the corresponding property. If $M_n(A)$ is finite for all $n \in \mathbb{N}$, then A is said to be* stably finite.

It is clear that e tracially finite implies e is finite, and that e properly infinite implies that e is infinite.

If $e \in A$ is properly infinite, then $e \otimes 1_n \precsim e$ for all $n \in \mathbb{N}$. Moreover, e is properly infinite if and only if there is a unital embedding $\mathcal{O}_\infty \to eAe$.

Any subprojection of a tracially finite projection is tracially finite, and any subprojection of a finite projection is again finite. Hence any projection that dominates an infinite projection is infinite. It is not true that a projection that dominates a properly infinite projection is again properly infinite. For example, if A is a properly infinite C^*-algebra and if B is a finite C^*-algebra, then $e = (1_A, 0) \in A \oplus B$ is properly infinite, whereas $f = (1_A, 1_B) \in A \oplus B$ is not properly infinite, although f dominates e. On the other hand we have the following:

Proposition 2.2 *Let A be a C^*-algebra, let $e, f \in A$ be projections and suppose that e is properly infinite, $e \precsim f$, and that f lies in the ideal of A generated by e. Then f is properly infinite.*

Let I be an ideal in A, and let $\pi\colon A \to A/I$ be the quotient mapping. If $e \in A$ is a properly infinite projection and $e \notin I$, then $\pi(e)$ is properly infinite.

Proof: The assumption that f lies in the ideal of A generated by e implies (by standard techniques, see [5]), that $f \precsim e \otimes 1_n$ for some n. Since e is properly infinite, $e \otimes 1_n \precsim e$. Hence $e \leq f \precsim e$. This implies that $f \oplus f \precsim e \oplus e \precsim e \precsim f$, and so f is properly infinite.

The second claim follows immediately from the definition. □

Example 2.3 (The Toeplitz algebra) The Toeplitz algebra, $\mathcal{T}$, is the C^*-algebra generated by the unilateral shift S on the Hilbert space ℓ^2. It contains the compact operators $\mathcal{K}$ on ℓ^2, and we have a short-exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0.$$

The unit $1_\mathcal{T}$ of $\mathcal{T}$ is infinite (because $1_\mathcal{T} = S^*S \sim SS^* < 1_\mathcal{T}$). But $C(\mathbb{T})$ is finite, and so $\pi(1)$ is finite — hence infiniteness doesn't pass to quotients. This also shows that $1_\mathcal{T}$ is not properly infinite (c.f. Proposition 2.2). Notice also, that $\mathcal{T}$ admits a trace (take any state on $C(\mathbb{T})$ and follow it by π). Hence $1_\mathcal{T}$ is weakly tracially finite.

Finiteness doesn't pass to quotients either. Actually, a tracially finite projection can become properly infinite in some quotient as the following example shows:

Example 2.4 Let A be a unital UHF-algebra with normalized trace τ. Consider the multiplier algebra $\mathcal{M}(A \otimes \mathcal{K})$. Find a sequence $\{p_n\}_{n=1}^{\infty}$ of non-zero mutually orthogonal projections in $A \otimes \mathcal{K}$ such that the sum $P = \sum_{n=1}^{\infty} p_n$ converges strictly in $\mathcal{M}(A \otimes \mathcal{K})$, and such that $\sum_{n=1}^{\infty} \tau(p_n) < \infty$. τ extends to an unbounded trace defined on the positive elements of $\mathcal{M}(A)$, and $\tau(P) = \sum_{n=1}^{\infty} \tau(p_n) < \infty$. Hence τ extends to a bounded trace on $P\mathcal{M}(A \otimes \mathcal{K})P$ $(= B)$. Since τ is faithful on $P(A \otimes \mathcal{K})P$ $(= I)$, and I is an essential ideal in B, it follows that τ is faithful on B. This shows that P is tracially finite.

Let $\pi \colon B \to B/I$ be the quotient mapping. By [12, Corollary 3.7], B/I is purely infinite and simple, and therefore $\pi(P)$ is properly infinite.

For more about ideals of multiplier algebras (and related results) see G. A. Elliott [9], H. Lin [14], and S. Zhang [20].

The equivalence of (i), (ii) and (iii) in the theorem below is essentially, but not explicitly, contained in K. Goodearl and D. Handelman's paper [10], and Uffe Haagerup's theorem in [11] is the bridge from quasi-traces to traces.

The theorem tells (at least in the exact case) when a unital C^*-algebra A admits a tracial state. Notice that if A is properly infinite, then A cannot admit a tracial state. As will be shown in Section 4, the reverse does not hold.

To each unital C^*-algebra A associate the triple $(K_0(A), K_0(A)^+, [1_A]_0)$, where

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in P_\infty(A)\}, \qquad K_0(A)^+ = \{[p]_0 \mid p \in P_\infty(A)\},$$

where $P_\infty(A)$ is the set of projections in all matrix algebras over A, and where $[1_A]_0 \in K_0(A)^+$ is a distinguished order unit of $(K_0(A), K_0(A)^+)$.

A *state* on $K_0(A)$ is a group homomorphism $f \colon K_0(A) \to \mathbb{R}$ which satisfies $f(K_0(A)^+) \subseteq \mathbb{R}^+$ and $f([1_A]_0) = 1$.

Theorem 2.5 (Goodearl-Handelman, Haagerup) *Let A be a unital C^*-algebra. The following conditions are equivalent:*

(i) *there is no state on $K_0(A)$,*

(ii) *no (non-zero) quotient of A is stably finite,*

(iii) *$M_n(A)$ is properly infinite for some $n \in \mathbb{N}$.*

If, in addition, A is exact, then these conditions are equivalent to

(iv) *A admits no tracial states.*

Proof: (i) $\Rightarrow$ (iii): Put $u = [1_A]_0 \in K_0(A)^+$. We show first that there exist $k, l \in \mathbb{N}$ with $k > l$ and $ku \leq lu$, if (i) holds. Indeed, assume to the contrary that $ku \leq lu$ implies $k \leq l$ for all $k, l \in \mathbb{N}$. Define $f_0 \colon \mathbb{Z}u \to \mathbb{R}$ to be $f_0(ku) = k$. Then f_0 is well-defined, positive, and $f_0(u) = 1$. But then, by [10, Theorem 3.2], f_0 would extend to a state on $K_0(A)$, in contradiction with (i).

Let e_n denote the unit of $M_n(A)$. Then $nu = [e_n]_0$, and thus $[e_k]_0 \leq [e_l]_0$. It follows that there exists $m \in \mathbb{N}$ such that $e_k \oplus e_m \precsim e_l \oplus e_m$. Put $n = l + m$ and $d = k - l > 0$. Then $e_{n+d} \precsim e_n$, and, consequently,

$$e_{n+rd} \sim e_{n+d} \oplus e_{(r-1)d} \precsim e_n \oplus e_{(r-1)d} \sim e_{n+(r-1)d} \precsim \cdots \precsim e_n,$$

for all $r \in \mathbb{N}$. Choose $r \in \mathbb{N}$ such that $rd \geq n$. Then $e_n \oplus e_n \precsim e_{n+rd} \precsim e_n$. This shows that $M_n(A)$ is properly infinite.

(iii) $\Rightarrow$ (ii): Assume $M_n(A)$ is properly infinite. Then every (non-zero) quotient of $M_n(A)$ is properly infinite by Proposition 2.2. Hence $M_n(A/I)$ is properly infinite for all proper ideals I of A, and consequently, A/I is not stably finite for any proper ideal I.

(ii) $\Rightarrow$ (i): We show that the existence of a state on $K_0(A)$ will imply that A/I is stably finite for some proper ideal I of A.

Assume for this purpose, that f is a state on $K_0(A)$. By [3, Theorem 3.3], f lifts to a (positive, normalized) quasi-trace τ on A. The set $I = \{x \in A \mid \tau(x^*x) = 0\}$ is a (proper) closed two-sided ideal in A, and τ factors through a faithful quasi-trace τ' on A/I. The quasi-trace τ' extends (by the definition of a quasi-trace) to a faithful quasi-trace on each matrix algebra $M_n(A/I)$. A unital C^*-algebra which admits a faithful quasi-trace must be finite (since if $v^*v = 1$, then $1 = \tau(v^*v) = \tau(vv^*) = 1 - \tau(1 - vv^*)$, which implies $\tau(1 - vv^*) = 0$ and hence $vv^* = 1$). Hence A/I is stably finite.

(iv) $\Rightarrow$ (i): Every trace τ induces a state on $K_0(A)$ by the formula $K_0(\tau)([e]_0 - [f]_0) = \tau(e) - \tau(f)$, where $e, f \in P_\infty(A)$, and with τ extended to all matrix algebras over A.

(i) $\Rightarrow$ (iv): As used in the proof of (ii) $\Rightarrow$ (i), a state f on $K_0(A)$ would extend to a quasi-trace on A, and as proved by U. Haagerup [11], any quasi-trace on an exact C^*-algebra is a trace. $\square$

The equivalence between (iii) and (iv) in Theorem 2.5 (that holds for exact C^*-algebras) can be restated as: e is weakly tracially finite if and only if $e \otimes 1_n$ is not properly infinite for all $n \in \mathbb{N}$.

As seen in Example 2.3, not all infinite projections are properly infinite. For simple C^*-algebras we have the following result of Joachim Cuntz:

Proposition 2.6 (Cuntz, [5, Proposition 2.2]) *In a simple C^*-algebra every infinite projection is properly infinite.*

3 Stable C^*-algebras

Recall that a C^*-algebra A is *stable* if $A \cong A \otimes \mathcal{K}$. Since $\mathcal{K} \cong \mathcal{K} \otimes \mathcal{K}$, it follows that $A \otimes \mathcal{K}$ is stable for every C^*-algebra A. A C^*-algebra is *σ-unital* if it admits a countable approximate unit (of positive elements). A C^*-algebra A is called *σ_{p}-unital* if it admits a countable approximate consisting of projections. In that case, the approximate unit can be chosen to consist of an increasing sequence of projections.

For a C^*-algebra A, let $F(A)$ be the set of elements $a \in A^+$ for which there exists $e \in A^+$ with $a = ae$ $(= ea)$. For $a, b \in A^+$ write $a \sim b$ if there exists $x \in A$ with $a = x^*x$ and $b = xx^*$. It is shown in [15, Theorem 3.5] that this defines an equivalence relation on A^+.

Theorem 3.1 ([13, Theorem 2.1 + Proposition 2.2]) *Let A be a σ-unital C^*-algebra. Then the following are equivalent:*

(i) *A is stable,*

(ii) *for each $a \in F(A)$ there exists $b \in A^+$ with $a \sim b$ and $ab = 0$,*

(iii) *for each $a \in F(A)$ and for each $\varepsilon > 0$ there exist $b, c \in A^+$ with $b \sim c$, $\|bc\| \leq \varepsilon$, and $\|a - b\| \leq \varepsilon$.*

Lemma 3.2 *Let A be a σ-unital C^*-algebra of stable rank one. Then A is stable if and only if for all $a \in F(A)$ there exist $b, c \in F(A)$ such that $a \sim b \sim c$ and $bc = 0$.*

Proof: The "only if" part follows immediately from (i) $\Rightarrow$ (ii) in Theorem 3.1.

We proceed to prove the "if" part. We use (iii) $\Rightarrow$ (i) in Theorem 3.1. Let $a \in F(A)$ be given. By assumption we can find $c', c'' \in A^+$ such that $a \sim c' \sim c''$ and $c'c'' = 0$. Hence there exists $x \in A$ with $x^*x = a$ and $xx^* = c'$. Since $\mathrm{sr}(A) = 1$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of invertible elements in $\tilde{A}$, A with a unit adjoined, with $x_n \to x$. Each x_n has a unitary polar decomposition $u_n|x_n|$ (with u_n a unitary in $\tilde{A}$). Now,

$$|x_n|^2 = x_n^* x_n \to a, \qquad u_n|x_n|^2 u_n^* = x_n x_n^* \to c'.$$

This entails that $u_n a u_n^* \to c'$.

Put $b_n = u_n^* c' u_n$ and $c_n = u_n^* c'' u_n$. Then $b_n, c_n \in A^+$, $b_n \to a$, $b_n c_n = 0$, and $b_n \sim c_n$. This shows that (iii) in Theorem 3.1 is satisfied, and so A is stable. □

Retain the notation $\mathcal{M}(A)$ for the multiplier algebra of A.

Lemma 3.3 *Assume that A is a simple C^*-algebra and that $\mathcal{M}(A)$ is infinite. Then for each $a \in F(A)$ there exist $b, c \in A^+$ such that $a \sim b \sim c$ and $bc = 0$.*

Proof: The set $F(A)$ is contained in the Pedersen ideal of A, which is an algebraically simple, essential ideal in $\mathcal{M}(A)$. Hence every non-zero algebraic ideal of $\mathcal{M}(A)$ contains $F(A)$. Let $a \in F(A)$ be given. Since $\mathcal{M}(A)$ is infinite, it contains a non-unitary isometry S. The algebraic ideal in $\mathcal{M}(A)$ generated by $1 - SS^*$ contains a. Therefore

$$a = \sum_{j=0}^{n} X_j (1 - SS^*) Y_j$$

for some $n \in \mathbb{N}$ and some $X_j, Y_j \in \mathcal{M}(A)$. Put $E_0 = 1 - SS^*$, and put $E_j = SE_{j-1}S^*$ for $j \geq 1$. Set

$$X = \sum_{j=0}^{n} X_j (S^*)^j E_j, \qquad Y = \sum_{j=0}^{n} E_j S^j Y_j,$$

and set $T = S^{n+1}$. Then $XY = a$, T is an isometry in $\mathcal{M}(A)$,

$$1 - TT^* = E_0 + E_1 + \cdots + E_n,$$

and $YY^*(1 - TT^*) = (1 - TT^*)YY^* = YY^*$.

Put $x_0 = X(YY^*)^{1/2}$. Then $x_0 x_0^* = a^2$, $x_0 \in A$, and $x_0^* x_0 \in (1 - TT^*)A(1 - TT^*)$. Let $u|x_0|$ be the polar decomposition for x_0 (with $u \in A^{**}$), and set $x = u|x_0|^{1/2} \in A$. Then $xx^* = a$ and $x^*x = (x_0^* x_0)^{1/2} \in (1 - TT^*)A(1 - TT^*)$.

Put $b = x^*x$ and $c = TaT^*$. Then $b, c \in A^+$, $bc = 0$, $a \sim b$, and $a \sim c$ (for the latter, use that $a = r^*r$ and $c = rr^*$ when $r = Ta^{1/2} \in A$). □

The following lemma is well-known:

Lemma 3.4 *Let A be a stable C^*-algebra. Then $B(H)$, the bounded operators on an infinite dimensional, separable Hilbert space H, embeds unitally into $\mathcal{M}(A)$. In particular, $\mathcal{M}(A)$ is properly infinite.*

Proof: Write $A = D \otimes \mathcal{K}$ for some C^*-algebra D. Represent D faithfully and non-degenerately on a Hilbert space H_1, and identify $\mathcal{K}$ with the compact operators on a Hilbert space H. Then $D \otimes \mathcal{K}$ is a subalgebra of $B(H_1)\bar{\otimes}B(H)$, and $\mathcal{M}(D \otimes \mathcal{K})$ is the normalizer (or the set of multipliers) of $D\otimes\mathcal{K}$ in $B(H_1)\bar{\otimes}B(H)$ (see [16, Section 3.12]). Clearly, $\mathbb{C}I_{H_1} \otimes B(H)$ normalizes $D \otimes \mathcal{K}$, and is therefore contained in $\mathcal{M}(D \otimes \mathcal{K})$. This gives a unital embedding of $B(H)$ into $\mathcal{M}(D\otimes\mathcal{K})$. Since the unit of $B(H)$ is properly infinite (because H is infinite-dimensional), it follows that the unit of $\mathcal{M}(D\otimes\mathcal{K})$ is properly infinite. □

Theorem 3.5 *Let A be a simple, σ-unital C^*-algebra of* $\mathrm{sr}(A) = 1$. *Then $\mathcal{M}(A)$ is finite if A is non-stable, and $\mathcal{M}(A)$ is properly infinite if A is stable.*

Proof: The first statement follows immediately from Lemma 3.2 and Lemma 3.3, and the second statement is contained in Lemma 3.4. □

If we omit the simplicity assumption on A (but keep the assumption on the stable rank), then we get:

Proposition 3.6 *Let A be a σ-unital C^*-algebra of* $\mathrm{sr}(A) = 1$. *Then $\mathcal{M}(A)$ is properly infinite if and only if A is stable.*

Proof: The "if" part follows from Lemma 3.4. Assume that $\mathcal{M}(A)$ is properly infinite. Then there exist isometries $S, T \in \mathcal{M}(A)$ with orthogonal range projections. Let $a \in F(A)$. Then $b = SaS^*$ and $c = TaT^*$ satisfy the conditions of Lemma 3.2. □

The condition on the stable rank in the two results above cannot be omitted — but perhaps relaxed. Indeed, if A is a (unital) properly infinite C^*-algebra, then $\mathcal{M}(A) = A$, and A is not stable. A properly infinite C^*-algebra has stable rank ∞ (c.f. [17, Proposition 6.5]).

Theorem 3.7 ([18, Theorem 5.3]) *There exists a simple, σ_{p}-unital, nuclear (and hence exact) C^*-algebra A with* $\mathrm{sr}(A) = 1$ *such that $M_2(A)$ is stable while A is not stable.*

4 Sums of finite projections

Let e, f be projections in a C^*-algebra A. Assume that e and f are finite (in some sense). Does it follow that $e \oplus f$, respectively, $e \oplus e$ are finite (in some, perhaps different, sense)? Consider the following four notions of finiteness:

- (TF) = tracially finite,
- (F) = finite,
- (WTF) = weakly tracially finite,
- (NPI) = not properly infinite.

(c.f. Definition 2.1). The following implications between these properties are immediate:

$$(\mathrm{TF}) \Rightarrow (\mathrm{F}) \Rightarrow (\mathrm{NPI}), \qquad (\mathrm{TF}) \Rightarrow (\mathrm{WTF}) \Rightarrow (\mathrm{NPI}). \tag{1}$$

None of the remaining 7 implications hold: To show this it suffices to show that

$$(\mathrm{F}) \not\Rightarrow (\mathrm{WTF}), \qquad (\mathrm{WTF}) \not\Rightarrow (\mathrm{F}). \tag{2}$$

The second non-implications in (2) follows from Example 2.3. The non-implication (F) $\not\Rightarrow$ (WTF), i.e. not all finite (unital) C^*-algebras admit a tracial state, is really the main point of this article, and it is verified below Example 4.3.

Using that traces on a C^*-algebra extend to all matrix algebras over it, one has:

Proposition 4.1 *Let e be a tracially finite, respectively, weakly tracially finite, projection in a C^*-algebra A. Then $e \oplus e$ is tracially finite, respectively, weakly tracially finite.*

To each of the four notions of finiteness one can consider their corresponding stabilized versions. For example, a projection $e \in A$ will be said to be *stably finite*, (SF), if $e \otimes 1_n$ is finite for all $n \in \mathbb{N}$. Define (STF), (SWTF) and (SNPI) similarly. Proposition 4.1 says that the properties (TF) and (WTF) are stable, i.e.,

$$(\mathrm{STF}) \Leftrightarrow (\mathrm{TF}), \qquad (\mathrm{SWTF}) \Leftrightarrow (\mathrm{WTF});$$

and by Theorem 2.5,

$$(\mathrm{SNPI}) \Leftrightarrow (\mathrm{WTF}),$$

for all exact C^*-algebras.

We have (TF) $\Rightarrow$ (SF) $\Rightarrow$ (F) for all C^*-algebras, and (SF) $\Rightarrow$ (WTF) for exact C^*-algebras (c.f. Theorem 2.5), but (SF) $\not\Rightarrow$ (TF). To see the latter, the unit of the C^*-algebra $\tilde{\mathcal{K}}$, the compacts with a unit adjoined, is stably finite but not tracially finite. For a more convincing example of a stably finite but not tracially finite projection, take the unit of the C^*-algebra $(C_0(\mathbb{R}) \otimes \mathcal{O}_2)^\sim$.

Except for the cases covered by Proposition 4.1, the sum of two "finite" projection will fail to be "finite" as the following two examples show:

Example 4.2 *There exist tracially finite projections e, f in some C^*-algebra such that $e \oplus f$ is properly infinite.*

Proof: Let B be the unique unital AF-algebra with dimension group

$$G = \mathbb{Q} \oplus \mathbb{Q} \ \ (= K_0(B)), \qquad G^+ = \{(s,t) \in \mathbb{Q} \oplus \mathbb{Q} \mid s > 0, \ t > 0\} \ \ (= K_0(B)^+),$$

and order unit $u = (1,1)$ $(= [1_B]_0)$. Since (G, G^+) is a simple dimension group, it follows that B is a simple C^*-algebra.

Let τ_1, τ_2 be the (extremal) tracial states on B given by $K_0(\tau_1)(s,t) = s$ and $K_0(\tau_2)(s,t) = t$. Find a sequence $\{p_n\}_{n=1}^\infty$ of projections in B with

$$\tau_1(p_n) = 2^{-n}, \qquad \tau_2(p_n) = 1 - 2^{-n}.$$

Let $\{e_{i,j}\}_{i,j \in \mathbb{N}}$ be a system of matrix units for the compacts $\mathcal{K}$, and put

$$P = \sum_{n=1}^\infty p_n \otimes e_{n,n} \in \mathcal{M}(B \otimes \mathcal{K}),$$

(the sum is strictly convergent). Then $1 - P = \sum_{n=1}^\infty (1_B - p_n) \otimes e_{n,n}$. As in Example 2.4, τ_1 extends to a bounded, faithful trace on $P\mathcal{M}(B \otimes \mathcal{K})P$, and τ_2 extends to a bounded, faithful trace on $(1-P)\mathcal{M}(B \otimes \mathcal{K})(1-P)$.

It follows that P and $1 - P$ are tracially finite. But $\mathcal{M}(B \otimes \mathcal{K})$ is properly infinite (c.f. Lemma 3.4), and so $P \oplus (1-P) \sim 1$ is properly infinite. $\square$

Example 4.3 *There exists a finite projection e in some C^*-algebra such that $e \oplus e$ is properly infinite.*

Proof: Let A be the C^*-algebra from Theorem 3.7. Put $B = \mathcal{M}(A)$. Then $M_2(B) \cong \mathcal{M}(M_2(A))$. By Theorem 3.5, B is finite and $M_2(B)$ is properly infinite. Let e be the unit of B. Then $e \oplus e$ is the unit of $M_2(B)$, and hence e is finite and $e \oplus e$ is properly infinite. $\square$

The projection e in Example 4.3 cannot be weakly tracially finite (WTF) (since a trace on $B = eBe$ would extend to a trace on $M_2(B)$, but $M_2(B)$ is properly infinite, and no properly infinite C^*-algebra can admit a trace). This proves the non-implication (F) $\not\Rightarrow$ (WTF).

Remark 4.4 Example 4.3 gives an indication that there might exist simple C^*-algebras that contain simultaneously infinite and non-zero finite projections.

In more detail, assume that A were a simple C^*-algebra with a non-zero finite projection e and an infinite projection f. Then $f \precsim e \otimes 1_n$ for some $n \in \mathbb{N}$. By Proposition 2.2 and Proposition 2.6, $e \otimes 1_n$ is properly infinite. Let $1 \leq m < n$ be chosen such that $e' = e \otimes 1_m$ is finite and $e \otimes 1_{m+1}$ is infinite. Then $e \otimes 1_{m+1} \precsim e' \oplus e'$, and therefore $e' \oplus e'$ is properly infinite.

In other words, if e finite had implied $e \oplus e$ not properly infinite, then it would follow that no simple C^*-algebra could contain finite and infinite projections. However, Example 4.3 shows that this algebraic approach to the problem does not work.

One cannot (immediately) from Example 4.3 construct a simple C^*-algebra with finite and infinite projections, although the following strategy seems tempting: If A is a finite unital C^*-algebra such that $M_2(A)$ is properly infinite, and if I is a maximal ideal in A, then A/I is simple and $M_2(A/I)$ is properly infinite (c.f. Proposition 2.2). But A/I need not be finite. Perhaps one can choose A in such a way that A/I does become finite, at least for some maximal ideal I.

Corollary 4.5 *Let $A * M_2(\mathbb{C})$ be the universal unital free product of a unital separable C^*-algebra A and $M_2(\mathbb{C})$. Let $f \in M_2(\mathbb{C}) \subseteq A * M_2(\mathbb{C})$ be a one-dimensional projection.*

Then f is not properly infinite in $A * M_2(\mathbb{C})$.

Proof: By Example 4.3 and its proof there is a unital finite C^*-algebra B such that $M_2(B)$ is the multiplier algebra of a stable C^*-algebra. Let $e \in M_2(B)$ correspond to the unit of $B \subset M_2(B)$ (so that e is a finite projection in $M_2(B)$), and let $\varphi_2 \colon M_2(\mathbb{C}) \to M_2(B)$ be a unital *-homomorphism with $\varphi_2(f) = e$ (where $f \in M_2(\mathbb{C})$ is the given one-dimensional projection).

By Lemma 3.4, $B(H)$ embeds unitally into $M_2(B)$, for some infinite dimensional separable Hilbert space H, and therefore we can find a unital embedding $\varphi_1 \colon A \to M_2(B)$. By universality of the free product, there exists a (unique) *-homomorphism $\varphi \colon A * M_2(\mathbb{C}) \to M_2(B)$ such that $\varphi|_A = \varphi_1$ and $\varphi|_{M_2(\mathbb{C})} = \varphi_2$. In particular, $\varphi(f) = e$. Proposition 2.2 now yields that f is not properly infinite. $\square$

We do not know if f in Corollary 4.5 is actually finite for all A, or, say, when $A = \mathcal{O}_2$.

The situation in reduced free products is different:

Proposition 4.6 *Let A be a unital properly infinite C^*-algebra, let ρ_1 be a faithful state on A, let ρ_2 be a faithful state on $M_2(\mathbb{C})$, and consider the reduced free product C^*-algebra*

$$(\mathfrak{A}, \rho) = (A, \rho_1) * (M_2(\mathbb{C}), \rho_2).$$

Let $f \in M_2(\mathbb{C}) \subseteq \mathfrak{A}$ be a one-dimensional projection. Then f is properly infinite.

Proof: Since A is properly infinite, it admits a unital embedding $\mathcal{O}_\infty \to A$. There is a sequence of mutually orthogonal non-zero projections in $\mathcal{O}_\infty \subseteq A$, and in this sequence there is a (necessarily properly infinite, full) projection $q \in A$ with $\rho_1(q) < \rho_2(f)$. The projections f and q are ρ-free by the construction of the reduced free product. By [7], ρ is faithful on $\mathfrak{A}$. Since $\rho(q) = \rho_1(q) < \rho_2(f) = \rho(f)$, it follows from [1] (see also [8, Proposition 1.1]) that $q \precsim f$ in $\mathfrak{A}$. By Proposition 2.2, this entails that f is properly infinite. □

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