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June 5, 2020

Differentiation and the Lebesgue-Radon-Nikodym Theorem

On the concept of differentiation of signed and complex measures

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Abstract

The main theme of this project is the concept of differentiation of a signed or complex measure with respect to a positive measure on the same σ-algebra. A central theorem is the Lebesgue-Radon-Nikodym Theorem, which proves the decomposition of a signed or complex measure into measures that are respectively absolutely continuous and singular with respect to a positive measure. The Radon-Nikodym Theorem follows directly and provides an abstract notion of the derivative of a signed or complex measure. The Lebesgue-Radon-Nikodym Theorem has many applications; one of which is the result that the dual space of $L^p(\mu)$, for $1 \leq p < \infty$ and a σ-finite positive measure μ, is isometrically isomorphic to $L^q(\mu)$, where q is the conjugate exponent to p, which can be obtained as a consequence of the Lebesgue-Radon-Nikodym Theorem for complex measures, and in particular, the Radon-Nikodym Theorem.

The project initializes with the theory and elementary properties of signed and complex measures. Following, the concept of differentiation of signed or complex measures is introduced in different successive levels of abstraction. This begins with the Lebesgue-Radon-Nikodym Theorem and the very abstract notion of the Radon-Nikodym derivative, and then letting $(X, \mathcal{A}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ leads to a more refined result of differentiation of signed or complex measures with respect to the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. The theory will lead to a proof of the Fundamental Theorem of Calculus for Lebesgue integrals, which derives from the special case of $n = 1$. This includes the theory of functions of bounded variation and their very significant connection to complex Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In the final part of the project, the theory is used to construct an example of a non-atomic measure, which is singular with respect to the Lebesgue measure. This includes the theory of the Cantor ternary set and the Cantor function. The example ties the theories from several parts of this project together in a very beautiful way.

Contents

1 Introduction

The goal of this project is to showcase the work I have done with my advisor, Mikael, in order to gain a general understanding of the concept of differentiation of measures.

The reader is assumed to be familiar with elementary notions from measure theory and functional analysis. In particular, the reader should be comfortable with measures and the theory behind, as well as bounded linear functionals on normed vector spaces, i.e., dual spaces. Standard textbooks and references are Schilling 2017 and Folland 1999. Moreover, the reader should have an understanding of the basic topological concepts presented in Munkres 2008. However, as the perspective of the project is not basically topological, these concepts are thoroughly referenced and explained. The project is mostly self-contained. When prerequisite results are needed, the results are stated with a reference. In particular, some results from functional analysis are stated without proof, as they are well-known to everyone who has done any fundamental functional analysis.

In the initial chapter, signed measures as well as the elementary properties and theory behind, are introduced. This section follows Folland 1999 (section 3.1) proving the Hahn Decomposition Theorem and the Jordan Decomposition Theorem, which leads to a general understanding of signed measures as a unique decomposition into positive measures.

In the following chapter, complex measures are presented. This is based on Rudin 1987 (chapter 6). As complex measures can be decomposed into a real and imaginary part, both of which are finite signed measures, the elementary properties follow more of less directly from the theory of signed measures.

This leads to the chapter in which the Lebesgue-Radon-Nikodym Theorem is proved, first for signed measures and next for complex measures. The chapter follows Folland 1999 (section 3.2) in the proof of the Lebesgue-Radon-Nikodym Theorem and the introduction of the Radon-Nikodym derivative as well as elementary properties hereof.

Next, an application of the Lebesgue-Radon-Nikodym Theorem is showcased, as the theory of bounded linear functionals on $L^p(\mu)$-spaces is presented. The chapter follows Rudin 1987 (chapter 6) and Schilling 2017 (chapter 21) in the proof of an isometrical isomorphism between the dual space, $(L^p(\mu))^*$, and the vector space, $L^q(\mu)$, for $1 \leq p < \infty$ and q the conjugate exponent to p. Moreover, this result is used to show that $L^p(\mu)$ is in fact reflexive for $1 < p < \infty$. The special case of $p = 2$ follows from the theory of Hilbert spaces, which is presented following Folland 1999 (section 5.5.).

The following chapter revolves around differentiation on an Euclidean space, $\mathbb{R}^n$. Following Folland 1999 (section 3.4), the definition of the pointwise derivative of a complex or signed measure with respect to the Lebesgue measure is introduced. The theory includes three successively sharper versions of the Fundamental Differentiation Theorem ending with the Lebesgue Differentiation Theorem. Moreover, it is proven that the pointwise derivative coincides with the Radon-Nikodym derivative under certain regularity conditions.

Thus, letting $n = 1$ and considering $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, leads to the theory of functions of bounded variation, which is presented following Folland 1999 (section 3.5). In this chapter, the goal is to examine functions of bounded variation and their role in the characterization of complex Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The chapter shows how the Fundamental Theorem of Calculus can be proved rather easily as a consequence of this particular theory.

Finally, the Cantor ternary set and the Cantor function are constructed based on various exercises from Schilling 2017 (chapter 6, 7 & 20). This chapter includes main theory and remarkable properties of the Cantor set and the Cantor function in order to handle this with the respect deserved. Personally, I think that this final chapter concludes the project very beautifully by showing how to construct an example of a non-atomic singular measure using the theory presented in the project.

2 Signed measures

The purpose of this chapter is to introduce and characterize signed measures. A general understanding of signed measures is obtained through two main examples, which by the Jordan Decomposition Theorem turn out to be the *only* examples. The theory of this chapter is based on Folland 1999 (section 3.1).

2.1 Definition and elementary properties

This section introduces the definition and elementary properties of signed measures, as well as some examples hereof.

Definition 2.1. Let $(X, \mathcal{A})$ be a measurable space. A *signed measure* is a map $\nu: \mathcal{A} \to [-\infty, \infty]$ such that

(i) $\nu(\varnothing) = 0$.

(ii) $\nu(\dot\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$ for every sequence of disjoint sets $\{E_i\}_{i\geq 1} \subseteq \mathcal{A}$.

(iii) ν assumes at most one of the values ∞ and $-\infty$.

Remark 2.2. One may notice that the definition of a signed measure is a generalization of measures allowing negative values. It is clear that a measure fulfils the definition, hence a measure is in particular a signed measure. To avoid confusion, measures shall forwardly be referred to as *positive measures*.

Example 2.3. Let μ_1, μ_2 be positive measures with at least one of them being finite. Then $\nu := \mu_1 - \mu_2$ is a signed measure.

Proof. (i) By the definition of positive measures, $\nu(\varnothing) = \mu_1(\varnothing) - \mu_2(\varnothing) = 0$.

(ii) Let $\{E_i\}_i \subseteq \mathcal{A}$ be a sequence of disjoint sets. Then by countable additivity of μ_1, μ_2,

$$\begin{aligned}\nu(\dot\bigcup_{i=1}^{\infty} E_i) &= \mu_1(\dot\bigcup_{i=1}^{\infty} E_i) - \mu_2(\dot\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu_1(E_i) - \sum_{i=1}^{\infty} \mu_2(E_i) \\ &= \sum_{i=1}^{\infty} \mu_1(E_i) - \mu_2(E_i) = \sum_{i=1}^{\infty} \nu(E_i),\end{aligned}$$

since at least one of the series $\sum_{i=1}^{\infty} \mu_1(E_i)$ and $\sum_{i=1}^{\infty} \mu_2(E_i)$ converges.

(iii) By assumption at most one of μ_1 and μ_2 assumes infinite values, thus $\nu = \mu_1 - \mu_2$ assumes at most one of the values ∞ and $-\infty$. Furthermore, note that ν assumes the value ∞ if and only if μ_1 is infinite, and similarly ν assumes $-\infty$ if and only if μ_2 is infinite. □

Definition 2.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. A measurable function $f: X \to [-\infty, \infty]$ is *extended μ-integrable* if $\int_X f^+ d\mu < \infty$ or $\int_X f^- d\mu < \infty$.

Example 2.5. The preceding definition gives rise to yet an example of a signed measure: If f is an extended μ-integrable function, then ν defined by $\nu(E) := \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$ for every $E \in \mathcal{A}$ is a signed measure. Note that f being extended μ-integrable implies that at least one of f^+, f^- lies in $L^1(\mu)$, where f^+, f^- are positive, measurable functions, so $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are positive measures with at least one of them being finite. Hence, this is simply a special case of Example 2.3 with $\mu_1(E) = \int_E f^+ d\mu$ and $\mu_2(E) = \int_E f^- d\mu$ for every $E \in \mathcal{A}$, thus, it is already proven that ν is a signed measure.

Remark 2.6. Not only are these some of the most obvious examples of signed measures; as a matter of fact, they are the *only* examples of a signed measure. In particular, every signed measure can be expressed in either one of these forms by the Jordan Decomposition Theorem, which is proven in a later section.

Proposition 2.7. *Let ν be a signed measure on $(X,\mathcal{A})$. Then the following properties hold.*

(i) If $\{E_i\}_{i\geq 1}\subseteq\mathcal{A}$ is an increasing sequence, then

$$\nu(\bigcup_{i=1}^{\infty}E_i)=\lim_{i\to\infty}\nu(E_i). \qquad \text{(Continuity from below)}$$

(ii) If $\{E_i\}_{i\geq 1}\subseteq\mathcal{A}$ is a decreasing sequence, and $\nu(E_1)$ is finite, then

$$\nu(\bigcap_{i=1}^{\infty}E_i)=\lim_{i\to\infty}\nu(E_i). \qquad \text{(Continuity from above)}$$

Proof. (i) Let $\{E_i\}_{i\geq 1}\subseteq\mathcal{A}$ be an increasing sequence. Set $E_0:=\varnothing$. Thus, the union may be written as a disjoint union, $\bigcup_{i=1}^{\infty}E_i=\biguplus_{i=1}^{\infty}(E_i\setminus E_{i-1})$. Then

$$\begin{aligned}\nu(\bigcup_{i=1}^{\infty}E_i)&=\sum_{i=1}^{\infty}\nu(E_i\setminus E_{i=1})=\lim_{n\to\infty}\sum_{i=1}^{n}\nu(E_i\setminus E_{i-1})\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\nu(E_i)-\nu(E_{i-1})=\lim_{n\to\infty}\nu(E_n).\end{aligned}$$

(ii) Let $\{E_i\}_{i\geq 1}\subseteq\mathcal{A}$ be a decreasing sequence. Set $F_i:=E_1\setminus E_i$ for every $i\geq 1$. Thus, $\{F_i\}_{i\geq 1}\subseteq\mathcal{A}$ is increasing, $\nu(E_1)=\nu(F_i\uplus E_i)=\nu(F_i)+\nu(E_i)$, and $\bigcup_{i=1}^{\infty}F_i=E_1\setminus(\bigcap_{i=1}^{\infty}E_i)$. Then by (i),

$$\begin{aligned}\nu(E_1)&=\nu(\bigcup_{i=1}^{\infty}F_i)+\nu(\bigcap_{i=1}^{\infty}E_i)=\lim_{i\to\infty}\nu(F_i)+\nu(\bigcap_{i=1}^{\infty}E_i)\\&=\lim_{i\to\infty}(\nu(E_1)-\nu(E_i))+\nu(\bigcap_{i=1}^{\infty}E_i)=\nu(E_1)-\lim_{i\to\infty}\nu(E_i)+\nu(\bigcap_{i=1}^{\infty}E_i),\end{aligned}$$

and since $\nu(E_1)$ is finite, subtracting it from both sides yields that $\nu(\bigcap_{i=1}^{\infty}E_i)=\lim_{i\to\infty}\nu(E_i)$. □

2.2 The Hahn Decomposition Theorem

In this section the theory of positive, negative and null sets is introduced in order to prove the Hahn Decomposition Theorem, which states that for any signed measure ν on $(X,\mathcal{A})$, the space X can be decomposed into disjoint positive and negative sets with respect to ν. This result proves to be very important in the theory of signed measures.

Definition 2.8. Let ν be a signed measure on $(X,\mathcal{A})$. Then a set $E\in\mathcal{A}$ is called *positive*, *negative* and *null*, respectively, if $\nu(F)\geq 0,\nu(F)\leq 0,\nu(F)=0$ for every $F\in\mathcal{A}$ with $F\subseteq E$.

Remark 2.9. The definition above yields that a null set is a set that is both positive and negative. Moreover, any measurable subset of a positive/negative/null set is positive/negative/null respectively. This follows directly from the definition.

Lemma 2.10. *The union of a countable family of positive or negative sets is positive or negative, respectively.*

Proof. The proof is given for positive sets only, as it follows analogously for negative sets. Let $\{P_i\}_{i\geq 1}$ be a countable family of positive sets, and set $Q_n:=P_n\setminus\bigcup_{i=1}^{n-1}P_i$ for $n\in\mathbb{N}$. Then clearly, $Q_n\subseteq P_n$ for every $n\in\mathbb{N}$, hence each Q_n is a positive set by Remark 2.9. Note also that $\dot\bigcup_{i=1}^{\infty} Q_i = \bigcup_{i=1}^{\infty} P_i$, where $\dot\bigcup_{i=1}^{\infty} Q_i$ is a disjoint union, since $Q_i \cap Q_j = \varnothing$ for each $i \neq j$. Now, let $E \in \mathcal{A}$ such that $E \subseteq \bigcup_{i=1}^{\infty} P_i$. Then $E = E \cap (\bigcup_{i=1}^{\infty} P_i) = E \cap (\dot\bigcup_{i=1}^{\infty} Q_i)$. Hence

$$\nu(E) = \nu\big(E \cap \big(\dot\bigcup_{i=1}^{\infty} Q_i\big)\big) = \nu\big(\dot\bigcup_{i=1}^{\infty} E \cap Q_i\big) = \sum_{i=1}^{\infty} \nu(E \cap Q_i) \geq 0,$$

since $E \cap Q_i \subseteq Q_i$. Thus, $\bigcup_{i=1}^{\infty} P_i$ is positive as wanted. □

Theorem 2.11 (The Hahn Decomposition Theorem). *Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist P and N, respectively positive and negative sets for ν, such that $P \cup N = X$, where $P \cap N = \varnothing$. Moreover, P and N are unique up to null sets, i.e., if P', N' is another such pair, then $P \triangle P' = N \triangle N'$ is null for ν.*

Proof. Assume without loss of generality that ν does not assume the value ∞. Note that this is sufficient, since otherwise one considers the signed measure $-\nu$, and the proof follows analogously. Let $m := \sup\{\nu(E) : E \in \mathcal{A} \text{ is a positive set}\}$. Then there exists a sequence $\{Q_i\}_{i\geq 1} \subseteq \mathcal{A}$ of positive sets with $\lim_{i\to\infty} \nu(Q_i) = m$. Let $P_n := \bigcup_{i=1}^{n} Q_i$ for each $n \in \mathbb{N}$. Thus, $\{P_n\}_{n\geq 1} \subseteq \mathcal{A}$ is an increasing sequence with $\lim_{n\to\infty} \nu(P_n) = m$. Now let $P := \bigcup_{n=1}^{\infty} P_n$. Then by Lemma 2.10, P is positive, and by Proposition 2.7 (i) (continuity from below), $\nu(P) = \nu(\bigcup_{n=1}^{\infty} P_n) = \lim_{n\to\infty} \nu(P_n) = m$. Note that in particular $0 \leq \nu(P) = m < \infty$. Now, the aim is show that $N := X \backslash P$ is a negative set. Assume by contradiction that N is not negative, i.e., there exists $E \subseteq N$ such that $\nu(E) > 0$. Let $E \subseteq N$ be such a set, and assume first that E is positive. Then $E \cup P$ is positive by Lemma 2.10, and $\nu(E \cup P) = \nu(E) + \nu(P) > m$, which is a contradiction. Hence, N cannot contain any positive, nonnull sets. Moreover, if $A \subseteq N$ such that $\nu(A) > 0$, there exists $B \subseteq A$ with $\nu(B) > \nu(A)$: Since A cannot be positive, there exists $C \subseteq A$ with $\nu(C) < 0$. Let $B := A \setminus C$. Then $\nu(B) = \nu(A) - \nu(C) > \nu(A)$ as wanted.

Define a sequence of sets in N inductively: Let $n_1 := \inf\{n \in \mathbb{N} : \exists B \subseteq N : \nu(B) > n^{-1}\}$ and let $A_1 \subseteq N$ be such a set, i.e., $\nu(A_1) > n_1^{-1}$. Similarly, let $n_2 := \inf\{n \in \mathbb{N} : \exists B \subseteq A_1 : \nu(B) > \nu(A_1) + n^{-1}\}$ and let A_2 be such a set. This is possible since for $A \subseteq N$ with $\nu(A) > 0$ there exists $B \subseteq A$ with $\nu(B) > \nu(A)$. Continuing this way, one obtains a decreasing sequence $\{A_i\}_{i\geq 1} \subseteq N$, for which $\nu(A_i) > \nu(A_{i-1}) + n_i^{-1} > \sum_{j=1}^{i} n_j^{-1}$. Let $A := \bigcap_{i=1}^{\infty} A_i$. By assumption $n_1^{-1} < \nu(A_1) < \infty$, so by Proposition 2.7 (ii) (continuity from below),

$$\infty > \nu(A) = \nu\big(\bigcap_{i=1}^{\infty} A_i\big) = \lim_{i\to\infty} \nu(A_i) \geq \lim_{i\to\infty} \sum_{j=1}^{i} n_j^{-1} = \sum_{j=1}^{\infty} n_j^{-1},$$

which implies that $n_j^{-1} \to 0$ as $j \to \infty$, or equivalently, $n_j \to \infty$ as $j \to \infty$. Once again, as $A \subseteq N$ with $\nu(A) > 0$, there exists $B \subseteq A$ such that $\nu(B) > \nu(A) + n^{-1}$ for some $n \in \mathbb{N}$. Note that $B \subseteq A$ implies that also $B \subseteq A_j$ for each $j \in \mathbb{N}$. By choosing j large enough, $n < n_j$, so

$$\nu(B) > \nu(A) + n^{-1} > \nu(A_{j-1}) + n_j^{-1},$$

but this contradicts with the construction of n_j. Hence, N cannot contain any set $E \in \mathcal{A}$ with $\nu(E) > 0$, i.e., N is negative.

Now, suppose P' and N' is another set of respectively positive and negative sets for ν with $P' \cup N' = X$. First, observe that $P \triangle P' = N \triangle N'$:

$$\begin{aligned} P \triangle P' &= (P \setminus P') \cup (P' \setminus P) \\ &= \big((X \setminus N) \setminus (X \setminus N')\big) \cup \big((X \setminus N') \setminus (X \setminus N)\big) \\ &= (N' \setminus N) \cup (N \setminus N') = N' \triangle N = N \triangle N'. \end{aligned}$$

Thus, it suffices to show that $P \triangle P'$ is null for ν. Since $P \setminus P' \subseteq P$ and also $P \setminus P' = P \setminus (X \setminus N') = P \cap N' \subseteq N'$, $P \setminus P'$ is both positive and negative, i.e., $P \setminus P'$ is null. By an analogous argument, $P' \setminus P$ is also null, and thus $P \triangle P'$ is null for ν as wanted. □

Remark 2.12. This decomposition of X into a disjoint union of a positive and negative set is called a *Hahn decomposition* of ν. As proven, it is unique up to null sets, however, it is usually not unique in general: Note that if U is a ν-null set and $U \subseteq P$, then $P' := P \setminus U$ is a positive set, $N' = N \cup U$ is a negative set, and also $P' \cup N' = X$ with $P' \cap N' = \varnothing$. Thus, ν-null sets can be transferred from P to N (or from N to P). Although, a Hahn decomposition of ν is not unique, it gives rise to a canonical representation of ν as the difference of two positive measures, as it shall be proven in the following.

2.3 The Jordan Decomposition Theorem

The purpose of this section is prove the Jordan Decomposition Theorem, which provides a complete characterization of the decomposition of signed measures into positive measures. The result builds upon the Hahn Decomposition Theorem from the preceding section.

Definition 2.13. Let ν and μ be signed measures on $(X, \mathcal{A})$. Then ν is *singular with respect to* μ, if there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, where E is null for ν, and F is null for μ. This is denoted $\nu \perp \mu$.

Remark 2.14. One might think of this as ν and μ being perpendicular, which agrees with the notation. Note that if ν is singular with respect to μ, then μ is also singular with respect to ν. For this reason, the term that ν and μ are mutually singular is often used. For an intuitive understanding, the definition translates to ν and μ 'living on disjoint sets'. The concept of 'support' of measures might spring to mind with this definition. One should be aware of this, as this understanding would consequently imply that no measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is singular with respect to the Lebesgue measure, as the Lebesgue measure is supported on the whole space, $\mathbb{R}$. This is however not the case.

Example 2.15. Let δ_x be the Dirac measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\delta_x(E) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases},$$

for every $E \in \mathcal{B}$. Then $\big(\mathbb{R}\setminus\{x\}\big) \cup \{x\} = \mathbb{R}$, where $\{x\}$ is a Lebesgue null set, as it is a singleton, and $\mathbb{R}\setminus\{x\}$ is a Dirac null set. Thus, every discrete measure living on singletons, such as the Dirac measure, is singular with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. This example is quite obvious, and for this reason it is also not very interesting. As it turns out, it is a lot more cumbersome to come up with an example of a non-atomic measure that is singular with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. However, there *are* examples hereof; one of which is displayed in the very last chapter, 'Singularity and the Lebesgue measure'.

Theorem 2.16 (The Jordan Decomposition Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. For the existence part, let $P \cup N = X$ be a Hahn decomposition of X, and define $\nu^+(E) := \nu(E \cap P)$ and $\nu^-(E) := -\nu(E \cap N)$ for every $E \in \mathcal{A}$. Thus,

$$\begin{aligned} \nu^+(E) - \nu^-(E) &= \nu(E \cap P) + \nu(E \cap N) \\ &= \nu((E \cap P) \cup (E \cap N)) = \nu(E). \end{aligned}$$

Note that for every $E \subseteq N$, it holds that $\nu^+(E) = \nu(E \cap P) = \nu(\varnothing) = 0$, hence N is null for ν^+; similarly, P is null for ν^-. Thus, $\nu^+ \perp \nu^-$ as wanted.

For the uniqueness, let μ^+, μ^- be another such pair of positive measures with $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. Let $E, F \in \mathcal{A}$ be such that $E \cup F = X$ with $\mu^+(E) = \mu^-(F) = 0$. Clearly,

E is ν-positive and F is ν-negative, hence $E \cup F = X$ is another Hahn decomposition, and therefore, $P \triangle E = N \triangle F$ is ν-null. Then for every $A \in \mathcal{A}$,

$$\begin{aligned}\mu^+(A) &= \mu^+(A \cap E) = \nu(A \cap E) \\ &= \nu\big((A \cap (E \cap P)) \cup (A \cap (E \setminus P))\big) = \nu\big(A \cap E \cap P\big) + \nu\big(A \cap (E \setminus P)\big) \\ &= \nu\big(A \cap E \cap P\big) + \nu\big(A \cap (P \setminus E)\big) = \nu\big((A \cap E \cap P) \cup (A \cap (P \setminus E))\big) \\ &= \nu(A \cap P) = \nu^+(A),\end{aligned}$$

since $A \cap (E \setminus P), A \cap (P \setminus E) \subseteq P \triangle E$. Thus, $\mu^+ = \nu^+$. Analogously, $\mu^- = \nu^-$. □

Definition 2.17. The measures ν^+ and ν^- are called the *positive* and *negative variation* of ν, respectively. The *total variation* of ν is defined as $|\nu| := \nu^+ + \nu^-$.

Remark 2.18. The total variation $|\nu|$ is a well-defined positive measure. A signed measure ν is said to be *finite*, respectively, *σ-finite*, if the total variation $|\nu|$ is finite, respectively σ-finite.

Example 2.19. Let ν be a signed measure and μ a positive measure on $(X, \mathcal{A})$. If ν is given by $\nu(E) := \int_E f d\mu$ for f an extended μ-integrable function, then the total variation of ν is given by $|\nu|(E) = \nu^+(E) + \nu^-(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$ for every $E \in \mathcal{A}$.

Proof. Define $P := \big\{A \subseteq X : f\,|_A \geq 0\big\}$ and $N := \big\{A \subseteq X : f\,|_A < 0\big\}$. Clearly, $P \cap N = \varnothing$, and since for every $A \subseteq X$, either $f\,|_A \geq 0$ or $f\,|_A < 0$, one obtains that $P \cup N = X$. Now, for every $E' \in \mathcal{A}$ with $E \subseteq P$, it holds that $\nu(E) = \int_E f d\mu \geq 0$, hence P is ν-positive. Similarly, one obtains that N is ν-negative. Thus, $P \cup N = X$ is a Hahn decomposition. Note that also $P' := \big\{A \subseteq X : f\,|_A > 0\big\}$ and $N' := \big\{A \subseteq X : f\,|_A \leq 0\big\}$ construct a Hahn decomposition, since $P \triangle P' = \big\{A \subseteq X : f\,|_A = 0\big\}$ is null for ν, and ν-null sets can be transferred from P to N and vice versa.

Now, let $\nu_1(E) := \int_E f^+ d\mu$ and $\nu_2(E) := \int_E f^- d\mu$ for every $E \in \mathcal{A}$. Note that ν_1 and ν_2 are well-defined, positive measures on $(X, \mathcal{A})$, since f^+ and f^- are positive, measurable functions. This definition of ν_1 and ν_2 implies that $\nu_1(N) = \int_N f^+ d\mu = 0$ and similarly, $\nu_2(P) = \int_P f^- d\mu = 0$. Thus, $\nu_1 \perp \nu_2$. Moreover, $\nu(E) = \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \nu_1(E) - \nu_2(E)$, for every $E \in \mathcal{A}$, entailing exactly that ν_1 is the positive variation of ν, and ν_2 the negative variation of ν. Thus, for every $E \in \mathcal{A}$ the total variation of ν is given by

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu.$$

□

The following proposition is based on exercises 2, 4, 5 and 7 (Folland 1999, section 3.1).

Proposition 2.20. *Let ν be a signed measure on $(X, \mathcal{A})$, and let $E \in \mathcal{A}$. Then the following properties hold.*

(i) *E is ν-null if and only if $|\nu|(E) = 0$.*

(ii) *If λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \mu^+$ and $\mu \geq \nu^-$.*

(iii) *If ν_1 and ν_2 are signed measures that both omit ∞ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.*

(iv) *$\nu^+(E) = \sup\big\{\nu(F) : F \in \mathcal{A}, F \subseteq E\big\}$ and $\nu^-(E) = -\inf\big\{\nu(F) : F \in \mathcal{A}, F \subseteq E\big\}$.*

(v) *$|\nu|(E) = \sup\big\{\sum_{i=1}^n |\nu(E_i)| : n \in \mathbb{N}, E_1, ..., E_n$ disjoint and $\bigcup_{i=1}^n E_i = E\big\}$.*

Proof. (i) Suppose first that $|\nu|(E) = \nu^+(E)+\nu^-(E) = 0$. Then for every $A \subseteq E$, monotonicity of positive measures yields that $0 = |\nu|(A) = \nu^+(A)+\nu^-(A)$ implying that $\nu^+(A) = \nu^-(A) = 0$. Hence, $\nu(A) = \nu^+(A) - \nu^-(A) = 0$, i.e., E is ν-null as wanted.

Conversely, suppose E is ν-null. Let $P \cupdot N = X$ be a Hahn decomposition of ν. Then $E = (E \cap P) \cupdot (E \cap N)$, and since $E \cap P \subseteq E$, one obtains that $0 = \nu(E \cap P) = \nu^+(E)$, and likewise, $\nu^-(E \cap N) = 0$. Hence, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ as wanted.

(ii) Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν such that $\nu^+ \perp \nu^-$, and let $X = P \cupdot N$ be a Hahn decomposition for ν. By assumption, $\nu = \nu^+ - \nu^- = \lambda - \mu$. Thus, $\nu^+(E) = \nu^+(E\cap P) = \nu(E\cap P) = \lambda(E\cap P) - \mu(E\cap P)$, for every $E \in \mathcal{A}$, hence $\lambda(E) \geq \lambda(E\cap P) = \nu^+(E) + \mu(E\cap P) \geq \nu^+(E)$. Similarly, it is obtained that $\mu \geq \nu^-$.

(iii) Assume without loss of generality that ν_1, ν_2 both omit the value $+\infty$. Let $\nu_1 = \nu_1^+ - \nu_1^-$, and $\nu_2 = \nu_2^+ - \nu_2^-$ be the Jordan decompositions for ν_1 and ν_2 respectively. By assumption, ν_1^+, ν_2^+ are finite positive measures. Thus, $\nu_1 + \nu_2 = \nu_1^+ + \nu_2^+ - (\nu_1^- + \nu_2^-)$ is a well-defined signed measure. Now, let $\nu_1 + \nu_2 = (\nu_1 + \nu_2)^+ - (\nu_1 + \nu_2)^-$ be the Jordan decomposition for $\nu_1 + \nu_2$. Then by (ii) $\nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$, and $\nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$, and thus,

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- = |\nu_1| + |\nu_2|.$$

(iv) Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition for ν. Then for every $F \in \mathcal{A}$ with $F \subseteq E$, it holds that $\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E)$, hence $\nu^+(E) \geq \sup\{\nu(F) : F \in \mathcal{A}, F \subseteq E\}$. For the other inequality, let $X = P \cupdot N$ be a Hahn decomposition for ν. Then $\nu^+(E) = \nu(E\cap P)$ for every $E \in \mathcal{A}$. Since $E \cap P \subseteq E$, it follows that $\nu^+(E) \leq \sup\{\nu(F) : F \in \mathcal{A}, F \subseteq E\}$. Thus, $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{A}, F \subseteq E\}$. The proof follows analogously for ν^-.

(v) As before, let $X = P \cupdot N$ be a Hahn decomposition for ν. For every $E \in \mathcal{A}$

$$\begin{aligned} |\nu|(E) &= \nu^+(E) + \nu^-(E) = \nu^+(E \cap P) + \nu^-(E \cap N) \\ &= |\nu(E \cap P)| + |\nu(E \cap N)|, \end{aligned}$$

since $\nu^+(E \cap N) = \nu^-(E \cap P) = 0$. This yields the inequality, $|\nu|(E) \leq \sup\{\sum_{i=1}^n |\nu(E_i)| : n \in \mathbb{N}, E_1, ..., E_n \text{ disjoint and } \dot\bigcup_{i=1}^n E_i = E\}$. For the reverse inequality, note that for every $E = \dot\bigcup_{i=1}^n E_i$, with $E_1, ..., E_n$ disjoint, it holds that

$$\begin{aligned} \sum_{i=1}^n |\nu(E_i)| &= \sum_{i=1}^n |\nu^+(E_i) - \nu^-(E_i)| \leq \sum_{i=1}^n |\nu^+(E_i)| + |\nu^-(E_i)| \\ &= \sum_{i=1}^n |\nu|(E_i) = |\nu|(\dot\bigcup_{i=1}^n E_i) = |\nu|(E). \end{aligned}$$

Hence, also $|\nu|(E) \geq \sup\{\sum_{i=1}^n |\nu(E_i)| : n \in \mathbb{N}, E_1, ..., E_n \text{ disjoint and } \dot\bigcup_{i=1}^n E_i = E\}$, thus the equality follows. □

2.4 Integration and absolute continuity of signed measures

In this section, integration with respect to signed measures is introduced, as well as the notion of absolutely continuity of signed measures with respect to positive measures.

Definition 2.21. Let ν be a signed measure. Then *integration with respect to* ν is defined as

$$\int f d\nu := \int f d\nu^+ - \int f d\nu^-,$$

for $f \in L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$, where $\int f d\nu^\pm = \int \Re(f) d\nu^\pm + i \int \Im(f) d\nu^\pm$.

The following proposition is based on exercise 3 (Folland 1999, section 3.1).

Proposition 2.22. *Let ν be a signed measure on $(X, \mathcal{A})$. Then the following properties hold.*

(i) $L^1(\nu) = L^1(|\nu|)$.

(ii) Let $f \in L^1(\nu)$. *Then* $\left|\int f d\nu\right| \leq \int |f| \, d|\nu|$.

(iii) Let $E \in \mathcal{A}$. *Then* $|\nu|(E) = \sup\left\{|\int_E f d\nu| : |f| \leq 1\right\}$.

Proof. (i) Since the total variation $|\nu|$ is a positive measure, it is clear that $f \in L^1(|\nu|)$ if and only if $\int |f| d|\nu| := \int |f| d\nu^+ + \int |f| d\nu^- < \infty$, i.e., $\int |f| d\nu^+ < \infty$ and $\int |f| d\nu^- < \infty$. Thus, $f \in L^1(|\nu|)$ if and only if $f \in L^1(\nu^+)$ and $f \in L^1(\nu^-)$, equivalently $f \in L^1(\nu^+) \cap L^1(\nu^-)$. Hence $L^1(|\nu|) = L^1(\nu)$ as wanted.

(ii) Let $f \in L^1(\nu)$. Then

$$\begin{aligned}\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.\end{aligned}$$

(iii) Let $E \in \mathcal{A}$. Then for every measurable function f with $|f| \leq 1$, it holds that

$$|\nu|(E) = \int_E d|\nu| \geq \int_E |f| d|\nu| \geq \left|\int_E f d\nu\right|,$$

hence $|\nu|(E) \geq \sup\left\{|\int_E f d\nu| : |f| \leq 1\right\}$. For the other inequality, let $X = P \cup N$ be a Hahn decomposition for ν. Then for every $E \in \mathcal{A}$,

$$\begin{aligned}|\nu|(E) &= \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) \\ &= \int_E \mathbb{1}_P d\nu - \int_E \mathbb{1}_N d\nu = \int_E \mathbb{1}_P - \mathbb{1}_N d\nu = \left|\int_E \mathbb{1}_P - \mathbb{1}_N d\nu\right|,\end{aligned}$$

and since, $|\mathbb{1}_P - \mathbb{1}_N| \leq 1$, the inequality $|\nu|(E) \leq \sup\left\{|\int_E f d\nu| : |f| \leq 1\right\}$ is obtained. □

Definition 2.23. Let ν be a signed measure and μ a positive measure on $(X, \mathcal{A})$. Then ν is *absolutely continuous with respect to* μ if $\nu(E) = 0$, whenever $\mu(E) = 0$ for every $E \in \mathcal{A}$. This is denoted $\nu \ll \mu$.

The following proposition is based on exercises 8 and 9 (Folland 1999, section 3.2).

Proposition 2.24. *Let let ν be a signed measure and μ a positive measure on $(X, \mathcal{A})$. Then the following properties hold.*

(i) $\nu \perp \mu$ *if and only if* $|\nu| \perp \mu$ *if and only if* $\nu^+ \perp \mu$ *and* $\nu^- \perp \mu$.

(ii) $\nu \ll \mu$ *if and only if* $|\nu| \ll \mu$ *if and only if* $\nu^+ \ll \mu$ *and* $\nu^- \ll \mu$.

(iii) Let $\{\nu_i\}_{i\geq 1}$ *be a sequence of positive measures. If* $\nu_i \perp \mu$ *for all* $i \in \mathbb{N}$, *then* $\sum_{i=1}^{\infty} \nu_i \perp \mu$, *and if* $\nu_i \ll \mu$ *for all* $i \in \mathbb{N}$, *then* $\sum_{i=1}^{\infty} \nu_i \ll \mu$.

Proof. (i) Assume $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ with $X = E \cup F$ such that $\mu(E) = 0$ and F is a ν-null set. Then by Proposition 2.20 (i), $|\nu|(F) = 0$, hence $|\nu| \perp \mu$.

Assume $|\nu| \perp \mu$, such that $\mu(E) = |\nu|(F) = 0$. Then

$$0 = |\nu|(F) = \nu^+(F) + \nu^-(F),$$

so $\nu^+(F) = 0$ and $\nu^-(F) = 0$, since ν^+,ν^- are positive measures. Hence, $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Assume $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then there exist E_1, F_1 with $X = E_1 \uplus F_1$ such that $\mu(E_1) = \nu^+(F_1) = 0$, and there exist E_2, F_2 with $X = E_2 \uplus F_2$ such that $\mu(E_2) = \nu^-(F_2) = 0$. Since countable unions of null sets are null sets, $\mu(E_1 \cup E_2) = 0$. Since $F_1 \cap F_2 \subset F_1, F_2$,

$$\nu(F_1 \cap F_2) = \nu^+(F_1 \cap F_2) - \nu^-(F_1 \cap F_2) = 0.$$

Now, clearly $X = (E_1 \cup E_2) \uplus (F_1 \cap F_2)$. Hence $\nu \perp \mu$, which completes the proof.

(ii) Assume $\nu \ll \mu$. If $\mu(E) = 0$ for some $E \in \mathcal{A}$, then $\nu(E) = 0$. Now, let $X = P \uplus N$ be a Hahn decomposition for ν, and let $\nu = \nu^+ - \nu^-$ be its Jordan decomposition. Suppose $\mu(E) = 0$. Then $E \cap P, E \cap N \subseteq E$, hence $\mu(E \cap P) = \mu(E \cap N) = 0$, which yields

$$0 = \nu(E \cap P) = \nu^+(E \cap P) = \nu^+(E),$$

and similarly for ν^-. Hence, $\nu^+(E) = \nu^-(E) = 0$, and thus, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Clearly, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ is equivalent to $|\nu| \ll \mu$, since

$$|\nu|(E) = \nu^+(E) + \nu^-(E),$$

hence $|\nu|(E) = 0$ if and only if $\nu^+(E) = \nu^-(E) = 0$.

Now, assume $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Suppose $\mu(E) = 0$ for some $E \in \mathcal{A}$. Then

$$\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0,$$

hence $\nu \ll \mu$, which completes the proof.

(iii) Let $\{\nu_i\}_{i \geq 1}$ be a sequence of positive measures. Assume $\nu_i \perp \mu$ for every $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, there exist $E_i, F_i \in \mathcal{A}$ such that $X = E_i \uplus F_i$ with $\mu(E_i) = 0$ and F_i a ν_i-null set. Define $E := \bigcup_{i=1}^{\infty} E_i$ and $F := \bigcap_{i=1}^{\infty} F_i$. Note that $F \subseteq F_i$ for each $i \in \mathbb{N}$, thus $E \cap F = \varnothing$. Then, also $X = E \uplus F$. And $\mu(E) = 0$, since the countable union of null sets a null set by Lemma 2.10. Also, $F \subseteq F_i$ yields that F is a ν_i-null set for each $i \in \mathbb{N}$. Thus, F is a $\sum_{i=1}^{\infty} \nu_i$-null sets. Hence, $\sum_{i=1}^{\infty} \nu_i \perp \mu$ as wanted.

Assume $\nu_i \ll \mu$ for each $i \in \mathbb{N}$. Suppose $\mu(E) = 0$ for some $E \in \mathcal{A}$. Then by assumption, $\nu_i(E) = 0$ for each $i \in \mathbb{N}$. Hence, $\sum_{i=1}^{\infty} \nu_i(E) = 0$, and thus $\sum_{i=1}^{\infty} \nu_i \ll \mu$. □

Remark 2.25. One may think of absolute continuity as being the antithesis of mutual singularity, as $\nu \perp \mu$ and $\nu \ll \mu$ yields that $\nu = 0$: Assume $\nu \perp \mu$ and $\nu \ll \mu$. Let $X = E \uplus F$. Then $\mu \perp \nu$ is equivalent to $\mu \perp |\nu|$ by the preceding proposition. Thus, suppose $\mu(E) = |\nu|(F) = 0$. Then $\mu(E) = 0$ yields that also $|\nu|(E) = 0$, implying exactly that $\nu = 0$.

The motivation behind the notion 'absolute continuity' is not immediately obvious, however it becomes more clear in the following theorem.

Theorem 2.26. *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{A})$. Then*

$$\nu \ll \mu \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : \mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon.$$

Proof. Note that by Proposition 2.24 (ii), $\nu \ll \mu$ if and only $|\nu| \ll \mu$, and for every $E \in \mathcal{A}$

$$|\nu(E)| = |\nu^+(E) + \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E).$$

Thus, it suffices to assume that $\nu = |\nu|$, i.e., ν is a positive measure.

Assume first that $\mu(E) = 0$, and given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$, whenever $\mu(E) < \delta$. Then $\mu(E) = 0 < \delta$ for every $\delta > 0$, so by assumption, $|\nu(E)| < \varepsilon$ for every $\varepsilon > 0$, and thus, $|\nu(E)| = 0$. Hence, $|\nu| \ll \mu$.

Conversely, assume by contradiction that there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $E_n \in \mathcal{A}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$. Now, define $F_k := \bigcup_{n=k}^{\infty} E_n$ and $F := \bigcap_{k=1}^{\infty} F_k$. Then $F \subseteq F_k$ for each $k \in \mathbb{N}$, hence

$$\mu(F) \leq \mu(F_k) \leq \sum_{n=k}^{\infty} \mu(E_n) < \sum_{n=k}^{\infty} 2^{-n} = 2^{1-k},$$

which yields that $\mu(F) < 2^{1-k}$ for every $k \in \mathbb{N}$, i.e., $\mu(F) = 0$. But $\nu(F_k) \geq \varepsilon$ for every $k \in \mathbb{N}$, hence by Proposition 2.7 (ii) (continuity from above),

$$\nu(F) = \nu(\bigcap_{k=1}^{\infty} F_k) = \lim_{k\to\infty} \nu(F_k) \geq \varepsilon,$$

since $\{F_k\}_{k\geq 1}$ is a decreasing sequence with $\nu(F_1)$ finite, as ν assumed to be finite. Thus, $\nu(F)$ cannot be equal to zero, which yields that ν cannot be absolutely continuous with respect to μ, which is a contradiction. □

Corollary 2.27. *Let μ be a positive measure, and let $f \in L^1(\mu)$. Then*

$$\forall \varepsilon > 0 \exists \delta > 0 : \mu(E) < \delta \Rightarrow \Big| \int_E f d\mu \Big| < \varepsilon.$$

Proof. Let $f \in L^1(\mu)$. Then $\int f d\mu := \int \Re(f) d\mu + i \int \Im(f) d\mu$, and $f \in L^1(\mu)$ if and only if $\Re(f) \in L^1(\mu)$ and $\Im(f) \in L^1(\mu)$. Define ν_1, ν_2 signed measures by

$$\nu_1(E) := \int_E \Re(f) d\mu, \quad \text{and} \quad \nu_2(E) := \int_E \Im(f) d\mu$$

for every $E \in \mathcal{A}$. Now clearly, $\mu(E) = 0$ implies that

$$\nu_1(E) := \int_E \Re(f) d\mu = 0, \quad \text{and} \quad \nu_2(E) := \int_E \Im(f) d\mu = 0,$$

hence $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. Note that ν_1 is finite if and only if $\Re(f) \in L^1(\mu)$, since for every $E \in \mathcal{A}$,

$$|\nu_1|(E) = \int_E \Re(f)^+ d\mu + \int_E \Re(f)^- d\mu = \int_E |\Re(f)| d\mu.$$

Similarly, for ν_2. Thus, ν_1, ν_2 are both finite. Then by the preceding theorem, given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|\nu_1(E)| = \Big| \int \Re(f) d\mu \Big| < \varepsilon,$$

whenever $\mu(E) < \delta_1$, and

$$|\nu_2(E)| = \Big| \int \Im(f) d\mu \Big| < \varepsilon,$$

whenever $\mu(E) < \delta_2$. Let $\delta := \min\{\delta_1, \delta_2\}$. Then $\mu(E) < \delta$ implies that

$$\Big| \int_E f d\mu \Big| = \Big| \int_E \Re(f) d\mu + i \int_E \Im(f) d\mu \Big| \leq \Big| \int_E \Re(f) d\mu \Big| + \Big| \int_E \Im(f) d\mu \Big| < 2\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, this completes the proof. □

This concludes the preliminary theory of signed measures. With this theory presented, the following chapter proceeds with the theory of complex measures, which utilizes signed measures.

3 Complex measures

The purpose of this chapter is to introduce complex measures, as well as the elementary theory behind. In particular, the concepts from the theory of signed measures are introduced according to complex measures. The theory of this chapter is based on Folland 1999 (section 3.3) and Rudin 1987 (chapter 6).

3.1 Complex measures and the total variation

Definition 3.1. Let $(X, \mathcal{A})$ be a measurable space. A *complex measure* is a map $\lambda: \mathcal{A} \to \mathbb{C}$ such that

(i) $\lambda(\varnothing) = 0$.

(ii) $\lambda(\dot\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \lambda(E_i)$ for every sequence of disjoint sets $\{E_i\}_{i\geq 1} \subseteq \mathcal{A}$.

Example 3.2. Note that complex measures are finite, hence every finite positive measure is in particular a complex measure. Thus, also the measure, λ, defined by $\lambda(E) = \int_E f d\mu$ for every $E \in \mathcal{A}$ for some positive measure μ and some $f \in L^1(\mu)$ is a complex measure.

Remark 3.3. The finiteness of complex measures requires the series from (ii) to be convergent. Moreover, permutations of the subscript do not change the value, hence every reordering of the series converge implying that the series is in fact absolutely convergent.

Definition 3.4. The *total variation* of a complex measure λ is defined by

$$|\lambda|(E) := \sup\Big\{ \sum_{i=1}^{\infty} |\lambda(E_i)| : E_1, E_2, ... \text{ disjoint, and } \dot\bigcup_{i=1}^{\infty} E_i = E \Big\}.$$

Remark 3.5. As is proven in the following, the total variation $|\lambda|$ is in fact a positive measure on $(X, \mathcal{A})$, and not only is it a measure, it is also finite. The proof of finiteness of the total variation requires a lemma.

Lemma 3.6. *Let $z_1, ..., z_N \in \mathbb{C}$. Then there exists $S \subseteq \{1, ..., N\}$ such that*

$$\Big|\sum_{k\in S} z_k\Big| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

Proof. Let $z_k = |z_k| e^{i\alpha_k}$ for $k = 1, ..., N$ and $N \in \mathbb{N}$. Let $S_\theta := \big\{k \in \{1, ..., N\} : \cos(\alpha_k - \theta) > 0\big\}$ for $\theta \in [-\pi, \pi]$. Then $|e^{-i\theta}| = 1$, and thus

$$\begin{aligned}\Big|\sum_{k\in S_\theta} z_k\Big| &= \Big|\sum_{k\in S_\theta} |z_k| e^{i(\alpha_k - \theta)}\Big| \geq \Re\Big(\sum_{k\in S_\theta} |z_k| e^{i(\alpha_k-\theta)}\Big) \\ &= \sum_{k\in S_\theta} |z_k| \cos(\alpha_k - \theta) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta),\end{aligned}$$

where $\cos^+(x) := \max\{\cos(x), 0\}$. Now, choose $\theta_0 \in [-\pi, \pi]$ to maximize the last sum, and let $S := S_\theta$. This sum is bigger than or equal to the average value, hence

$$\begin{aligned}\Big|\sum_{k\in S} z_k\Big| \geq \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta_0) &\geq \sum_{k=1}^{N} |z_k| \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^+(\alpha - \theta) d\theta \\ &= \sum_{k=1}^{N} |z_k| \frac{1}{2\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^+(\alpha - \theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.\end{aligned}$$

□

Proposition 3.7. *The total variation $|\lambda|$ of a complex measure λ on $(X, \mathcal{A})$ is a finite positive measure on $(X, \mathcal{A})$.*

Proof. Note that the only partition of $\varnothing \in \mathcal{A}$ is $\dot\bigcup_{i=1}^{\infty} \varnothing$, hence $|\lambda|(\varnothing) = \sup\left\{ \sum_{i=1}^{\infty} |\lambda(\varnothing)|\right\} = 0$ is clear, since $\lambda(\varnothing) = 0$. Now, it is proven that $|\lambda|$ is countably additive. Let $E \in \mathcal{A}$ be given. Let $\{E_i\}_{i\geq 1}$ be a partition of E such that $E = \dot\bigcup_{i=1}^{\infty} E_i$. For every $i \in \mathbb{N}$, one may choose $a_i \in \mathbb{R}$ such that $a_i < |\lambda|(E_i)$. Then for each E_i, there is a partition $\{A_{ij}\}_{j\geq 1}$ such that

$$a_i < \sum_{j=1}^{\infty} |\lambda(A_{ij})|.$$

Now, $\{A_{ij}\}_{j\geq 1}$ is a partition of E_i, so $\{A_{ij}\}_{i,j\geq 1}$ is in particular a partition of E, hence

$$\sum_{i=1}^{\infty} a_i \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\lambda(A_{ij})| \leq |\lambda|(E).$$

Taking the supremum over all possible choices of $\{a_i\}_{i\geq 1}$ thus yields that $\sum_{i=1}^{\infty} |\lambda|(E_i) \leq |\lambda|(E)$. It then suffices to show that also $\sum_{i=1}^{\infty} |\lambda|(E_i) \geq |\lambda|(E)$. Let $\{E'_j\}_{j\geq 1}$ be an arbitrary partition of E. Then for any fixed $i \in \mathbb{N}$, $\{E'_j \cap E_i\}_{j\geq 1}$ satisfies that

$$\dot\bigcup_{j=1}^{\infty} (E'_j \cap E_i) = \dot\bigcup_{j=1}^{\infty} E'_j \cap E_i = E \cap E_i = E_i,$$

thus $\{E'_j \cap E_i\}_{j\geq 1}$ is a partition of E_i. Then

$$\begin{aligned}\sum_{j=1}^{\infty} |\lambda(E'_j)| &= \sum_{j=1}^{\infty} \Big|\sum_{i=1}^{\infty} \lambda(E'_j \cap E_i)\Big| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\lambda(E'_j \cap E_i)| \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\lambda(E'_j \cap E_i)| \leq \sum_{i=1}^{\infty} |\lambda|(E_i).\end{aligned}$$

Now, since the partition $\{E'_j\}_{j\geq 1}$ of E was arbitrarily chosen, $\sum_{j=1}^{\infty} |\lambda(E'_j)| \leq \sum_{i=1}^{\infty} |\lambda|(E_i)$ holds for any partition of E, and thus $|\lambda|(E) \leq \sum_{i=1}^{\infty} |\lambda|(E_i)$, which completes the proof of countable additivity of $|\lambda|$. Thus, $|\lambda|$ is indeed a positive measure. To prove that $|\lambda|$ is finite, suppose by contradiction that there exists an $E \in \mathcal{A}$ such that $|\lambda|(E) = \infty$. Let $a := \pi + \pi|\lambda(E)|$. Since $\infty = |\lambda|(E) > a$, a partition $\{E_i\}_{i\geq 1}$ can be chosen such that $\sum_{i=1}^{N} |\lambda(E_i)| > a$ for some $N \in \mathbb{N}$. Now, $\lambda(E_i) \in \mathbb{C}$, hence by Lemma 3.6 there is a set $S \subseteq \{1, \ldots, N\}$ such that for $A := \dot\bigcup_{i\in S} E_i$,

$$|\lambda(A)| = \Big|\sum_{i\in S} \lambda(E_i)\Big| \geq \frac{1}{\pi} \sum_{i\in S} |\lambda(E_i)| > \frac{a}{\pi} > 1.$$

Now, let $B := E \setminus A$. Then $|\lambda(B)| = |\lambda(E) - \lambda(A)| \geq |\lambda(A)| - |\lambda(E)| > \frac{a}{\pi} - |\lambda(E)| = 1$. Thus, $E = A \,\dot\cup\, B$ and $|\lambda(A)|, |\lambda(B)| > 1$. Since $|\lambda|$ is a positive measure, and in particular holds the property of countable additivity, $\infty = |\lambda|(E) = |\lambda|(A \,\dot\cup\, B) = |\lambda|(A) + |\lambda(B)|$, hence at least one of $|\lambda|(A)$ and $|\lambda(B)|$ must assume the value ∞. Therefore, if $|\lambda|(X) = \infty$, there exist $A_1, B_1 \in \mathcal{A}$ such that $X = A_1 \,\dot\cup\, B_1$ where $|\lambda(A_1)|, |\lambda(B_1)| > 1$ and $|\lambda|(B_1) = \infty$ without loss of generality. So by this argument, also $B_1 = A_2 \,\dot\cup\, B_2$ for some $A_2, B_2 \in \mathcal{A}$ with $|\lambda(A_2)|, |\lambda(B_2)| > 1$ and $|\lambda|(B_2) = \infty$. Continuing this way, one obtains a disjoint collection of countably infinitely many sets $\{A_i\}_{i\geq 1}$ with $|\lambda(A_i)| > 1$ for every $i \in \mathbb{N}$. Now, since λ is a complex measure, thus, in particular, is countably additive,

$$\lambda\Big(\dot\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \lambda(A_i),$$

where $\lambda(\dot\bigcup_{i=1}^{\infty} A_i) \in \mathbb{C}$, so the series must converge. But $|\lambda(A_i)| > 1$ for every $i \in \mathbb{N}$ implies that $\lambda(A_i) \nrightarrow 0$ as $i \to \infty$, so the series cannot converge, thus a contradiction. Hence, it must hold that $|\lambda|(X) < \infty$ as wanted. $\square$

Remark 3.8. Note that $|\lambda(E)| \leq |\lambda|(E)$ for every $E \in \mathcal{A}$. Thus, the fact that $|\lambda|$ is finite, yields that $|\lambda(E)| \leq |\lambda|(E) \leq |\lambda|(X) < \infty$ for every $E \in \mathcal{A}$, hence the range of λ is bounded. This is sometimes referred to as λ being of *bounded variation*.

Definition 3.9. Let λ and μ be complex measures on $(X, \mathcal{A})$. Define

$$(\lambda + \mu)(E) := \lambda(E) + \mu(E)$$
$$(c\lambda)(E) := c(\lambda(E))$$

for every $E \in \mathcal{A}$ and $c \in \mathbb{C}$.

Remark 3.10. With the definition above, $\lambda + \mu$ and $c\lambda$ are in fact complex measures, and thus, the space $C := \{\lambda : \mathcal{A} \to \mathbb{C} \text{ complex measure}\}$ is a vector space. This is in particular a result of complex measures being finite. Moreover, if one defines $\|\lambda\| := |\lambda|(X)$, this forms a well-defined norm, hence C becomes a normed vector space.

3.2 Mutual singularity and absolute continuity

This section is devoted to defining the concepts known from the theory of signed measures according to complex measure. As it turns out, there is a natural way of doing this.

Notation. For a complex measure λ, the real and imaginary parts are denoted $\lambda_{\mathfrak{R}}$ and $\lambda_{\mathfrak{I}}$, respectively. Thus, every complex measure can be written (uniquely) as the decomposition, $\lambda = \lambda_{\mathfrak{R}} + i\lambda_{\mathfrak{I}}$, where $\lambda_{\mathfrak{R}}, \lambda_{\mathfrak{I}}$ are finite signed measures.

From this notation, the concepts of signed measures generalize easily:

Definition 3.11. Let λ be a complex measure. Then *integration with respect to* λ is defined by

$$\int f d\lambda := \int f d\lambda_{\mathfrak{R}} + i \int f d\lambda_{\mathfrak{I}}$$

for $f \in L^1(\lambda) := L^1(\lambda_{\mathfrak{R}}) \cap L^1(\lambda_{\mathfrak{I}})$.

Definition 3.12. Let λ and μ be complex measures on $(X, \mathcal{A})$. Then λ is *singular with respect to* μ, denoted $\lambda \perp \mu$, if $\lambda_i \perp \mu_j$ for every $i, j = \mathfrak{R}, \mathfrak{I}$.

Definition 3.13. Let λ be a complex measure and μ a positive measure on $(X, \mathcal{A})$. Then λ is *absolutely continuous with respect to* μ, denoted $\lambda \ll \mu$, if $\lambda_{\mathfrak{R}} \ll \mu$, and $\lambda_{\mathfrak{I}} \ll \mu$.

Proposition 3.14. *Let λ_1, λ_2 be complex measures and μ a positive measure on $(X, \mathcal{A})$. Then the following properties hold.*

(i) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(ii) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

(iii) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.

(iv) If $\lambda \ll \mu$ then $|\lambda| \ll \mu$.

(v) If $\lambda_1 \ll \mu$ and $\lambda_1 \perp \mu$, then $\lambda_1 = 0$.

Proof. The proofs are similar to the proofs for signed measures. □

Thus, the elementary theory of complex measures is presented, and the following chapter proceeds with the proof of The Lebesgue-Radon-Nikodym Theorem for signed and complex measures, respectively.

4 The Lebesgue-Radon-Nikodym Theorem

The purpose of this chapter is to prove the Lebesgue-Radon-Nikodym Theorem, which provides a complete picture of the relationship between a signed or complex measure and a given positive measure. The theory of this chapter is based on Folland 1999 (section 3.2).

4.1 The proof of the Lebesgue-Radon-Nikodym Theorem

In this section, the Lebesgue-Radon-Nikodym Theorem is proved for signed and complex measures, respectively. This commences with a lemma needed in order to prove the theorem for signed measures. From here on, the theorem is proven for complex measures.

Notation. Let ν be a signed measure on $(X, \mathcal{A})$ defined by $\nu(E) := \int_E f d\mu$ for a positive measure μ and an extended μ-integrable function, f. This relationship between ν and μ is from now on be described with the notation $d\nu = f d\mu$.

Lemma 4.1. *Let ν and μ be finite positive measures on $(X, \mathcal{A})$. Then either $\nu \perp \mu$, or there exist $\varepsilon > 0$ and $E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\nu(A) \geq \varepsilon\mu(A)$ for every $A \in \mathcal{A}$ with $A \subseteq E$, i.e., E is a positive set with respect to $\nu - \varepsilon\mu$.*

Proof. For each $n \in \mathbb{N}$, let $X = P_n \uplus N_n$ be a Hahn decomposition for the signed measure $\nu - n^{-1}\mu$. Define $P := \bigcup_{n=1}^{\infty} P_n$ and $N := \bigcap_{n=1}^{\infty} N_n = X \setminus P$. Then N is $(\nu - n^{-1}\mu)$-negative for every $n \in \mathbb{N}$. Hence, $0 \leq \nu(N) \leq n^{-1}\mu(N)$, and since it holds for every $n \in \mathbb{N}$, this implies that $\nu(N) = 0$. Now, either $\mu(P) = 0$ or $\mu(P) > 0$, as μ is a positive measure. If $\mu(P) = 0$, then $\mu(P) = \nu(X \setminus P) = \nu(N) = 0$ with $X = P \uplus N$, and thus $\nu \perp \mu$. Conversely, if $\mu(P) > 0$, then $0 < \mu(P) = \mu(\bigcup_{n=1}^{\infty} P_n) \leq \sum_{n=1}^{\infty} \mu(P_n)$, hence $\mu(P_n) > 0$ for some $n \in \mathbb{N}$, and P_n is a $(\nu - n^{-1}\mu)$-positive set. This completes the proof with $\varepsilon = n^{-1}$ and $E = P_n$. $\square$

Theorem 4.2 (The Lebesgue-Radon-Nikodym Theorem for signed measures)**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures ψ and ρ on $(X, \mathcal{A})$ such that*

$$\psi \perp \mu, \qquad \rho \ll \mu, \qquad \nu = \psi + \rho.$$

Moreover, $d\rho = f d\mu$ for an extended μ-integrable function $f: X \to \mathbb{R}$, which is unique μ-a.e.

Proof. Case I: Suppose ν and μ are finite positive measures. Define a set $\mathcal{F}$ by

$$\mathcal{F} := \Big\{ f: X \to [0, \infty] \text{ measurable} : \int_E f d\mu \leq \nu(E), \ \forall E \in \mathcal{A} \Big\}.$$

Note that $0 \in \mathcal{F}$, so $\mathcal{F}$ is non-empty. Let $f, g \in \mathcal{F}$, and define a function $h: X \to [0, \infty]$ by $h(x) := \max\{f(x), g(x)\}$. Then h is measurable. Let $A := \{x \in X : f(x) > g(x)\}$. Then

$$\int_E h d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E),$$

hence $h \in \mathcal{F}$. Now, let $a := \sup \{ \int_X f d\mu : f \in \mathcal{F} \}$. Then $a \leq \nu(X) < \infty$, since ν is finite by assumption. Choose a sequence $\{f_n\}_{n \geq 1}$ of functions in $\mathcal{F}$ such that $\lim_{n \to \infty} \int_X f_n = a$. Define g_n by $g_n(x) = \max\{f_1(x), ..., f_n(x)\}$ for each $n \in \mathbb{N}$. Then $g_n \in \mathcal{F}$ by the previous argument. Moreover, $g_n \geq f_n$ for every $n \in \mathbb{N}$, so $\int_X g_n d\mu \geq \int_X f_n d\mu$, and thus, $\lim_{n \to \infty} \int_X g_n d\mu = a$. Let f be a function defined by $f(x) := \sup_n f_n(x)$. Since g_n converges to f pointwise as $n \to \infty$, i.e., $\lim_{n \to \infty} g_n = \sup_n f_n = f$, and $g_n \leq g_{n+1}$ for every n, the Monotone Convergence Theorem (Theorem 2.14 Folland 1999) yields that

$$\int_X f d\mu = \int_X \lim_{n \to \infty} g_n d\mu = \lim_{n \to \infty} \int_X g_n d\mu = a.$$

Furthermore, $f \in \mathcal{F}$, since $g_n \in \mathcal{F}$ for every $n \in \mathbb{N}$. Now, $\int_X f d\mu = a < \infty$ implies that $f < \infty$ μ-a.e., so f can be taken to be real-valued everywhere, i.e., $f: X \to [0,\infty)$. Also, $f \in L^1(\mu)$, hence f is in particular extended μ-integrable, as wanted. Thus, define measures ρ and ψ by $d\rho := f d\mu$ and $d\psi := d\nu - f d\mu$, i.e., $\psi = \nu - \rho$. Note that $f \in L^1(\mu)$ with f being positive yields that $\rho(E) = \int_E f d\mu$ is finite and positive for every $E \in \mathcal{A}$, so ρ is a finite positive measure. Moreover, $f \in \mathcal{F}$ yields that ψ too is a positive measure, since $\psi(E) = \nu(E) - \int_E f d\mu \geq 0$, for every $E \in \mathcal{A}$, and ψ is finite, since ν and ρ are finite measures. It thus suffices to show that $\psi \perp \mu$. Assume by contradiction that $\psi \not\perp \mu$. Then by Lemma 4.1, there exists $\varepsilon > 0$ and $E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\psi(A) \geq \varepsilon\mu(A) = \varepsilon\mu(E \cap A)$ for every $A \subseteq E$. If $A \in \mathcal{A}$ is arbitrary, then $A = B \cup C$, where $B \subseteq E$ and $C \cap E = \varnothing$, hence by additivity of the positive measures ψ, μ,

$$\psi(A) = \psi(B) + \psi(C) \geq \psi(B) \geq \varepsilon\mu(B) = \varepsilon(\mu(B \cap E) + \mu(C \cap E)) = \varepsilon\mu(E \cap A),$$

so $\psi(A) \geq \varepsilon\mu(E \cap A)$ for every $A \in \mathcal{A}$, or equivalently, $d\nu - f d\mu = d\psi \geq \varepsilon \mathbb{1}_E d\mu$. Thus, $d\nu \geq (f + \varepsilon\mathbb{1}_E) d\mu$, implying that $\nu(A) \geq \int_A (f + \varepsilon\mathbb{1}_E) d\mu$ for every $A \in \mathcal{A}$, hence $f + \varepsilon\mathbb{1}_E \in \mathcal{F}$. But

$$\int_X f + \varepsilon\mathbb{1} d\mu = \int_X f d\mu + \int_X \varepsilon\mathbb{1}_E d\mu = a + \varepsilon\mu(E) > a,$$

since $\mu(E) > 0$, which contradicts the definition of a. Hence, it must hold that $\psi \perp \mu$, and thus, $d\nu = d\psi + f d\mu = d\psi + d\rho$, where $\psi \perp \mu$ and $\rho \ll \mu$, as wanted.

For uniqueness, suppose that also $d\nu = d\psi' + f' d\mu$, where $\psi' \perp \mu$. Then

$$d\psi - d\psi' = f' d\mu - f d\mu = (f' - f) d\mu.$$

Note that since $d\psi, d\psi'$ and $f d\mu, f' d\mu$ are finite measures, the above is a well-defined finite signed measure. By assumption $\psi \perp \mu$ and $\psi' \perp \mu$, hence following the proof of Proposition 2.24 (iii) analogously, one obtains that also $\psi - \psi' \perp \mu$. Also clearly, $(f' - f) d\mu \ll d\mu$, which means that the measure is both singular and absolutely continuous with respect to μ, hence $d\psi - d\psi' = (f' - f) d\mu = 0$. This implies exactly that $\psi = \psi'$ and $f = f'$ μ-a.e. by Theorem 2.23 (Folland 1999). This proves uniqueness of ψ and ρ, when ψ and ρ are finite.

Case II: Suppose ν and μ are σ-finite positive measures. Then X can be written as a countable disjoint union of non-empty sets, each of which has finite measure under ν and μ: By the assumption of ν being σ-finite, $X = \biguplus_{i=1}^{\infty} B_i$, where $\nu(B_i) < \infty$, and similarly by the assumption of μ being σ-finite, $X = \biguplus_{i=1}^{\infty} B_i'$, where $\mu(B_j') < \infty$, so $X = \biguplus_{i=1}^{\infty} \biguplus_{j=1}^{\infty} B_i \cap B_j'$, thus, by re-indexing one obtains that, $X = \biguplus_{i=1}^{\infty} A_i$ with $\nu(A_i), \mu(A_i) < \infty$. Now define ν_i and μ_i for every $i \geq 1$ by $\nu_i(E) := \nu(E \cap A_i)$ and $\mu_i(E) := \mu(E \cap A_i)$ for every $E \in \mathcal{A}$. Note that each ν_i and μ_i are finite positive measures. Then by the first part of this proof, each ν_i can be decomposed as $d\nu_i = d\psi_i + f_i d\mu_i$ for (unique) positive measures $d\psi_i$, and $f_i d\mu_i$, where $f_i d\mu_i \ll d\mu_i$ and $\psi_i \perp \mu_i$. Now since, $\nu_i(X \setminus A_i) = \mu_i(X \setminus A_i) = 0$,

$$\psi_i(X \setminus A_i) = \nu_i(X \setminus A_i) - \int_{X \setminus A_i} f_i d\mu_i = 0$$

and it is justified to assume that $f_i |_{X \setminus A_i} = 0$. Define $\psi := \sum_{i=1}^{\infty} \psi_i$ and $f := \sum_{i=1}^{\infty} f_i$. Then

$$\begin{aligned}\nu(E) &= \sum_{i=1}^{\infty} \nu_i(E \cap A_i) = \sum_{i=1}^{\infty} \psi_i(E \cap A_i) + \sum_{i=1}^{\infty} \int_{E \cap A_i} f_i d\mu_i \\ &= \psi(E) + \int_E \sum_{i=1}^{\infty} f_i d\mu = \psi(E) + \int_E f d\mu,\end{aligned}$$

for every $E \in \mathcal{A}$, where the third equality follows from $f |_{X \setminus A_i} = 0$ and $\mu_i(E) = \mu(E \cap A_i)$. Thus, $d\nu = d\psi + f d\mu$. Moreover, defining ρ by $d\rho := f d\mu$, ψ and ρ are σ-finite measures by construction, so Proposition 2.24 (iii) yields that $\psi = \sum_{i=1}^{\infty} \psi_i \perp \sum_{i=1}^{\infty} \mu_i = \mu$ as wanted.

For uniqueness, suppose that also $d\nu = d\psi' + f'd\mu$, where $\psi' \perp \mu$. Let $X = \dot{\bigcup}_{i=1}^{\infty} A_i$ with $\nu(A_i) < \infty$ as before. Then $d\psi + fd\mu = d\nu = d\psi' + f'd\mu$, and thus for every $A_i \in \mathcal{A}$ and $i \in \mathbb{N}$,

$$\infty > \nu(A_i) = \psi(A_i) + \int_{A_i} fd\mu = \psi'(A_i) + \int_{A_i} f'd\mu$$

which implies that $\psi(A_i) - \psi'(A_i) = \int_{A_i}(f' - f)d\mu$. Define $\kappa_i(E) := \psi(E \cap A_i) - \psi'(E \cap A_i) = \int_{E \cap A_i}(f' - f)d\mu$ for each $i \in \mathbb{N}$ and for every $E \in \mathcal{A}$. Then κ_i is a well-defined (finite) signed measure. By previous arguments, $\kappa_i \perp \mu$ and also $\kappa_i \ll \mu$, hence $\kappa_i = 0$ for every $i \in \mathbb{N}$. Then using countable additivity,

$$\begin{aligned} 0 = \sum_{i=1}^{\infty} \kappa_i(E) = \sum_{i=1}^{\infty} \psi(E \cap A_i) - \psi'(E \cap A_i) = \sum_{i=1}^{\infty} \int_{E \cap A_i} (f' - f)d\mu \\ = \psi(E) - \psi'(E) = \int_E (f' - f)d\mu \end{aligned}$$

for every $E \in \mathcal{A}$, implying that $\psi = \psi'$ and $f = f'$ μ-a.e.

To complete the proof, suppose ν is a σ-finite signed measure and μ a σ-finite positive measure. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Suppose without loss of generality that ν^+ is finite and ν^- σ-finite. Then there exist unique finite positive measures ψ^+ and ρ^+ such that $\nu^+ = \psi^+ + \rho^+$, where $\psi^+ \perp \mu$ and $\rho^+ \ll \mu$, and there exist unique σ-finite positive measures ψ^- and ρ^- such that $\nu^- = \psi^- + \rho^-$, where $\psi^- \perp \mu$ and $\rho^- \ll \mu$. Hence,

$$\nu = \nu^+ - \nu^- = \psi^+ + \rho^+ - (\psi^- + \rho^-) = \psi^+ - \psi^- + (\rho^+ - \rho^-),$$

where $\psi := \psi^+ - \psi^-$ and $\rho := \rho^+ - \rho^-$ are well-defined σ-finite signed measures. By Proposition 2.24 (i),(ii), $\psi^+ \perp \mu$ and $\psi^- \perp \mu$ implies that also $\psi \perp \mu$, and $\rho^+ \ll \mu$ and $\rho^- \ll \mu$ implies that $\rho \ll \mu$. The decomposition is unique by the uniqueness of the decompositions $\nu = \nu^+ - \nu^-$ and $\nu^+ = \psi^+ + \rho^+$ and $\nu^- = \psi^- + \rho^-$. □

Theorem 4.3 (The Lebesgue-Radon-Nikodym Theorem for complex measures)**.** *Let λ be a complex measure and μ a σ-finite positive measure on $(X, \mathcal{A})$. Then there exist unique complex measures ψ and ρ such that*

$$\psi \perp \mu, \qquad \rho \ll \mu, \qquad \lambda = \psi + \rho,$$

where $d\rho = fd\mu$ for a unique $f \in L^1(\mu)$.

Proof. The proof follows by applying Theorem 4.2 to the real and imaginary part of λ, since $\lambda = \lambda_{\mathfrak{R}} + i\lambda_{\mathfrak{I}}$, where $\lambda_{\mathfrak{R}}, \lambda_{\mathfrak{I}}$ are finite signed measures. Uniqueness follows as before. □

Remark 4.4. The noticeable difference in the Lebesgue-Radon-Nikodym Theorem for signed and complex measures is that $f \in L^1(\mu)$, when λ, thus also ρ, is a complex measure.

Definition 4.5. The decomposition of a complex or σ-finite signed measure, $\nu = \psi + \rho$, into measures that are respectively absolutely continuous and singular with respect to μ, is called the *Lebesgue decomposition* of ν with respect to μ.

4.2 The Radon-Nikodym derivative

The Lebesgue-Radon-Nikodym Theorem gives rise to an abstract notion of the derivative of a signed or complex measures with respect to a positive measure. For ease, the following results are stated and proved only for σ-finite signed measures, but generalize to complex measures by applying them to the real and imaginary parts respectively.

Theorem 4.6 (The Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{A})$ with $\nu \ll \mu$. Then $d\nu = f d\mu$ for some extended μ-integrable function $f: X \to \mathbb{R}$. Moreover, f is unique μ-a.e.*

Proof. The result follows from Theorem 4.2: Let $d\nu = d\psi + f d\mu$ be the Lebesgue decomposition of ν with respect to μ. Then $f d\mu \ll d\mu$, and by assumption $\nu \ll \mu$. Thus, $\mu(E) = 0$ implies that $\int_E f d\mu = 0$ and also $\nu(E) = 0$ for every $E \in \mathcal{A}$. Then, $\psi(E) = \nu(E) - \int_E f d\mu = 0$ for every $E \in \mathcal{A}$ with $\mu(E) = 0$. Hence, $\psi \ll \mu$. But also, $\psi \perp \mu$, implying that $\psi = 0$. Thus, $d\nu = f d\mu$ for some $f: X \to \mathbb{R}$ extended μ-integrable function, and f is unique μ-a.e. □

Definition 4.7. The class of functions equal to f μ-a.e. is called the *Radon-Nikodym derivative* of ν with respect to μ. This is denoted $d\nu = \frac{d\nu}{d\mu} d\mu$.

Nonexample 4.8. The Lebesgue decomposition and the Radon-Nikodym derivative can only be guaranteed to exist for σ-finite measures. Let μ be the counting measure and m the Lebesgue measure on $([0,1], \mathcal{B}([0,1]))$. Then $m \ll \mu$, but $dm \neq f d\mu$ for any f. Moreover, μ has no Lebesgue decomposition with respect to m.

Proof. If $\mu(E) = 0$ for some $E \in \mathcal{B}([0,1])$, then $E = \varnothing$, and hence, $m(E) = 0$. Thus, $m \ll \mu$. Assume by contradiction that $dm = f d\mu$ for some $f: [0,1] \to [0, \infty]$. Then, since the Lebesgue measure is non-atomic, $0 = m(\{x\}) = \int_{\{x\}} f(x) d\mu = f(x)$, so $f = 0$. But then $dm = 0 d\mu$, which implies that $m = 0$. This is a contradiction, hence $dm \neq f d\mu$ for any f. Now, assume by contradiction that $\mu = \psi + \rho$ with $\psi \perp m$ and $\rho \ll m$. Then for every $x \in [0,1]$ it holds that $m(\{x\}) = 0$, which implies that $\rho(\{x\}) = 0$, thus $\psi(\{x\}) = \mu(\{x\}) = 1$. Then $\psi = \mu$, hence $m \ll \mu = \psi$. Thus, $\psi \perp m$ yields that $m = 0$, but this is a contradiction. □

Remark 4.9. If ν_1 and ν_2 are σ-finite signed measures with $d\nu_1 = f_1 d\mu$ and $d\nu_2 = f_2 d\mu$, then

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

In general, this very abstract notion of a derivative can be shown to fulfil many of the known properties of derivatives; some of which are proven in the following.

Proposition 4.10. *Let ν be a σ-finite signed measure and μ, ψ σ-finite positive measures on $(X, \mathcal{A})$, such that $\nu \ll \mu$ and $\mu \ll \psi$. Then the following properties hold.*

(i) Let $g \in L^1(\mu)$. Then $g \frac{d\nu}{d\mu} \in L^1(\mu)$ and $\int_X g d\nu = \int_X g \frac{d\nu}{d\mu} d\mu$.

(ii) It holds that $\nu \ll \psi$ and $\frac{d\nu}{d\psi} = \frac{d\nu}{d\mu} \frac{d\mu}{d\psi}$, ψ-a.e.

Proof. It suffices to prove the result for σ-finite positive measures by considering the Jordan decomposition of σ-finite signed measures. The assumption that $\nu \ll \mu$ and $\mu \ll \psi$ yields that $\nu = \frac{d\nu}{d\mu} d\mu$ and $\mu = \frac{d\mu}{d\psi} d\psi$ by Theorem 4.6 (The Radon-Nikodym Theorem).

(i) Suppose $g = \mathbb{1}_E$ for some $E \in \mathcal{A}$. Then

$$\int_X \mathbb{1}_E d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_X \mathbb{1}_E \frac{d\nu}{d\mu} d\mu$$

for every $E \in \mathcal{A}$. Thus, by linearity, (i) therefore holds for every simple function u. Now, suppose g is a nonnegative measurable function. Then g is the pointwise limit of simple functions, i.e., $\lim_{n \to \infty} u_n = g$. By the Monotone Convergence Theorem (Theorem 2.14, Folland 1999),

$$\int_X g d\nu = \lim_{n \to \infty} \int_X u_n d\nu = \lim_{n \to \infty} \int_X u_n \frac{d\nu}{d\mu} d\mu = \int_X \lim_{n \to \infty} u_n \frac{d\nu}{d\mu} d\mu = \int_X g \frac{d\nu}{d\mu} d\mu.$$

Now, suppose $g \in L^1(\mu)$. Then $g = g^+ - g^-$, where g^+ and g^- are nonnegative measurable functions, hence by linearity

$$\int_X g d\nu = \int_X g^+ d\nu - \int_X g^- d\nu = \int_X g^+ \frac{d\nu}{d\mu} d\mu - \int_X g^- \frac{d\nu}{d\mu} d\mu$$
$$= \int_X (g^+ - g^-) \frac{d\nu}{d\mu} d\mu = \int_X g \frac{d\nu}{d\mu} d\mu.$$

And so if $g \in L^1(\mu)$, then also $g\frac{d\nu}{d\mu} \in L^1(\mu)$, and (i) holds.

(ii) That $\nu \ll \psi$ follows from the assumptions $\nu \ll \mu$ and $\mu \ll \psi$: If $\psi(E) = 0$ for some $E \in \mathcal{A}$, then $\mu(E) = 0$, which implies that also $\nu(E) = 0$. Hence, $\nu \ll \psi$, so by Theorem 4.6 (The Radon-Nikodym Theorem), $d\nu = \frac{d\nu}{d\psi} d\psi$. The result now follows from part (i) by replacing ν and μ with ψ and μ and letting $g = \mathbb{1}_E \frac{d\nu}{d\mu}$ for $E \in \mathcal{A}$, such that

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_X \mathbb{1}_E \frac{d\nu}{d\mu} d\mu = \int_X \mathbb{1}_E \frac{d\nu}{d\mu} \frac{d\mu}{d\psi} d\psi$$

for every $E \in \mathcal{A}$, and thus, $\frac{d\nu}{d\mu} = \frac{d\nu}{d\mu}\frac{d\mu}{d\psi}$ ψ-a.e. as wanted. □

Corollary 4.11. *If $\mu \ll \psi$ and $\psi \ll \mu$, then $\frac{d\psi}{d\mu}\frac{d\mu}{d\psi} = 1$ a.e. with respect to either μ or ψ.*

Proof. The result follows directly from Proposition 4.10, as $\mu \ll \psi$ implies that $d\mu = \frac{d\mu}{d\psi} d\psi$ by Theorem 4.6 (The Radon-Nikodym Theorem), so $\int_E 1 d\mu = \mu(E) = \int_E \frac{d\mu}{d\psi} d\psi = \int_E \frac{d\mu}{d\psi}\frac{d\psi}{d\mu} d\mu$, for every $E \in \mathcal{A}$, hence $\frac{d\mu}{d\psi}\frac{d\psi}{d\mu} = 1$ μ-a.e. Similarly, one obtains that $\frac{d\mu}{d\psi}\frac{d\psi}{d\mu} = 1$ ψ-a.e. □

Lemma 4.12. *Let ν be a signed or complex measure and μ a positive measure on $(X, \mathcal{A})$ with $\nu \ll \mu$ such that $d\nu = f d\mu$. Then the total variation of ν is given by $d|\nu| = |f| d\mu$.*

Proof. For signed measures, the equality has already been established in Example 2.19. Suppose ν is a complex measure. Since $\sum_{i=1}^\infty |\nu(E_i)| = \sum_{i=1}^\infty |\int_{E_i} f d\mu| \leq \sum_{i=1}^\infty \int_{E_i} |f| d\mu = \int_E |f| d\mu$ for every $E \in \mathcal{A}$ and every partition $\{E_i\}_{i\geq 1} \subseteq \mathcal{A}$, it is clear that $|\nu|(E) \leq \int_E |f| d\mu$. To prove the reverse inequality, let $\varepsilon > 0$ be given. By Theorem 2.26 (Folland 1999), there is a simple function $u \in L^1(\mu)$ such that $\int |f - u| d\mu < \varepsilon$. Then there exist disjoint sets $E_1, ..., E_n \in \mathcal{A}$ such that $E = \bigcup_{i=1}^n E_i$ and $u \mid_E = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ for $c_i \in \mathbb{C}$. Then

$$|\nu|(E) \geq \sum_{i=1}^n |\nu(E_i)| = \sum_{i=1}^n \left| \int_{E_i} f d\mu \right| \geq \sum_{i=1}^n \left| \int_{E_i} u d\mu \right| - \sum_{i=1}^n \left| \int_{E_i} u - f d\mu \right|$$
$$= \int_E |u| d\mu - \sum_{i=1}^n \left| \int_{E_i} u - f d\mu \right| \geq \int_E |u| d\mu - \sum_{i=1}^n \int_{E_i} |u - f| d\mu$$
$$= \int_E |u| d\mu - \int_E |u - f| d\mu \geq \int_E |u| d\mu - \varepsilon \geq \int_E |f| d\mu - 2\varepsilon,$$

with equality in line two, since $\sum_{i=1}^n \left| \int_{E_i} u d\mu \right| = \sum_{i=1}^n |c_i| \mu(E_i) = \int_E \sum_{i=1}^n |c_i| \mathbb{1}_{E_i} d\mu = \int_E |u| d\mu$, and where the last inequality follows from $\left| \int_E |u| - |f| d\mu \right| \leq \int_E \big||u| - |f|\big| d\mu \leq \int_E |u - f| d\mu \leq \varepsilon$, so $\int_E |u| - |f| d\mu \geq -\varepsilon$. Hence, $\varepsilon > 0$ being arbitrary completes the proof.

□

Remark 4.13. The preceding lemma provides an alternative definition of the total variation measure, namely the measure that satisfies that if $d\nu = f d\mu$, then $d|\nu| = |f| d\mu$. With this result, it can easily be verified that for a signed or complex measure ν with Lebesgue decomposition $d\nu = d\psi + f d\mu$, the total variation of ν is given by $d|\nu| = d|\psi| + |f| d\mu$.

The following chapter shows a connection between the Lebesgue-Radon-Nikodym Theorem and the dual space of $L^p(\mu)$, as the Radon-Nikodym Theorem is used to give a complete characterization of the bounded linear functionals on $L^p(\mu)$.

5 Bounded linear functionals on L^p

The purpose of this chapter is to prove that the dual space of $L^p(\mu)$, denoted $(L^p(\mu))^*$, is isometrically isomorphic to $L^q(\mu)$ for μ a σ-finite positive measure, $1 \leq p < \infty$ and q the conjugate exponent to p. This result can be obtained as a consequence of the Lebesgue-Radon-Nikodym Theorem. The theory of this chapter is based on Rudin 1987 (chapter 6) and Schilling 2017 (chapter 21).

5.1 The dual space of L^p

In this section, it is proven that $(L^p(\mu))^*$ is isometrically isomorphic to $L^q(\mu)$ for every $1 \leq p < \infty$, and as a result that $L^p(\mu)$ is reflexive for every $1 < p < \infty$.

Remark 5.1. Let $g \in L^q(\mu)$, and let $\Phi_g : L^p(\mu) \to \mathbb{C}$ be defined by $\Phi_g(f) := \int_X fg d\mu$ for $f \in L^p(\mu)$. Clearly, Φ_g is linear by linearity of the integral. Note that by Hölder's inequality,

$$|\Phi_g(f)| = \left| \int_X fg d\mu \right| \leq \|fg\|_1 \leq \|g\|_q \|f\|_p,$$

hence $\|\Phi_g\| = \sup\left\{|\Phi_g(f)| : \|f\|_p = 1\right\} \leq \|g\|_q$, and Φ_g is in fact bounded and linear, i.e., $\Phi_g \in (L^p(\mu))^*$. Hence, the space of such functionals Φ_g for $g \in L^q(\mu)$ is a subset of $(L^p(\mu))^*$. The question is whether every $\Phi \in (L^p(\mu))^*$ is of this form for some $g \in L^q(\mu)$, and whether this representation is unique. The following theorem answers this question. The theorem requires some preliminary lemmas, which are stated and/or proved in the following. To begin with, the definition of an isometrical isomorphism is given.

Definition 5.2. Let X and Y be normed vector spaces over a field, $\mathbb{K}$, and let $T : X \to Y$ be a linear map. Then T is an *isometry*, if $\|Tx\| = \|x\|$ for every $x \in X$. Moreover, T is a an *isomorphism*, if T is invertible with bounded inverse. Thus, T is an *isometrical isomorphism*, if T is an isomorphism that is also an isometry.

Lemma 5.3. *(Theorem 1.40 Rudin 1987). Let μ be a finite positive measure on $(X, \mathcal{A})$, let $f \in L^1(\mu)$, and let S be a closed set in the complex plane. If $A_E(F) = \frac{1}{\mu(E)} \int_E f d\mu \in S$ for every $E \in \mathcal{A}$ with $\mu(E) > 0$, then $f(x) \in S$ for μ-a.e. $x \in X$.*

Lemma 5.4. *Let $(X, \mathcal{A})$ be a measurable space, and let $f : X \to \mathbb{C}$ be a measurable function. Then there exists a measurable function $\alpha : X \to \mathbb{C}$ such that $|\alpha| = 1$ and $|f| = \alpha f$.*

Proof. Let $E := \left\{x \in X : f(x) = 0\right\}$. Define $\alpha : X \to \mathbb{C}$ by $\alpha(x) := \frac{|f(x) + \mathbb{1}_E|}{f(x) + \mathbb{1}_E}$ for every $x \in X$. Then

$$\alpha(x) = \begin{cases} 1, & x \in E \\ \frac{|f(x)|}{f(x)}, & x \notin E \end{cases}.$$

Since $\{0\}$ is a measurable set, and $E = f^{-1}(\{0\})$, E is measurable. Thus, α being measurable follows from f being measurable, and the map $z \mapsto \frac{|z|}{z}$ being continuous on $\mathbb{C} \setminus \{0\}$. Now it is clear that α satisfies that $|\alpha| = 1$, and $|f| = \alpha f$, which completes the proof. □

Lemma 5.5. *Let μ be a σ-finite positive measure on $(X, \mathcal{A})$. Then there exists $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.*

Proof. By assumption, $X = \bigcup_{i=1}^\infty E_i$ for sets $E_i \in \mathcal{A}$ for which $\mu(E_i) < \infty$. Define

$$w_n(x) := \begin{cases} 0, & x \in X \setminus E_n \\ \frac{2^{-n}}{1 + \mu(E_n)}, & x \in E_n \end{cases},$$

and let $w(x) := \sum_{n=1}^\infty w_n(x)$. Then clearly, $w \in L^1(\mu)$, and $0 < w(x) < 1$ as wanted. □

Theorem 5.6 (The Dual Space of L^p Characterization Theorem)**.** *Let μ be a σ-finite positive measure on $(X, \mathcal{A})$, and let Φ be a bounded linear functional on $L^p(\mu)$, i.e., $\Phi \in (L^p(\mu))^*$. Then there exists a unique $g \in L^q(\mu)$, where q is the exponent conjugate to p, such that*

$$\Phi(f) = \int_X fg d\mu$$

for $f \in L^p(\mu)$. Moreover $\|\Phi\| = \|g\|_q$.

Proof. For the uniqueness, suppose that g and g' both satisfy that $\Phi(f) = \int_X fgd\mu = \int_X fg'd\mu$ for every $f \in L^p(\mu)$. Then $0 = \int_X f(g-g')d\mu$ for every $f \in L^p(\mu)$, thus in particular,

$$0 = \int_X \mathbb{1}_E(g-g')d\mu = \int_E g - g'd\mu$$

for every $E \in \mathcal{M}$ with $\mu(E) < \infty$ such that $\mathbb{1}_E \in L^p(\mu)$. Thus, since μ is σ-finite by assumption, X can be covered with at most countably many of such sets implying that $g - g' = 0$ μ-a.e. and thus, $g = g'$ μ-a.e. This completes the uniqueness of $g \in L^q(\mu)$.

Now, existence is proven. Note that if $\|\Phi\| = 0$, hence $\Phi = 0$, then $0 = g \in L^q(\mu)$ satisfies that $\Phi(f) = \int_X fgd\mu = 0$ for every $f \in L^p(\mu)$, and moreover, $\|g\|_q = 0 = \|\Phi\|$. Therefore, assume now that $\|\Phi\| > 0$. The proof is given by splitting into two cases of μ.

Case I: Suppose μ is finite, i.e., $\mu(X) < \infty$. Define $\lambda: \mathcal{A} \to \mathbb{C}$ by $\lambda(E) := \Phi(\mathbb{1}_E)$ for every $E \in \mathcal{A}$, which is well-defined, since $\Phi: L^p(\mu) \to \mathbb{C}$, and $\mathbb{1}_E \in L^p(\mu)$ for every $E \in \mathcal{A}$, since μ is assumed to be finite. It is now proven that λ is in fact a complex measure. Clearly, $\lambda(\emptyset) = \Phi(\mathbb{1}_\emptyset) = \Phi(0) = 0$ by linearity of Φ. Now, let $A, B \in \mathcal{A}$ be disjoint such that $A \cap B = \emptyset$. Then $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B$, hence $\lambda(A \cup B) = \lambda(A) + \lambda(B)$ for every such pair, i.e., λ is additive. To prove that λ is countably additive, let $E = \dot\bigcup_{n=1}^\infty E_n$ be a countable union of disjoint sets, and let $A_k = \dot\bigcup_{n=1}^k E_n$. Then $\|\mathbb{1}_E - \mathbb{1}_{A_k}\|_p = \mu(E \setminus A_k)^{1/p} \to \mu(\emptyset)^{1/p} = 0$ as $k \to \infty$, since $1 \le p < \infty$, and clearly, $A_k \to E$ as $k \to \infty$. By continuity of Φ, it thus holds that $|\Phi(\mathbb{1}_{A_k}) - \Phi(\mathbb{1}_E)| \to 0$ as $k \to \infty$, i.e., $\Phi(\mathbb{1}_{A_k}) \to \Phi(\mathbb{1}_E)$ as $k \to \infty$. Hence,

$$\sum_{n=1}^k \lambda(E_n) = \lambda(A_k) \to \lambda(E) = \lambda(\dot\bigcup_{n=1}^\infty E_n) \quad \text{as } k \to \infty,$$

which yields that $\sum_{n=1}^\infty \lambda(E_n) = \lambda(E)$ as wanted. Thus, λ is a complex measure. Now, the claim is that λ is absolutely continuous with respect to μ. For $E \in \mathcal{A}$, then $0 = \mu(E) = \int_X \mathbb{1}_E d\mu$ yields that $\|\mathbb{1}_E\|_p = 0$, which implies that $\lambda(E) = \Phi(\mathbb{1}_E) = 0$ by boundedness of Φ. Then by Theorem 4.6 (The Radon-Nikodym Theorem) applied to the complex measure λ, there exists $g \in L^1(\mu)$ such that

$$\Phi(\mathbb{1}_E) = \lambda(E) = \int_E gd\mu = \int_X \mathbb{1}_E gd\mu$$

for every $E \in \mathcal{A}$. Thus, by linearity of Φ, it holds that $\Phi(u) = \int_X ugd\mu$ for every simple function $u \in L^p(\mu)$. Now, since the set of simple functions in $L^p(\mu)$ is dense in $L^p(\mu)$, every $f \in L^p(\mu)$ is the limit of simple functions $u_n \in L^p(\mu)$. Thus, by the Dominated Convergence Theorem (Theorem 2.24 Folland 1999),

$$\lim_{n\to\infty} \int_X u_n gd\mu = \int_X \lim_{n\to\infty} u_n gd\mu = \int_X fg\mu,$$

implying that $\Phi(f) = \Phi(\lim_{n\to\infty} u_n) = \int_X fgd\mu$ by continuity of Φ.

Now, it is proven that $g \in L^q(\mu)$ and $\|\Phi\| = \|g\|_q$. Note that by Hölder's inequality, $|\Phi(f)| \le \|f\|_p \|g\|_q$, which establishes the inequality, $\|\Phi\| \le \|g\|_q$. The reverse inequality is split into two cases; $p = 1$ and $1 < p < \infty$. If $p = 1$, then

$$\left| \int_E gd\mu \right| = |\Phi(\mathbb{1}_E)| \le \|\Phi\| \|\mathbb{1}_E\|_1 = \|\Phi\| \mu(E)$$

for every $E \in \mathcal{A}$. Thus, since μ is assumed to be finite, and $g \in L^1(\mu)$, Lemma 5.3 yields that $|g| \le \|\Phi\|$ μ-a.e., implying that $\|g\|_\infty := \inf\{c > 0 : \mu(\{|g| \ge c\}) = 0\} \le \|\Phi\|$, hence $g \in L^\infty(\mu)$ and $\|g\|_\infty = \|\Phi\|$ as wanted. Now, suppose $1 < p < \infty$. Since $g \in L^1(\mu)$ is a complex, measurable function, there exists a measurable function $\alpha: X \to \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha g = |g|$ by Lemma 5.4. Let $E_n := \{x \in X : |g(x)| \le n\}$. Define $f: X \to \mathbb{C}$ by $f := \mathbb{1}_{E_n}|g|^{q-1}\alpha$. Note that $|f|^p = |\alpha|^p|g|^{p(q-1)} = |g|^q$ on E_n for every $n \in \mathbb{N}$, hence f is bounded, and thus since μ is assumed to be finite, $f \in L^p(\mu)$. Then using Hölder's inequality, and the fact that p and q are conjugate exponents,

$$\begin{aligned}\int_{E_n} |g|^q d\mu &= \int_X \mathbb{1}_{E_n}|g|^q d\mu = \int_X \mathbb{1}_{E_n}|g|^{q-1}|g| d\mu = \int_X \mathbb{1}_{E_n}|g|^{q-1}\alpha g d\mu \\ &= \int_X fg d\mu = \Phi(f) \le \|\Phi\|\|f\|_p \le \|\Phi\|\Big(\int_{E_n} |g|^q d\mu\Big)^{1/p} = \|\Phi\|\Big(\int_{E_n} |g|^q d\mu\Big)^{1-1/q}\end{aligned}$$

implying that $\big(\int_{E_n} |g|^q d\mu\big)^{1/q} \le \|\Phi\|$ and thus $\int_{E_n} |g|^q d\mu \le \|\Phi\|^q$ for every $n \in \mathbb{N}$. Thus, since $\mathbb{1}_{E_n}|g|^q$ is increasing with $\mathbb{1}_{E_n}|g|^q \to \mathbb{1}_X|g|^q$ as $n \to \infty$, the Monotone Convergence Theorem (Theorem 2.14 Folland 1999) yields that

$$\int_{E_n} |g|^q d\mu \to \int_X |g|^q d\mu = \|g\|_q^q$$

as $n \to \infty$, implying that $\|g\|_q^q \le \|\Phi\|^q$, hence $\|g\|_q \le \|\Phi\|$. Thus, $\|g\|_q = \|\Phi\|$, and $g \in L^q(\mu)$ as wanted. This completes the existence part of the proof in the case, where μ is finite.

Case II: Suppose $\mu(X) = \infty$, but μ is σ-finite. Then by Lemma 5.5, there exists $w \in L^1(\mu)$ such that $0 < w < 1$ for every $x \in X$. Define $d\tilde{\mu} := w d\mu$. Then $\tilde{\mu}$ is clearly a finite positive measure on $(X, \mathcal{A})$. Now, let $\psi: L^p(\tilde{\mu}) \to L^p(\mu)$ be defined by $\psi(F) := w^{1/p}F$ for every $F \in L^p(\tilde{\mu})$. Then ψ is clearly linear. Moreover, it is an isometry, since

$$\|F\|_p = \Big(\int_X |F|^p d\tilde{\mu}\Big)^{1/p} = \Big(\int_X |F|^p w d\mu\Big)^{1/p} = \Big(\int_X |w^{1/p}F|^p d\mu\Big)^{1/p} = \|w^{1/p}F\|_p.$$

Then $\Psi: L^p(\tilde{\mu}) \to \mathbb{C}$ defined by $\Psi(F) := \Phi(w^{1/p}F)$ is a bounded linear function on $L^p(\tilde{\mu})$, i.e., $\Psi \in (L^p(\tilde{\mu}))^*$, with $\|\Psi\| = \|\Phi\|$. Now, by the first part of this proof, Case I, since $\tilde{\mu}$ is finite, there exists a $G \in L^q(\tilde{\mu})$ such that $\Psi(F) = \int_X FG d\tilde{\mu}$ and $\|G\|_q = \|\Psi\|$. Suppose $p = 1$. Let $g = G$. Then $\|g\|_\infty = \|G\|_\infty = \|\Psi\| = \|\Phi\|$, hence $\|g\|_q = \|\Phi\|$, and so $g \in L^q(\mu)$. Moreover,

$$\Phi(f) = \Psi(w^{-1/p}f) = \int_X w^{-1/p} fG d\tilde{\mu} = \int_X w^{-1} fgw d\mu = \int_X fg d\mu$$

as wanted. Conversely, suppose $1 < p < \infty$. Let $g := w^{1/q}G$. Then

$$\|g\|_q^q = \int_X |g|^q d\mu = \int_X |w^{1/q}G|^q d\mu = \int_X |G|^q d\tilde{\mu} = \|G\|_q^q = \|\Psi\|^q = \|\Phi\|^q,$$

hence $g \in L^q(\mu)$ and $\|g\|_q = \|\Phi\|$ as wanted. Moreover, $G d\tilde{\mu} = Gw d\mu = w^{1-1/q} g d\mu = w^{1/p} g d\mu$, thus

$$\Phi(f) = \Psi(w^{-1/p}f) = \int_X w^{-1/q} fG d\tilde{\mu} = \int_X w^{-1/p} f w^{1/p} g d\mu = \int_X fg d\mu,$$

completing the proof. □

Remark 5.7. If Φ and g are related as in Theorem 5.6, then Φ is denoted by Φ_g. The map $L^q(\mu) \ni g \mapsto \Phi_g \in (L^p(\mu))^*$ is an isometry, since $\|\Phi_g\| = \|g\|_q$, and is an isomorphism, since the inverse $(L^p(\mu))^* \ni \Phi_g \mapsto g \in L^q(\mu)$ exists by the one to one correspondence between g and Φ_g and is clearly also bounded. Thus, Theorem 5.6 proves that $(L^p(\mu))^*$ is in fact isometrically isomorphic to $L^q(\mu)$. From this result, the question arises: *For which* $1 \le p < \infty$ *is the space* $L^p(\mu)$ *reflexive?*. This question is answered in the following, which initializes with some prerequisite theory behind reflexivity. In particular, the following proposition is stated in order to establish notation.

Proposition 5.8 (Theorem 5.8 (d) Folland 1999)**.** *Let X be a normed vector space. Let $f \in X$, and define $\hat{f}: X \to \mathbb{C}$ by $\hat{f}(\Phi) = \Phi(f)$ for every $\Phi \in X^*$. Then $\Lambda: f \mapsto \hat{f}$ is a linear isometry from X into X^{**}.*

Definition 5.9. A Banach space X is called *reflexive* if $\Lambda(X) = X^{**}$, i.e., if Λ is surjective.

Remark 5.10. The map Λ is isometric, so $x \in X$ is usually identified with the image $\Lambda(x) = \hat{x} \in X^{**}$, and thus, X being reflexive corresponds to X and X^{**} being isometrically isomorphic.

Theorem 5.11. *$L^p(\mu)$ is reflexive for every $1 < p < \infty$.*

Proof. Let $\varphi \in (L^p(\mu))^{**}$, $\varphi: (L^p(\mu))^* \to \mathbb{C}$ be given. The goal is to show that there exists $f \in L^p(\mu)$ such that $\Lambda(f) = \hat{f} = \varphi$. Define a map $\tilde{\varphi}: L^q(\mu) \to \mathbb{C}$ by $\tilde{\varphi}(g) := \varphi(\Phi_g)$ for every $g \in L^q(\mu)$. This is well-defined, since the map $L^q(\mu) \ni g \mapsto \Phi_g \in (L^p(\mu))^*$ is an (isometrical) isomorphism by Theorem 5.6. Note that since the map $L^q(\mu) \ni g \mapsto \Phi_g \in (L^p(\mu))^*$ is linear,

$$\tilde{\varphi}(\alpha_1 g_1 + \alpha_2 g_2) = \varphi(\Phi_{\alpha_1 g_1 + \alpha_2 g_2}) = \varphi(\alpha_1 \Phi_{g_1} + \alpha_2 \Phi_{g_2}) = \alpha_1 \varphi(\Phi_{g_1}) + \alpha_2 \varphi(\Phi_{g_2})$$

for every $\alpha_1, \alpha_2 \in \mathbb{C}$ and $g_1, g_2 \in L^q(\mu)$, hence $\tilde{\varphi}$ is linear. Moreover, the isometrical isomorphism $L^q(\mu) \ni g \mapsto \Phi_g \in (L^p(\mu))^*$ is in particular surjective and $\|g\|_q = \|\Phi_g\|$, hence

$$\begin{aligned} \|\tilde{\varphi}\| &= \sup\left\{\|\tilde{\varphi}(g)\| : \|g\|_q = 1\right\} = \sup\left\{\|\varphi(\Phi_g)\| : \|g\|_q = 1\right\} \\ &= \sup\left\{\|\varphi(\Phi_g)\| : \|\Phi_g\| = 1\right\} = \|\varphi\|, \end{aligned}$$

thus, since φ is bounded, so is $\tilde{\varphi}$. Hence, $\tilde{\varphi} \in (L^q(\mu))^*$. Now, by Theorem 5.6, there exists a unique $f \in L^p(\mu)$ such that $\tilde{\varphi}(g) = \int_X fg d\mu$ for every $g \in L^q(\mu)$. Then for every $g \in L^q(\mu)$,

$$\hat{f}(\Phi_g) = \Phi_g(f) = \int_X fg d\mu = \tilde{\varphi}(g) = \varphi(\Phi_g),$$

hence $\hat{f} = \varphi$, which completes the proof. □

Remark 5.12. The proof of the preceding theorem uses that for $1 < p < \infty$, the dual space of $L^q(\mu)$, for q the exponent conjugate, is isometrically isomorphic to $L^p(\mu)$, applying Theorem 5.6 to $L^q(\mu)$. Thus, as for $p = 1$, the exponent conjugate is $q = \infty$, and so Theorem 5.6 does not apply to $L^\infty(\mu)$, i.e., $(L^\infty(\mu))^*$ is not isometrically isomorphic to $L^1(\mu)$. For this reason, the result cannot be used to prove that $L^1(\mu)$ and $L^\infty(\mu)$ are reflexive.

5.2 The dual space of L^2

This section follows Folland 1999 (section 5.5) and displays the theory of dual spaces and reflexitivity of Hilbert spaces. Unlike for any other $1 \leq p < \infty$, $L^2(\mu)$ is a Hilbert space, hence the results from the preceding section follow from this. Let $\mathcal{H}$ denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem 5.13 (The Riesz Representation Theorem (Theorem 5.25, Folland 1999))**.** *For every $F_y \in \mathcal{H}^*$, there exists a unique $y \in \mathcal{H}$ such that $F_y(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$.*

Remark 5.14. In other words, the preceding theorem states that $\mathcal{H}^* = \{F_y : y \in \mathcal{H}\}$. The Cauchy-Schwartz' inequality yields that $|\langle x, y \rangle| \leq \|x\| \|y\|$ with equality if (and only if) $x = ay$ for some $a \in \mathbb{C}$. Thus, for every $y \in \mathcal{H}$, one may choose $x = \frac{1}{\|y\|} y \in \mathcal{H}$. Then $\|x\| = 1$, and $|\langle x, y \rangle| = \|x\| \|y\| = \|y\|$ and thus $\|F_y\| = \sup\left\{|\langle x, y \rangle| : \|x\| = 1\right\} = \|y\|$. Hence, the map $\mathcal{H} \ni y \mapsto F_y \in \mathcal{H}^*$ is in fact a conjugate-linear isometrical isomorphism implying that $\mathcal{H} \cong \mathcal{H}^*$, i.e., $\mathcal{H}$ is isometrically isomorphic to its dual space.

Proposition 5.15. *$\mathcal{H}$ is a reflexive Banach space.*

Remark 5.16. The proof of the preceding proposition follows similarly to the proof of Theorem 5.11. The results show that Hilbert spaces, $\mathcal{H}$, possesses a very strong form of reflexivity, where not only is $\mathcal{H}$ isomorphic to $\mathcal{H}^{**}$; $\mathcal{H}$ is isometrically isomorphic to $\mathcal{H}^{*}$. In particular, the self-duality of $L^2(\mu)$ can be used to prove the Lebesgue-Radon-Nikodym Theorem, as seen in the following, where the result is proven for finite positive measures using Theorem 5.13 (The Riesz Representation Theorem).

The following theorem is based on exercise 18 (Folland 1999, section 6.2).

Theorem 5.17. *Let ν and μ be finite positive measures on $(X,\mathcal{A})$. Then there exist (unique) finite positive measures ψ and ρ such that*

$$\psi \perp \mu, \qquad \rho \ll \mu, \qquad \nu = \psi + \rho.$$

Moreover, $d\rho = f d\mu$ for some $f \in L^1(\mu)$.

Proof. Define a finite, positive measure λ on $(X,\mathcal{A})$ by $\lambda := \nu + \mu$, and let $L^2(\lambda) := \{f\colon X \to \mathbb{R} \text{ measurable} : \|f\|_2 < \infty\}$ and $(L^2(\lambda))^* := \{T\colon L^2(\lambda) \to \mathbb{R} : T \text{ is bounded and linear}\}$. Let $g \in L^2(\lambda)$, and let $\Phi\colon L^2(\lambda) \to \mathbb{R}$ be defined by $\Phi(g) := \int_X g d\nu$. Then by the triangle inequality for integrals and Hölder's inequality,

$$\Big|\int_X g d\nu\Big| \le \int_X |g| d\nu \le \int_X |g| d\lambda \le \lambda(X)^{1/2}\|g\|_2 < \infty,$$

so Φ is bounded, hence $\Phi \in (L^2(\lambda))^*$. Then by Theorem 5.13 (The Riesz Representation Theorem), there exists a unique $h \in L^2(\lambda)$ such that $\Phi(g) = \int_X g d\nu = \langle g, h\rangle = \int_X gh d\lambda$. Thus,

$$\int_X g d\nu = \int_X gh d\lambda = \int_X gh d\nu + \int_X gh d\mu,$$

which yields that $\int_X g(1-h) d\nu = \int_X gh d\mu$. Now, it is proven that $0 \le h(x) \le 1$ for λ-a.e. $x \in X$. Let $E := \{x \in X : h(x) < 0\}$. Then $\nu(E) = \int_E d\nu \le \int_E 1 - h d\nu = \int_E h d\mu \le 0$, hence $\nu(E) = 0$, as ν is assumed to be positive. Then also, $\mu(E) = 0$, so $\lambda(E) = \nu(E) + \mu(E) = 0$. Let $F := \{x \in X : h(x) > 1\}$. Then $\mu(F) = \int_F d\mu \le \int_F h d\mu = \int_F 1 - h d\nu \le 0$, hence by similar arguments, $\mu(F) = \nu(F) = 0$, thus $\lambda(F) = 0$. Then $0 \le h(x) \le 1$ for λ-a.e. $x \in X$, so h can be taken as $h\colon X \to [0,1]$. Now, let $A := \{x \in X : h(x) < 1\}$ and $B := \{x \in X : h(x) = 1\}$ such that $X = A \cup B$. Define positive finite measures ρ and ψ by $\rho(E) := \nu(E \cap A)$ and $\psi(E) := \nu(E \cap B)$ for every $E \in \mathcal{A}$. Then $\nu(E) = \psi(E) + \rho(E)$ for every $E \in \mathcal{A}$, hence the goal is to show that $\psi \perp \mu$ and $\rho \ll \mu$. Clearly, $\psi(A) = \nu(\varnothing) = 0$, and

$$\mu(B) = \int_B d\mu = \int_B h d\mu = \int_B 1 - h d\nu = 0,$$

which proves that $\psi \perp \mu$. To prove $\rho \ll \mu$, it suffices to prove that $d\rho = f d\mu$. Let $A_n := \{x \in X : h(x) < 1 - \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Now, μ being finite implies that $L^2(\mu) \supseteq L^\infty(\mu)$ by Proposition 6.12 (Folland 1999). Thus, $(1-h)^{-1}\mathbb{1}_{A_n}$ being bounded, implies that $(1-h)^{-1}\mathbb{1}_{A_n} \in L^\infty(\mu) \subseteq L^2(\mu)$ for every $n \in \mathbb{N}$. As $\{A_n\}_{n\ge1}$ is decreasing with $\bigcap_{n=1}^\infty A_n = A$, and $\{(1-h)^{-1}\mathbb{1}_{A_n}\}_{n\ge1}$ is increasing with $\lim_{n\to\infty}(1-h)^{-1}\mathbb{1}_{A_n} = (1-h)^{-1}\mathbb{1}_A$, the Monotone Convergence Theorem (Theorem 2.14 Folland 1999) and continuity from above yields that

$$\begin{aligned}\rho(E) = \nu(E \cap A) &= \lim_{n\to\infty} \nu(E \cap A_n) = \lim_{n\to\infty} \int_{E\cap A_n} d\nu = \lim_{n\to\infty} \int_E (1-h)^{-1}(1-h)\mathbb{1}_{A_n} d\nu \\ &= \lim_{n\to\infty} \int_E h(1-h)^{-1}\mathbb{1}_{A_n} d\mu = \int_E \lim_{n\to\infty} h(1-h)^{-1}\mathbb{1}_{A_n} d\mu = \int_E h(1-h)^{-1}\mathbb{1}_A d\mu,\end{aligned}$$

thus, setting $f := h(1-h)^{-1}\mathbb{1}_A$ completes the argument. As ρ is a positive, finite measure, it is clear that $f \in L^1(\mu)$ as wanted. The uniqueness follows as in Theorem 4.2. □

This wraps up the chapter showing how the theory of bounded linear functionals on $L^p(\mu)$ is linked to the Lebesgue-Radon-Nikodym Theorem.

6 Differentiation on a Euclidean space

In this chapter, the concept of differentiation of signed or complex measures is examined in the case $(X, \mathcal{A}, \mu) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), m)$, where m denotes the Lebesgue measure. The theory of this chapter is based on Folland 1999 (section 3.4).

6.1 The Hardy-Littlewood maximal function and the Maximal Theorem

The purpose of this section is to introduce the Hardy-Littlewood maximal function and to prove the Maximal Theorem, which will be used to prove theorems leading to the Lebesgue Differentation Theorem. The section begins with a definition of the pointwise derivative of a signed or complex measure with respect to the Lebesgue measure.

Definition 6.1. Let ν be a signed or complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When the limit exists, the *pointwise derivative* of ν with respect to m is defined as

$$F(x) := \lim_{r \to 0} \frac{\nu(B(x,r))}{m(B(x,r))},$$

where $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ is the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$.

Remark 6.2. The open balls, $B(x,r)$, have a very nice behaviour of 'shrinking to x' when $r \to 0$. For this reason, $B(x,r)$ is chosen in the definition of the pointwise derivative. One could also replace $B(x,r)$ by other suitable sets which 'shrink nicely to x' as $r \to 0$. The definition of 'shrinking nicely to x' is to be examined later on.

Example 6.3. Let ν be a signed or complex measure with $\nu \ll m$ such that $d\nu = f dm$ by Theorem 4.6 (The Radon-Nikodym Theorem). Then $\nu(B(x,r)) = \int_{B(x,r)} f dm$, which implies that

$$\frac{\nu(B(x,r))}{m(B(x,r))} = \frac{\int_{B(x,r)} f dm}{m(B(x,r))},$$

i.e., this is simply the average value of f on $B(x,r)$. Thus, when the limit exists,

$$F(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{m(B(x,r))} = \lim_{r \to 0} \frac{\int_{B(x,r)} f dm}{m(B(x,r))} = f(x),$$

so one would hope that with the definition above, $F = f$ m-a.e. As it turns out, this is the case, when $\nu(B(x,r))$ is finite for every $x \in \mathbb{R}^n$ and $r > 0$. This may be considered to be a generalization of the Fundamental Theorem of Calculus, which yields that the derivative of $\int f dm$ is exactly f.

The section is now proceeded with some prerequisite results needed to prove the Maximal Theorem beginning with a result regarding the regularity properties of the Lebesgue measure.

Theorem 6.4 (Theorem 2.40 a., Folland 1999)**.** *Let $E \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\begin{aligned} m(E) &= \inf\{m(U) : U \text{ open, } U \supseteq E\} && \text{(Outer regularity)} \\ &= \sup\{m(K) : K \text{ compact, } K \subseteq E\}. && \text{(Inner regularity)} \end{aligned}$$

Lemma 6.5. *Let C be a collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B \in C} B$. If $m(U) > c$ for some $c > 0$, then there exist disjoint $B_1, ..., B_k \in C$ such that $\sum_{i=1}^k m(B_i) > 3^{-n} c$.*

Proof. Assume $m(U) > c$. Then by Theorem 6.4, $m(U) = \sup\{m(K) : K \subseteq U, K \text{ compact}\}$ implying that there exist a compact set $K \subseteq U = \bigcup_{B \in C} B$ such that also $m(K) > c$. Clearly, U is an open covering of K, hence by compactness of K, there exists a finite subcover

$A_1, ..., A_m$ such that $K \subseteq \bigcup_{i=1}^m A_i$, where $A_i := B(x_i, r_i)$ is the open ball with center x_i and radius $r_i > 0$. Now, choose $B_1 := \{A_i : r_i \geq r_j \text{ for every } j = 1, .., m\}$, i.e., let B_1 be the A_i with the largest radius for $i \in \{1, ..., m\}$. Choose B_2 to be the largest (meaning with largest radius) of the remaining A_i's that is also disjoint from B_2. Choose B_3 to be the largest of the remaining A_i's that is disjoint from both B_1 and B_2. Continue this way until the list of A_i's is exhausted. Now, if $A_i \neq B_j$ for every $j \in \{1, ..., k\}$, there must exist a B_j such that $A_i \cap B_j \neq \varnothing$. Moreover, if j is the smallest integer with this property, then the radius of A_i must be smaller than or equal to B_j, since otherwise $A_i = B_j$. If $B_j = B(x_j, r_j)$, let $B_j^* := B(x_j, 3r_j)$, such that B_j^* is the open ball concentric with B_j with radius thrice as big as the radius of B_j. Then $A_i \subseteq B_j^*$. But then $K \subseteq \bigcup_{i=1}^m A_i \subseteq \bigcup_{j=1}^k B_j^*$, and thus,

$$c < m(K) \leq \sum_{j=1}^{k} m(B_j^*) = 3^n \sum_{j=1}^{k} m(B_j),$$

implying exactly that $\sum_{j=1}^k m(B_j) > 3^{-n} c$ as wanted. □

Definition 6.6. Let $f: \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function. Then f is *locally integrable* with respect to m, if $\int_K |f(x)| dm < \infty$ for every compact set $K \in \mathcal{B}(\mathbb{R}^n)$. Let $L^1_{loc}(m)$ denote the space of such functions.

Definition 6.7. Let $f \in L^1_{loc}(m)$, and let $x \in \mathbb{R}^n$ and $r > 0$. The *average value of f on $B(x,r)$* is the function $A_r f: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ defined by

$$A_r f(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dm(y).$$

Definition 6.8. Let $A_1, ..., A_n$ be topological spaces. A function $f: A_1 \times \cdots \times A_n \to \mathbb{C}$ is *jointly continuous* if f is continuous with respect to the product topology on $A_1 \times \cdots \times A_n$.

Lemma 6.9. *Let $f \in L^1_{loc}(m)$. Then $A_r f$ is jointly continuous in $(x,r) \in \mathbb{R}^n \times \mathbb{R}_+$.*

Proof. Note that the product topology on $\mathbb{R}^n \times \mathbb{R}_+$ equals the standard metric topology on $\mathbb{R}^{n+1}$. Also, note that $m(B(x,r)) = r^n m(B(0,1))$. Moreover, $\mathbb{1}_{B(x,r)} \to \mathbb{1}_{B(x_0,r_0)}$ as $(x,r) \to (x_0, r_0)$ on $\mathbb{R}^n \setminus S(x_0, r_0)$, where $S(x_0, r_0) := \{y \in \mathbb{R}^n : |y - x_0| = r_0\}$. Thus, $\mathbb{1}_{B(x,r)} \to \mathbb{1}_{B(x_0,r_0)}$ m-a.e., since $m(S(x_0,r_0)) = 0$. Moreover, if $r < r_0 + \frac{1}{2}$ and $|x - x_0| < \frac{1}{2}$, one obtains that $|\mathbb{1}_{B(x,r)}| \leq |\mathbb{1}_{B(x_0, r_0+1)}|$. Now, $f \in L^1_{loc}(m)$ implies that $\mathbb{1}_{B(x,r)} f \in L^1(m)$, hence by the Dominated Convergence Theorem (Theorem 2.24 Folland 1999),

$$\lim_{(x,r)\to(x_0,r_0)} \int \mathbb{1}_{B(x,r)}(y) f(y) dm(y) = \int \mathbb{1}_{B(x_0,r_0)}(y) f(y) dm(y) = \int_{B(x_0,r_0)} f(y) dm(y),$$

thus $\int_{B(x,r)} f(y) dm(y)$ is continuous in $(x,r) \in \mathbb{R}^n \times \mathbb{R}_+$, which proves that

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dm(y) = \frac{1}{m(B(0,1)) r^n} \int_{B(x,r)} f(y) dm(y)$$

is jointly continuous, as wanted. □

Definition 6.10. Let $f \in L^1_{loc}$. Then the *Hardy-Littlewood maximal function*, is the function $Hf: \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$Hf(x) := \sup_{r>0} \{A_r |f|(x)\} = \sup_{r>0} \left\{ \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dm(y) \right\}.$$

Remark 6.11. By the preceding lemma, $(Hf)^{-1}((a,\infty)) = \bigcup_{r>0}(A_r|f|)^{-1}((a,\infty))$ is open, since A_rf is continuous with respect to the product topology, and arbitrary unions of open sets are open, hence Hf is a Borel measurable function.

Theorem 6.12 (The Maximal Theorem). *There exists a constant $C > 0$ such that for every $f \in L^1(m)$ and every $\alpha > 0$,*

$$m\big(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}\big) \le \frac{C}{\alpha}\int_{\mathbb{R}^n}|f(x)|dm(x) = \frac{C}{\alpha}\|f\|_1.$$

Proof. Let $E_\alpha := \big\{x \in \mathbb{R}^n : Hf(x) > \alpha\big\}$. For every $x \in E_\alpha$, one may choose $r_x > 0$ such that $A_{r_x}|f|(x) > \alpha$. Then the open balls $B(x, r_x)$ cover E_α, i.e., $E_\alpha \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_{x_i})$. Thus, by Lemma 6.5, if $m(E_\alpha) > c$, then there exist disjoint $B(x_1, r_{x_1}), ..., B(x_k, r_{x_k})$, i.e., $x_1, ..., x_k \in E_\alpha$, such that $\sum_{i=1}^{k} m(B(x_i, r_{x_i}) > 3^{-n}c$. Then

$$c < 3^n\sum_{i=1}^{k} m(B(x_i, r_{x_i}) \le 3^n\sum_{i=1}^{k}\frac{1}{\alpha}\int_{B(x_i,rx_i)}|f(y)|dm(y) \le \frac{3^n}{\alpha}\int_{\mathbb{R}^n}|f(y)|dm(y),$$

and thus, letting $c \to m(E_\alpha)$, one obtains that $m(E_\alpha) \le \frac{3^n}{\alpha}\int_{\mathbb{R}^n}|f(y)|dm(y)$, as wanted. □

This concludes the section with the proof of the Maximal Theorem, from which the following section proceeds to prove the Lebesgue Differentiation Theorem.

6.2 The Lebesgue Differentiation Theorem

In this section, three consecutive stronger versions of the Fundamental Differentiation Theorem are presented, ending with the Lebesgue Differentiation Theorem. The purpose of this section of is prove that the pointwise derivative of a signed or complex measure ν with respect to m is in fact equal to the Radon-Nikodym derivative, $\frac{d\nu}{dm}$, m-a.e. under certain assumptions. The section is initialized with a lemma needed to prove the first theorem.

Lemma 6.13. *(Theorem 2.41 (Folland 1999)). Let $f \in L^1(m)$. Then for every $\varepsilon > 0$, there exists a continuous function g such that $\int_X |f - g|dm < \varepsilon$.*

Theorem 6.14. *Let $f \in L^1_{loc}(m)$. Then $\lim_{r\to 0} A_rf(x) = f(x)$ for m-a.e. $x \in \mathbb{R}^n$.*

Proof. Note that it suffices to prove that $A_rf(x) \to f(x)$ as $r \to 0$ for almost every $x \in \mathbb{R}^n$ with $|x| \le N$ for some $N \in \mathbb{N}$, since for every $x \in \mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that $|x| \le N$. Thus, assume $|x| \le N$. Assume also that $r \le 1$, which is justified as $r \to 0$. Then for $|y| \le N + r \le N + 1$ the values of $A_rf(x)$ depend only on $f(y)$. Thus, by replacing f with $f\mathbb{1}_{B(0,N+1)}$, one may assume that $f \in L^1(m)$. Now, by Lemma 6.13, given $\varepsilon > 0$, there exists a continuous function g such that $\int_X |f - g|(x)dm(x) < \varepsilon$. Continuity of g implies that for every $x \in \mathbb{R}^n$ and for every $\delta > 0$, there exists an $r > 0$, such that $|g(y) - g(x)| < \delta$, whenever $|y - x| < r$. Hence,

$$|A_rg(x) - g(x)| = \frac{1}{m(B(x,r))}\Big|\int_{B(x,r)} g(y) - g(x)dm(y)\Big| < \delta$$

implying that $A_rg(x) \to g(x)$ as $r \to 0$ for every $x \in \mathbb{R}^n$. Then

$$\begin{aligned}\limsup_{r\to 0}|A_rf(x) - f(x)| &= \limsup_{r\to 0}|A_rf(x) - f(x) - (A_rg(x) - g(x)) + (A_rg(x) - g(x))| \\ &= \limsup_{r\to 0}|A_r(f - g)(x) + (g - f)(x) + A_rg(x) - g(x)| \\ &\le H(f - g)(x) + |f(x) - g(x)| + \delta.\end{aligned}$$

Let $P_\alpha := \{x \in \mathbb{R}^n : \limsup_{r\to 0} |A_r f(x) - f(x)| > \alpha\}$, $E_\alpha := \{x \in \mathbb{R}^n : H(f-g)(x) > \alpha\}$ and $F_\alpha := \{x \in \mathbb{R}^n : |f(x) - g(x)| > \alpha\}$. Now, the goal is to show that P_α is a Lebesgue null set. Note that $P\alpha \subseteq E_{\frac{\alpha}{2}} \cup F_{\frac{\alpha}{2}}$, where

$$\tfrac{\alpha}{2} m(F_{\frac{\alpha}{2}}) = \int_{F_{\frac{\alpha}{2}}} \tfrac{\alpha}{2} dm(x) \leq \int_{F_{\frac{\alpha}{2}}} |f(x) - g(x)| dm(x) < \varepsilon,$$

hence by Theorem 6.12,

$$m(P_\alpha) \leq m(E_{\frac{\alpha}{2}}) + m(F_{\frac{\alpha}{2}}) \leq \tfrac{2C}{\alpha} \int |f(x) - g(x)| dm(x) + \tfrac{2}{\alpha}\varepsilon \leq \left(\tfrac{2C}{\alpha} + \tfrac{2}{\alpha}\right)\varepsilon.$$

Thus, $m(P_\alpha) = 0$ for every $\alpha > 0$, and $\limsup_{r\to 0} |A_r f(x) - f(x)| = 0$ for every $x \notin \bigcup_{n=1}^{\infty} P_{1/n}$. Hence, it is concluded that $A_r f(x) \to f(x)$ for m-a.e. $x \in \mathbb{R}^n$, as wanted. □

Remark 6.15. The preceding theorem yields that for $f \in L^1_{loc}(m)$, and for m-a.e. $x \in \mathbb{R}^n$,

$$0 = \lim_{r\to 0} A_r f(x) - f(x) = \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r))} f(y) - f(x) dm(y).$$

But, as proven in the following theorem, something even stronger holds.

Theorem 6.16. *Let* $L_f := \{x \in \mathbb{R}^n : \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r))} |f(y) - f(x)| dm(y) = 0\}$. *If* $f \in L^1_{loc}(m)$, *then* $m(\mathbb{R}^n \setminus L_f) = 0$.

Proof. Let $c \in \mathbb{C}$ be arbitrary. By applying Theorem 6.14 to $g(x) := |f(x) - c|$, it is concluded that

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c| dm(y) = |f(x) - c|$$

for m-a.e. $x \in \mathbb{R}^n$, i.e., every $x \in \mathbb{R}^n \setminus P_c$, where P_c is a Lebesgue null set. Let D denote a countable dense subset of $\mathbb{C}$, and let $P := \bigcup_{c\in D} P_c$. Then $m(P_c) = 0$ yields that also $m(P) = 0$. Let $x \notin P$, and let $\varepsilon > 0$ be given. Then one may choose $c \in D$ with $|f(x) - c| < \varepsilon$ such that

$$|f(y) - f(x)| = |f(y) - (f(x) - c) + (f(x) - c) - f(x)| \leq |f(y) - c| + \varepsilon$$

implying that

$$\limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \leq |f(x) - c| + \varepsilon < 2\varepsilon$$

hence $\limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) = 0$ for every $x \notin P$, which completes the proof. □

The preceding theory considers families of open balls $B(x,r)$. However, these may be replaced by families of more general sets, which 'shrink nicely to x', as is defined below.

Definition 6.17. A family $\{E_r\}_{r>0}$ of sets $E_r \in \mathcal{B}(\mathbb{R}^n)$ *shrinks nicely* to $x \in \mathbb{R}^n$ as $r \to 0$, if the following conditions hold.

(i) $E_r \subseteq \overline{B(x,r)}$ for each $r > 0$.

(ii) There exists an $\alpha > 0$, independent of r, such that $m(E_r) > \alpha m(B(x,r))$.

Example 6.18. Let $U \in \mathcal{B}(\mathbb{R}^n)$ with $U \subseteq B(0,1)$ and $m(U) > 0$, and let $E_r := \{x + ry : y \in U\}$ for $r > 0$. Then the family $\{E_r\}_{r>0}$ shrinks nicely to x as $r \to 0$.

Proof. (i) Clearly, $E_r \subseteq \overline{B(x,r))}$, since $|y| < 1$ implies that $|x - x + ry| < r$.

(ii) Let $\alpha := \frac{m(U)}{m(B(0,1))+1}$. Then since the Lebesgue measure is invariant under translation,

$$m(E_r) = r^n m(U) = r^n m(B(0,1))\frac{m(U)}{m(B(0,1))} = m(B,r)\frac{m(U)}{m(B(0,1))} > \alpha m(B,r).$$

□

Remark 6.19. This example shows that for a family $\{E_r\}_{r>0}$ that shrinks nicely to x, the sets E_r need not contain x itself, as one may notice that if $0 \notin U$, then $x \notin E_r$ for any $r > 0$.

Theorem 6.20 (The Lebesgue Differentiation Theorem)**.** *Let $f \in L^1_{loc}$. Then for every $x \in L_f$,*

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) = 0, \quad \textit{and} \quad \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dm(y) = f(x)$$

for every family $\{E_r\}_{r>0}$ that shrinks nicely to x as $r \to 0$.

Proof. By assumption $E_r \subseteq \overline{B(x,r)}$ and $m(E_r) > \alpha m(B(x,r))$ for some $\alpha > 0$, hence

$$\begin{aligned}\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(E_r)} \int_{B(x,r)} |f(y) - f(x)| dm(y)\\ &\leq \frac{1}{\alpha m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y),\end{aligned}$$

thus by Theorem 6.16, $\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) = 0$ for every $x \in L_f$. Applying the triangle inequality of integrals, one obtains that

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) - f(x) dm(y) = 0,$$

hence

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dm(y) = f(x)$$

for $x \in L_f$ as wanted. In particular, the equalities hold for m-a.e. $x \in \mathbb{R}^n$. □

Thus, the proof of the Lebesgue Differentiation Theorem is concluded. The section is now proceeded with the definition of regular Borel measures needed in order to prove the main result of the section, namely that the pointwise derivative of a signed or complex measure ν with respect to m, where ν has Lebesgue descomposition $d\nu = d\psi + f dm$, is equal to f m-a.e., if ν is regular.

Definition 6.21. Let ν be a positive Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then ν is *regular* if the following conditions hold.

(i) ν is finite on every compact $K \in \mathcal{B}(\mathbb{R}^n)$.

(ii) $\nu(E) = \inf\{\nu(U) : U \text{ open}, E \subseteq U\}$ for every $E \in \mathcal{B}(\mathbb{R}^n)$. (Outer regularity)

Remark 6.22. Condition (i) actually implies condition (ii). However, as this shall not be proven here, condition (ii) is assumed explicitly. Also, note that condition (i) implies that every regular measure is σ-finite.

Example 6.23. Let $f : \mathbb{R}^n \to [0, \infty]$ be Borel measurable, and let ν be a positive measure defined by $d\nu := f dm$. Then ν is regular if and only if $f \in L^1_{loc}(m)$.

Proof. Note that $\nu(K) = \int_K |f(x)| dm(x)$ for every compact $K \in \mathcal{B}(\mathbb{R}^n)$, thus condition (i) is clearly equivalent to $f \in L^1_{loc}(m)$. Now assume condition (i) holds. Suppose $E \in \mathcal{B}(\mathbb{R}^n)$ is bounded. Then given $\delta > 0$, there is a bounded, open set $U \supseteq E$ such that $m(U) < m(E) + \delta$ by Theorem 6.4. Then $m(U \setminus E) < \delta$. By Corollary 2.27 for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\int_E f dm| < \varepsilon$ whenever $\mu(E) < \delta$. Thus, there is $U \supseteq E$ satisfying that $m(U \setminus E) < \delta$, which implies that $|\int_{U \setminus E} f dm| < \varepsilon$, which then again implies that $\nu(U) = \int_U f dm < \int_E f dm + \varepsilon = \nu(E) + \varepsilon$. Now $E \subseteq U$ implies that also $\nu(E) \leq \nu(U)$, and the result follows. Suppose now that E is unbounded. Let $E_i = (E \cap B(0,i)) \setminus \bigcup_{j=1}^{i} B(0,j)$ for each $i \in \mathbb{N}$, such that $E = \dot{\bigcup}_{i=1}^{\infty} E_i$, where E_i is bounded. Thus, given δ, there exists a bounded, open set $U_i \supseteq E_i$ such that $m(U_i \setminus E_j) < \delta$ for each E_i. Hence, given $2^{-i}\varepsilon > 0$, there is an open set $U_i \supseteq E_i$ such that $|\int_{U_i \setminus E_i} f dm(y)| < 2^{-i}\varepsilon$, which implies that $\int_{U_i} f dm < \int_{E_i} f dm + 2^{-i}\varepsilon$. By continuity from above, $\nu(E) = \nu(\dot{\bigcup}_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \int_{E_i} f dm$. Letting $U = \bigcup_{i=1}^{\infty} U_i$, one obtains $U \supseteq E$ open, and

$$\nu(U) \leq \sum_{i=1}^{\infty} \nu(U_i) = \sum_{i=1}^{\infty} \int_{U_i} f dm \leq \sum_{i=1}^{\infty} \int_{E_i} f dm + 2^{-i}\varepsilon = \nu(E) + \varepsilon$$

as wanted. □

Proposition 6.24. *Let λ, μ be positive Borel measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. If $\lambda + \mu$ is regular, then λ and μ are regular.*

Proof. (i) Let $K \in \mathcal{B}(\mathbb{R}^n)$ be compact. Then $\infty > (\lambda + \mu)(K) = \lambda(K) + \mu(K)$, which clearly implies that $\lambda(K), \mu(K) < \infty$.

(ii) By the assumption that $\lambda + \mu$ is a regular measure, given $\varepsilon_i > 0$, there exist an open set $U_i \in \mathcal{B}(\mathbb{R}^n)$ such that $A \subseteq U_i$ and $(\lambda + \mu)(U_i) < (\lambda + \mu)(A) + \varepsilon_i$ for every $A \in \mathcal{B}(\mathbb{R}^n)$. Now, let $\{U_i\}_{i \geq 1}$ be a decreasing sequence such that

$$\lim_{i \to \infty} (\lambda + \mu)(U_i) = (\lambda + \mu)(A)$$

and $(\lambda + \mu)(U_i) \geq (\lambda + \mu)(A)$ for every $i \in \mathbb{N}$. Note that $\mu(A) \leq \mu(U_i)$ and $\lambda(A) \leq \lambda(U_i)$ by monotonicity of the positive measures μ and λ, hence

$$\begin{aligned} \varepsilon_i > (\lambda + \mu)(U_i) - (\lambda + \mu)(A) &= \lambda(U_i) + \mu(U_i) - \lambda(A) - \mu(A) \\ &= \lambda(U_i) - \lambda(A) \geq 0, \end{aligned}$$

hence, $\lambda(U_i) < \lambda(A) + \varepsilon_i$, and thus $\lim_{i \to \infty} \mu(U_i) = \mu(A)$, as wanted. Similarly for μ. □

Definition 6.25. Let ν be a signed or complex Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then ν is *regular*, if the total variation $|\nu|$ is regular.

Theorem 6.26 (The Pointwise Derivative Theorem)**.** *Let ν be a regular signed or complex Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, and let $d\nu = d\psi + f dm$ be the Lebesgue decomposition of ν with respect to m. Then*

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for m-a.e. $x \in \mathbb{R}^n$ and every family $\{E_r\}_{r > 0}$ that shrinks nicely to x as $r \to 0$.

Proof. By Lemma 4.12 and the following remark, $d|\nu| = d|\psi| + |f| dm$. Hence Proposition 6.24 yields that $|\psi|$ and $|f| dm$ are regular, since $|\nu|$ is regular by assumption. In particular, $|f| dm$ being regular yields that $f \in L^1_{loc}(m)$ by Example 6.23. Note that

$$\frac{\psi(E_r)}{m(E_r)} = \frac{\nu(E_r) - \int_{E_r} f(y) dm(y)}{m(E_r)} = \frac{\nu(E_r)}{m(E_r)} - \frac{1}{m(E_r)} \int_{E_r} f(y) dm(y).$$

Therefore, by Theorem 6.20 it suffices to show that if ψ is regular and $\psi \perp m$, then $\frac{\psi(E_r)}{m(E_r)} \to 0$ as $r \to 0$, when $\{E_r\}_{r>0}$ shrinks nicely to x. Furthermore, note that

$$\left|\frac{\psi(E_r)}{m(E_r)}\right| \le \frac{|\psi|(E_r)}{m(E_r)} \le \frac{|\psi|(B(x,r))}{m(E_r)} \le \frac{|\psi|(B(x,r))}{\alpha m(B(x,r))},$$

for some $\alpha > 0$. Thus, it suffices to assume $E_r = B(x,r)$ and that ψ is a positive measure. Let $A \in \mathcal{B}(\mathbb{R}^n)$ such that $\psi(A) = m(\mathbb{R}^n \setminus A) = 0$ using the assumption that $\psi \perp m$. Define $F_k := \left\{x \in A : \limsup_{r\to 0} \frac{\psi(B(x,r))}{m(B(x,r))} > \frac{1}{k}\right\}$ for each $k \in \mathbb{N}$. The goal is to show that $m(F_k) = 0$ for every $k \in \mathbb{N}$. By regularity of ψ, given $\varepsilon > 0$, there exists an open set $U_\varepsilon \supseteq A$ such that $\psi(U_\varepsilon) < \varepsilon$, since $\psi(A) = 0$. Now each $x \in A$ is the center of an open ball $B(x,r) \subseteq U_\varepsilon$ such that $\psi(B(x,r)) > \frac{m(B(x,r))}{k}$. Let $V_\varepsilon := \bigcup_{x\in F_k} B(x,r)$. Suppose by contradiction that $m(V_\varepsilon) > c$ for some $c > 0$. Then by Lemma 6.5, there exist disjoint open balls $B(x_1,r_1),...,B(x_l,r_l)$ in V_ε such that $\sum_{i=1}^{l} m(B_i) > 3^{-n}c$. Then

$$\begin{aligned} c &< 3^n \sum_{i=1}^{l} m(B(x_i,r_i)) \le 3^n k \sum_{i=1}^{l} \psi(B(x_i,r_i)) = 3^n k \psi(\dot{\bigcup_{i=1}^{l}} B(x_i,r_i)) \\ &\le 3^n k \psi(V_\varepsilon) \le 3^n k \psi(U_\varepsilon) \le 3^n k\varepsilon, \end{aligned}$$

where the inequalities in the second line follows from $\dot{\bigcup}_{i=1}^{l} B(x_i,r_i) \subseteq V_\varepsilon$, monotonicity of ψ, and $B(x,r) \subseteq U_\varepsilon$ for every $x \in F_k$, which implies that $V_\varepsilon \subseteq U_\varepsilon$. Thus, $m(V_\varepsilon) \le 3^n k\varepsilon$, since the above yields that for every $c > 0$ with $m(V_\varepsilon) > c$, it holds that $c < 3^n k\varepsilon$, hence if $m(V_\varepsilon) > 3^n k\varepsilon$, then $3^n k\varepsilon < 3^n k\varepsilon$, but this is a contradiction. Since $\varepsilon > 0$ was arbitrarily chosen and since $F_k \subseteq V_\varepsilon$, one obtains that $m(V_\varepsilon) = 0$ implying that also $m(F_k) = 0$ for every $k \in \mathbb{N}$, as wanted. □

This concludes the theory of differentiation on a Euclidean space proving that the pointwise derivative of a signed or complex measure with respect to the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is in fact equal to f m-a.e. Moreover, if $\nu \ll m$ such that $d\nu = \frac{d\nu}{d\mu} d\mu$, this indeed proves that the pointwise derivative agrees with the Radon-Nikodym derivative, $\frac{d\nu}{dm}$, m-a.e. as wanted. This leads to the theory of functions of bounded variation, where complex Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are to be considered. This theory will lead to a proof of the Fundamental Theorem of Calculus.

7 Functions of bounded variation and complex Borel measures

This chapter revolves around functions of bounded variation, which turn out to play a significant role in the characterization of complex Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. One purpose of the chapter is to prove the ultimate version of the Fundamental Theorem of Calculus for Lebesgue integrals. The theory of this chapter is based on Folland 1999 (section 3.5).

7.1 Lebesgue-Stieltjes measures

In this section, Lebesgue-Stieltjes measures are introduced as a preliminary to the theory of functions of bounded variation, as increasing and right-continuous functions correspond to positive Borel measures the same way functions of normalized bounded variation correspond to complex Borel measures. The section is based on Schilling 2017 (chapter 6) and Folland 1999 (section 1.5).

Theorem 7.1 (Carathéodory)**.** *Let $\mathscr{S} \subseteq \mathcal{P}(X)$ be a semiring, and let $\mu: \mathscr{S} \to [0, \infty]$ be a premeasure. Then μ has an extension to a measure μ on $\sigma(\mathscr{S})$. Furthermore, if $\mathscr{S}$ contains an exhausting sequence $\{S_n\}_{n\geq 1} \subseteq \mathscr{S}$, i.e., $\{S_n\}_{n\geq 1}$ is increasing with $\bigcup_{n=1}^{\infty} S_n = X$, such that $\mu(S_n) < \infty$ for every $n \in \mathbb{N}$, then the extension is unique.*

Theorem 7.2 (Lebesgue-Stieltjes measures)**.** *Let $F: \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. Then*

$$\mu_F((a, b]) \coloneqq F(b) - F(a), \qquad \text{for every } a \leq b \in \mathbb{R},$$

has a unique extension to a positive Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Conversely, if μ_F is a Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and μ_F is finite on all bounded Borel sets, then

$$F(x) \coloneqq \begin{cases} \mu_F((0, x]), & x \geq 0 \\ -\mu_F((x, 0]), & x < 0 \end{cases}$$

is increasing and right-continuous.

Proof. Let $\mathscr{S} \coloneqq \{(a, b], a \leq b \in \mathbb{R}\}$ such that $\sigma(\mathscr{S}) = \mathcal{B}(\mathbb{R})$. It is proven that $\mathscr{S}$ is a semi-ring.

(i) Since $(a, a] = \varnothing$ for every $a \in \mathbb{R}$, it is clear that $\varnothing \in \mathscr{S}$.

(ii) Let $S, T \in \mathscr{S}$. If $S = \varnothing$ or $T = \varnothing$, then $S \cap T = \varnothing \in \mathscr{S}$. Thus, suppose $S = (a_1, b_1]$, and $T = (a_2, b_2]$ are non-empty, and assume without loss of generality that $a_1 \leq a_2$. If also, $b_1 \leq a_2$, then $S \cap T = \varnothing \in \mathscr{S}$. On the other hand, if $a_2 < b_1$, then $S \cap T = (a_2, \min\{b_1, b_2\}] \in \mathscr{S}$.

(iii) Let $S, T \in \mathscr{S}$. If $S = \varnothing$, then $S \setminus T = \varnothing$, which can be written as a finite disjoint union of the set $\varnothing \in \mathscr{S}$. If $T = \varnothing$, then $S \setminus T = S = \varnothing \uplus S$. Now, suppose $S = (a_1, b_1], T = (a_2, b_2] \neq \varnothing$. Note that if $S \subseteq T$, then $S \setminus T = \varnothing \in \mathscr{S}$. Otherwise,

$$\begin{aligned} S \setminus T = (a_1, b_1] \setminus (a_2, b_2] &= (a_1, b_1] \cap ((-\infty, a_2] \cup (b_2, \infty)) \\ &= (a_1, \min\{b_1, a_2\}] \uplus (\max\{a_1, b_2\}, b_1], \end{aligned}$$

with $(a_1, \min\{b_1, a_2\}] \in \mathscr{S}$ and $(\max\{a_1, b_2\}, b_1] \in \mathscr{S}$.

By Theorem 7.1, it thus suffices to check that ν_F is a premeasure on $\mathscr{S}$ in order to prove existence. Note that $\mathscr{S}$ is not a σ-algebra, hence ν_F is not a measure.

(i) It is clear that $\nu_F(\varnothing) = \nu_F((a, a]) = F(a) - F(a) = 0$.

(ii) Let $\{S_n\}_{n\geq 1}$ be a sequence of disjoint sets $S_n = (a_n, b_n] \in \mathscr{S}$ with $\biguplus_{n=1}^{\infty} S_n = (a, b] = S \in \mathscr{S}$. Let $\delta_1, \delta_2 > 0$ be given. Observe that $\bigcup_{n=1}^{\infty} (a_n, b_n + \delta_1) \supset [a + \delta_2, b]$ is an open cover of the set $[a+\delta_2, b]$. Now, $[a+\delta_2, b]$ is closed and bounded, hence compact by the Heine-Borel Theorem, thus there exists a finite open subcover, i.e., there exists $N \in \mathbb{N}$ such that $\bigcup_{n=1}^N (a_n, b_n+\delta_1) \supset [a+\delta_2, b]$, implying that $\bigcup_{n=1}^N (a_n, b_n+\delta_1] \supset (a+\delta_2, b]$. The goal is to show that $\nu_F((a,b]) = \sum_{n=1}^\infty \nu_F((a_n, b_n])$ by showing $\lim_{N\to\infty}(\nu_F((a,b]) - \sum_{n=1}^N \nu_F((a_n,b_n])) = 0$. Now, choose $a_n' \leq a_n$ and $b_n' \geq b_n$ such that $a_n' = a$ for some $n \in \{1, .., N\}$, and likewise $b_n' = b$ for some $n \in \{1, ..., N\}$. Then $(a,b] = \dot\bigcup_{n=1}^N (a_n', b_n'] \supseteq \dot\bigcup_{n=1}^N (a_n, b_n]$. Thus,

$$\begin{aligned}
0 &= \nu_F((a,b]) - \sum_{n=1}^N \nu_F((a_n', b_n']) \leq \nu_F((a,b]) - \sum_{n=1}^N \nu_F((a_n, b_n]) \\
&= \nu_F((a+\delta_2, b]) + \nu_F((a, a+\delta_2]) - \Big(\sum_{n=1}^N (\nu_F((a_n, b_n+\delta_1]) - \sum_{n=1}^N \nu_F((b_n, b_n+\delta_1])\Big) \\
&\leq \nu_F((a, a+\delta_2]) + \sum_{n=1}^N \nu_F((b_n, b_n+\delta_1]),
\end{aligned}$$

since $(a+\delta_2, b] \subseteq \bigcup_{n=1}^N (a_n, b_n+\delta_1]$, hence $\nu_F((a+\delta_2, b]) - \sum_{n=1}^N \nu_F((a_n, b_n+\delta_1]) \leq 0$. Now, by right-continuity of F, given $\varepsilon > 0$, one may choose $\delta_1 > 0$ and $\delta_2 > 0$ such that $\nu_F((a, a+\delta_2) = F(a) - F(a+\delta_2) < \frac{\varepsilon}{2}$ and $\nu_F((b_n, b_n+\delta_1]) = F(b_n) - F(b_n+\delta_1) < \frac{\varepsilon}{2^{n+1}}$. Then

$$0 \leq \nu_F((a,b]) - \sum_{n=1}^N \nu_F((a_n, b_n] \leq \nu_F((a, a+\delta_2]) + \sum_{n=1}^N \nu_F((b_n, b_n+\delta_1]) < \frac{\varepsilon}{2} + \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}}$$

hence letting $N \to \infty$, one obtains that $0 \leq \nu_F((a,b]) - \sum_{n=1}^\infty \nu_F((a_n, b_n]) \leq \varepsilon$, and since $\varepsilon > 0$ was arbitrary, this completes the proof of existence.

For uniqueness, note that the sequence $\{(-n,n]\}_{n\geq 1} \subseteq \mathscr{S}$ is an exhausting sequence, since $(-n,n] \subseteq (-(n+1), n+1]$ and $\bigcup_{n=1}^\infty (-n,n] = \mathbb{R}$. This completes the uniqueness of μ.

Now, let μ_F be a Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then F is increasing, since for $y < x < 0$, then $(x,0] \subseteq (y,0]$, hence $F(y) = -\mu_F((y,0]) \leq -\mu_F((x,0]) = F(x)$, and for $0 \leq y < x$, then $(0,y] \subseteq (0,x]$, hence $F(y) = \mu_F((0,y]) \leq \mu_F((0,x]) = F(x)$ by monotonicity of the positive measure μ_F and for $y < 0 \leq x$, it holds that $F(y) = -\mu_F((y,0]) \leq \mu_F((0,x])$. Let $x \geq 0$. Then $\{(0, x+\frac{1}{n}]\}_{n\geq 1}$ is a decreasing sequence of sets $(0, x+\frac{1}{n}] \in \mathcal{B}(\mathbb{R})$ with $\bigcap_{n=1}^\infty (0, x+\frac{1}{n}] = (0,x]$, and $\mu_F((0, x+1]) < \infty$, since μ is assumed to be finite on bounded sets, hence for $y \in [x, x+1]$, by continuity from above,

$$\lim_{y\to x^+} F(y) = \lim_{n\to\infty} \mu_F((0, x+\tfrac{1}{n}]) = \mu_F(\bigcap_{n=1}^\infty (0, x+\tfrac{1}{n}]) = \mu_F((0,x]) = F(x).$$

Similarly, if $x < 0$, then $\{(x+\frac{1}{n}, 0]\}_{n\geq 1}$ is an increasing sequence with $\bigcup_{n=1}^\infty (x, 0]$, hence for $y \in [x, x+1]$, by continuity from below,

$$\lim_{y\to x^+} F(y) = \lim_{n\to\infty} \mu_F((0, x+\tfrac{1}{n}]) = \mu_F(\bigcup_{n=1}^\infty E_n) = \mu_F((x,0]) = F(x).$$

This completes the proof.

□

Remark 7.3. The measure μ_F is called the *Lebesgue-Stieltjes measure associated to F*. If μ is a finite Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $\mu = \mu_F$ with $F(x) = \mu((-\infty, x])$, and F is the cumulative distribution function of μ. Note that if $\tilde{F}$ is the function specified in Theorem 7.2, then $F(x) = \tilde{F}(x) + \mu((-\infty, 0])$, i.e., F differs from $\tilde{F}$ by a constant $\mu((-\infty, 0])$. The results from the previous section about differentiation on Euclidean spaces apply in particular to $\mathbb{R}$, and thus, by the correspondence between positive Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and increasing, right-continuous functions from Theorem 7.2, these results are in particular results about differentiation and integration of such functions. Lebesgue-Stieltjes measures hold some nice regularity properties, which are to be examined.

Proposition 7.4. *Let μ be a Lebesgue-Stieltjes measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Then*

$$\mu(E) = \inf\{\mu(U) : U \text{ open, } U \supseteq E\} \qquad \text{(Outer regularity)}$$
$$= \sup\{\mu(K) : K \text{ compact, } K \subseteq E\}. \qquad \text{(Inner regularity)}$$

Proof. It is clear that for every $E \in \mathcal{B}(\mathbb{R})$, it holds that

$$\mu(E) = \inf\Big\{\sum_{i=1}^{\infty} F(b_i) - F(a_i) : E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]\Big\} = \inf\Big\{\sum_{i=1}^{\infty}\mu((a_i,b_i]) : E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]\Big\}.$$

To prove outer regularity of μ, it is proven that $\mu(E) = \inf\{\sum_{i=1}^{\infty}\mu(a_i,b_i)) : E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)\}$ and then this is generalized to the wanted equality. Let $E \in \mathcal{B}(\mathbb{R})$ and suppose that $E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)$. Note that $(a_i,b_i) = \dot\bigcup_{n=1}^{\infty}\big(a_i - \frac{(b_i-a_i)}{n} + (b_i-a_i), b_i - \frac{(b_i-a_i)}{n+1}\big]$ for each $i \in \mathbb{N}$. Thus,

$$\mu\Big(\bigcup_{i=1}^{\infty}(a_i,b_i)\Big) = \sum_{i=1}^{\infty}\mu((a_i,b_i)) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\mu\Big(\big(a_i - \tfrac{(b_i-a_i)}{n} + (b_i-a_i), b_i - \tfrac{(b_i-a_i)}{n+1}\big]\Big) \geq \mu(E).$$

Conversely, given $\varepsilon > 0$, there exists $\{(a_i,b_i]\}_{i\geq 1}$ with $E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]$ and $\sum_{i=1}^{\infty}\mu((a_i,b_i]) \leq \mu(E)+\varepsilon$. Then by right-continuity of F, one may choose $\delta_i > 0$ such that $F(b_i+\delta_i) - F(b_i) \leq \varepsilon + 2^{-i}$. Then $E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i+\delta_i)$ and

$$\sum_{i=1}^{\infty}\mu((a_i,b_i+\delta_i)) \leq \sum_{i=1}^{\infty}\mu((a_i,b_i]) + \varepsilon \leq \mu(E) + 2\varepsilon,$$

which proves the equality with open intervals. Let $U = \bigcup_{i=1}^{\infty}(a_i,b_i)$. Then U is open with $E \subseteq U$ and $\mu(U) \leq \mu(E)+\varepsilon$. Since also $\mu(U) \geq \mu(E)$, this proves that μ is outer regular.

To prove that μ is inner regular, suppose first that $E \in \mathcal{B}(\mathbb{R})$ is bounded. Then if E is also closed, E is compact, in which case the equality is obvious. Suppose E is not closed. Let $\varepsilon > 0$ be given. Then one may choose $U \supseteq \overline{E}\setminus E$ open such that $\mu(E) \leq \mu(\overline{E}\setminus E) + \varepsilon$. Now, set $K := \overline{E}\setminus U$. Then K is clearly compact and $K \subseteq E$. Thus,

$$\mu(K) = \mu(\overline{E}) - \mu(U) = \mu(\overline{E}\setminus E) + \mu(E) - \mu(U) \geq \mu(E) - \varepsilon,$$

which proves the equality in the case, where $E \in \mathcal{B}(\mathbb{R})$ is bounded. Suppose E is unbounded. Let $E_i := E \cap (i,i+1]$ for each $i \in \mathbb{N}$ such that $E = \dot\bigcup_{i=-\infty}^{\infty} E_i$. Then each E_i is bounded, so for every $\varepsilon > 0$ there exists a compact set $K_i \subseteq E_i$ such that $\mu(K_i) \geq \mu(E_i) - \varepsilon 2^{-|i|}$. Let $H_n := \dot\bigcup_{i=-n}^{n} K_i$. Then H_n is compact, $H_n \subseteq E$ and

$$\mu(E) \geq \mu(H_n) = \sum_{i=-n}^{n}\mu(K_i) \geq \sum_{i=-n}^{n}\mu(E_i) - \varepsilon 2^{-|i|} = \mu\Big(\dot\bigcup_{i=-n}^{n} E_i\Big) - \varepsilon,$$

and since $\{(\dot\bigcup_{i=-n}^{n} E_i)\}_{n\geq 1}$ is an increasing sequence with $\mu(E) = \lim_{n\to\infty}\mu(\bigcup_{i=-n}^{n} E_i)$, the equality follows as wanted. □

Remark 7.5. As the Lebesgue measure, m, on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is the Lebesgue-Stieltjes measure given by $m((a,b]) = F(a) - F(b)$, where $F(x) = x$ for every $x \in \mathbb{R}$, this also proves Theorem 6.4 in the case where $n = 1$. Moreover, as every right-continuous and increasing function is bounded on closed and bounded intervals, i.e., compact subsets of $\mathbb{R}$, it is clear that Lebesgue-Stieltjes measures are finite on compact sets $K \in \mathcal{B}(\mathbb{R})$, thus, the preceding proposition proves that Lebesgue-Stieltjes measures are in fact regular.

7.2 Functions of bounded variation

Let $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$. In this section, functions of bounded variation are introduced, as well as the theory behind.

Theorem 7.6. *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function, and let $G : \mathbb{R} \to \mathbb{R}$ be defined by $G(x) = F(x+) := \lim_{y \to x^+} F(y)$ for every $x \in \mathbb{R}$. Then*

(i) The set $A := \{x \in \mathbb{R} : F \text{ is discontinuous}\}$ is countable.

(ii) F and G are differentiable m-a.e. with $F' = G'$ m-a.e.

Proof. (i) Note that F is continuous in $x \in \mathbb{R}$ if and only if $F(x+) = F(x-)$. The assumption that F is increasing yields that the intervals $(F(x-), F(x+))$ are disjoint for every $x \in \mathbb{R}$. Moreover, if $|x| < N$ for some $N \in \mathbb{N}$, the intervals are contained in the interval $(F(-N), F(N))$. Thus,

$$\sum_{|x|<N} (F(x+) - F(x-)) \leq F(N) - F(-N) < \infty,$$

which implies that $A_N := \{x \in (-N, N) : F(x+) \neq F(x-)\}$ is countable, since if $F(x+) \neq F(x-)$, i.e., $F(x+) - F(x-) > 0$ for uncountably many $x \in (-N, N)$, there would exist $n \in \mathbb{N}$ such that $S_n := \{x \in (-N, N) : (F(x+) - F(x-)) \geq \frac{1}{n}\}$ is infinite, and thus $\sum_{|x|<N}(F(x+) - F(x-)) \geq \sum_{x \in S_n} \frac{1}{n} = \infty$. Since A_N is countable for every $N \in \mathbb{N}$, A is countable as wanted.

(ii) The assumption that F is increasing together with the definition of $G(x) := F(x+)$ yields that G is increasing and right-continuous. Moreover, $G(x) = F(x)$ for every $x \in \mathbb{R} \setminus A$. Thus, if μ_G is the Lebesgue-Stieltjes measure associated to G, then

$$G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]), & h \geq 0 \\ -\mu_G((x+h, x]), & h < 0 \end{cases}.$$

Note that the families $\{E_r\}_{r>0} = \{(x-r, x]\}_{r>0}$ and $\{E_r\}_{r>0} = \{(x, x+r]\}_{r>0}$ shrink nicely to $x \in \mathbb{R}$ as $r \to 0$: Clearly, $(x-r, x], (x, x+r] \subseteq \overline{B(x, r)}$ for every $r > 0$, and $m(B(x, r)) = 2r$, so $m((x-r, x]) = r = m((x, x+r]) > \frac{2}{3}r = \frac{1}{3}m(B(x, r))$. Thus, letting $|h| = r \to 0$ yields that

$$G'(x) = \lim_{|h| \to 0} \frac{G(x+h) - G(x)}{|h|} = \lim_{r \to 0} \frac{\mu_G(E_r)}{m(E_r)}.$$

Since μ_G is regular, Theorem 6.26 yields that G' exists for m-a.e. $x \in \mathbb{R}$. Now, let $H(x) := G(x) - F(x)$. Thus, the goal is to show that H' exists and equals zero m-a.e. Note that $H(x) = G(x) - F(x) = F(x+) - F(x)$, and thus, the assumption that F is increasing yields that $F(x+) \geq F(x)$, hence $H(x) \geq 0$. Let $A' := \{x \in \mathbb{R} : H(x) > 0\}$. Then $A' \subseteq A$. Now, let $\{x_i\}_{i \geq 1}$ be an enumeration of the $x \in \mathbb{R}$ for which $H \neq 0$. Then $H(x_i) > 0$ for every x_i, and

$$\sum_{|x_i|<N} H(x_i) = \sum_{|x_i|<N} F(x_i+) - F(x_i) \leq F(N) - F(-N) < \infty,$$

since $(F(x_i), F(x_i+))$ are disjoint intervals contained in the interval $(F(-N), F(N))$ for every $|x_i| < N$. Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\mu(E) := \sum_{i \geq 1} H(x_i)\delta_{x_i}(E)$, where δ_{x_i} is the Dirac measure. Then μ is finite on compact sets $K \subseteq (-N, N)$, and $\mu((-\infty, 0]) = \sum_{x_i \leq x} H(x_i)$, where $\sum_{x_i \leq x} H(x_i)$ is clearly increasing and right-continuous, thus μ is regular by Theorem 7.2. Moreover, $m(A') = 0$, since $A' \subseteq A$, and $\mu(\mathbb{R} \setminus A') = 0$, so $\mu \perp m$. Then

$$\begin{aligned} \lim_{|h| \to 0} \left| \frac{H(x+h) - H(x)}{h} \right| &\leq \lim_{|h| \to 0} \frac{H(x+h) + H(x)}{|h|} \leq \lim_{|h| \to 0} \frac{\mu((x-2|h|, x+2|h|))}{|h|} \\ &= \lim_{|h| \to 0} 4\frac{\mu((x-2|h|, x+2|h|))}{4|h|} = \lim_{|h| \to 0} 4\frac{\mu((x-2|h|, x+2|h|))}{m((x-2|h|, x+2|h|))} = 0, \end{aligned}$$

for m-a.e. $x \in \mathbb{R}$ by Theorem 6.26, letting $r = 2|h|$, since the family $\{(x-r, x+r)\}_{r>0}$ shrinks nicely to x as $r \to 0$. Hence, H' exists and $H' = 0$ for m-a.e. $x \in \mathbb{R}$, as wanted. □

Definition 7.7. Let $F: \mathbb{R} \to \mathbb{C}$. The *total variation function of F* is defined by

$$T_F(x) := \sup\Big\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\Big\}.$$

Remark 7.8. Note that $T_F: \mathbb{R} \to [0, \infty]$ is an increasing function. Adding more subdivision points in the sum only increases the value of the sums in the definition of T_F, thus for $a < b$, one may assume that a is always a subdivision point. Hence, the total variation of F on $[a, b]$ is defined as follows.

Definition 7.9. The *total variation of F* on $[a, b]$ is defined by

$$T_F(b) - T_F(a) = \sup\Big\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b\Big\}.$$

Remark 7.10. In the case where $T_F(b) = T_F(a) = \infty$, the definition should be interpreted as $T_F(b) = T_F(a) + \sup\Big\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b\Big\}$.

Definition 7.11. Let $F: \mathbb{R} \to \mathbb{C}$. Then F is of *bounded variation* on $\mathbb{R}$, if $T_F(\infty) := \lim_{x\to\infty} T_F(x) < \infty$, and F is of bounded variation on $[a, b]$ if $T_F(b) - T_F(a) < \infty$. Moreover, let $BV := \big\{F : \mathbb{R} \to \mathbb{C} : T_F(\infty) < \infty\big\}$ and $BV([a, b]) := \big\{F : [a, b] \to \mathbb{C} : T_F(b) - T_F(a) < \infty\big\}$.

Remark 7.12. If $F \in BV$, then restricting F to $[a, b]$ yields that $F\big|_{[a,b]} \in BV([a, b])$. Conversely, if $F \in BV([a, b])$, then extending F by setting $F(x) := F(a)$ for every $x < a$ and $F(x) := F(b)$ for every $x > b$ yields that $F \in BV$.

Example 7.13. The following are examples of functions of bounded variation.

(i) Let $F: \mathbb{R} \to \mathbb{R}$ be bounded and increasing. Then $F \in BV$.

(ii) Let $F, G \in BV$, and let $a, b \in \mathbb{C}$. Then $aF + bG \in BV$.

(iii) Let $F: \mathbb{R} \to \mathbb{R}$ be differentiable with F' bounded. Then $F \in BV([a, b])$ for every $-\infty < a < b < \infty$.

Proof. (i) Consider $T_F(x)$ for $x \in \mathbb{R}$. Since F is increasing,

$$\begin{aligned} T_F(x) &= \sup\Big\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots x_n = x\Big\} \\ &= \sup\Big\{\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) : n \in \mathbb{N}, -\infty < x_0 < \cdots x_n = x\Big\} \\ &= \sup\Big\{F(x) - F(x_0)\Big\} = F(x) - F(-\infty). \end{aligned}$$

Now, since F is bounded, there exists $M \in \mathbb{N}$ such that $|F(x)| \le M$ for all $x \in \mathbb{R}$. Then $T_F(x) = F(x) - F(-\infty) \le 2M < \infty$, hence $T(\infty) < \infty$.

(ii) Consider the sums in the definition of T_{aF+bG}:

$$\begin{aligned} &\sum_{i=1}^{n} |aF(x_i) + bG(x_i) - (aF(x_{i-1}) + bG(x_{i-1}))| \\ \le &\sum_{i=1}^{n} |aF(x_i) - aF(x_{i-1})| + \sum_{i=1}^{n} |bG(x_i) - bG(x_{i-1})| \\ = &|a| \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |b| \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})|, \end{aligned}$$

hence $T_{aF+bG}(x) \le |a|T_F(x) + |b|T_G(x)$ for every $x \in \mathbb{R}$. Thus, $F, G \in BV$ yields that $T_{aF+bG}(\infty) \le |a|T_F(\infty) + |b|T_G(\infty) < \infty$.

(iii) Let $x \in [a,b]$. For every $[x_{i-1}, x_i] \subseteq [a,b]$, the Mean Value Theorem yields that there exists $c \in [x_{i-1}, x_i]$ such that $|F(x_i) - F(x_{i-1})| = |F'(c)||x_i - x_{i-1}|$. Moreover, since F' is bounded, there exists $M \in \mathbb{N}$ such that $|F(x_i) - F(x_{i-1})| = |F'(c)||x_i - x_{i-1}| \le M|x_i - x_{i-1}|$ for every $c \in \mathbb{R}$. Thus,

$$\begin{aligned} T_F(b) - T_F(a) &= \sup\Big\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, a = x_0 < \dots < x_n = b \Big\} \\ &\le \sup\Big\{ \sum_{i=1}^{n} M|x_i - x_{i-1}| : n \in \mathbb{N}, a = x_0 < \dots < x_n = b \Big\} = M(b-a) < \infty. \end{aligned}$$

□

Lemma 7.14. *Let $F:\mathbb{R} \to \mathbb{R}$ be in BV. Then $T_F \pm F$ are bounded and increasing functions.*

Proof. Suppose $x < y \in \mathbb{R}$. Let $\varepsilon > 0$ be given, and choose a partition $x_0 < \dots < x_n = x$ such that $\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \ge T_F(x) - \varepsilon$. Then $T_F(y)$ can be approximated by $\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|$, so

$$\begin{aligned} T_F(y) \pm F(y) &\ge \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \pm F(y) \\ &= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \pm (F(y) - F(x) + F(x)) \\ &\ge \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \pm F(x) \ge T_F(x) - \varepsilon \pm F(x), \end{aligned}$$

hence $T_F(y) \pm F(y) \ge T_F(x) \pm F(x)$, so $T_F \pm F$ are increasing. Then for $x < y \in \mathbb{R}$,

$$|F(y) - F(x)| \le T_F(y) - T_F(x) \le T_F(\infty) - T_F(-\infty) < \infty,$$

which implies that F is bounded, and hence $T_F \pm F$ are bounded. □

Theorem 7.15. *The following hold.*

(i) $F \in BV$ if and only if $\mathfrak{R}(F) \in BV$ and $\mathfrak{I}(F) \in BV$.

(ii) Let $F:\mathbb{R} \to \mathbb{R}$. Then $F \in BV$ if and only if F is the difference of two bounded and increasing functions.

(iii) Let $F \in BV$. Then the limits $F(x+)$ and $F(x-)$ exists for every $x \in \mathbb{R}$, and also the limits $F(\pm\infty)$ exists.

(iv) Let $F \in BV$. Then $A := \{x \in \mathbb{R} : F \text{ is discontinuous}\}$ is a countable set.

(v) Let $F \in BV$ and $G(x) := F(x+)$ for every $x \in \mathbb{R}$. Then F',G' exist, and $F' \overset{m-a.e.}{=} G'$.

Proof. (i) If $F = \mathfrak{R}(F) + i\mathfrak{I}(F) \in BV$, then $|\mathfrak{R}(F)| \le |F(x)|$ for every $x \in \mathbb{R}$ yields that $T_{\mathfrak{R}(F)}(x) \le T_F(x)$ for every $x \in \mathbb{R}$, hence $T_{\mathfrak{R}(F)}(\infty) \le T_F(\infty) < \infty$, so $\mathfrak{R}(F) \in BV$. Similarly for $\mathfrak{I}(F)$. Conversely, if $\mathfrak{R}(F), \mathfrak{I}(F) \in BV$, then $|F(x)| \le |\mathfrak{R}(F)(x)| + |\mathfrak{I}(F)(x)|$ for every $x \in \mathbb{R}$ yields that $T_F(x) \le T_{\mathfrak{R}(F)}(x) + T_{\mathfrak{I}(F)}(x)$ for every $x \in \mathbb{R}$, hence $T_F(\infty) \le T_{\mathfrak{R}(F)}(\infty) + T_{\mathfrak{I}(F)}(\infty) < \infty$.

(ii) Suppose $F = G - H$, where $G, H:\mathbb{R} \to \mathbb{R}$ are bounded and increasing. Then by Example 7.13 (i), $G, H \in BV$, and thus by Example 7.13 (ii), $F = G - H \in BV$. Conversely, suppose

$F:\mathbb{R} \to \mathbb{R}$ and $F \in BV$. Then $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$, where $T_F \pm F$ are bounded and increasing by Lemma 7.14.

(iii) Let $F \in BV$. Then also $\Re(F), \Im(F) \in BV$ by (i), hence by (ii), each of the functions $\Re(F), \Im(F): \mathbb{R} \to \mathbb{R}$ is the difference of two bounded and increasing functions, i.e., $\Re(F) = F_1 - F_2$ and $\Im(F) = F_3 - F_4$. Then $F = F_1 - F_2 + i(F_3 - F_4)$, where F_i bounded and increasing for $i \in \{1, ..., 4\}$. Therefore, the limits $F_i(x+)$ and $F_i(x-)$ exist and are finite for every $x \in \mathbb{R}$ and $i \in \{1, ..., 4\}$. Moreover, F_i being bounded yields that $F_i(\pm\infty) < \infty$ for $i \in \{1, ..., 4\}$, so

$$F(x+) = F_1(x+) - F_2(x+) + i(F_3(x+) - F_4(x+)),$$

so $F(x+)$ exists. Similarly for $F(x-)$. And also

$$F(\pm\infty) = F_1(\pm\infty) - F_2(\pm\infty) + i(F_3(\pm\infty) - F_4(\pm\infty)),$$

so the limits $F(\pm\infty)$ exists.

(iv) As in (iii), $F \in BV$ yields that $F = F_1 - F_2 + i(F_3 - F_4)$, where each F_i is bounded and increasing. Then by Theorem 7.6 (i), the sets $A_i := \{x \in \mathbb{R} : F_i \text{ is discontinuous}\}$ are countable for $i \in \{1, .., 4\}$. Now, clearly, $A := \{x \in \mathbb{R} : F \text{ is discontinuous}\} \subseteq \bigcup_{i=1}^4 A_i$, and thus, since every finite union of countable sets is countable, $\#A \leq \# \bigcup_{i=1}^4 A_i$ yields that A is countable as wanted.

(v) For $F \in BV$ with $F = F_1 - F_2 + i(F_3 - F_4)$ for F_i bounded and increasing, define $G_i(x) := F_i(x+)$ for each $i \in \{1, ..., 4\}$. Thus, $G(x) = F(x+) = F_1(x+) - F_2(x+) + i(F_3(x+) - F_4(x+)) = G_1(x+) - G_2(x+) + i(G_3(x+) - G_4(x+))$, and by Theorem 7.15 (i), $G_i' = F_i'$ m-a.e. for each $i = 1, ..., 4$. Hence, $G' = F'$ m-a.e. as wanted. ☐

Definition 7.16. Let $F:\mathbb{R} \to \mathbb{R}$ be in BV. The representation $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ is called the *Jordan decomposition* of F, and $\frac{1}{2}(T_F \pm F)$ are called the positive, respectively negative, variations of F.

Proposition 7.17. *Let $F:\mathbb{R} \to \mathbb{R}$ be in BV. Then the positive and negative variations are*

$$\tfrac{1}{2}(T_F \pm F)(x) = \sup\Big\{\sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} : n \in \mathbb{N} : -\infty < x_0 < \cdots < x_n = x\Big\} \pm \tfrac{1}{2}F(-\infty).$$

Proof. Note that $x^{\pm} := \max(\pm x, 0) = \frac{1}{2}(|x| \pm x)$ for $x \in \mathbb{R}$. Thus, since $\sum_{i=1}^n F(x_i) - F(x_{i-1}) = F(x_n) - F(x_0)$, one obtains that

$$\begin{aligned}
1/2(T_F \pm F)(x) &= \sup\Big\{\sum_{i=1}^{n} 1/2(|F(x_i) - F(x_{i-1})| \pm F(x)) : -\infty < x_0 < \cdots < x_n = x\Big\} \\
&= \sup\Big\{\sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} \pm 1/2F(x_0) : -\infty < x_0 < \cdots < x_n = x\Big\} \qquad (*) \\
&= \sup\Big\{\sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} : -\infty < x_0 < \cdots < x_n = x\Big\} \pm \tfrac{1}{2}F(-\infty). \qquad (**)
\end{aligned}$$

To justify that $(*) = (**)$, let $\varepsilon > 0$ be given. There exists x_0 such that $F(-\infty) - \varepsilon < F(x_0) < F(-\infty) + \varepsilon$. Assume without loss of generality that x_0 is a subdivision point. Then for every partition,

$$\sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} \pm 1/2F(-\infty) \leq \sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} \pm 1/2F(x_0) \pm \frac{\varepsilon}{2} \leq (*) + \frac{\varepsilon}{2},$$

$$\sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))^{\pm} \pm 1/2F(x_0) \leq \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})^{+} \pm 1/2F(-\infty) \pm \frac{\varepsilon}{2} \leq (**) + \frac{\varepsilon}{2},$$

for every partition, which implies that also $(*) \leq (**)$. ☐

7.3 Characterization of complex Borel measures

The theory of functions of bounded variation rises the question: *Which functions of bounded variation correspond to a complex Borel measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*?* In this section, an answer to this question is provided. Initially, functions of normalized bounded variation are introduced.

Definition 7.18. Define the space of functions of *normalized bounded variation* by

$$NBV := \big\{F \in BV : F \text{ is right-continuous and } F(-\infty) = 0\big\}.$$

Example 7.19. Let $F \in BV$. Then $G: \mathbb{R} \to \mathbb{C}$ defined by $G(x) := F(x+) - F(-\infty)$ lies in NBV.

Proof. Since $F(-\infty)$ is just a constant, non-dependent of $x \in \mathbb{R}$, it is clear from Theorem 7.15 that $G \in BV$. Moreover, G is right-continuous by definition, since $\lim_{y\to x^+} G(y) = F(x+) - F(-\infty) = G(x)$, and $G(-\infty) = \lim_{y\to-\infty} F(y+) - F(-\infty) = F(-\infty) - F(-\infty) = 0$. □

Lemma 7.20. *Let $F \in BV$. Then $T_F(-\infty) = 0$. Moreover, if F is right-continuous, then so is T_F.*

Proof. Let $\varepsilon > 0$ be given. Let $x \in \mathbb{R}$, and choose $-\infty < x_0 < \cdots < x_n = x$ such that $\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon$. By definition,

$$T_F(x) - T_F(x_0) = \sup\Big\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, x_0 < \cdots < x_n = x\Big\},$$

hence, $T_F(x) - T_F(x_0) \geq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \geq T_F(x) - \varepsilon$, which implies that $T_F(x_0) \leq \varepsilon$. Then T_F being increasing yields that $T_F(y) \leq \varepsilon$ for every $y \in \mathbb{R}$ with $y < x_0$. Therefore, T_F being a positive function yields that $T(-\infty) = 0$. Now, suppose that F is right-continuous. Let $\varepsilon > 0$ be given, and let $x \in \mathbb{R}$. Moreover, let $\alpha := T_F(x+) - T_F(x)$, and choose $\delta > 0$ such that $|F(x+h) - F(x)| < \varepsilon$ and $T_F(x+h) - T_F(x+) < \varepsilon$, whenever $0 < h < \delta$. The goal is to prove that $\alpha = 0$. Now, for $0 < h < \delta$, there exists a partition $x = x_0 < \cdots < x_n = x + h$ such that

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \geq {}^3\!/\!_4(T_F(x+h) - T_F(x)) \geq {}^3\!/\!_4(T(x+) - T_F(x)) = {}^3\!/\!_4\alpha,$$

hence $\sum_{i=2}^n |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}\alpha - |F(x_1) - F(x)| \geq \frac{3}{4}\alpha - \varepsilon$. Likewise, there exists a partition $x = t_0 < \cdots < t_m = x_1$ such that $\sum_{i=1}^m |F(t_i) - F(t_{i-1})| \geq \frac{3}{4}\alpha$. Then $x = t_0 < \cdots < t_m < x_2 < \cdots < x_n = x + h$ is a partition of $[x, x+h]$, and thus

$$\begin{aligned}
\alpha + \varepsilon > T_F(x+h) - T_F(x) &= T_F(x+h) - T_F(x) + T_F(x_1) - T_F(x) - (T_F(x_1) - T_F(x)) \\
&\geq \sum_{i=1}^m |F(t_i) - F(t_{j-1})| + \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \\
&\geq \tfrac{3}{4}\alpha + \tfrac{3}{4}\alpha - \varepsilon = \tfrac{3}{2}\alpha - \varepsilon,
\end{aligned}$$

hence $\alpha < 4\varepsilon$, and therefore, $\varepsilon > 0$ being arbitrary yields that $\alpha = 0$ as wanted. □

Theorem 7.21. *Let μ be a complex Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $F: \mathbb{R} \to \mathbb{C}$ is defined by $F(x) := \mu((-\infty, x])$, then $F \in NBV$. Conversely, if $F \in NBV$, there exists a unique complex Borel measure μ_F such that $\mu_F((-\infty, x]) = F(x)$. Moreover, $|\mu_F| = \mu_{T_F}$.*

Proof. Any complex measure can be decomposed as $\mu = \mu_{\mathfrak{R}} + i\mu_{\mathfrak{I}} = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-)$, where $\mu_{\mathfrak{R}}, \mu_{\mathfrak{I}}$ are finite signed measures, i.e., $\mu_i^\pm$ is a finite positive measure for each $i \in \{1, 2\}$. Suppose $F_i^\pm$ is defined by $F_i^\pm(x) := \mu_i^\pm((-\infty, x])$. Then $F_i^\pm$ is increasing and right-continuous by Theorem 7.2 and the following remark. Moreover, $F_i^\pm(-\infty) = \mu_i^\pm(\varnothing) = 0$, and $F_i^\pm(\infty) =$

$\mu_i^\pm(\mathbb{R}) < \infty$. Thus, every $F_i^\pm$ is increasing and bounded, hence $F = F_1^+ - F_1^- + i(F_2^+ - F_2^-)$ is of bounded variation by Theorem 7.15 (i) and (ii). Now, F is clearly also right-continuous, and $F(-\infty) = F_1^+(-\infty) - F_1^-(-\infty) + i(F_2^+(-\infty) - F_2^-(-\infty)) = 0$, hence $F \in NBV$.

Conversely, let $F \in NBV$. By Theorem 7.15 (i), $\mathfrak{R}(F), \mathfrak{I}(F) \in BV$, thus by (ii),

$$F = \tfrac{1}{2}(T_{\mathfrak{R}(F)} + \mathfrak{R}(F)) - \tfrac{1}{2}(T_{\mathfrak{R}(F)} - \mathfrak{R}(F)) + i(\tfrac{1}{2}(T_{\mathfrak{I}(F)} + \mathfrak{I}(F)) - \tfrac{1}{2}(T_{\mathfrak{I}(F)} - \mathfrak{I}(F))).$$

It is clear that F being right-continuous and $F(-\infty) = 0$ implies that $\mathfrak{R}(F)$ and $\mathfrak{I}(F)$ are right-continuous, and $\mathfrak{R}(F)(-\infty) = \mathfrak{I}(F)(-\infty) = 0$. Then by Lemma 7.20, $T_{\mathfrak{R}(F)}(-\infty) = T_{\mathfrak{I}(F)}(-\infty) = 0$, and $T_{\mathfrak{R}(F)}$ and $T_{\mathfrak{R}(F)}$ are right-continuous. Moreover, $T_{\mathfrak{R}(F)} \pm \mathfrak{R}(F)$ and $T_{\mathfrak{I}(F)} \pm \mathfrak{I}(F)$ are increasing by Lemma 7.14. This implies that $F = F_1^+ - F_1^- + i(F_2^+ - F_2^-)$, where $F_i^\pm \in NBV$, and in particular, $F_i^\pm$ is right-continuous and increasing for each $i \in \{1,2\}$. Thus by Theorem 7.2, each $F_i^\pm$ gives rise to a unique finite positive Borel measure, $\mu_{F_i^\pm}$ with $F_i^\pm(x) = \mu_{F_i^\pm}((-\infty, x])$, such that $F(x) = \mu_F((-\infty, x])$, and μ_F is unique by the uniqueness of the decomposition of a complex measure into its real and imaginary part, as well as the uniqueness of the Jordan decomposition of these.

To complete the proof, let $\mu_{T_F}((-\infty, x]) = T_F(x)$. The goal is to prove that $|\mu_F| = \mu_{T_F}$. By definition of the total variation,

$$\begin{aligned}\mu_{T_F}((-\infty, x]) = T_F(x) &= \sup\Big\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\Big\} \\ &= \sup\Big\{\sum_{i=1}^n |\mu((x_{i-1}, x_i])| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\Big\} \\ &\le \sup\Big\{\sum_{i=1}^\infty |\mu(E_i)| : n \in \mathbb{N}, \dot{\bigcup_{i=1}^\infty} E_i = (-\infty, x]\Big\} = |\mu_F|((-\infty, x]),\end{aligned}$$

hence $\mu_{T_F}((-\infty, x]) \le |\mu_F|((-\infty, x])$ for every $x \in \mathbb{R}$. To prove the other inequality, let $(a, b] \in \mathcal{B}(\mathbb{R})$ for $a < b$. Then

$$|\mu_F((a, b])| = |F(b) - F(a)| \le T_F(b) - T_F(a) = \mu_{T_F}((a, b]).$$

Let $S := \big\{\dot\bigcup_{i=1}^n (a_i, b_i] : n \in \mathbb{N}, -\infty \le a_i \le b_i \le \infty\big\}$. Then S is a ring over $\mathcal{P}(\mathbb{R})$, as S is closed under differences and pairwise unions. Moreover, in the proof of Theorem 7.2 it has been obtained that $\mathscr{S} = \big\{(a, b] : a \le b \in \mathbb{R}\big\}$ is a semi-ring, thus, S being the collection of finite disjoint unions of elements in $\mathscr{S}$ yields that S is a ring. By countable additivity,

$$\Big|\mu_F(\dot{\bigcup_{i=1}^n}(a_i, b_i])\Big| = \Big|\sum_{i=1}^n \mu_F((a_i, b_i])\Big| \le \sum_{i=1}^n |\mu_F((a_i, b_i])| \le \sum_{i=1}^n \mu_{T_F}((a_i, b_i]) = \mu_{T_F}(\dot{\bigcup_{i=1}^n}(a_i, b_i]),$$

hence $|\mu_F(E)| \le \mu_{T_F}(E)$ for every $E \in S$. Now, let $\mathcal{M} := \big\{E \in \mathcal{B}(\mathbb{R}) : |\mu_F(E)| \le \mu_{T_F}(E)\big\}$, and let $\{E_i\}_{i \ge 1}$ be an increasing sequence in $\mathcal{M}$. Then by continuity from below,

$$|\mu_F(\bigcup_{i=1}^\infty E_i)| = \lim_{i\to\infty} |\mu_F(E_i)| \le \lim_{i\to\infty} \mu_{T_F}(E_i) = \mu_{T_F}(\bigcup_{i=1}^\infty E_i),$$

so $\bigcup_{i=1}^\infty E_i \in \mathcal{M}$. Let $\{E_i\}_{i \ge 1}$ be a decreasing sequence in $\mathcal{M}$. Since μ_F is a complex measure and thus finite, and also μ_{T_F} is a finite measure, since $T_F(-\infty) < \infty$, continuity from above yields that also

$$|\mu_F(\bigcap_{i=1}^\infty E_i)| = \lim_{i\to\infty} |\mu_F(E_i)| \le \lim_{i\to\infty} \mu_{T_F}(E_i) = \mu_{T_F}(\bigcap_{i=1}^\infty E_i),$$

so $\bigcap_{i=1}^\infty E_i \in \mathcal{M}$. Hence, $\mathcal{M}$ is closed under countable monotone unions and intersections, so $\mathcal{M}$ is a monotone class. Thus, if $M(S)$ is the smallest monotone class containing S, then $S \subseteq \mathcal{M}$ yields that $M(S) \subseteq \mathcal{M}$. By Monotone Class Theorem for rings (Theorem 7.1, Berezansky, Sheftel, and Us 1996), $M(S) = \sigma_R(S)$, where $\sigma_R(S)$ is the σ-ring generated by S. Since $\sigma_R(S)$ is closed under countable unions, one obtains $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n,n] \in \sigma_R(S)$, and therefore, $\sigma_R(S) = \sigma(S)$, where $\sigma(S)$ is the σ-algebra generated by S. Hence $\mathcal{B}(\mathbb{R}) = \sigma(S) \subseteq \mathcal{M}$, and thus, it holds for every $E \in \mathcal{B}(\mathbb{R})$ that $|\mu_F(E)| \le \mu_{T_F}(E)$. Then

$$\begin{aligned} |\mu_F|((-\infty,x]) &= \sup\Big\{\sum_{i=1}^{\infty}|\mu_F(E_i)| : (-\infty,x] = \dot{\bigcup_{i=1}^{\infty}} E_i\Big\} \\ &\le \sup\Big\{\sum_{i=1}^{\infty}\mu_{T_F}(E_i) : (-\infty,x] = \dot{\bigcup_{i=1}^{\infty}} E_i\Big\} = \mu_{T_F}((-\infty,x]), \end{aligned}$$

which proves the other inequality, hence $|\mu_F| = \mu_{T_F}$ as wanted. □

Remark 7.22. The preceding theorem answers the question posed earlier, that is, functions of normalized bounded variation correspond to complex Borel measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. It turns out that there is a direct link between the positive and negative variations of $F \in NBV$ defined in Definition 7.16 and the positive and negative variations of the corresponding Borel measure: If $F \in NBV$ is a real-valued function such that $\mu_F((-\infty,x]) = F(x)$ is a finite signed Borel measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, then the positive and negative variations of μ_F is given by $\mu_F^{\pm} = \mu_{\frac{1}{2}(T_F \pm F)}$. This follows, since the total variation of μ_F is given by $|\mu_F| = \mu_T$. The next question to arise is: *Which functions of normalized bounded variation correspond to complex Borel measures that are singular, respectively, absolutely continuous with respect to Lebesgue measure?*

Lemma 7.23. *Let $F \in NBV$, and let $\mu_F((-\infty,x]) = F(x)$. Then μ_F is regular.*

Proof. Recall that μ_F is regular if $|\mu_F|$ is regular. By Theorem 7.15, $|\mu_F| = \mu_{T_F}$, where $\mu_{T_F}((-\infty,x]) = T_F(x)$ for T_F increasing and right-continuous by Lemma 7.20. Thus, by Proposition 7.4, μ_{T_F} is regular, and hence, so is μ_F. □

Proposition 7.24. *Let $F \in NBV$. Then the following properties hold.*

(i) F is differentiable with $F' \in L^1(m)$.

(ii) $\mu_F \perp m$ if and only if $F' = 0$ m-a.e.

(iii) $\mu_F \ll m$ if and only if $F(x) = \int_{-\infty}^{x} F'(t)dm(t)$.

Proof. (i) Let $F \in NBV$ and μ_F the corresponding complex Borel measure from Theorem 7.21 such that $\mu_F((-\infty,x]) = F(x)$. Let $d\mu_F = d\lambda + f dm$ be the Lebesgue decomposition of μ with respect to m. Consider the difference quotient of F.

$$\frac{|F(x+h)-F(x)|}{|h|} = \begin{cases} \frac{\mu_F((x,x+h])}{m((x,x+h])}, & h > 0 \\ \frac{\mu_F((x+h,x])}{m((x+h,x])}, & h < 0 \end{cases}.$$

Since the families $\{E_r\}_{r>0} = \{(x-r,x]\}_{r>0}$ and $\{E_r\}_{r>0} = \{(x,x+r]\}_{r>0}$ shrink nicely to $x \in \mathbb{R}$ as $r \to 0$, letting $|h| = r \to 0$ yields that

$$F'(x) = \lim_{|h|\to 0}\frac{|F(x+h)-F(x)|}{|h|} = \lim_{r\to 0}\frac{\mu_F(E_r)}{m(E_r)} = f(x)$$

for m-a.e. $x \in \mathbb{R}$ by Theorem 6.26, since μ_F is regular by Lemma 7.23. Moreover, as μ_F is a complex measure, Theorem 4.3 (The Lebesgue-Radon-Nikodym Theorem for complex measures) yields that $F' \overset{m\text{-a.e.}}{=} f \in L^1(m)$.

(ii) Suppose $\mu_F \perp m$ such that $\mathbb{R} = A \cup B$, where $\mu_F(A) = m(B) = 0$. Now, for every $x \in \mathbb{R}$, it holds that either $x \in A$ or $x \in B$, thus for a family $\{E_r\}_{r>0}$, which shrinks nicely to x as $r \to 0$, one eventually obtains that either $E_r \subseteq A$ or $E_r \subseteq B$. So for every $x \in A$,

$$F'(x) = \lim_{r\to 0} \frac{\mu_F(E_r)}{m(E_r)} = 0,$$

as $\mu_F(E_r) = 0$ as $r \to 0$, so $\{x \in \mathbb{R} : F' \neq 0\} \subseteq B$ for a Lebesgue null set B, hence $F' = 0$ m-a.e.

Conversely, suppose $F' = 0$ m-a.e. Let $A := \{x \in \mathbb{R} : F' = 0\}$ and let $B := \mathbb{R} \setminus A$. By assumption, $F' = 0$ m-a.e., hence $m(B) = 0$, and for every $x \in A$,

$$0 = F'(x) = \lim_{r\to 0} \frac{\mu_F(E_r)}{m(E_r)},$$

hence $\mu_F(A) = 0$. Thus, $\mu_F \perp m$.

(iii) Suppose $\mu_F \ll m$. By Theorem 4.6 (The Radon-Nikodym Theorem), $d\mu_F = f dm$ for $f \in L^1(\mu)$. Also, by part (i), $F' = f$ m-a.e., which yields that

$$F(x) = \mu((-\infty, x]) = \int_{(-\infty,x]} f(t)dm(t) = \int_{(-\infty,x]} F'(t)dm(t).$$

Conversely, suppose $F(x) = \int_{-\infty}^{x} F'(t)dm(t)$. Then $\mu_F((-\infty, x]) = F(x) = \int_{-\infty}^{x} F'(t)dm(t)$, so $d\mu_F = F'dm$, hence $\mu_F \ll m$. □

Definition 7.25. Let $F: \mathbb{R} \to \mathbb{R}$. Then F is *absolutely continuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\sum_{i=1}^{N} |F(b_i) - F(a_i)| < \varepsilon$, whenever $\sum_{i=1}^{N}(b_i - a_i) < \delta$ for a finite set of disjoint intervals $(a_1, b_1), ..., (a_N, b_N)$, i.e., $a_1 < b_1 < a_2 < \cdots < a_N < b_N$.

Remark 7.26. If F is absolutely continuous, then F is also uniformly continuous: Let $N = 1$ and $\varepsilon > 0$, then for every δ, one obtains that $|F(b) - F(a)| < \varepsilon$, whenever $|b - a| < \delta$. Thus, absolute continuity is stronger than uniform continuity. Moreover, if F is everywhere differentiable with F' bounded, then F is absolutely continuous by the Mean Value Theorem: If $|F'| \leq M$ for some $M \in \mathbb{N}$, then for every $\varepsilon > 0$, one may choose $\delta := \frac{\varepsilon}{M}$, such that $\sum_{n=1}^{N} |F(b_n) - F(a_n)| \leq M \sum_{n=1}^{N} |b_n - a_n| \leq M\delta = \varepsilon$, whenever $\sum_{n=1}^{N} |b_n - a_n| \leq \delta$. In fact, F being absolutely continuous is equivalent to the corresponding complex Borel measure μ_F being absolutely continuous with respect to the Lebesgue measure, as seen in the following.

Proposition 7.27. *Let $F \in NBV$. Then F is absolutely continuous if and only if $\mu_F \ll m$.*

Proof. Suppose $\mu_F \ll m$. Define $E := \bigcup_{i=1}^{N}(a_i, b_i]$ for some $N \in \mathbb{N}$, and let $\varepsilon > 0$ be given. Then by Theorem 2.26, there exists $\delta > 0$ such that $m(E) = \sum_{i=1}^{N}(b_i - a_i) < \delta$ yields that $|\mu_F(E)| = |\sum_{i=1}^{N} \mu_F(a_i, b_i)| = |\sum_{i=1}^{N} F(b_i) - F(a_i)| < \varepsilon$, hence F is absolutely continuous. Conversely, suppose that F is absolutely continuous. Let $E \in \mathcal{B}(\mathbb{R})$ such that $m(E) = 0$, and let $\varepsilon > 0$ be given. By absolute continuity of F, one may choose $\delta > 0$ such that $\sum_{i=1}^{N} |F(b_i) - F(a_i)| < \varepsilon$, whenever $\sum_{i=1}^{N}(b_i - a_i) < \delta$ for disjoint intervals $(a_1, b_1), ..., (a_N, b_N)$. By Theorem 6.4, $m(E) = \inf\{m(U) : U \supseteq E \text{ open}\}$, so there exists a decreasing sequence $\{U_i\}_{i\geq 1}$ of open sets $U_i \supseteq E$ such that $m(U_1) < \delta$, thus $m(U_i) < \delta$ for every $i \geq 0$. Then by continuity from above, $\lim_{i\to\infty} \mu_F(U_i) = \mu_F(E)$. Now, every open subset of $\mathbb{R}$ can be written as a disjoint union of countably many open intervals, thus, let $U_i = \bigcup_{k=1}^{\infty}(a_i^k, b_i^k)$. Then $\delta > m(U_i) = \sum_{k=1}^{\infty}(b_i^k - a_i^k) \geq \sum_{k=1}^{N}(b_i^k - a_i^k)$ for every $N \in \mathbb{N}$, hence

$$\sum_{k=1}^{N} |\mu_F((a_i^k, b_i^k))| = \sum_{k=1}^{N} |\mu_F((a_i^k, b_i^k])| = \sum_{k=1}^{N} |F(b_i^k) - F(a_i^k)| < \varepsilon,$$

for every $N \in \mathbb{N}$ by absolute continuity of F. Thus, letting $N \to \infty$ yields that $\mu_F(U_i) = \lim_{N\to\infty} \sum_{k=1}^{N} |\mu_F((a_i^k, b_i^k))| \leq \varepsilon$, hence $|\mu_F(E)| \leq \varepsilon$. Now, since $\varepsilon > 0$ was chosen arbitrarily, this shows that $\mu_F(E) = 0$, and thus, $\mu_F \ll m$. □

Corollary 7.28. *Let $f \in L^1(m)$, and let $F: \mathbb{R} \to \mathbb{C}$ be defined by $F(x) := \int_{-\infty}^x f(t)dm(t)$ for every $x \in \mathbb{R}$. Then $F \in NBV$, F is absolutely continuous, and $f = F'$ m-a.e. Conversely, if $F \in NBV$ is absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)dm(t)$.*

Proof. The assumption $f \in L^1(m)$ yields that $d\mu := f dm$ defines a complex measure. Then

$$\mu((-\infty, x]) = \int_{(-\infty,x]} f(t)dm(t) = F(x),$$

hence $F \in NBV$ by Theorem 7.21. Now clearly, $\mu \ll m$, hence F is absolute continuous by Proposition 7.27. Moreover, Proposition 7.24 yields that

$$\int_{-\infty}^x F'(t)dm(t) = F(x) = \int_{-\infty}^x f(t)dm(t),$$

which implies exactly that $F' = f$ m-a.e. Conversely, if $F \in NBV$ is absolutely continuous, and $\mu_F((-\infty, x]) = F(x)$, then $\mu_F \ll m$ by Proposition 7.27. Moreover, F is differentiable with $F' \in L^1(m)$, and $F(x) = \int_{-\infty}^x F'(t)dm(t)$ by Proposition 7.24. □

This winds up the complete characterization of complex Borel measures is given by the theory of functions of bounded variation. The following section proceeds with the proof of the Fundamental Theorem of Calculus, which utilizes the preceding theory.

7.4 The Fundamental Theorem of Calculus

This section concludes the chapter with the proof of the Fundamental Theorem of Calculus for Lebesgue integrals; a result which can be obtained almost directly from the preceding theory of functions of bounded variation. The section is initialized with a lemma.

Lemma 7.29. *Let $F: \mathbb{R} \to \mathbb{C}$ be absolutely continuous on $[a, b]$. Then $F \in BV([a, b])$.*

Proof. Let $\varepsilon = 1$ be given, and choose $\delta > 0$ such that $\sum_{i=1}^N |F(b_i) - F(a_i)| < 1$, whenever $\sum_{i=1}^N (b_i - a_i) < \delta$ for disjoint intervals $(a_1, b_1), ..., (a_N, b_N) \subset [a, b]$. Choose $N := \inf\{n \in \mathbb{N} : n \geq \frac{b-a}{\delta}\}$, such that $\delta \geq \frac{b-a}{N}$. Let $a = x_0 < \cdots < x_n = b$ be any partition of $[a, b]$. Then by (possibly) adding more subdivision points, the intervals (x_{i-1}, x_i) can be collected into at most N groups of consecutive intervals such that $\sum_{i=1}^{k_j} (x_i - x_{i-1}) < \delta$ for $j \in \{1, ..., N\}$, i.e., the sum of the intervals lengths in each group is at most δ. Then $\sum_{i=1}^{k_j} |F(x_i) - F(x_{i-1})| < 1$, hence

$$\begin{aligned} T_F(b) - T_F(a) &= \sup\Big\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : a = x_0 < \cdots < x_n = b \Big\} \\ &= \sup\Big\{ \sum_{j=1}^N \sum_{i=1}^{k_j} |F(x_i) - F(x_{i-1})| : a = x_0 < \cdots < x_{k_N} = b \Big\} \leq N. \end{aligned}$$

□

Theorem 7.30 (Fundamental Theorem of Calculus for Lebesgue Integrals)**.** *Let $F: [a, b] \to \mathbb{C}$ for $-\infty < a < b < \infty$. Then the following are equivalent.*

(i) F is absolutely continuous on $[a, b]$.

(ii) $F(x) - F(a) = \int_a^x f(t)dm(t)$ for some $f \in L^1(m)$.

(iii) F' exists for m-a.e. $x \in [a, b]$ with $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)dm(t)$.

Proof. Assume without loss of generality that $F(a) = 0$. Note that this is justified, since otherwise, one may consider the function $\tilde{F}(x) = F(x) - F(a)$. Moreover, it follows trivially that (iii) implies (ii), thus, it suffices to prove that (ii) implies (i) and (i) implies (iii). To prove the latter, note first that the assumption that F is absolutely continuous on $[a,b]$ yields that $F \in BV([a,b])$ by Lemma 7.29. Now, expand F such that $F(x) = F(a) = 0$ for every $x < a$ and $F(x) = F(b)$ for every $x > b$. Then $F: \mathbb{R} \to \mathbb{C}$, and by Remark 7.12, $F \in BV$. By assumption, F is continuous, hence F is in particular right-continuous, and $F(-\infty) = 0$, so $F \in NBV$. Thus, Corollary 7.28 yields exactly that F' exists m-a.e. and $F' \in L^1(m)$, so $F' \in L^1([a,b], m)$, and $F(x) = \int_{-\infty}^{x} F'(t)dm(t)$, hence $F(x) - F(a) = \int_a^x F'(t)dm(t)$. This proves that (i) implies (iii). Now, to prove that (ii) implies (i), let $f(t) = 0$ for every $t \notin [a,b]$, such that $F(x) = \int_{-\infty}^{x} f(t)dm(t)$. Then, by Corollary 7.28, F is absolutely continuous as wanted. □

The following decomposition of complex Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is sometimes important. Moreover, it provides a nice transition into the final chapter.

Definition 7.31. Let μ be a complex Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then μ is *discrete*, if there exists a countable subset, $\{x_i\}_{i\geq 1} \subseteq \mathbb{R}^n$, and there exists $c_i \in \mathbb{C}$, such that $\sum_{i=1}^{\infty} |c_i| < \infty$ and $\mu = \sum_{i=1}^{\infty} c_i \delta_{x_i}$. Conversely, μ is *continuous* or *non-atomic* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^n$.

Lemma 7.32. *Let μ be a complex Borel measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then μ can be decomposed as $\mu = \mu_d + \mu_c$, where μ_d is discrete, and μ_c is continuous.*

Proof. Define $E \coloneqq \{x \in \mathbb{R}^n : \mu(\{x\}) \neq 0\}$. Let $F \subseteq E$ be any countable subset. Then $\sum_{x\in F} \mu(\{x\}) = \mu(F)$, so the series is convergent, and, in particular, absolutely convergent. The claim is now that $E_n \coloneqq \{x \in E : |\mu(\{x\})| > \frac{1}{n}\}$ is finite for every $n \in \mathbb{N}$. Suppose by contradiction that $E_n \coloneqq \{x \in E : |\mu(\{x\})| > \frac{1}{n}\}$ is infinite. Then there exists a countably infinite subset, $F_n' \subseteq E_n$, i.e., there exists a countable subset $F' \subseteq E$ such that $F_n' \coloneqq \{x \in F' : |\mu(\{x\})| > \frac{1}{n}\} \subseteq E_n$ is countably infinite. However, as the series $\sum_{x\in F'} \mu(\{x\})$ converges absolutely, this is a contradiction. Thus, $E = \bigcup_{n=1}^{\infty} E_n$ being a countable union of finite sets yields that E itself is countable. Now, define measures $\mu_d(A) \coloneqq \mu(A \cap E)$ and $\mu_c(A) \coloneqq \mu(A \setminus E)$ for every $A \in \mathcal{B}(\mathbb{R})$. Then μ_d is discrete, μ_c is continuous, and $\mu(A) = \mu(A \cap E) + \mu(A \setminus E) = \mu_d(A) + \mu_c(A)$ for every $A \in \mathcal{B}(\mathbb{R})$, which completes the proof. □

Remark 7.33. If μ is discrete, then μ is concentrated on a countable subset $\{x_i\}_{i\geq 1} \subseteq \mathbb{R}^n$, i.e., $\mu(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty}\{x_i\}) = 0$, where $m(\bigcup_{i=1}^{\infty}\{x_i\}) = 0$, as countable sets have Lebesgue measure zero. Therefore, it is clear that if μ is discrete, then $\mu \perp m$. On the other hand, if $\mu \ll m$, then $m(\{x\}) = 0$ for every $x \in \mathbb{R}^n$ yields that also $\mu(\{x\}) = 0$, hence μ is continuous. Now, if μ is a complex Borel measure with Lebesgue decomposition $\mu = \psi + f dm$, where $f dm \ll m$ and $\psi \perp m$, then $f dm$ is continuous, and $\psi = \psi_d + \psi_c$ by Lemma 7.32, thus $\psi \perp m$ yields that also $\psi_d \perp m$ and $\psi_c \perp m$. Therefore, every complex Borel measure can be written as

$$\mu = \mu_{ac} + \mu_{sc} + \mu_d,$$

where $\mu_{ac} \ll m$, μ_{sc} is continuous, but $\mu_{sc} \perp m$, and μ_d is discrete. This leads to the existence of a nonzero, singular continuous measure μ_{sc} on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which by Proposition 7.24 and Proposition 7.27 corresponds to functions $F \in NBV$ such that F is not absolutely continuous, but F is differentiable with $F' = 0$ m-a.e. The existence of such a measure is to be examined in the last chapter, which is devoted to answering the question: *Are there any non-atomic measures, which are singular with respect to the Lebesgue measure?*

8 Singularity and the Lebesgue measure

Throughout this chapter, let m denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let $\mathbb{R}$ be equipped with the standard metric topology. The purpose of this section is to give an example of an non-atomic measure, which is singular with respect to the Lebesgue measure. The theory of this section is based on exercises 6.8, 7.12 & 20.9 from Schilling 2017.

8.1 The Cantor ternary set and the Cantor function

In this section, the Cantor ternary set and the Cantor function is introduced, as well as theory behind.

Definition 8.1. Let $C_0 := [0,1]$. Define the *Cantor ternary set*, denoted $\mathcal{C}$, by

$$\mathcal{C} := \bigcap_{n=1}^{\infty} C_n,$$

where $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$ for $n \geq 1$ (with the convention that $\frac{C_n}{3} := \{\frac{x}{3} : x \in C_n\}$).

Remark 8.2. Let $\mathcal{C} \subseteq [0,1]$ be equipped with the subspace topology. The definition of each set, C_n, corresponds to the construction of iteratively removing the open middle third from the initial set C_0, i.e., $C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ etc. Thinking of the construction this way provides a picture of a decreasing nested sequence $\{C_n\}_{n\geq 1}$ with $C_1 \supset C_2 \supset \cdots$. Let $\mathcal{P}([0,1])$ be the power set of $[0,1]$ equipped with the discrete topology. Define a map $\Phi : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ by $\Phi(A) := \frac{1}{3}A \cup (\frac{1}{3}A + \frac{2}{3})$. for every $A \in \mathcal{P}([0,1])$. Then

$$\Phi(C_0) = \Phi([0,1]) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] = C_1,$$

thus, $C_1 \subset C_0$ yields that $C_2 = \Phi(C_1) \subset \Phi(C_0) = C_1$. Continuing this way, one obtains that $C_{n+1} = \Phi(C_n) \subset \Phi(C_{n-1}) = C_n$ for every $n \in \mathbb{N}$, and thus, $\{C_n\}_{n\geq 1}$ is a decreasing sequence with $C_1 \supset C_2 \supset \cdots$. For an intuitive understanding, the first five iterations are sketched in the figure below.

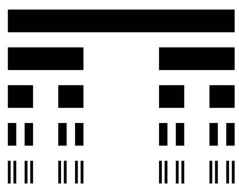

Figure 1: The Cantor set

Lemma 8.3. *The Cantor set, $\mathcal{C}$, has Lebesgue measure zero.*

Proof. Note that every C_n is a Borel set, which consists of 2^n disjoint intervals, each of which has length 3^{-n}. Thus, for each $n \in \mathbb{N}$, it holds that $m(C_n) = 2^n 3^{-n}$, or similarly, by translation invariance of the Lebesgue measure that

$$m(C_n) = m(\Phi(C_{n-1})) = \tfrac{2}{3} m(\Phi(C_{n-2})) = \tfrac{2}{3}\tfrac{2}{3} m(\Phi(C_{n-3})) = \cdots = \left(\tfrac{2}{3}\right)^{n-1} m(\Phi(C_0)) = \left(\tfrac{2}{3}\right)^n.$$

Furthermore, $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$, where $\{C_n\}_{n\geq 1}$ is a decreasing sequence of Borel sets with $m(C_0) = m([0,1]) = 1 < \infty$, thus by continuity from above,

$$m(\mathcal{C}) = m\Big(\bigcap_{n=1}^{\infty} C_n\Big) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\tfrac{2}{3}\right)^n = 0$$

as wanted. □

Proposition 8.4. *The Cantor set, $\mathcal{C}$, holds the following properties.*

(i) $\mathcal{C}$ is metrizable.

(ii) $\mathcal{C}$ is compact.

(iii) $\mathcal{C}$ has no isolated points.

(iv) $\mathcal{C}$ is totally disconnected.

Proof. (i) The Cantor set $\mathcal{C}$ is very clearly metrizable, as it is a subspace of $\mathbb{R}$ given the subspace topology, i.e., the standard metric topology. Note that since each C_n is a finite union of closed intervals, C_n itself is closed. Hence, $\mathcal{C}$ is closed in the metric topology on $\mathbb{R}$, since arbitrary intersections of closed sets are closed. Thus, $\mathcal{C}$, being a closed subset of the complete metric space $\mathbb{R}$, is in particular a complete metric space.

(ii) Observe that the Cantor set is bounded in the region $[0,1]$, since for every $a, b \in \mathcal{C}$, $|b - a| \leq 1 < 1 + \varepsilon$ for every $\varepsilon > 0$. Now, since $\mathcal{C}$ is a closed and bounded subspace of $\mathbb{R}$, $\mathcal{C}$ is compact by the Heine-Borel Theorem. This result also follows from the construction of $\Phi\colon \mathbb{P}([0,1]) \to \mathcal{P}([0,1])$: Every map from a discrete topological space is continuous, since the preimage of every set is open in the discrete topology, hence Φ is a continuous map, and thus, C_0 being compact yields that $\Phi(C_0) = C_1$ is compact, which then again yields that $\Phi(C_1) = C_2$ is compact etc. Then, $\mathcal{C}$ being an intersection of compact sets yields that $\mathcal{C}$ is compact. Furthermore, every finite intersection of sets from the nested sequence, $\{C_n\}_{n\geq 1}$ is non-empty. Thus, by compactness of $\mathcal{C}$ and the finite intersection property, $\mathcal{C}$ is non-empty.

(iii) Let $\varepsilon > 0$ be given. Choose $n \in \mathbb{N}$ large enough so that $3^{-n} < \varepsilon$. Let $x \in \mathcal{C} = \bigcap_{i=1}^{\infty} C_i$. Then $x \in C_n$. Let $J_n^1, ..., J_n^{2^n}$ denote the 2^n intervals, each of length 3^{-n}, which make up C_n arranged in increasing order of their endpoints, i.e., $C_n = \bigsqcup_{k=1}^{2^n} J_n^k$. Then $x \in C_n$ yields that there exists $k \in \{1, ..., 2^n\}$ such that $x \in J_n^k$. Let $J_n^k = [a_k, b_k]$. Then for some $j \in \{1, ..., 2^{n+1}\}$,

$$J_n^k \cap \Phi(C_n) = [a_k, b_k] \cap C_{n+1} = J_{n+1}^j \sqcup J_{n+1}^{j+1} = [a_j, b_j] \sqcup [a_{j+1}, b_{j+1}] = [a_k, b_j] \sqcup [a_{j+1}, b_k],$$

i.e., the map Φ preserves endpoints, so the endpoints are never removed in the iteration. Thus, pick $y = a_k$ or $y = b_k$ such that $y \neq x$ with $y \in C_n$ for every $n \in \mathbb{N}$, hence $y \in \mathcal{C}$. Then $|x - y| \leq 3^{-n} < \varepsilon$. This proves that every neighbourhood of $x \in \mathcal{C}$ contains at least one other point $y \in \mathcal{C}$, so x is not an isolated point, hence $\mathcal{C}$ has no isolated points.

(iv) The claim is that $\mathcal{C}$ is totally disconnected, i.e., the only connected subspaces of $\mathcal{C}$ are the one-point sets. Let $a, b \in \mathcal{C}$ be distinct points in $\mathcal{C}$. Choose $n \in \mathbb{N}$ so large that $|b - a| > 3^{-n}$. Then there exists a point c, which lies between a and b, such that $c \notin C_n$, hence $c \notin \mathcal{C}$. This shows that any subspace of $\mathcal{C}$ containing two points a, b has separation and therefore, it not connected. Thus, $\mathcal{C}$ is totally disconnected. □

Theorem 8.5. *(Brouwer 1910). The Cantor set is uniquely determined up to homeomorphism by the properties given in Proposition 8.4.*

Remark 8.6. As a result of the preceding theorem, any set homeomorphic to the Cantor ternary set can be referred to as *a Cantor set.*

Example 8.7. $X \coloneqq \prod_{i=1}^{\infty} \{0,1\}$ given the product topology is a Cantor set.

Proof. It suffices to prove that X holds the properties from Proposition 8.4.

(i) Let each $\{0,1\}$ be equipped with the discrete topology, and let X be equipped with the product topology. Then $\{0,1\}$ is metrizable, since the discrete topology is induced by the discrete metric. Thus, $X = \prod_{i=1}^{\infty} \{0,1\}$ being a countable product of metric spaces implies that X itself is metrizable (Munkres 2008, p. 129).

(ii) Each $\{0,1\}$ is a finite topological space, hence $\{0,1\}$ is compact. Thus, X is compact by Tychonoff's Theorem (Theorem 37.3, Munkres 2008).

(iii) Assume by contradiction that $x \in X$ is an isolated point, that is, $\{x\} \subseteq X$ is open. Now, every open set in the product topology is a union of sets, $\prod_{i=1}^{\infty} U_i$, where $U_i \subseteq \{0,1\}$ is open and $U_i = \{0,1\}$ for all but finitely many $i \in \mathbb{N}$. Therefore, any singleton $\{x\} \subseteq X$ cannot be open in the product topology, as $\{x\} = \prod_{i=1}^{\infty}\{x_i\}$ for $\{x_i\} = \{0\},\{1\} \neq \{0,1\}$ for every $i \in \mathbb{N}$. Thus, X has no isolated points.

(iv) Every topological space equipped with the discrete topology is totally disconnected, so $\{0,1\}$ is totally disconnected. This may be proven directly, as a connected space is an open subset that cannot be represented as the union of disjoint, non-empty open subsets. Thus, $\{0,1\} = \{0\} \cup \{1\}$ is not connected, hence the only connected subsets of $\{0,1\}$ are the one-point sets, $\{0\}$,$\{1\}$. This implies that also X is totally disconnected, since if $C \subseteq X$ is a connected subspace, which is not a one-point set, then $x \neq y \in C$ for some $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$. Then $x_i \neq y_i$ for some $i \in \mathbb{N}$. Now, let $p_i : X \to \{0,1\}$ be the i'th projection. Since the projection is continuous, and connectedness is preserved under continuous images, $p_i(C) \subseteq \{0,1\}$ is connected. But $\{x_i, y_i\} \subseteq p_i(C)$, and $x_i \neq y_i$ implies that $\{x_i, y_i\} \neq \{0\},\{1\}$, which is a contradiction. Thus, X is totally disconnected. □

Definition 8.8. Let $C_n = \bigcup_{k=1}^{2^n} J_n^k$ with $J_n^k = [a_k, b_k]$, and let $I_n^1, ..., I_n^{2^n-1}$ denote the $2^n - 1$ intervals, which make up $[0,1] \setminus C_n$, arranged in increasing order of their endpoints. Define a sequence $\{F_n\}_{n \geq 1}$ of functions $F_n : \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := \begin{cases} 0, & \text{if } x \leq 0 \\ k2^{-n}, & \text{if } x \in I_n^k, 1 \leq k \leq 2^n - 1 \\ (\frac{3}{2})^n x + k2^{-n} - (\frac{3}{2})^n b_k, & \text{if } x \in J_n^k, 1 \leq k \leq 2^n \\ 1, & \text{if } x \geq 1 \end{cases}.$$

Remark 8.9. The definition of $F_n(x)$, when $x \in J_n^k$ for $1 \leq k \leq 2^n$, corresponds to interpolating linearly, as seen on the sketch below. Thus, by definition, each F_n is a continuous function.

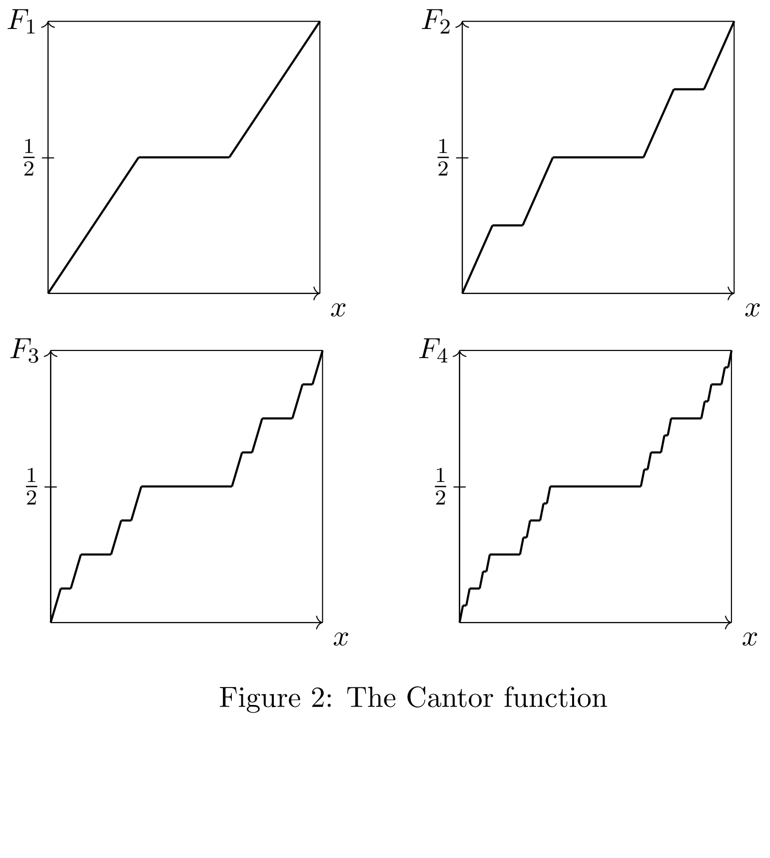

Figure 2: The Cantor function

Proposition 8.10. *Let $\{F_n\}_{n\geq 1}$ be the sequence defined in Definition 8.8. Then the following hold.*

(i) $\{F_n\}_{n\geq1}$ *converges uniformly to a continuous function,* $F: \mathbb{R} \to \mathbb{R}$*, which is called the* Cantor function.

(ii) $F \in NBV$.

(iii) F *is differentiable* m*-a.e. with* $F' = 0$ m*-a.e.*

(iv) F *is not absolutely continuous.*

Proof. (i) Since $\mathbb{R}$ is a complete metric space, it suffices to show that the sequence $\{F_n\}_{n\geq1}$ is Cauchy. By definition, one may observe that

$$|F_n(x) - F_{n+1}(x)| \leq \tfrac{1}{2}2^{-n} = \frac{1}{2^{n+1}},$$

for every $x \in \mathbb{R}$ and for every $n \in \mathbb{N}$. Let $0 < \varepsilon < 1$ be given. Set $N := -\frac{\log(\varepsilon)}{\log(2)}$. Then for every $n, m \geq N$ with $n < m$, and for every $x \in \mathbb{R}$,

$$|F_n(x) - F_m(x)| \leq \sum_{k=n}^{m-1} \frac{1}{2^{k+1}} = \frac{1}{2^n} - \frac{1}{2^m} \leq \frac{1}{2^N} = \varepsilon.$$

Thus, $\{F_n\}_{n\geq1}$ is Cauchy with respect to the uniform norm. Hence, by completeness of $\mathbb{R}$, the sequence $\{F_n\}_{n\geq1}$ converges uniformly to F, and as each F_n is continuous, so is the uniform limit, F.

(ii) That F is monotonically increasing is inherited directly from the pointwise limit, since for every $x < y \in \mathbb{R}$, it holds that $F_n(x) \leq F_n(y)$ for every $n \in \mathbb{N}$. Thus,

$$F(x) = \lim_{n\to\infty} F_n(x) \leq \lim_{n\to\infty} F_n(y) = F(y)$$

for every $x, y \in \mathbb{R}$ with $x < y$. Moreover, as $|F(x)| \leq 1$ for every $x \in \mathbb{R}$, it is clear that F is bounded. Thus, by Example 7.13 (i), $F \in BV$. Also, F is continuous, so F is in particular right-continuous, and $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$, as $F(x) = 0$ for every $x \leq 0$. Thus, $F \in NBV$ as wanted.

(iii) Let $x \in \mathbb{R} \setminus \mathcal{C}$. Then either $x \in (-\infty, 0)$, $x \in (1, \infty)$, or $x \in [0,1] \setminus \mathcal{C}$. Note that if $x \in (-\infty, 0)$, then $F_n(x) = 0$ for every $n \in \mathbb{N}$, and $F(x) = F_n(x)$, which yields that $F_n'(x) = F'(x)$. Similarly, for $x \in (1, \infty)$, where $F_n(x) = 1$ for every $n \in \mathbb{N}$. Thus, , $F'(x) = F_n'(x) = 0$ for every $x \in (-\infty, 0) \cup (1, \infty)$. Now, suppose $x \in [0,1] \setminus \mathcal{C}$. Note that each I_n^k is open, thus there exists $n, k \in \mathbb{N}$ such that $x \in I_n^k$, and hence $F_n(x) = F(x)$. Then for every $x \in \mathbb{R} \setminus \mathcal{C}$, $F(x) = F_n(x)$ for some n, which imply once again that $F'(x) = F_n'(x)$. By definition, $F_n(x)$ is constant on each I_n^k, which yields $F'(x) = F_n'(x) = 0$ for every $x \in [0,1] \setminus \mathcal{C}$, and hence also for every $x \in \mathbb{R} \setminus \mathcal{C}$. Thus, F' exists and equals zero for every $x \in \mathbb{R} \setminus \mathcal{C}$, and since $m(\mathcal{C}) = 0$, F' exists m-a.e. as wanted.

(iv) It is proved that F is not absolutely continuous using Definition 7.25. Let $0 < \varepsilon < 1$ be given. The goal is to prove that there exists a finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ such that $\sum_{k=1}^N b_k - a_k < \delta$, but $\sum_{k=1}^N |F(b_k) - F(a_k)| \nleq \varepsilon$. Let $I_n^k = (a_k, b_k)$ for each $k \in \{1, \ldots, 2^n - 1\}$. The sets $I_n^1, \ldots, I_n^{2^n-1}$ are disjoint, so $a_1 < b_1 < a_2 < \cdots < a_{2^n-1} < b_{2^n-1}$. Thus, with the convention that $b_0 := 0$, the intervals $(b_0, a_1), \ldots, (b_{2^n}, a_{2^n-1})$ form a set of finite disjoint intervals. Then by Lemma 8.3,

$$\sum_{k=1}^{2^n-1} a_k - b_{k-1} = m(C_n) \to m(\mathcal{C}) = 0$$

as $n \to \infty$. Thus, for every $\delta > 0$, one may choose $n \in \mathbb{N}$ such that $\sum_{k=1}^{2^n-1} a_k - b_{k-1} < \delta$. Now consider $\sum_{k=1}^{2^n-1} |F(a_k) - F(b_{k-1})|$. By definition, F is constant on each $I_n^k = (a_k, b_k)$, i.e., $F(a_k) = F(b_k)$ for every $k \in \{1, ..., 2^n - 1\}$. Hence

$$\sum_{k=1}^{2^n-1} |F(a_k) - F(b_{k-1})| = F(a_{2^n-1}) - F(b_0) = F(a_{2^n-1}) - F(0).$$

But $F(a_{2^n-1}) - F(0) \to F(1) - F(0) = 1$ as $n \to \infty$, hence

$$\sum_{k=1}^{2^n-1} |F(a_k) - F(b_{k-1})| \nleq \varepsilon,$$

for $\varepsilon < 1$, which proves that F cannot be absolutely continuous, as wanted. □

This concludes the theory behind the Cantor set and the Cantor function. With this theory presented, the following section proceeds to define the Cantor measure utilizing the preceding results.

8.2 The Cantor measure

In this section, the Cantor measure is defined. This builds upon the preceding section, in particular, the Cantor function, which is proven to be of normalized bounded variation and thus, gives rise to a Borel measure.

Definition 8.11. Define a finite positive Borel measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\mu((-\infty, x]) = F(x)$$

for every $x \in \mathbb{R}$. This measure is called the *Cantor measure.*

Remark 8.12. As $F \in NBV$ by Proposition 8.10 (ii), the definition above does in fact define a unique (complex) Borel measure by Theorem 7.21. One may note that the Cantor measure is real and moreover positive, since F is a real-valued positive function. Thus, the Cantor measure is a well-defined finite, positive Borel measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The Cantor measure can also be viewed in the light of Theorem 7.2, since $F: \mathbb{R} \to \mathbb{R}$ is continuous and monotonically increasing, i.e., the Cantor measure is actually the Lebesgue-Stieltjes measure associated to F. The Cantor measure has continuous distribution function, F, hence the Cantor measure is non-atomic, since $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. The construction of this non-atomic measure provides a very nice way of showing that the Cantor set, $\mathcal{C}$, is uncountable, as an application of the following lemma.

Lemma 8.13. *Let μ be a non-atomic measure on $(X, \mathcal{A})$. Then every countable set is a μ-null set.*

Proof. Let C be a countable set, and let $\{c_1, c_2, c_3, ...\}$ be an enumeration of C. Note that this enumeration is finite if and only if C is finite. Since for every $x \in X$, every singleton $\{x\} \in \mathcal{A}$, hence also $C = \bigcup_{n=1}^{\infty} \{c_n\} \in \mathcal{A}$. Then

$$\mu(C) = \mu(\bigcup_{n=1}^{\infty} \{c_n\}) = \sum_{n=1}^{\infty} \mu(\{c_n\}) = \sum_{n=1}^{\infty} 0 = 0$$

as wanted. □

This lemma yields almost directly that the Cantor set is uncountable, as proven in the following theorem.

Theorem 8.14. *The Cantor set, $\mathcal{C}$, is uncountable.*

Proof. Assume by contradiction that $\mathcal{C}$ is countable. Thus, since the Cantor measure is a non-atomic measure, the assumption yields that $\mu(\mathcal{C}) = 0$. Now $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$, where $C_n = [0,1] \setminus \dot\bigcup_{k=1}^{2^n-1} I_n^k$, and $I_n^k = (a_k, b_k)$. Thus,

$$\mu(C_n) = \mu\Big(\dot\bigcup_{k=1}^{2^n-1} [b_{k-1}, a_k] \Big) = \sum_{k=1}^{2^n-1} F(a_k) - F(b_{k-1}).$$

Then by continuity from above,

$$\mu(\mathcal{C}) = \mu\Big(\bigcap_{n=1}^{\infty} C_n \Big) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \sum_{k=1}^{2^n-1} F(a_k) - F(b_{k-1}) = 1.$$

But $\mu(\mathcal{C}) = 1 \neq 0$, hence $\mathcal{C}$ cannot be countable, and thus, $\mathcal{C}$ is uncountable as wanted. □

Remark 8.15. From a topological point of view, the preceding result follows from Proposition 8.4, by making use of the fact that the Cantor set $\mathcal{C}$ is non-empty, compact and metrizable, hence Hausdorff, and has no isolated points. Thus, the result follows directly from Theorem 27.7 (Munkres 2008), which states that every non-empty compact Hausdorff space with no isolated points is uncountable.

Theorem 8.16. *The Cantor measure is singular with respect to the Lebesgue measure.*

Proof. By Proposition 8.10, the Cantor function F is differentiable with $F' = 0$ m-a.e., which by Proposition 7.24 implies exactly that $\mu \perp m$. The result is also not very difficult to prove directly from the definition of mutual singularity: By Lemma 8.3, $m(\mathcal{C}) = 0$, thus, the goal is to prove that $\mu(\mathbb{R} \setminus \mathcal{C}) = 0$, since then $\mathbb{R} = (\mathbb{R} \setminus \mathcal{C}) \cup \mathcal{C}$ with $m(\mathcal{C}) = \mu(\mathbb{R} \setminus \mathcal{C}) = 0$. Note that

$$\mu(\mathbb{R}) = \mu(\bigcup_{n=1}^{\infty} (-\infty, n])) = \lim_{n\to\infty} \mu((-\infty, n]) = \lim_{n\to\infty} F(n) = 1,$$

by continuity from below. Since also $\mu(\mathcal{C}) = 1$, it follows that $\mu(\mathbb{R} \setminus \mathcal{C}) = 0$, as wanted. □

The preceding result ties together this project by providing an answer to the question: *Are there any non-atomic measures, which are singular with respect to the Lebesgue measure?*, using the theory of differentiation of measures and functions of bounded variation. Moreover, one might notice that the preceding result gives rise to a very nice way of showing that the Cantor function is not absolutely continuous without even using the definition of absolute continuity of functions: Assume by contradiction that the Cantor function is absolutely continuous. This is equivalent to the corresponding measure, the Cantor measure, being absolutely continuous with respect to the Lebesgue measure. However, as the Cantor measure is shown to be singular with respect to the Lebesgue measure, this would imply that the Cantor measure equals zero. But this is a contradiction, as $\mu(\mathcal{C}) = 1 \neq 0$, and thus, the Cantor measure cannot be absolutely continuous with respect to the Lebesgue measure, implying that the Cantor function is not absolutely continuous. This concludes another example of how the theory of several chapters in this project can be used to show some very beautiful mathematical results.

Bibliography

Berezansky, Yurij M, Zinovij G Sheftel, and Georgij F Us (1996). *Functional Analysis*. Vol. 1. Birkhäuser.

Brouwer, LEJ (1910). "On the structure of perfect sets of points". In: *KNAW, Proceedings*. Vol. 12, pp. 785–794.

Folland, Gerald B (1999). *Real Analysis: Modern Techniques and Their Applications*. 2nd edition. John Wiley & Sons.

Munkres, James R (2008). *Topology*. 2nd edition. Pearson.

Rudin, Walter (1987). *Real and complex analysis*. 3rd edition. McGraw-Hill.

Schilling, René L (2017). *Measures, Integrals and Martingales*. 2nd edition. Cambridge University Press.