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Kirchberg's Theorems

and

Purely Infinite C^*-algebras

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Abstract

In this exposition we examine the Cuntz algebras and, more generally, purely infinite C^*-algebras and prove the results known as Kirchberg's theorems. There are three theorems, namely (1) any separable, exact C^*-algebra admits an embedding in $\mathcal{O}_2$, (2) $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ if and only if A is a unital, simple, separable and nuclear C^*-algebra and (3) if A is a simple, separable and nuclear C^*-algebra then $A \otimes \mathcal{O}_\infty \cong A$ if and only if A is purely infinite.

Contents

Introduction

The primary goal of this exposition is to prove Kirchberg's theorems, namely that any separable, exact C^*-algebra admits an embedding in $\mathcal{O}_2$, that $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ if and only if A is a unital, simple, separable and nuclear C^*-algebra and finally that if A is unital, simple, separable and nuclear C^*-algebra then $A \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$ if and only if A is purely infinite. First however, we need to introduce the Cuntz algebras and purely infinite C^*-algebras, and this is done in Chapter 2. In Chapter 3 we prove Kirchberg's theorems, and to aid us in that goal we also introduce the concept of ultrapowers.

This exposition presupposes a basic level of knowlegde concerning C^*-algebras, corresponding roughly to Chapters 2–3 in [17], and the basic theory of tensor products of C^*-algebras, corresponding roughly to the Sections 3.1–3.4 in [5]. There is also a lot of results from different areas of operator algebra that are required throughout this exposition, and most of the sections in Chapter 1 are devoted to listing those results. We do not prove any of these statements, nor do we sketch the proofs, but references are given (most of the statements can be found in [20, Section 4.1 and Section 6.1], but almost no proofs are given there). Chapter 2 of this exposition is based on [11] Sections V.4 and V.5, with few exceptions and separate references are given in those cases. Chapter 3 is based on [20] Sections 6.2–7.2, although a few statements and corresponding proofs are based on [15] (these exceptions are explicitly singled out). Throughout this exposition though, we will need an abundance of results that did not quite fit in, hence the extensive bibliography. Following the references the reader should be able to track down all relevant proofs, should he be so inclined. There is also a collection of appendices concerning: filters, hereditary subalgebras, crossed products and quasidiagonal C^*-algebras. These are meant to be very brief introductions to the subjects for the reader who is unfamiliar with the topics (the exception being quasi-diagonal C^*-algebras, since this appendix exists solely to prove one lemma and go through one example).

Notation

The letters A, B, C, D are reserved for C^*-algebras in this exposition. The spectrum of an element $a \in A$ in a C^*-algebra A will be denoted $\sigma(a)$. We let A_+ denote the cone of positive elements in A and and $\mathrm{Aut}(A)$ the group of automorphisms on A. The $n \times n$ matrices with entries in a C^*-algebra A will be denoted $M_n(A)$ whereas the $n \times n$ matrices with complex entries will simply be denoted M_n. When $A \subseteq B$ is a sub-C^*-algebra of the C^*-algebra B, we will simply refer to A as a subalgebra of B. When we say that A is a unital subalgebra of the unital C^*-algebra B, we mean that A contains the unit of B unless otherwise specified. The C^*-algebra of elements in B that commute with A is called the relative commutant of A in B and is denoted $B \cap A'$. A closed, two-sided ideal I in a C^*-algebra A will be refered to as an ideal, and we will point out if we are considering other types of ideal (i.e., algebraic ideals, left- or right ideals). The unitary group in a unital C^*-algebra A will be denoted $\mathcal{U}(A)$. If $p, q \in A$ are projections, then p and q are said to be equivalent, written $p \sim q$, if they are Murray-von Neumann equivalent, i.e., there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

The letters $\mathcal{H}$ and $\mathcal{K}$ will always refer til Hilbert spaces, and we let $B(\mathcal{H})$ denote the bounded linear operators on the Hilbert space $\mathcal{H}$. A *-homomorphism $\pi : A \to B(\mathcal{H})$ will simply be refered to as a representation of A on $\mathcal{H}$, and if π is a representation of A on $\mathcal{H}$ then π is said to be faithful if it is injective. We let $\mathbb{K}$ denote the compact operators on an infinite dimensional, separable Hilbert space.

CHAPTER 1

Preliminaries

Before we dive into the theory of this chapter, a cautionary note; the fact that the chapter is called 'Preliminaries' should not lead one to the conclusion that we only state trivial facts here. On the contrary, there are quite a few highly non-trivial, very deep and beautiful results in this chapter. The placement of these statements simply reflects that they will be needed in the current exposition, although we do not have time to delve into proofs.

Unitary Equivalence

Definition 1.0.1. Let A and B be C^*-algebras with B unital. Then two *-homomorphisms $\varphi, \psi : A \to B$ are said to be **unitarily equivalent** in B, written $\varphi \sim_u \psi$, if there exists a unitary $u \in \mathcal{U}(B)$ such that $u^*\varphi(a)u = \psi(a)$, for all $a \in A$.

Keeping the setup, φ and ψ are said to be **approximately unitarily equivalent** in B, written $\varphi \approx_u \psi$, if for each finite set $F \subseteq A$ and $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(B)$ such that $\|u^*\varphi(a)u - \psi(a)\| < \varepsilon$ for all $a \in F$.

Note that in order to check that two *-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent it is sufficient to check on a generating subset of A, i.e., a subset $A_0 \subseteq A$ such that $C^*(A_0) = A$. If it holds that for each finite subset $F \subseteq A_0$ and $\varepsilon > 0$ there exists a unitary $u \in B$ such that

$$\|u^*\varphi(a)u - \psi(a)\| < \varepsilon$$

then $\varphi \approx_u \psi$.

The next proposition essentially states that approximately unitary equivalence behaves as it should, in the separable setting.

Proposition 1.0.2. *Let A be a separable C^*-algebra and B a unital C^*-algebra. Then any pair of *-homomorphisms $\varphi, \psi : A \to B$ are approximately unitarily equivalent if and only if there exists a sequence of unitaries $(u_n) \subseteq \mathcal{U}(B)$ such that*

$$\|u_n^*\varphi(a)u_n - \psi(a)\| \to 0$$

for all $a \in A$.

Proof. Let $(a_k)_{k=1}^{\infty} \subseteq A$ be a dense sequence. For each $n \in \mathbb{N}$ choose a unitary $u_n \in B$ such that

$$\|u_n^* \varphi(a_k) u_n - \psi(a_k)\| < \frac{1}{n}$$

for all $1 \leq k \leq n$. Then for each $k \in \mathbb{N}$ it follows that

$$\lim_{n \to \infty} \|u_n^* \varphi(a_k) u_n - \psi(a_k)\| = 0.$$

For arbitrary $a \in A$ and $\varepsilon > 0$ choose $k \in \mathbb{N}$ such that $\|a - a_k\| < \varepsilon/3$ and $n_0 \in \mathbb{N}$ such that $\|u_n^* \varphi(a_k) u_n - \psi(a_k)\| < \varepsilon/3$ for all $n \geq n_0$. Then

$$\|u_n^* \varphi(a) u_n - \psi(a)\| \leq \|u_n^* \varphi(a - a_k) u_n\| + \|u_n^* \varphi(a_k) u_n - \psi(a_k)\| + \|\psi(a - a_k)\| < \varepsilon$$

for all $n \geq n_0$, which completes the proof. $\square$

The next proposition indicates why approximately unitary equivalence of *-homomorphisms is of interest.

Proposition 1.0.3. *Let A and B be unital C^*-algebras. Suppose there exists a pair of *-homomorphisms $\varphi_0 : A \to B$, $\psi_0 : B \to A$ such that $\varphi_0 \circ \psi_0 \approx_u id_B$ and $\psi_0 \circ \varphi_0 \approx_u id_A$. Then $A \cong B$ and there exist isomorphisms $\varphi : A \to B$, $\psi : B \to A$ with $\varphi^{-1} = \psi$ such that $\varphi \approx_u \varphi_0$ and $\psi \approx_u \psi_0$.*

Proof. See [20, Corollary 2.3.4]. $\square$

Definition 1.0.4. If $A \subseteq B$ is a subalgebra of a unital C^*-algebra B then an automorphism $\tau \in \mathrm{Aut}(A)$ is said to be **inner** in B if $\tau \sim_u \mathrm{id}_A$ in B. Similarly, $\tau \in \mathrm{Aut}(A)$ is said to be **approximately inner** in B if $\tau \approx_u \mathrm{id}_A$ in B.

Keeping the setup of this definition, when $B = A$ or it is clear from the context which ambient C^*-algebra we consider, we will omit the reference to B and simply say that τ is inner or approximately inner.

Matrix Algebras and the Compact Operators

The reader is assumed to be familiar with the construction of matrix algebras and they will therefore not be introduced here. Instead, we will concentrate on a few facts about matrix algebras that will be needed in the following.

As the reader will recall, for any matrix $[a_{ij}]_{i,j=1}^n \in M_n(A)$ we have the norm estimates:

$$\max_{1 \leq i,j \leq n} \|a_{ij}\| \leq \left\| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right\| \leq \sum_{i,j=1}^{n} \|a_{ij}\| \tag{1.1}$$

However, in case $[a_{ij}]_{i,j=1}^n \in M_n(A)$ is a diagonal matrix we can actually compute the norm. Namely:

$$\left\| \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix} \right\| = \max_{1 \leq i \leq n} \|a_i\|. \tag{1.2}$$

To see this simply represent A faithfully on $\mathcal{H}$ and use the definiton of the norm on $M_n(A)$. The above equality also holds when the columns or rows of the matrix are permuted. Although this is a fact of general interest, we will usually be in need of a similar norm estimate in a slightly more general setting: Let A be a unital C^*-algebra and $p_1, ..., p_n$ be projections in A such that $\sum_{i=1}^n p_i = 1$. It follows that the projections are orthogonal, and that for each $a \in A$ we have the identity

$$a = \sum_{i,j=1}^{n} p_i a p_j.$$

If we represent A faithfully on some Hilbert space $\mathcal{H}$, then the above equality along with the fact that the p_i's are orthogonal implies that we can view A as the matrix-algebra

$$B := \left\{ [p_i a p_j]_{i,j=1}^n \mid a \in A \right\} \subseteq B\left(\bigoplus_{j=1}^{n} \mathcal{H}_j \right),$$

where $\mathcal{H}_i = p_i(\mathcal{H})$. This is a slighty different setup than the one considered before, but a lot of the considerations carry over, for instance the equality (1.2):

$$\|\sum_{i=1}^{n} p_i a p_i\| = \max_{1\le i\le n} \|p_i a p_i\|.$$

Indeed, the assumptions on the projections $p_1, ..., p_n$ imply that each unit vector $\xi \in \mathcal{H}$ can be uniquely decomposed as a sum $\xi = \sum_{i=1}^n \xi_n$ where $\sum_{i=1}^n \|\xi_i\|^2 = \|\xi\|^2 = 1$. Hence

$$\begin{aligned} \left\| (\sum_{i=1}^{n} p_i a p_i)(\xi) \right\|^2 &= \sum_{i=1}^{n} \|(p_i a p_i)(\xi_i)\|^2 \\ &\le \max_{1\le i\le n} \|p_i a p_i\|^2 \sum_{i=1}^{n} \|\xi_i\|^2 \\ &= \max_{1\le i\le n} \|p_i a p_i\|^2 \end{aligned}$$

yielding one inequality. The other is easily obtained by evaluating in the right unit vectors.

The above considerations even holds if the columns in the above matrix are permuted, i.e. the elements in the sum can be of the form $p_i a p_j$ as long as each projection occurs no more than once on each side of a. These considerations will be used implicitly in the following sections and we will simply say something along the lines of "by performing a matrix trick", to allude to the above considerations.

We will quite frequently use the fact that $M_n \otimes A \cong M_n(A)$ whenever A is a C^*-algebra. It is an easy exercise to check that the map $M_n \otimes A \to M_n(A)$ given by $\sum_{i,j=1}^n e_{ij} \otimes a_{ij} \mapsto [a_{ij}]_{i,j=1}^n$ is indeed an isomorphism. Given a linear map $\rho\colon A \to B$ we let the inflated map $\rho_n : M_n(A) \to M_n(B)$ be given by letting ρ act on each entry, i.e., $\rho_n([a_{ij}]) = [\rho(a_{ij})]$. Given the isomorphism $M_n \otimes A \cong A$ we often simply write $1_{M_n} \otimes \rho$ for this inflation. Note that if ρ happens to be a *-homomorphism, then so is ρ_n for each $n \in \mathbb{N}$. Similarly if ρ is injective or surjective, then so is ρ_n. However it is not true that if ρ is a positive linear map, i.e., $\rho(A_+) \subseteq B_+$, then so is ρ_n, and this leads to the notion of completely positive maps. However, before we get to that, we briefly examine the compact operators.

The main fact concerning the compact operators we are interested in here, is that we can find an increasing sequence $A_n \subseteq \mathbb{K}$ of subalgebras such that $A_n \cong M_n(\mathbb{C})$ and

$$\overline{\bigcup_{n=1}^{\infty} A_n} \cong \mathbb{K}.$$

To see this, consider a fixed infinite dimensional, separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ and let $\mathbb{K} \subseteq B(\mathcal{H})$ denote the compact operators. For each $i, j \in \mathbb{N}$ let $E_{ij} \in \mathbb{K}$ be the operator given by $E_{ij}(\xi) = \langle \xi, e_j \rangle e_i$. Then it is easy to see that $E_{ij}E_{kl} = \delta_{ik}E_{il}$, where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise, and $E_{ij}^* = E_{ji}$. We therefore deduce that for each $n \in \mathbb{N}$ the subalgebra $\mathbb{K} \supseteq \operatorname{span}\{E_{ij} \mid 1 \leq i, j \leq n\} =: A_n$ is isomorphic to $M_n(\mathbb{C})$. Each A_n is contained in $\mathbb{K}$ since each A_n consists entirely of finite rank operators. Furthermore, all operators of rank 1 are of the form $\langle \cdot, \eta \rangle \xi$ for some $\xi, \eta \in \mathcal{H}$ and therefore we deduce that $\overline{\bigcup_{n=1}^{\infty} A_n}$ contains all operators of rank 1. It follows that

$$\overline{\bigcup_{n=1}^{\infty} A_n} \cong \mathbb{K}.$$

See [17] for a more detailed overview, and proofs of the statements made here. A slighty more elegant route towards this result would be to prove that the projections $p_i = \sum_{j=1}^{i} E_{jj}$ constitutes an approximate unit for $\mathbb{K}$ (see, for instance, [17, ex. 6.3.2]).

Operatorspaces and Completely Positive Maps

In this section we introduce the notion of operator spaces and completely positive and completely bounded maps as well as list various extension results that will be needed in the sections to come.

Definition 1.0.5. Let A be a C^*-algebra. If E is a closed linear subspace of A then E is called an **operator space**.

If moreover A is unital and E is self-adjoint and contains the unit of A, then E is called an **operator system**.

Given an operator system $E \subseteq A$, we say that $a \in E$ is positive, if it is positive in A. Thus it makes sense to talk about bounded linear maps between operator spaces and positive maps between operator systems. However, we are usually interested in stronger properties of linear maps between operator spaces and operator systems. These are the properties of being completely bounded and completely positive.

Given an operator space E, resp. an operator system, we can construct the $n \times n$ matrices with entries in E to obtain a new operator space, resp. operator system. This is done in a fashion completely similar to what can be done for C^*-algebras. Everything in the construction of matrix algebras carries over to operator spaces, resp. operator systems, in particular we can still consider the inflation of a map $\rho : E \to F$ between operator spaces, resp. operator systems, in the same way as for C^*-algebras. Crucially, if $E \subseteq A$ is an operator space, resp. operators system, in the C^*-algebra A, then we consider $M_n(E) \subseteq M_n(A)$ in the obvious way. Via this inclusion $M_n(E)$ inherits a norm, and if E is an operator system, it also inherits positive elements in the same manner as before.

We adopt the notation $M_n \otimes E := M_n(E)$, when E is an operator system. Correspondingly, the inflation of a linear map $\rho : E \to F$ between operator systems is often denoted $1_{M_n} \otimes \rho$. The main point of interest for the inflations is whether positivity of ρ carries over to the inflations ρ_n and whether the norm of ρ_n is uniformely bounded on n. Thus we reach the following definition:

Definition 1.0.6. A linear map $\rho : E \to F$ between operator spaces E and F is said to be **completely bounded** if

$$\sup\{\|\rho_n\| \mid n \in \mathbb{N}\} < \infty,$$

in which case we let $\|\rho\|_{cb} = \sup\{\|\rho_n\| \mid n \in \mathbb{N}\}$.

A linear map $\rho : E \to F$ between operator systems is said to be **n-positive** if ρ_n is positive, and it is said to be **completely positive** if it is n-positive for all $n \in \mathbb{N}$.

We use the following abbreviations; completely positive is abbreviated cp. and unital, complete positive will be abbreviated ucp. It follows from the next proposition that any ucp. map ρ is automatically completely bounded and $\|\rho\|_{cb} = 1$.

Proposition 1.0.7. *Let S be an operator system and B a C^*-algebra. If $\rho : S \to B$ is completely positive then ρ is completely bounded with $\|\rho\|_{cb} = \|\rho(1)\|$.*

Next we turn our attention to Stinespring's Theorem. It will not be needed in the following, but it is simply too beautiful to ignore. A proof may be found in [22]

Theorem 1.0.8 (Stinespring)**.** *Let A be a unital C^*-algebra, $\mathcal{H}$ a Hilbert space and $\rho : A \to B(\mathcal{H})$ a completely positive map. Then there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : A \to B(\mathcal{K})$ and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\rho(1)\| = \|V\|^2$ such that*

$$\rho(a) = V^*\pi(a)V$$

for all $a \in A$.

Finally we list a selection of important extension results.

Theorem 1.0.9 (Arveson)**.** *Let E be an operator system in a unital C^*-algebra A and $\mathcal{H}$ a Hilbert space. Each ucp. map $\rho : E \to B(\mathcal{H})$ extends to a ucp. map $\overline{\rho} : A \to B(\mathcal{H})$.*

Proof. See [18] Theorem 6.5 □

Theorem 1.0.10 (Wittstock)**.** *Let E be an operator space in a unital C^*-algebra A and $\mathcal{H}$ a Hilbert space. Each unital, completely bounded map $\rho : E \to B(\mathcal{H})$ extends to a unital, completely bounded map $\overline{\rho} : A \to B(\mathcal{H})$ with $\|\overline{\rho}\|_{cb} = \|\rho\|_{cb}$.*

Proof. See [18] Theorem 7.2 □

As was the case with Stinespring's Theorem, we will not exactly need these two theorems in the current exposition, but they are simply to important to ignore. However, we will need the following variation of Wittstock's Theorem. It is proven in [15, Lemma 1.6] that this result follows from Wittstock's Theorem.

Lemma 1.0.11. *Let E be an operator system in a unital C^*-algebra A and $\mathcal{H}$ a Hilbert space. For every self-adjoint, unital, completely bounded map $\rho\colon E \to B(\mathcal{H})$, there is a ucp. map $\overline{\rho} : A \to B(\mathcal{H})$ with $\|\overline{\rho}|_E - \rho\|_{cb} \leq \|\rho\|_{cb} - 1$.*

Nuclear and Exact C^*-algebras.

As the notions of exactness and nuclearity for C^*-algebras play an important part in Kirchberg's theorems we give a brief introduction to the subject here.

Definition 1.0.12. A C^*-algebra A is said to be **nuclear** if for each C^*-algebra B, the canonical surjection

$$A \otimes_{\max} B \to A \otimes_{\min} B$$

is injective, i.e., if $A \otimes_{\min} B = A \otimes_{\max} B$.

Another way of phrasing the above definition is that there is a unique C^*-norm on the algebraic tensor product $A \odot B$ for all C^*-algebras B. Thus, when A is nuclear we will let $A \otimes B$ denote the completion of $A \odot B$ with respect to the unique C^*-norm. Correspondingly, when dealing with tensor products of nuclear C^*-algebras we will freely use properties of both the minimal and maximal tensor product without specifying which we are considering.

Proposition 1.0.13 (Permanence)**.** *Nuclearity enjoys the following permanence properties:*

(i) All abelian C^*-algebras are nuclear.

(ii) If A is nuclear and G is an amenable group acting on A, then $A \rtimes G$ is nuclear

(iii) If A and B are nuclear, then $A \otimes B$ is nuclear

(iv) If B is a hereditary subalgebra of a nuclear C^*-algebra A, then B is nuclear.

(v) Any inductive limit of nuclear C^*-algebras is again nuclear.

Proof. For (i) and (ii) see [5, Proposition 2.4.2] resp. [5, Theorem 4.2.6].
For (iii) and (iv) see [2, Proposition IV.3.1.1] resp. [2, Corollary IV.3.1.14].
For (v) see [17, Theorem 6.3.10]. □

One can also characterize nuclear C^*-algebras via nuclear maps;

Definition 1.0.14. A completely positive contraction $\rho : A \to B$ between C^*-algebras A and B is called **nuclear** if for every finite subset $F \subseteq A$ and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and completely positive contractions σ and η making the diagram

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr, dashed, "\sigma"'] & & B \\ & M_n(\mathbb{C}) \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

commute within ε on F, i.e., $\|\rho(a) - (\eta \circ \sigma)(a)\| < \varepsilon$ for all $a \in F$.

If ρ is a nuclear ucp. map then the maps σ and η can be chosen to be ucp. as well (see [5]).
The following theorem will be used quite a lot in this exposition, often without reference.

Theorem 1.0.15 (Choi-Effros)**.** *A is a nuclear C^*-algebra if and only if $id_A : A \to A$ is nuclear.*

Proof. See [8] □

Note that this theorem also implies that if E is a operator system, A is nuclear and $\rho : E \to A$ is a ucp. map then ρ is automatically nuclear.

Theorem 1.0.16 (Choi-Effros Lifting Theorem)**.** *Let A be a unital C^*-algebra, I be an ideal in A, $\pi : A \to A/I$ be the quotient mapping and E be a separable operator system. For each nuclear ucp. map $\rho : E \to A/I$ there is a ucp. map $\lambda : E \to A$ such that $\rho = \pi \circ \lambda$, i.e., ρ has a ucp. lift;*

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ E \arrow[ur, dashed, "\lambda"] \arrow[r, "\rho"'] & A/I \end{tikzcd}$$

Proof. [7] □

In the proof of Kirchbergs embedding theorem we will need a more general version of this result. This relies on the notion of approximately injective C^*-algebras.

Definition 1.0.17. A C^*-algebra A is called **approximately injective** if it satisfies the following condition: given finite dimensional operator systems $E_1 \subseteq E_2 \subseteq B(\mathcal{H})$, then for any cp. map $\rho : E_1 \to A$ and $\varepsilon > 0$ there exists a cp. map $\overline{\rho} : E_2 \to A$ such that $\|\overline{\rho}|_{E_1} - \rho\| < \varepsilon$.

It is proven in [13] Proposition 4.3 that any nuclear C^*-algebra is approximately injective.

Lemma 1.0.18. *Let A and B be unital C^*-algebras with A separable, let J be an ideal in B which is approximately injective, and $\varphi : A \to B/J$ be an injective *-homomorphism. Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and suppose that the induced map of algebraic tensor products*

$$A \odot B(\mathcal{H}) \to (B \odot B(\mathcal{H}))/(J \odot B(\mathcal{H}))$$

extends to a *-homomorphism

$$\overline{\varphi} : A \otimes_{\min} B(\mathcal{H}) \to (A \otimes_{\min} B(\mathcal{H}))/(J \otimes_{\min} B(\mathcal{H})).$$

Then there is a ucp. map $\rho : A \to B$ which lifts φ.

Proof. See [15] Lemma 2.4, for a proof of the fact that the above statement follows from Theorem 3.4 in [13]. □

Definition 1.0.19. A C^*-algebra A is said to be **exact** if the functor $A \otimes_{\min} -$ is exact, i.e. if there always exists a canonical isomorphism

$$(A \otimes_{\min} B)/(A \otimes_{\min} J) \cong A \otimes_{\min} (B/J)$$

for all C^*-algebras B and ideals J in B.

Another way of phrasing the above definition is that the kernel of the canonical surjection $\pi : A \otimes_{\min} B \to A \otimes_{\min} (B/J)$ is exactly $A \otimes_{\min} J$ for all C^*-algebras B and ideals $J \subseteq B$. It follows directly from the definitions that any nuclear C^*-algebra is exact since the functor $A \otimes_{\max} -$ is exact for any C^*-algebra A.

Proposition 1.0.20 (Permanence)**.** *Exactness enjoys the following permanence properties:*

(i) Every subalgebra of an exact C^*-algebra is again exact.

(ii) If A and B are exact C^*-algebras then so is $A \otimes_{\min} B$.

(iii) Every quotient of an exact C^*-algebra is exact.

(iv) If A is an exact C^*-algebra and G is an amenable, locally compact group acting on A then $A \rtimes G$ is exact.

Proof. See [14] Proposition 7.1. □

Remark 1.0.1. One should not make the mistake of thinking that any subalgebra of a nuclear C^*-algebra is nuclear. Indeed, Kirchberg's embedding theorem, which will be proved in section 3.2, states that any separable, exact C^*-algebra is isomorphic to a subalgebra of the nuclear C^*-algebra $\mathcal{O}_2$. Therefore, since the free group on two generators $\mathbb{F}_2$ is a countable, exact group, the reduced group C^*-algebra $C_r^*(\mathbb{F}_2)$ is isomorphic to a subalgebra of $\mathcal{O}_2$. However, $\mathbb{F}_2$ is not amenable, which implies that $C_r(\mathbb{F}_2)$ is not nuclear. (see [5] Proposition 5.1.8, Theorem 2.6.8 and Example 2.6.7 for a proof of the statements made here).

One can also characterize exact C^*-algebras via. nuclear maps;

Theorem 1.0.21 (Kirchberg)**.** *A C^*-algebra is exact if and only if it admits a nuclear embedding into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proof. See [24] Theorem 9.1. □

Corollary 1.0.22. *Let A be an exact, unital, separable C^*-algebra, E a finite dimensional operator system in A and $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$, a ucp. map $\sigma : E \to M_n(\mathbb{C})$ and a unital, completely bounded map $\eta : \sigma(E) \to E$ such that $\eta \circ \sigma = id_E$ and $\|\eta\|_{cb} \leq 1 + \varepsilon$.*

Proof. See [20] Corollary 6.1.12. □

Note that the relation $\eta \circ \sigma = \mathrm{id}_E$ implies that η is a self-adjoint map.

CHAPTER 2

Purely Infinite C^*-algebras

In this part of the exposition we introduce purely infinite C^*-algebras. As we shall see, this is a class of particularly well-behaved C^*-algebras and it will be important when proving Kirchberg's theorems. We also study the Cuntz algebras, partly as concrete examples of simple and purely infinite C^*-algebras, but mainly for the benefit of the next chapter, as they play a large part in the main theorems of that chapter.

2.1 Cuntz Algebras

For each $2 \leq n < \infty$ let $\mathcal{O}_n$ be the C^*-algebra generated by n isometries $s_1, ..., s_n$ satisfying the relation

$$\sum_{i=1}^{n} s_i s_i^* = 1 \tag{2.1}$$

and the following universal property: whenever $t_1, ..., t_n$ is another set of isometries satisfying (2.1), there exists a unique homomorphism $\rho : \mathcal{O}_n \to C^*(t_1, ..., t_n)$ such that $\rho(s_i) = t_i$ for each $1 \leq i \leq n$ (For a general introduction to C^*-algeras see [2, Section II.8.3]). These algebras are called Cuntz algebras. As we shall see $\mathcal{O}_n$ is simple for all $n \geq 2$. Thus if $t_1, ..., t_n$ is another set of isometries that satisfy (2.1) then $\mathcal{O}_n \cong C^*(t_1, ..., t_n)$. First however, we need to show that such C^*-algebras actually exist. We do not give all the details, but simply outline the ideas. Let $\mathcal{H} = l^2(\mathbb{N})$ and $S_i \in B(\mathcal{H})$ given by

$$S_i(e_k) = e_{kn+i}.$$

Then it is easily seen that the S_i's satisfy the relation (2.1), i.e., that these relations can actually be realised. It would be tempting to simply define $\mathcal{O}_n := C^*(S_1, ..., S_n)$, but this C^*-algebra does not necesarily have the desired universal property. Instead we construct $\mathcal{O}_n$ as follows; consider the free *-algebra F_n generated by $s_1, ..., s_n$, i.e.,

$$F_n := \{p(s_1, ..., s_n, s_1^*, ..., s_n^*) \mid p \text{ is a non-commutative polynomium in } 2n \text{ variables}\}.$$

By a representation of the relations, we mean a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : F_n \to B(\mathcal{H})$ is a *-homomorphism such that the elements $\pi(s_1), ..., \pi(s_n)$ satisfy the desired relations. For each $x \in F_n$ we then let

$$\|x\|_c = \sup\{\|\pi(x)\| \mid \pi \text{ is a representation of the relation (2.1)}\}.$$

Then $\|\cdot\|_c$ is a well-defined C^*-seminorm on F_n (since the s_i's are required to be isometries, this places an upper bound on $\|\pi(x)\|$ for any representation π), and by taking the quotient with the kernel of $\|\cdot\|_c$ and completing with respect to the induced norm we therefore obtain a C^*-algebra with the desired universal property (since we have a non-zero representation of the relations, this will not be the 0 C^*-algebra).

We need to get a better grip on the algebraic structure in the $\mathcal{O}_n$'s so consider $\mathcal{O}_n$ for some fixed n. Since

$$\sum_{i=1}^{n} s_i s_i^* = 1$$

it follows that the projections $s_i s_i^*$ have pairwise orthogonal ranges, hence that

$$s_j^* s_i = s_j^*(s_j s_j^*)(s_i s_i^*)s_i = 0$$

when $j \neq i$. Let W_k^n denote the words of length k in the letters $\{1, ..., n\}$ and if $\mu = i_1 i_2 \cdots i_k \in W_k^n$ let

$$s_\mu = s_{i_1} s_{i_2} \cdots s_{i_k}.$$

If $\mu, \nu \in W_k^n$ then $s_\mu^* s_\nu \neq 0$ if and only if $\mu = \nu$, in which case $s_\mu^* s_\nu = 1$. Let $W^n = \cup_{k \geq 1} W_k^n$ and for $\mu \in W^n$ let $|\mu|$ denote the length of the word. We summarize these observations in the following lemma:

Lemma 2.1.1. *Let $\mu, \nu \in W^n$ be given with $|\mu| = k$, $|\nu| = l$, and assume that $s_\mu^* s_\nu \neq 0$.*

(i) If $k = l$ then $\mu = \nu$ and $s_\mu^* s_\nu = 1$.

(ii) If $k < l$ there is a word $\nu' \in W_{l-k}^n$ such that $\nu = \mu\nu'$ and $s_\mu^* s_\nu = s_{\nu'}$

(iii) If $k > l$ there is a word $\mu' \in W_{k-l}^n$ such that $\mu = \nu\mu'$ and $s_\mu^* s_\nu = s_{\mu'}^*$

It clearly follows from this lemma that

$$\operatorname{span}\{s_\mu s_\nu^* \mid |\nu|, |\mu| < \infty\}$$

is a *-algebra and therefore dense in $\mathcal{O}_n$. In the following, $\mathfrak{F}_k^n$ and $\mathfrak{F}^n$ will denote the *-algebras (it follows from the next proposition that they are actually C^*-algebras).

$$\mathfrak{F}_k^n = \operatorname{span}\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k\}, \quad \mathfrak{F}^n = \overline{\cup_{k\geq 1} \mathfrak{F}_k^n}$$

Proposition 2.1.2. *The *-algebra $\mathfrak{F}_k^n$ is isomorphic to M_{n^k} and $\mathfrak{F}^n$ is the UHF-algebra of type n^∞.*

Proof. Since $s_\mu s_\nu^* s_{\mu'} s_{\nu'}^* = \delta_{\nu\mu'} s_\mu s_{\nu'}^*$ by lemma 2.1.1, where $\delta_{\nu\mu'}$, as usual, is defined by

$$\delta_{\nu\mu'} = \begin{cases} 1 & \text{when } \nu = \mu' \\ 0 & \text{otherwise,} \end{cases}$$

it follows that the set $\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k\}$ constitutes a set of matrix units for $\mathfrak{F}_k^n$. Hence $\mathfrak{F}_k^n$ is isomorphic to M_{n^k}. Furthermore these algebras are unital for each k with unit

$$\sum_{\mu \in W_k^n} s_\mu s_\mu^*.$$

Using that

$$\sum_{i=1}^{n} s_{\mu i} s_{\nu i}^* = s_\mu \left(\sum_{i=1}^{n} s_i s_i^* \right) s_\nu = s_\mu s_\nu^*$$

we see that we have unital embeddings $\mathfrak{F}_k^n \to \mathfrak{F}_{k+1}^n$ for each k, in this case inclusions, which implies that $\mathfrak{F}^n$ is isomorphic to the UHF-algebra of type n^∞. $\square$

In the proof of the next theorem we need to integrate Banach-space valued functions. This falls somewhat outside the scope of this exposition, but for a friendly introduction to the subject see [1]. Here we will simply note that if X is a Banach space and $f : [a, b] \to X$ is a continuous function, we can define the Riemann integral of f in a manner completely similar to the real or complex case. Alternatively, when X is a C^*-algebra one can simply view the construction as an existence proof for an element $x \in X$, denoted $x := \int_a^b f(t)\mathrm{d}t$ such that for all states κ on X we have

$$\kappa(x) = \int_a^b (\kappa \circ f)(t)dt,$$

where the last integral is the usual Riemann integral of a continuous function[1].

Recall that if $B \subseteq A$ is a subalgebra and both A and B are unital (although their units might differ), then $\Phi : A \to B$ is said to be an expectation onto B if Φ is positive, idempotent and unital. If in addtion we have $\Phi(x^*x) = 0$ if and only if $x = 0$ then Φ is said to be faithful.

Theorem 2.1.3. *There exists a faithful expectation $\Phi : \mathcal{O}_n \to \mathfrak{F}^n$.*

Proof. For any $\lambda \in \mathbb{T}$ the operators λs_i still satisfy the Cuntz relation, and thus, by the universal property of $\mathcal{O}_n$, there exists a unique homomorphism $\rho_\lambda : \mathcal{O}_n \to \mathcal{O}_n$ such that $\rho_\lambda(s_i) = \lambda s_i$. Note that for each λ it holds that $\rho_\lambda(s_\mu s_\nu^*) = \lambda^{|\mu|-|\nu|} s_\mu s_\nu^*$, and that, by the universal property of $\mathcal{O}_n$, $\rho_{\lambda^{-1}} = (\rho_\lambda)^{-1}$. Hence each ρ_λ is an automorphism. Furthermore this implies that for each $x \in \mathrm{span}\{s_\mu s_\nu^*\}$ the map $\lambda \mapsto \rho_\lambda(x)$, henceforth denoted by f_x, is continuous. Let $x \in \mathcal{O}_n$, $\varepsilon > 0$ and a sequence $(\lambda_k)_{k=1}^\infty \subseteq \mathbb{T}$ such that $\lambda_k \to \lambda$ be given. Choose $y \in \mathrm{span}\{s_\mu s_\nu^*\}$ such that $\|x - y\| < \varepsilon/3$ and next n such that $\|\rho_{\lambda_n}(y) - \rho_\lambda(y)\| < \varepsilon/3$. Then, since each ρ_λ is a contraction, it follows that

$$\begin{aligned} \|\rho_\lambda(x) - \rho_{\lambda_n}(x)\| &= \|(\rho_\lambda(x) - \rho_\lambda(y)) - (\rho_{\lambda_n}(x) - \rho_{\lambda_n}(y)) - (\rho_{\lambda_n}(y) - \rho_\lambda(y))\| \\ &\leq \|x - y\| + \|x - y\| + \|\rho_{\lambda_n}(y) - \rho_\lambda(y)\| < \varepsilon, \end{aligned}$$

hence f_x is continuous for any $x \in \mathcal{O}_n$.

Using the comments made earlier, it is now easy to deduce that the map $\Phi : \mathcal{O}_n \to \mathcal{O}_n$ defined by

$$\Phi(x) = \int_0^1 f_x(e^{2\pi i t})\mathrm{d}t$$

is unital, positive, faithful and contractive using that for each state κ on $\mathcal{O}_n$ we have

$$\kappa(\Phi(x)) = \int_0^1 (\kappa \circ f_x)(e^{2\pi i t})\mathrm{d}t.$$

For instance, to prove that Φ is unital, note that ρ_λ is unital for all $\lambda \in \mathbb{T}$ and therefore;

$$\begin{aligned} \kappa(\Phi(1)) &= \int_0^1 \kappa(f_1(e^{2\pi i t}))\mathrm{d}t \\ &= \int_0^1 1\mathrm{d}t = 1. \end{aligned}$$

Since this is true for all states on $\mathcal{O}_n$ and the states separate points we see that $\Phi(1) = 1$. The other statements can be proved in a similar fashion.

[1] In fact this is also true when X is merely a Banach algebra, and states are replaced with linear functionals.

Now to see that Φ is idempotent, note that for each state κ

$$\begin{aligned}\kappa(\Phi(s_\mu s_\nu^*)) &= \int_0^1 \kappa\left(e^{2\pi i t(|\mu|-|\nu|)} s_\mu s_\nu^*\right) \mathrm{d}t \\ &= \kappa(s_\mu s_\nu^*) \int_0^1 e^{2\pi i t(|\mu|-|\nu|)} \mathrm{d}t\end{aligned}$$

and hence

$$\Phi(s_\mu s_\nu^*) = \begin{cases} 0 & \text{if } |\mu| \neq |\nu| \\ s_\mu s_\nu^* & \text{if } |\mu| = |\nu|. \end{cases}$$

Since $\mathrm{span}\{s_\mu s_\nu^*\}$ is dense in $\mathcal{O}_n$ it follows that Φ maps $\mathcal{O}_n$ onto $\mathfrak{F}^n$. Furthermore, the above calculation also shows that if $x \in \mathfrak{F}_k^n$ then $\Phi(x) = x$ and hence Φ is idempotent. $\square$

Note that in the last step of the proof we actually proved that if $x \in \mathrm{span}\{s_\mu s_\nu^* \mid |\mu|, |\nu| < n\}$ and the maximal length of the words involved in the expansion of x is m then Φ maps x into $\mathfrak{F}_m^n$. This observation will be helpful later.

The following result gives a more algebraic way of computing $\Phi(x)$ in certain cases, which is very handy. The proof is not very difficult, but slightly technical and is therefore omitted. Those readers interested may find a proof in [11, Lemma V.4.4, Lemma V.4.5].

Lemma 2.1.4. *For each positive integer m, there exists an isometry $w \in \mathcal{O}_n$ commuting with $\mathfrak{F}_m^n$ such that $\Phi(y) = w^* y w$ for every y in*

$$\mathrm{span}\{s_\mu s_\nu^* \mid \max\{|\mu|, |\nu|\} \leq m\}.$$

Now, as promised earlier, we will show that the Cuntz algebras are simple, and in particular also show that if $t_1, ..., t_n$ is another set of isometries satisfying the Cuntz relation, then $\mathcal{O}_n$ is isomorphic to $C^*(t_1, ..., t_n)$. However, as we shall see in the next chapter, this result implies that they are also purely infinite.

Theorem 2.1.5. *If $x \neq 0$ belongs to $\mathcal{O}_n$ then there are elements $a, b \in \mathcal{O}_n$ such that $axb = 1$.*

Proof. Since $x \neq 0$ it follows that $x^* x \neq 0$ and hence $\|\Phi(x^* x)\| \neq 0$. Scaling x we may assume that $\|\Phi(x^* x)\| = 1$. Then choose a positive element y in $\mathrm{span}\{s_\mu s_\nu^*\}$ such that $\|x^* x - y\| < \frac{1}{4}$. That y can be chosen to be positive is seen from the following argument. Let $(z_n) \subseteq \mathrm{span}\{s_\mu s_\nu^*\}$ be given such that $z_n \to x$. Then $z_n^* z_n \to x^* x$ hence there exists N such that $\|x^* x - z_N^* z_N\| < \frac{1}{4}$. Choose $y = z_N^* z_N$. Since Φ is contractive it follows that

$$\|\Phi(x^* x)\| - \|\Phi(y)\| \leq \|\Phi(x^* x) - \Phi(y)\| < \frac{1}{4},$$

which implies that $\|\Phi(y)\| > \frac{3}{4}$. Let m be the maximum length of the words involved in the expansion of y. Then by Lemma 2.1.4 there exists an isometry $w \in \mathcal{O}_n$ such that $\Phi(y) = w^* y w$. Since $\Phi(y) \in \mathfrak{F}_m^n$, which by Proposition 2.1.2 is a full matrix algebra, and $w^* y w = \Phi(y)$ is positive, it is diagonizable via an unitary matrix. From this diagonalization one obtains a projection $p \in \mathfrak{F}_k^n$ such that

$$p\Phi(y) = \Phi(y)p = \|\Phi(y)\|p > \frac{3}{4}p$$

and a unitary u such that $upu^* = e_{11} = s_1^m(s_1^*)^m$. To see this, let u be a unitary such that

$$u^*\Phi(y)u = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \lambda_m \end{pmatrix},$$

and arrange that $0 \leq \lambda_m \leq \cdots \leq \lambda_1$. Then it follows that

$$\Phi(y) = \sum_{i=1}^{n} \lambda_i p_i$$

where the p_i's are mutually orthogonal one-dimensional projections. Letting $p := p_1$ and u be the diagonalizing unitary the desired properties are obtained. Set

$$z := \|\Phi(y)\|^{-1/2}(s_1^*)^m upw^*.$$

Then, since

$$zz^* = \|\Phi(y)\|^{-1/2}(s_1^*)^m upw^*wpu^*s_1^m\|\Phi(y)\|^{-1/2} = \|\Phi(y)\|^{-1} \cdot 1$$

we have $\|z\|^2 \leq \frac{4}{3}$. Furthermore

$$\begin{aligned} zyz^* &= \|\Phi(y)\|^{-1}(s_1^*)^m up(w^*yw)pu^*s_1^m \\ &= \|\Phi(y)\|^{-1}(s_1^*)^m\|\Phi(y)\|upu^*s_1^m = (s_1^*)^m e_{11}s_1^m = 1. \end{aligned}$$

All this work finally pays off as

$$\begin{aligned} \|1 - zx^*xz\| &= \|z(y - x^*x)z^*\| \\ &\leq \|z\|^2\|y - x^*x\| < \frac{4}{3}\frac{1}{3} < 1 \end{aligned}$$

and therefore zx^*xz^* is invertible. Setting $b := z^*(zx^*xz^*)^{-1/2}$ we obtain

$$(b^*x^*)xb = 1$$

completing the proof. □

Corollary 2.1.6. *For each $n \in \mathbb{N}$ the Cuntz algebra $\mathcal{O}_n$ is simple. Furthermore $\mathcal{O}_n$ embeds in $\mathcal{O}_2$ for each $n \in \mathbb{N}$.*

Proof. The first statement is an immediate consequence of Theorem 2.1.5. For the second part, let $n \geq 2$ be fixed and s_1, s_2 be the standard generators for $\mathcal{O}_2$. Then we can define isometries $t_1, ..., t_n$ by

$$t_k = \begin{cases} s_1^{k-1}s_2 & \text{when } 1 \leq k \leq n-1 \\ s_1^n & \text{when } k = n. \end{cases}$$

Clearly t_k is an isometry for each $1 \leq k \leq n$. Furthermore

$$\begin{aligned} 1 - \sum_{k=1}^{n-1} t_kt_k^* &= 1 - \sum_{k=1}^{n-1} s_1^{k-1}s_2s_2^*s_1^{(k-1)*} \\ &= 1 - \sum_{k=1}^{n} s_1^{k-1}s_1^{(k-1)*} - s_1^ks_1^{k*} \\ &= s_1^ns_1^{*n} = t_{n+1}t_{n+1}^*, \end{aligned}$$

where we have utilized the relation $s_2s_2^* = 1 - s_1s_1^*$. Since the isometries $t_1, ..., t_n$ satisfy the Cuntz relation and $\mathcal{O}_n$ is simple for each $n \geq 2$ the desired result follows. □

It follows directly from Proposition 2.1.2 that M_n can be embedded in $\mathcal{O}_2$ for all $n \in \mathbb{N}$.

2.2 Simple, Infinite C^*-algebras.

Definition 2.2.1. A projection p is said to be **infinite** if it is equivalent to a proper subprojection of itself, i.e., there exists a partial isometri $s \in \mathcal{A}$ such that $p = s^*s$ and $ss^* < p$. If there exist projections $q_1, q_2 \in \mathcal{A}$ such that $p \sim q_1 \sim q_2$ and $q_1 + q_2 \leq p$ then p is said to be **properly infinite**.

These notions of infinity for projections leads to the following notions of infinity for C^*-algebras, although these definitions are only really appropriate in the specified setting (the definitions can be found in [20, Definition 1.1.1] and differ from the corresponding definitions in [11]).

Definition 2.2.2. A simple C^*-algebra is said to be **infinite**, if it contains an infinite projection.
A unital C^*-algebra is said to be **properly infinite** if the unit is properly infinite.

Note that a simple, unital C^*-algebra is infinite if and only if the unit is an infinite projection. This is perhaps easiest to realise by proving that if p is a finite projection, i.e., not infinite, and $q \leq p$, then q is also finite.

Proposition 2.2.3. *Let A be a simple C^*-algebra and $q \in A$ an infinite projection. Then there exists partial isometries t_i for all $i \geq 1$, such that $t_i^*t_i = q > \sum_{j=1}^n t_jt_j^*$ for all $n \geq 1$. In particular q is properly infinite.*

We skip some details in this proof as it is mostly routine calculations.

Proof. Let $s \in A$ be given such that $p := ss^* < q = s^*s$. By passing to $B = qAq$ it may be arranged that B is unital with unit $1 = q = s^*s$. Furthermore, since B is a hereditary subalgebra of A it is simple. Finally q is infinite in B since $qs = sq = s$, i.e., $s \in B$ and therefore $q \sim p$ in B.

Since $1 - p \in B$ is non-zero and B contains no algebraic ideals, being a simple and unital C^*-algebra, it follows that there exists a finite number of elements $y_i, z_i \in B$ such that

$$\sum_{i=1}^n y_i(1-p)z_i = 1_A.$$

Using that $y_i(1-p)z_i + z_i^*(1-p)y_i^* \leq y_i(1-p)y_i^* + z_i^*(1-p)z_i$ we see that

$$2 \cdot 1_A \leq \sum_{i=1}^n (y_i(1-p)y_i^* + z_i^*(1-p)z_i) =: a.$$

Hence a is invertible and setting $x_i = (y_i^* + z_i)a^{-1/2}$, we obtain that

$$\sum_{i=1}^n x_i^*(1-p)x_i = 1.$$

Set $t_1 = \sum_{i=1}^n s^{i-1}(1-p)x_i$. Note that when $i \neq j$ it holds that $(1-p)s^{*j}s^i(1-p) = 0$ and $(1-p)s^{*i}s^i(1-p) = 1-p$. In particular the $s^i(1-p)$'s are partial isometries with orthogonal range projections. Then one checks that

$$t_1^*t_1 = \sum_{i=1}^n x_i^*(1-p)x_i = 1,$$

in other words that t_1 is an isometry. Furthermore,

$$\left(\sum_{i=1}^{n} s^{i-1}(1-p)s^{*(i-1)}\right) t_1t_1^* = t_1t_1^*,$$

which implies that $t_1t_1^* \leq \sum_{i=1}^n s^{i-1}(1-p)s^{*(i-1)}$. Hence

$$t_1t_1^* \leq 1 - s^n s^{*n}.$$

Let $t_i = s^{n(i-1)}t_1$ for $i \geq 2$. Then $t_i^*t_i = 1$ and using the above computations it follows that

$$t_it_i^* \leq s^{n(i-1)}s^{*n(i-1)} - s^{ni}s^{*ni}.$$

It follows that the $t_it_i^*$ are pairwise orthogonal projections with

$$\sum_{i=1}^{k} t_it_i^* = 1 - s^{nk}s^{*nk} < 1,$$

thus desired conclusion has been reached. □

Remark 2.2.1. In general one can show that if $p \in A$ is a properly infinite projection in some C^*-algebra then there is a sequence $(t_n)_{n=1}^{\infty}$ of partial isometries in A with orthogonal range projections such that $t_n^*t_n = p$. We only sketch the argument here as the considerations are quite routine: Let $s_1, s_2 \in A$ be partial isometries such that $s_1^*s_1 = s_2^*s_2 = p$ and $s_1s_1^* + s_2s_2^* \leq p$. For each $k \in \mathbb{N}$ let $t_k = s_1^k s_2$. The assumption that $s_1s_1^* + s_2s_2^* \leq p$ ensures that $s_1^*s_2 = 0 = s_2^*s_1$ and from this one deduces that $t_k^*t_k = p$ and $t_nt_n^*t_mt_m^* = 0$ when $n \neq m$.

Lemma 2.2.4. *If p and q are projections in a simple C^*-algebra A, and p is infinite, then q is equivalent to a subprojection of p.*

Proof. Choose elements z_i such that $q = \sum_{i=1}^n z_ipz_i^*$, the existence of which is dealt with in the end of the proof. By Proposition 2.2.3 there exist partial isometries s_i, for $1 \leq i \leq n$, such that $\sum_{i=1}^n s_is_i^* \leq p$ and $p = s_j^*s_j$, for all $1 \leq j \leq n$. Note that this implies $s_j^*s_i = 0$ when $j \neq i$. Let $t = \sum_{i=1}^n z_ips_i^*$. Then

$$tt^* = \sum_{j=1}^{n}\sum_{i=1}^{n} z_ips_i^*s_jpz_j^* = \sum_{j=1}^{n} z_ipz_i^* = q,$$

hence t is a partiel isometry. Furthermore $pt^*t = t^*tp = t^*t$ and thus t^*t is a subprojection of p.

Now to prove the first statement of this proof. Since A is simple there exists elements $x_i, y_i \in A$ such that $\|q - \sum_{i=1}^n x_ipy_i\| < \frac{1}{2}$. Therefore

$$\begin{aligned} q &\leq \sum_{i=1}^{n}(qx_ipy_iq + qy_i^*px_i^*q) \\ &\leq \sum_{i=1}^{n}(qx_ipx_i^*q + qy_i^*py_iq) =: a \leq cq \end{aligned}$$

where

$$c = \sum_{i=1}^{n}(\|x_i\|^2 + \|y_i\|^2) \geq \left\| \sum_{i=1}^{n}(qx_ipx_i^*q + qy_i^*py_iq) \right\|.$$

Noting that $qa = aq = a$, we obtain that $qC^*(a) = C^*(a)$, and therefore the existence of approximate units imply that $q \in C^*(a)$. Since q is a projection this means that $q = 1_B(a)$ for some Borel set $B \subseteq \sigma(a) \subseteq [0, c]$. Thus the above calculations yields that $B = [1, c] \cap \sigma(a)$, since $1_b \leq \iota \leq c1_B$, where ι is given by $\iota(t) = t$, on $[0, c]$. Since 1_B is continuous on $\sigma(a)$ it follows that $f : \sigma(a) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ x^{-1/2} & \text{when } 1 \leq x \leq c \end{cases}.$$

is continuous. Hence

$$q = f(a)af(a) = \sum_{i=1}^{n} (f(a)qx_i p x_i^* q f(a) + f(a)qy_i^* p y_i q f(a)).$$

□

Before stating the next theorem we need the following definiton:

Definition 2.2.5. A C^*-algebra A is said to have real rank zero, written $RR(A) = 0$, if every self-adjoint element of A is the norm limit of self-adjoint elements with finite spectrum.

Theorem 2.2.6. *Let A be a simple C^*-algebra. Then the following are equivalent:*

(i) Every non-zero hereditary subalgebra of A is infinite.

(ii) A is not isomorphic to $\mathbb{C}$ and for every pair of non-zero elements $a, b \in A$, there exist elements x, y such that $xay = b$.

(iii) A is not isomorphic to $\mathbb{C}$ and for every pair of non-zero positive elements $a, b \in A$ and $\varepsilon > 0$ there exists an element $x \in A$ with $\|x\| \leq (\|b\|/\|a\|)^{1/2} + \varepsilon$ such that $xax^* = b$.

(iv) A is of real rank zero and every non-zero projection in A is properly infinite.

(v) A has no non-zero abelian quotiens and for every pair of positive elements $a, b \in A$, there is a sequence $(x_n)_{n=1}^{\infty} \subseteq A$ such that $x_n^* a x_n \to b$.

Proof. We only prove equivalence of conditions $(i), (ii), (iii)$ in the unital case. Note however, that since $pAp \subseteq A$ is a hereditary subalgebra of A and therefore simple, the second statement in (iv) follows directly from (i) and Theorem 2.2.3. See [16] for a proof of the full proposition.

(iii)⇒(ii): Let $a \in A$ be an arbitrary non-zero element. Then there exists an element $z \in A$ such that $(za^*)az^* = 1$. Hence, if we let $x = za^*$ and $y = z^*b$ it follows that $xay = b$.

(ii)⇒(i): Let $a \in A$ be positive and non-zero. There exists $x, y \in A$ such that $xa^{1/2}y = 1$, hence

$$1 = xa^{1/2}yy^*a^{1/2}x^* \leq \|y\|^2 xax^*.$$

It follows that $z := xax^*$ is invertible, implying $(z^{-1/2}x)a(x^*z^{-1/2}) = 1$. Now let $B \subseteq A$ be a hereditary subalgebra and $b \in A$ be a non-invertible, positive element. Such an element exists since B is not isomorphic to $\mathbb{C}$. Choose a (necesarily non-invertible) $x \in A$ such that $xbx^* = 1$. Let $s := b^{1/2}x^*$. Then s is a non-unitary isometry. Indeed, if s were invertible then

$$1 = ss^* = b^{1/2}x^*xb^{1/2} \leq \|x\|^2 b$$

and hence b would be invertible. Furthermore $p := ss^* \in B$, since $b^{1/2} \in B$. Finally we see that p is infinite. Indeed, $sp \in B$ and $(sp)^*(sp) = p$ while sps^* is a subprojection of p orthogonal to $s(1-p)s^* \neq 0$, implying that sps^* is a proper subprojection of p.

$(i) \Rightarrow (iii)$: Let $a \in A_+$ be given such that $\|a\| = 1$ and $0 < \varepsilon < \frac{1}{2}$. Let $f : [0,1] \to [0,1]$ be given by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1-\varepsilon \\ 1 - \varepsilon^{-1}(1-t) & \text{if } 1-\varepsilon \leq t \leq 1 \end{cases}.$$

Consider the hereditary subalgebra of A generated by $f(a)$, i.e. $\overline{f(a)Af(a)}$. We may assume that A is a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Hence if we let q denote $1_{[1-\varepsilon,1]}(a)$, then using the Borel function calculus we obtain that $q \in B(\mathcal{H})$ is a projection such that $qf(a) = f(a)q = f(a)$ and therefore for any $x \in \overline{f(a)Af(a)}$ $qx = xq = x$. By assumption there is an infinite projection $p \in \overline{f(a)Af(a)}$ and, as we have just argued $p \leq q$. Since $(1-\varepsilon)q \leq a$ it follows that $p(1-\varepsilon) \leq pap$. Lemma 2.2.4 implies that 1_A is equivalent to a subprojection of p, so let s be an isometry such that $ss^* \leq p$. It follows from the constructions made that

$$s^*as = s^*paps \geq s^*(1-\varepsilon)ps = (1-\varepsilon)1.$$

Thus, letting $b = (s^*as)^{1/2}$, we obtain

$$b^{-1}s^*asb^{-1} = 1$$

and $\|sb^{-1/2}\| \leq (1-\varepsilon)^{-1/2} < 1 + \varepsilon$ proving the statement when $\|a\| = 1$.

In the general situation let $a \in A_+$ and $\varepsilon > 0$ be given. Choose $x \in A$ such that

$$x\frac{a}{\|a\|}x^* = 1$$

and $\|x\| < 1 + \varepsilon\|a\|^{1/2}$. Then

$$\frac{x}{\|a\|^{1/2}}a\frac{x^*}{\|a\|^{1/2}} = 1$$

and

$$\left\| \frac{x}{\|a\|^{1/2}} \right\| < \frac{1+\varepsilon\|a\|^{1/2}}{\|a\|^{1/2}} = \|a\|^{-1/2} + \varepsilon,$$

completing the proof. □

Definition 2.2.7. A simple C^*-algebra is said to be **purely infinite** if it satisfies one of the equivalent conditions in Theorem 2.2.6.

Note that any unital, simple, purely infinite C^*-algebra A is automatically properly infinite by Theorem 2.2.3, and any C^*-algebra that statisfies condition (ii) or (iii) in Theorem 2.2.6 is automatically simple. One can extend the definition to include non-simple C^*-algebras, but since all the relevant C^*-algebras in this exposition are simple we will not go into that. We state the following without proof (see [11, Theorem V.7.3])

Theorem 2.2.8. *Let A be a C^*-algebra. Then $RR(A) = 0$ if and only if every hereditary subalgebra of A has an approximate unit consisting of projections.*

Note that it follows from this theorem along with Theorem 2.2.6 part (iv), that the set of projections in a purely infinite C^*-algebra span a dense subset. Furthermore, Theorem 2.2.6 part (i) implies that every non-zero projection is infinite. Thus purely infinite C^*-algebras have indeed earned their name, in the sense that there is an abundance of projections and they are all infinite (apart from the zero projection).

Purely infinite C^*-algebras enjoy the following permanence properties (amongst others).

Proposition 2.2.9 (Permanence Properties)**.** *Let A be a simple C^*-algebra.*

(i) If A is purely infinite then so is $M_n(A)$ for all $n \in \mathbb{N}$.

(ii) If A is purely infinite, then so is $A \otimes \mathbb{K}$.

(iii) If A is simple and separable and B is simple and purely infinite, then $A \otimes_{\min} B$ is simple and purely infinite.

Proof. Since $A \otimes_{\min} B$ is simple whenever A and B are simple (see [23, Chap. IV, Corollary 4.21]), we can freely use Theorem 2.2.6 throughout the proof.

Proof of (i): Let $B \subseteq M_n \otimes A$ be a non-zero hereditary subalgebra and $a \in B$ a non-zero positive element. Let $\{e_{ij}\}$ be some system of matrix units for M_n and write $a = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij}$. Since a is non-zero and positive it follows that there exists some $1 \leq i \leq n$ such that $a_{ii} \in A$ is a non-zero, positive element. Thus there exists an infinite projection $p \in \overline{a_{ii}Aa_{ii}}$, and furthermore there exists $x \in A$ such that $p = x^*a_{ii}x$. If we let $y \in M_n(A)$ be the matrix $e_ii \otimes x$, then $y^*ay = e_{ii} \otimes p$ is an infinite projection. Let $v = a^{1/2}y$. Then $vv^* \in \overline{aM_n(A)a}$ and vv^* is infinite since v^*v is infinite (it is not difficult to show that if p and q are equivalent projections in a C^*-algebra D then $pDp \cong qDq$ and that p is infinite if and only if pDp is infinite).

Proof of (ii): Since there exists an increasing sequence of subalgebras $B_n \subseteq \mathbb{K}$ such that $\overline{\bigcup_{n=1}^{\infty} A_n} \cong \mathbb{K}$ and $B_n \cong M_n$, we see that there exists an increasing sequence of subalgebras $A_n \subseteq A \otimes \mathbb{K}$ such that $\overline{\bigcup_{n=1}^{\infty} A_n} \cong A \otimes \mathbb{K}$ and $A_n \cong M_n(A)$. Now let a, b be a pair of non-zero positive elements in $A \otimes \mathbb{K}$ and choose sequences of non-zero positive elements $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \subseteq A \otimes \mathbb{K}$ such that $a_n, b_n \in A_n$, $a_n \to a$ and $b_n \to b$. Since each A_n is simple and purely infinite, by (i), it follows that for each $n \in \mathbb{N}$ there exists $x_n \in A_n$ such that $x_n a_n x_n^* = b_n$ and $\|x_n\| \leq 2\left(\frac{\|b_n\|}{\|a_n\|}\right)^{1/2}$. We can choose $n_0 \in \mathbb{N}$ such that $\|a_n\| \geq \|a\|/2$ and $\|b_n\| \leq 2\|b\|$ whenever $n \geq n_0$ and thus $\|x_n\| \leq 4\left(\frac{\|b\|}{\|a\|}\right)^{1/2}$ when $n \geq n_0$. A standard $\varepsilon/2$-argument shows that $x_n a x_n^* \to b$ and by part (v) of Theorem 2.2.6 this completes the proof ($A \otimes \mathbb{K}$ has no non-zero abelian quotients since A is non-abelian and $A \otimes \mathbb{K}$ is simple).

Proof of (iii): This is a consequence of [20, Theorem 4.1.10], which states that $A \otimes_{\min} B$ is simple and purely infinite whenever A and B are simple C^*-algebras with A not stably finite and B not Type I. Namely, a simple, separable C^*-algebra is of type I if and only if it is isomorphic to a full matrix algebra or $\mathbb{K}$ (see [2, p. 326–327]), and therefore the result follows from (i) and (ii). □

We will not prove the following proposition, but it does give a quite concrete picture of the K_0-group of a purely infinite C^*-algebra. Furthermore, the second statement implies that every pair of non-zero projections in $\mathcal{O}_2$ are equivalent, since $K_0(\mathcal{O}_2) = 0$ (see [20, p. 74–75]).

Proposition 2.2.10. *If A is a simple, purely infinite C^*-algebra, then*

$$K_0(A) = \{[p]_0 \mid p \in A \text{ is a non-zero projection}\}.$$

Moreover, if p and q are non-zero projections such that $[p]_0 = [q]_0$ then $p \sim q$.

Proof. See [10]. □

2.2.1 Cuntz Algebras Revisited

This section contains no proofs, only references to such. The main point of this section is to collect all the results concerning the Cuntz algebras that we will need later.

Definition 2.2.11. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by an infinite sequence of isometries $(s_n)_{n=1}^{\infty}$ with orthogonal range projections $s_1s_1^*, s_2s_2^*, \ldots$.

See [20, p. 70], for a justification that such a universal C^*-algebra exists.

Theorem 2.2.12. *For each $2 \leq n \leq \infty$ the Cuntz algebra $\mathcal{O}_n$ is unital, simple, nuclear and purely infinite.*

Proof. We already proved that $\mathcal{O}_n$ is unital, simple and purely infinite when $2 \leq n < \infty$. The proof for $\mathcal{O}_\infty$ runs along similar lines and may be found in Cuntz's original paper [9]. A proof of the fact that $\mathcal{O}_n$ is nuclear for all $2 \leq n \leq \infty$ may also be found in this article (or rather a proof that $\mathcal{O}_n \otimes \mathbb{K}$ is isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}$ and then the desired result follows from Propostion 1.0.13). □

The next series of statements are essentially a set of uniqueness theorems for some of the Cuntz-algebras. They will be indespensable in the chapter on Kirchbergs theorems.

Proposition 2.2.13. *Let A be a C^*-algebra and $p \in A$ a projection. There exists a unital embedding $\iota : \mathcal{O}_\infty \to pAp$ if and only if p is properly infinite.*

Proof. This follows from Theorem 2.2.12 along with Remark 2.2.1. □

Theorem 2.2.14. *Let A be a unital, simple, purely infinite C^*-algebra. Then any pair of unital *-homomorphisms $\varphi, \psi : \mathcal{O}_2 \to A$ are approximately unitarily equivalent.*

Proof. See [20, Theorem 5.1.1]. □

Theorem 2.2.15. *Let A be a unital, simple, purely infinite C^*-algebra, n be an even integer, $\varphi, \psi : \mathcal{O}_n \to A$ be unital *-homomorphisms, and $u \in A$ be the unitary element $\sum_{i=1}^n \varphi(s_i)\psi(s_i)^*$. Then φ and ψ are approximately unitarily equivalent if and only if $[u]_1 \in (n-1)K_1(A)$.*

Proof. See [20, Theorem 5.1.2]. □

The next result is contained in the first tensor product theorem, i.e., Theorem 3.3.1, but it will be a key ingredient in the next chapter. Not to worry though, this result can be proved without any reference to the tensor product theorem.

Theorem 2.2.16. *There is an isomorphism $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$.*

Proof. See [20, Theorem 5.2.1]. □

CHAPTER 3

Kirchberg's Theorems

3.1 Ultrapowers

In this section we aim to introduce ultrapowers of a C^*-algebra and we prove a few basic results that will be used in the sections to come. However, before diving into the theory of ultrapowers, we make a few short remarks concerning filters in the setting of sequences. None of the statements made here are particularly deep, but they are very useful and hence they are included for the sake of completeness (and to avoid repeating the same basic remarks at the beginning of each proof).

For any filter ω on $\mathbb{N}$ and any sequence (x_n) of real numbers we define

$$\limsup_{\omega} x_n = \inf_{X \in \omega} \sup_{n \in X} x_n.$$

As one would expect this defines a positive, subadditive map $l^\infty(\mathbb{R}) \to \mathbb{R}$ for any filter, i.e. $\limsup_\omega (x_n + y_n) \leq \limsup_\omega (x_n) + \limsup(y_n)$ and $\limsup_\omega x_n \geq 0$ whenever (x_n) is a sequence of positive numbers. Both these statements are easy to prove using the definition. From these two facts we immediatly deduce that if $(x_n) \subseteq \mathbb{R}$ is a sequence converging to $x \in \mathbb{R}$ along ω then $\limsup_\omega (x_n) = x$.

If we assume that ω is free then

$$\limsup_{\omega} x_n \leq \limsup_{n \to \infty} x_n.$$

This follows from the fact that for any $k \in \mathbb{N}$ there exists $X \in \omega$ such that $X \subseteq \{k, k+1, \dots\}$. Namely, since ω is free, for each $n \leq k$ there exists $X_n \in \omega$ such that $n \notin X_n$. Thus $X = \bigcap_{i=1}^k X_n \subseteq \{k, k+1, \dots\}$. It also follows from this observation, that if $(x_n)_{n=1}^\infty \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ in the usual sense, then (x_n) also converges to x along any free filter ω.

We can not quite deduce that if (x_n) converges to x along some filter ω then (x_n) converges to x in the usual sense (indeed any sequence in a compact space converges along any ultrafilter). However we can deduce that if ω is free, and (x_n) converges to x along ω then (x_n) has a subsequence that converges to x. Indeed, let $X_n = \{k \in \mathbb{N} \mid |x_k - x| < \frac{1}{n}\}$ which by definition is in ω. Note that $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$. Since ω is free we see that each X_n is infinite and we may therefore choose an increasing sequence of integers $(k_n)_{n=1}^\infty$ such that $k_n \in X_n$. By construction the sequence (x_{k_n}) converges to x in the usual sense.

Now we are ready to introduce ultrapowers. Let $l^\infty(A)$ denote the C^*-algebra of bounded sequences in A, i.e., $\{x : \mathbb{N} \to A \mid \sup_{n\in\mathbb{N}} \|x(n)\| < \infty\}$ with the supremum norm $\|(x_n)_{n=1}^\infty\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\|$ and pointwise operations.

Lemma 3.1.1. *Let A be a C^*-algebra, ω a filter on $\mathbb{N}$, and $c_\omega(A)$ be the set of those $(a_n)_{n=1}^\infty$ in $l^\infty(A)$ for which $\lim_\omega \|a_n\| = 0$. Then $c_\omega(A)$ is an ideal in $l^\infty(A)$.*

Proof. It is clear that for any $(a_n) \in l^\infty(A)$ it holds that $\limsup_\omega \|a_n^*\| = \lim_\omega \|a_n\| = 0$. Therefore $c_\omega(A)^* = c_\omega(A)$. Let $(a_n), (b_n) \in l^\infty(A)$. Then

$$0 \leq \limsup_\omega \|a_n + b_n\| \leq \lim_\omega \|a_n\| + \lim_\omega \|b_n\| = 0.$$

Hence $c_\omega(A)$ is stable under addition. Let $(a_n) \in c_\omega(A)$, $(b_n) \in l^\infty(A)$ and $\|b\|_\infty = \sup_{n\in\mathbb{N}} \|b_n\|$. Then

$$\limsup_\omega \|b_n a_n\| \leq \lim_\omega \|b\|_\infty \|a_n\| = 0,$$

and similarly $\lim_\omega \|a_n b_n\| = 0$, hence $c_\omega(A)$ is an algebraic ideal in $l^\infty(A)$. Now let $(a^k)_{k=1}^\infty$ be a sequence in $c_\omega(A)$ converging to $a = (a_n)_{n=1}^\infty \in l^\infty(A)$. Write $a^k = (a_n^k)_{n=1}^\infty$. Let $\varepsilon > 0$ be given and choose $l \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \|a_n - a_n^l\| < \varepsilon$. Then

$$\begin{aligned} |\limsup_\omega \|a_n\| - \lim_\omega \|a_n^l\|| &\leq \lim_\omega \|a_n - a_n^l\| \\ &\leq \sup_{n\in\mathbb{N}} \|a_n - a_n^l\| < \varepsilon \end{aligned}$$

and since $a^l \in c_\omega(A)$ this implies that $\lim_\omega \|a_n\| = 0$. Hence $c_\omega(A)$ is an ideal in $l^\infty(A)$. □

Definition 3.1.2. Let A be a C^*-algebra and ω a filter on $\mathbb{N}$. Set

$$A_\omega = l^\infty(A)/c_\omega(A).$$

We call A_ω the **ultrapower of A** with respect to the filter ω. Let $\pi_\omega : l^\infty(A) \to A_\omega$ be the quotient mapping and $\delta_A : A \to l^\infty(A)$ the diagonal map, i.e., $\delta_A(a) = (a, a, a, \dots)$. Define an embedding $\iota_A : A \to A_\omega$ by $\iota_A = \pi_\omega \circ \delta_A$.

We usually omit any reference to ι and simply identify A with a subalgebra of A_ω. Similarly when $B \subseteq A$ is a subalgebra we will adopt the convention that $B \subseteq A_\omega$. Furthermore, to ease notation, we often omit the reference to π_ω when dealing with ultrapowers, i.e., for some sequence $(a_n)_{n=1}^\infty$ we will simply write $(a_n)_{n=1}^\infty \in A_\omega$ instead of $\pi_\omega((a_n)_{n=1}^\infty)$.

When ω is the filter consisting of all co-finite subsets of $\mathbb{N}$, i.e., subsets with finite complement, then we write A_∞. Note that convergence along this filter is simply convergence in the usual sense. In particular, the comments made in the beginning of this section imply that for each C^*-algebra A and free filter ω, we have a commutative diagram:

$$\begin{tikzcd} l^\infty(A) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_\infty"'] & l^\infty(A) \arrow[d, "\pi_\omega"] \\ A_\infty \arrow[r, "\pi"] & A_\omega \end{tikzcd} \tag{3.1}$$

where π is the surjective *-homomorphism induced by π_ω. We refer to π as the canonical surjection.

Part of the point of considering A_ω as opposed to $l^\infty(A)$, is that the norm of an element $(a_n) \in A_\omega$ is $\lim_\omega \|a_n\|$, at least when ω is an ultrafilter. This is essentially the statement of the next lemma. The second statement is of a more technical nature and will be necessary later.

Lemma 3.1.3. *Let A be a C^*-algebra and ω a filter on $\mathbb{N}$. For each $a = (a_n)_{n\in\mathbb{N}} \in l^\infty(A)$ it holds that $\|\pi_\omega(a)\| = \limsup_\omega \|a_n\|$; and if ω is an ultrafilter then $\|\pi_\omega(a)\| = \lim_\omega \|a_n\|$.*

Let $a^1, \dots, a^k$ be a finite number of elements in $l^\infty(A)$. Write $a^j = (a^j_n)_{n=1}^\infty$, let $\varepsilon > 0$ and assume that $\|\pi_\omega(a^j)\| < \varepsilon$ for all j. Then there is a subset $X \in \omega$ such that $\|a^j_n\| < \varepsilon$ for all j and all $n \in X$.

Proof. Let $\nu : A_\omega \to \mathbb{R}_+$ be given by $\nu(\pi_\omega(a)) = \limsup_\omega \|a_n\|$, when $a = (a_n)_{n=1}^\infty \in l^\infty(A)$. To see that this is well-defined note that if $\pi_\omega(a) = \pi_\omega(b)$ then $\limsup_\omega |\|a_n\| - \|b_n\|| \le \limsup_\omega \|a_n - b_n\| = 0$ hence $\limsup_\omega \|a_n\| = \limsup_\omega \|b_n\|$ which implies that $\nu(a) = \nu(b)$. Furthermore, it is easy to see that ν is a seminorm that satisfies $\nu(\pi_\omega(ab)) \le \nu(\pi_\omega(a))\nu(\pi_\omega(b))$ and $\nu(\pi_\omega(a^*a)) = \nu(\pi_\omega(a))^2$ for all $a, b \in l^\infty(A)$. To see that ν is actually a norm let $(a_n)_{n=1}^\infty \in l^\infty(A)$ be given such that $\nu(a) = 0$, i.e., $\limsup_\omega \|a_n\| = 0$. Then for each $\varepsilon > 0$ there exists $X \in \omega$ such that $\sup_{n\in X} \|a_n\| < \varepsilon$. In other words $\|a_n\| < \varepsilon$ for all $n \in X$ and hence $\lim_\omega \|a_n\| = 0$. Thus ν is a pre-C^*-norm on A_ω. Since A_ω is a C^*-algebra it follows that ν is the unique C^*-norm on A_ω. Assuming that ω is an ultrafilter on $\mathbb{N}$, then for any $(a_n)_{n=1}^\infty \in l^\infty(A)$, $(\|a_n\|)_{n=1}^\infty$ is a sequence in the compact space $[0, \sup_{n\in\mathbb{N}} \|a_n\|]$ and therefore converges along ω. Hence $\nu(\pi_\omega(a)) = \lim_\omega \|a_n\|$, finishing the proof of the first statement.

Let $a^1, \dots, a^k$ be given as stated in the lemma. Then it follows from the above that $\limsup_\omega \|a^j\| < \varepsilon$ for all j. Hence there exists $X_1, ..., X_k \in \omega$ such that $\sup_{n\in X_j} \|a^j_n\| < \varepsilon$. Letting $X = X_1 \cap \dots \cap X_k \neq \emptyset$ we obtain that $X \in \omega$ and $\|a^j_n\| < \varepsilon$ for all $1 \le j \le k$ and $n \in X$, thus proving the second statement. □

Lemma 3.1.4. *Let A be a C^*-algebra and ω a filter on $\mathbb{N}$.*

(i) Each projection in A_ω lifts to a projection in $l^\infty(A)$.

(ii) If A is unital then each isometry in A_ω lifts to an isometry in $l^\infty(A)$.

(iii) If A is unital then each unitary element in A_ω lifts to a unitary element in $l^\infty(A)$.

Proof. (i): Let $p \in A_\omega$ be a projection and choose positive $a = (a_n)_{n=1}^\infty \in l^\infty(A)$ such that $\pi_\omega(a) = p$. By the previous lemma, this implies that

$$\limsup_\omega \|a_n^2 - a_n\| = 0.$$

For each $n \in \mathbb{N}$ let $F_n \in \omega$ be the set $F_n = \{k \in \mathbb{N} \mid \|a_k^2 - a_k\| < \frac{1}{n^2}\}$. It therefore follows from the continuous functional calculus that

$$\sigma(a_k) \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right],$$

for all $k \in F_n$. Note that $F_n \supseteq F_{n+1}$ for all $n \ge 1$ and hence

$$\mathbb{N} = (\mathbb{N}\backslash F_1) \sqcup \left(\bigcap_{n=1}^\infty F_n\right) \sqcup \left(\bigsqcup_{n=1}^\infty F_n \backslash F_{n+1}\right).$$

For each $n \in \mathbb{N}$ let $A_n = \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$ and note that 1_{A_n} is continuous on $\sigma(a_k)$ when $k \in F_n$. Define a new element $b = (b_n)_{n=1}^\infty \in l^\infty(A)$ by the following recipe: if $k \in \bigcap_{n=1}^\infty F_n$, let $b_k = a_k$ and if $k \in F_n$ but $k \notin F_{n+1}$ then set $b_k = 1_{A_n}(a_k)$. Clearly b is a projection in $l^\infty(A)$ and by construction $\limsup_\omega \|a_n - b_n\| = 0$ hence $\pi_\omega(b) = \pi_\omega(a) = p$.

(ii): Let $s \in A_\omega$ be an isometry and choose $a = (a_n)_{n=1}^\infty \in l^\infty(A)$ such that $\pi_\omega(a) = s$. Then

$$\limsup_\omega \|a_n^* a_n - 1\| = 0.$$

Choose $X \in \omega$ such that $\|a_n^* a_n - 1\| < 1$, i.e., such that $a_n^* a_n$ is invertible for all $n \in X$. Let $b_n = 1$ when $n \notin X$ and $b_n = a_n(a_n^* a_n)^{-1/2}$ when $n \in X$. Then $b = (b_n)_{n=1}^\infty$ is an isometry in $l^\infty(A)$ and we will prove that $\pi_\omega(b) = \pi_\omega(a)$. Let $0 < \varepsilon < \frac{1}{2}$ be given and choose $X_1 \in \omega$ such that $\|a_n^* a_n - 1\| < \frac{\varepsilon}{\|a\|_\infty}$. Then $X_2 = X \cap X_1 \in \omega$ and

$$\begin{aligned}\|b_n - a_n\| &= \|a_n((a_n^* a_n)^{-1/2} - 1)\| \\ &\leq \|a_n\|_\infty \|(a_n^* a_n)^{-1/2} - 1\| \\ &< \varepsilon\end{aligned}$$

and thus $\limsup_\omega \|b_n - a_n\| = 0$. The computations that show $\|a_n^* a_n - 1\| < \varepsilon \Rightarrow \|(a_n^* a_n)^{-1/2} - 1\| < \varepsilon$ is a standard application of the functional calculus and is therefore omitted.

(iii): Let $u \in A_\omega$ be a unitary element and choose $a = (a_n)_{n=1}^\infty \in l^\infty(A)$ such that $\pi_\omega(a) = u$. Then

$$\limsup_\omega \|a_n^* a_n - 1\| = \limsup_\omega \|a_n a_n^* - 1\| = 0.$$

Choose $X \in \omega$ such that $\|a_n^* a_n - 1\| < 1$ and $\|a_n a_n^* - 1\| < 1$, i.e., $a_n^* a_n$ and $a_n a_n^*$ are both invertible, for all $n \in X$. Then a_n has a left and right inverse, namely $(a_n^* a_n)^{-1} a_n^*$ and $a_n^*(a_n a_n^*)^{-1}$ respectively, and must therefore be invertible for all $n \in X$. Hence, there exists a unitary $v_n \in A$ such that $a_n = v_n|a_n|$ where $|a_n| = (a_n^* a_n)^{1/2}$. Set $b_n = 1$ when $n \notin X$ and $b_n = v_n$ when $n \in X$. Then $b = (b_n)_{n=1}^\infty$ is a unitary element in $l^\infty(A)$. Furthermore if we let $0 < \varepsilon < \frac{1}{2}$ be given and choose X_1 such that $\|a_n^* a_n - 1\| < \frac{\varepsilon}{\|a\|_\infty}$. Then for all $n \in X \cap X_1$ it follows that

$$\begin{aligned}\|b_n - a_n\| &= \|a_n((a_n^* a_n)^{-1/2} - 1)\| \\ &\leq \|a\|_\infty \|(a_n^* a_n)^{-1/2} - 1\| \\ &< \varepsilon.\end{aligned}$$

Hence $\limsup_\omega \|b_n - a_n\| = 0$ which implies that $\pi_\omega(a) = \pi_\omega(b)$. □

From the next lemma one deduces that if ω is a free filter, A is separable and $\varphi, \psi : A \to B$ are two *-homomorphisms, then $\varphi \approx_u \psi$ in B if and only if $\iota \circ \varphi \sim_u \iota \circ \psi$ in B_ω.

Lemma 3.1.5. *Let A, B be C^*-algebras with B unital and $\varphi, \psi : A \to B$ *-homomorphisms. Let ω be a free filter on $\mathbb{N}$ and $\iota : B \to B_\omega$ the inclusion mapping.*

(i) If $\iota \circ \varphi \approx_u \iota \circ \psi$ in B_ω then $\varphi \approx_u \psi$ in B.

(ii) If A is separable and if $\iota \circ \varphi \approx_u \iota \circ \psi$ in B_ω, then $\iota \circ \varphi \sim_u \iota \circ \psi$ in B_ω.

Proof. (i): Let $F \subseteq A$ be finite and $\varepsilon > 0$ be given. By assumption there exists a unitary element $u \in B_\omega$ such that

$$\|u(\iota \circ \varphi)(a)u^* - (\iota \circ \psi)(a)\| < \varepsilon$$

for all $a \in F$. Let $v = (v_n)_{n=1}^\infty \in l^\infty(B)$ be a unitary element that lifts u. Then by Lemma 3.1.3

$$\limsup_\omega \|v_n \varphi(a) v_n^* - \psi(a)\| < \varepsilon$$

and hence by the second statement in Lemma 3.1.3 it follows that there exists $X \in \omega$ such that $\|v_n \varphi(a) v_n^* - \psi(a)\| < \varepsilon$ for all $a \in F$ and $n \in X$. Since ω is a filter, $X \neq \emptyset$ so there exists $v_n \in B$ such that $\|v_n \varphi(a) v_n^* - \psi(a)\| < \varepsilon$ for all $a \in F$, and thus the statement has been proven.

(ii). By (i) $\varphi \approx_u \psi$ in B, and since A is separable this implies that there exists a sequence of unitaries $(v_n)_{n=1}^\infty \in B$ such that $\|v_n^* \varphi(a) v_n - \psi(a)\| \to 0$. Set $v = (v_n)_{n=1}^\infty \in l^\infty(B)$ and $u = \pi_\omega(v)$. Then, since ω is free

$$\begin{aligned}\|u(\iota \circ \varphi)(a)u^* - \psi(a)\| &= \limsup_\omega \|v_n \varphi(a) v_n^* - \psi(a)\| \\ &\leq \limsup_{n\to\infty} \|v_n \varphi(a) v_n^* - \psi(a)\| = 0,\end{aligned}$$

for all $a \in A$, which completes the proof. □

Note that we did not actually use the assumption that ω was free in the proof of (i). The same case could be made against the following proposition. However, the ultrapower is mainly of interest in the case where ω is a free ultrafilter and this will be our excuse. One could however delete the assumption in each of these statements and easily obtain a generalization of the results.

Proposition 3.1.6. *If A is a simple, purely infinite C^*-algebra, then so is A_ω, for any free ultrafilter ω.*

Proof. Let a, b be non-zero, positive elements in A_ω and choose positive elements $a_n, b_n \in A$ such that

$$\pi_\omega\left((a_n)_{n=1}^\infty\right) = a \quad \text{and} \quad \pi_\omega((b_n)_{n=1}^\infty) = b.$$

Since ω is an ultrafilter it follows that

$$\lim_\omega \|a_n\| = \|a\| > 0 \quad \text{and} \quad \lim_\omega \|b_n\| = \|b\| > 0.$$

In particular, there exists $X \in \omega$ such that

$$\|a_n\| \geq \frac{\|a\|}{2} \quad \text{and} \quad 0 < \|b_n\| \leq 2\|b\|,$$

for all $n \in X$. Since A is purely infinite and simple there exists $y_n \in A$ such that $y_n^* a_n y_n = b_n$ and

$$\|y_n\| \leq 2\left(\frac{\|b_n\|}{\|a_n\|}\right)^{1/2} \leq 4\frac{\|b\|}{\|a\|}.$$

Set $y_n = 0$ for $n \notin X$. Then $y = (y_n)_{n=1}^\infty \in l^\infty(A)$ and $\pi_\omega(y)^* a \pi_\omega(y) = b$ since

$$\|y_n^* a_n y_n - b_n\| = 0$$

for all $n \in X$. Hence condition (iii) of Theorem 2.2.6 is satisfied and A_ω is therefore simple and purely infinite. $\square$

3.2 Kirchberg's Embedding Theorem

3.2.1 Crossed Product Results

The purpose of this section is to prove Corollary 3.2.5. It may be advisable upon first read-through to skip this section entirely, only pausing to take in the statement of Corollary 3.2.5 and then return once the rest of this exposition has been consumed. Those unfamiliar with crossed products should take a quick detour to Appendix B, for a (very) brief introduction to the subject. Let us note that if A is a unital C^*-algebra and $\tau \in \mathrm{Aut}(A)$, there exists a unitary $u \in A$ such that $uau^* = \tau(a)$ and $A \rtimes_\tau \mathbb{Z} = C^*(A, u)$. We say that u implements τ.

Lemma 3.2.1. *Let A be a unital C^*-algebra, $\tau \in Aut(A)$ and (φ, v) a covariant homomorphism from (A, τ) into a unital C^*-algebra B, with φ injective and unital. Then there exists an injective homomorphism $\psi : A \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes B$ determined by $\psi(a) = 1 \otimes \varphi(a)$ for $a \in A$ and $\psi(u^n) = z^n \otimes v^n$ for $n \in \mathbb{Z}$, where $z \in C(\mathbb{T})$ denotes the canonical unitary generator and $u \in A \rtimes_\tau \mathbb{Z}$ implements τ.*

Proof. Since (φ, v) is a covariant homomorphism so is $(\overline{\varphi}, \overline{v})$ into $C(\mathbb{T}) \otimes B$ where $\overline{\varphi}(a) = 1 \otimes \varphi(a)$ and $\overline{v} = z \otimes v$. Hence the existence of ψ follows immediately (see Appendix B).

Let $\pi_0 : B \to B(\mathcal{H}_0)$ be a faithful representation of B on the Hilbert space $\mathcal{H}_0$, and λ a faithful representation of $C(\mathbb{T})$ on $l^2(\mathbb{Z})$ as in Lemma B.1.3, where $C(\mathbb{T})$ is thought of as the crossed product

$\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z}$. Then $\sigma := (\lambda \otimes \pi_0) \circ \psi$ is a representation of $A \rtimes_\tau \mathbb{Z}$ on the Hilbert space $\mathcal{H} := l^2(\mathbb{Z}) \otimes \mathcal{H}_0$. Let π be the representation of $A \rtimes_\tau \mathbb{Z}$ obtained from Lemma B.1.3 when applied to $\pi_0 \circ \varphi$. Then π is a faithful representation of $A \rtimes_\tau \mathbb{Z}$ on the Hilbert space $l^2(\mathbb{Z}) \otimes \mathcal{H}_0$. Injectivity of ψ will be proved by finding a unitary $w \in B(l^2(\mathbb{Z}) \otimes \mathcal{H}_0)$ such that $w\pi(x)w^* = \sigma(x)$ for all $x \in A \rtimes_\tau \mathbb{Z}$, which, since π is faithful, completes the proof.

Once we have unraveled the various maps involved, this proof becomes easy. So, let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}_0$, $(\delta_n)_{n\in\mathbb{Z}}$ be the canonical orthonormal basis for $l^2(\mathbb{Z})$ and consider the orthonormal basis $(\delta_n \otimes e_i)_{(n,i)\in\mathbb{Z}\times I}$ for $\mathcal{H}$. First we note that $\sigma(a) = 1_{B(l^2(\mathbb{Z}))} \otimes (\pi_0 \circ \varphi)(a)$, just by definition of σ, and hence

$$\sigma(a)(\delta_n \otimes e_i) = \delta_n \otimes (\pi_0 \circ \varphi)(a)(e_i)$$

for all $n \in \mathbb{Z}$ and $i \in I$. Similarly we see that

$$\pi(a)(\delta_n \otimes e_i) = \delta_n \otimes (\pi_0 \circ \varphi \circ \tau^{-n})(a)(e_i)$$

for all $n \in \mathbb{Z}$ and $i \in I$. Straight from the definitions we also get that

$$\pi(u^n) = \lambda_n \otimes 1 \quad \text{and} \quad \sigma(u^n) = \lambda_n \otimes \pi_0(v^n).$$

Now we are ready to define $w \in \mathcal{U}(B(l^2(\mathbb{Z}) \otimes \mathcal{H}_0))$. Namely, by letting

$$w(\delta_n \otimes e_i) = \delta_n \otimes \pi_0(v^n)(e_i),$$

for all $n \in \mathbb{Z}$ and $i \in I$. It is clear that $w^*(\delta_n \otimes e_i) = \delta_n \otimes \pi_0(v^{-n})(e_i)$. All that is left is to check that this unitary does what it is supposed to do:

$$\begin{aligned} w\pi(a)w^*(\delta_n \otimes e_i) &= \delta_n \otimes \pi_0(v^n\varphi(\tau^{-n}(a))v^{-n})(a)(e_i) \\ &= \delta_n \otimes \pi_0(\varphi((\tau^n \circ \tau^{-n})(a)))(e_i) \\ &= \delta_n \otimes (\pi_0 \circ \varphi)(a)(e_i) \\ &= \sigma(a)(\delta_n \otimes e_i) \end{aligned}$$

and

$$\begin{aligned} w\pi(u^k)w^*(\delta_n \otimes e_i) &= w(\lambda(k)(\delta_n) \otimes \pi_0(v^{-n})(e_i)) \\ &= w(\delta_{n+k} \otimes \pi_0(v^{-n})(e_i) \\ &= \delta_{n+k} \otimes \pi_0(v^k)(e_i) \\ &= \sigma(u^k)(\delta_n \otimes e_i) \end{aligned}$$

for all $a \in A$ and $k \in \mathbb{Z}$. Hence σ is unitarily equivalent to π as claimed. □

Lemma 3.2.2. *Let A be a unital C^*-algebra, B a unital subalgebra of A and $\tau \in Aut(B)$. Suppose that τ is approximately inner in A, let z denote the canonical unitary generator for $C(\mathbb{T})$ and $u \in B \rtimes_\tau \mathbb{Z}$ the canonical unitary that implements τ on B. If we let $\pi_\infty : l^\infty(A) \to A_\infty$ denote the projection, $\delta : B \to l^\infty(A)$ the diagonal map and choose $(v_n)_{n\in\mathbb{N}} \subseteq A$ such that $\lim_{n\to\infty} v_n b v_n^* = \tau(b)$ for all $b \in B$, then the maps*

$$b \mapsto 1 \otimes \pi_\infty(\delta(b)) \quad \textit{and} \quad u \mapsto z \otimes \pi_\infty(v_1, v_2, ...)$$

define an injective homomorphism $\varphi : B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes A_\infty$. Moreover, for any unital C^*-algebra C, this homomorphism extends continuously to an injective homomorphism

$$(B \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \to C(\mathbb{T}) \otimes ((l^\infty(A) \otimes_{\min} C)/(c_0(A) \otimes_{\min} C)).$$

Proof. The first thing we prove is that the last part of the statement follows from the first part. Since $(B \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \cong (B \otimes_{\min} C) \rtimes_{\tau\otimes \mathrm{id}} \mathbb{Z}$ (see Appendix B), the first part of the proposition, applied to both B and $B \otimes_{\min} C$, yields that φ extends continuously to an injective homomorphism

$$\overline{\varphi} : (B \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \to C(\mathbb{T}) \otimes (l^\infty(A \otimes_{\min} C)/c_0(A \otimes_{\min} C)),$$

since

$$\lim_{n\to\infty} (v_n \otimes 1)^*(b \otimes c)(v_n \otimes 1) = \tau(b) \otimes c,$$

for all $b \in B$, $c \in C$ and therefore $\tau \otimes \mathrm{id}$ is approximately inner in $A \otimes_{\min} C$. Now to complete this stage of the proof we show that

$$l^\infty(A) \otimes_{\min} C \subseteq l^\infty(A \otimes_{\min} C) \quad \text{and} \quad c_0(A) \otimes_{\min} C \cong c_0(A \otimes_{\min} C).$$

The last isomorphism is easy; let D be any C^*-algebra and $\psi : c_0(\mathbb{C}) \otimes_{\min} D \to c_0(D)$ be given by $\psi(\delta_n \otimes d) = \psi_{n,d}$, where $\psi_{n,d}(k) = d$ if $k = n$ and $\psi_{n,d}(k) = 0$ otherwise. It is easy to check that this map extends to an isomorphism $c_0(\mathbb{C}) \otimes_{\min} D \to c_0(D)$. Hence, in the present setting we observe that

$$c_0(A) \otimes_{\min} C \cong c_0(\mathbb{C}) \otimes_{\min} A \otimes_{\min} C \cong c_0(A \otimes_{\min} C).$$

Let σ be a faithful representation of A on $\mathcal{H}$ and π a faithful representation of C on $\mathcal{K}$. Then we get a faithful representation $\hat{\sigma}$ of $l^\infty(A)$ on $l^2(\mathbb{N}) \otimes \mathcal{H}$ by letting

$$\hat{\sigma}((a_n)_{n\in\mathbb{N}})(\delta_i \otimes e_j) = \delta_i \otimes \sigma(a_j)(e_j).$$

Similarly we obtain a faithful representation $\widehat{\sigma \otimes \pi}$ of $l^\infty(A \otimes_{\min} C)$ on $l^2(\mathbb{N}) \otimes \mathcal{H} \otimes \mathcal{K}$. Consider the injective homomorphism $\gamma : l^\infty(A) \odot C \to l^\infty(A \otimes_{\min} C)$ given by $\gamma((a_n)_{n\in\mathbb{N}} \otimes c) = (b_n \otimes c)_{n\in\mathbb{N}}$. To prove the desired inclusion it suffices to show that the diagram

$$\begin{tikzcd}
l^\infty(A) \odot C \arrow[rr, "\gamma"] \arrow[dr, "\hat{\sigma}\otimes\pi"'] & & l^\infty(A \otimes_{\min} C) \arrow[dl, "\widehat{\sigma\otimes\pi}"] \\
& B(l^2(\mathbb{N}) \otimes \mathcal{H}_1 \otimes \mathcal{H}_2). &
\end{tikzcd}$$

is commutative. This, though, is just a matter of computation, which is left to the reader. Now, collecting all the partial results proven above, all that's left to check is that the image of $\overline{\varphi}$ is contained in $C(\mathbb{T}) \otimes (l^\infty(A) \otimes_{\min} C)/(c_0(A) \otimes_{\min} C)$, when this C^*-algebra is identified with the image of the inclusion:

$$C(\mathbb{T}) \otimes (l^\infty(A) \otimes_{\min} C)/(c_0(A) \otimes_{\min} C) \to C(\mathbb{T}) \otimes (l^\infty(A \otimes_{\min} C)/c_0(A \otimes_{\min} C)).$$

Unravelling the various identifications made (again details are left to the reader), we see that

$$\overline{\varphi}(bu^n \otimes c) = z^n \otimes (\delta(b \otimes c) + c_0(B \otimes_{\min} C))$$

which is of course contained in the desired inclusion. We have therefore obtained an extension of φ extends to an injective *-homomorphism

$$(B \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \to C(\mathbb{T}) \otimes ((l^\infty(A) \otimes_{\min} C)/(c_0(A) \otimes_{\min} C)).$$

Now we prove the first part of the lemma, which is a lot easier. If we let $\iota : A \to A_\infty$ be the canonical inclusion and $v = \pi_\infty((v_1, v_2, ...)) \in A_\infty$, then by assumption

$$v\iota(b)v^* = \iota(\tau(b)),$$

hence $(\iota|_B, v)$ is a covariant homomorphism of (B,τ) into A_∞. Since ι is injective, the maps given by

$$b \mapsto 1 \otimes \iota(b) \quad \text{and} \quad u \mapsto z \otimes v$$

define an injective homomorphism $B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes A_\infty$ by Lemma 3.2.1. Translating appropriately we obtain the first statement of the lemma. □

Note that during the proof we actually proved that $l^\infty(A) \otimes_{\min} C$ embeds into $l^\infty(B \otimes_{\min} C)$ for any pair of C^*-algebras A, C. We state this as a separate proposition for future reference.

Proposition 3.2.3. *Let A and B be C^*-algebras. Then there is an embedding $\iota : l^\infty(A) \otimes_{\min} B \to l^\infty(A \otimes_{\min} B)$. If A and B are both unital, this embedding can be chosen to be unital.*

Lemma 3.2.4. *Let A be a separable, nuclear, unital C^*-algebra, B a unital subalgebra and $\tau \in Aut(B)$ approximately inner in A. Then the homomorphism $\varphi : B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes A_\infty$ from Lemma 3.2.2 has a ucp. lift $B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes l^\infty(A)$.*

Proof. This follows from Lemma 1.0.18 using the lemmas we have proven so far: the ideal $C(\mathbb{T}) \otimes c_0(A) = C(\mathbb{T}) \otimes c_0(\mathbb{C}) \otimes A$ in $C(\mathbb{T}) \otimes l^\infty(A)$ is nuclear, since all the algebras are nuclear. Hence it is approximately injective. Since $C(\mathbb{T})$ is nuclear, it follows that

$$(C(\mathbb{T}) \otimes l^\infty(A))/(C(\mathbb{T}) \otimes c_0(A)) \cong C(\mathbb{T}) \otimes (l^\infty(A)/c_0(A)).$$

This shows that we are in the situation of Lemma 1.0.18, and the required extension

$$(B \rtimes_\tau \mathbb{Z}) \otimes_{\min} B(\mathcal{H}) \to C(\mathbb{T}) \otimes (l^\infty(A) \otimes_{\min} B(\mathcal{H}))/(c_0(A) \otimes_{\min} B(\mathcal{H})))$$

follows from the Lemma 3.2.2 with $C = B(\mathcal{H})$. □

Corollary 3.2.5. *Let A be a separable, nuclear, unital C^*-algebra, B a unital subalgebra, $\tau \in Aut(B)$ approximately inner in A and ω a free ultrafilter on $\mathbb{N}$. Let $(v_n)_{n=1}^\infty \subseteq A$ be a sequence of unitaries such that $\|\tau(b) - v_n b v_n^*\| \to 0$ for all $b \in B$, $u \in A$ the unitary that implements τ and $z \in C(\mathbb{T})$ the canonical unitary generator. Then the *-homomorphism $\varphi : B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes A_\omega$ given by $\varphi(b) = 1 \otimes b$ and $\varphi(u) = z \otimes \pi_\omega((v_1, v_2, ...))$ has a ucp. lift $\rho : B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes l^\infty(A)$, i.e., there is a commutative diagram:*

$$\begin{tikzcd} & C(\mathbb{T}) \otimes l^\infty(A) \arrow[d, "\pi_\omega"] \\ B \rtimes_\tau \mathbb{Z} \arrow[ur, "\rho"] \arrow[r, "\varphi"] & C(\mathbb{T}) \otimes A_\omega . \end{tikzcd}$$

where φ is an injective *-homomorphism and ρ is a ucp. map.

Proof. For every free ultrafilter ω on $\mathbb{N}$ we have a commutative diagram:

$$\begin{tikzcd} l^\infty(A) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_\infty"'] & l^\infty(A) \arrow[d, "\pi_\omega"] \\ A_\infty \arrow[r, "\pi"'] & A_\omega \end{tikzcd}$$

where π denotes the canonical surjection. Thus we obtain a commutative diagram from Lemma 3.2.4:

$$\begin{tikzcd} & C(\mathbb{T}) \otimes l^\infty(A) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi_\infty"] & C(\mathbb{T}) \otimes l^\infty(A) \arrow[d, "\mathrm{id}\otimes\pi_\omega"] \\ B \rtimes_\tau \mathbb{Z} \arrow[ur, "\rho"] \arrow[r, "\varphi"] & C(\mathbb{T}) \otimes A_\infty \arrow[r, "\mathrm{id}\otimes\pi"'] & C(\mathbb{T}) \otimes A_\omega. \end{tikzcd}$$

To complete the proof we only need to check that $(\mathrm{id} \otimes \pi) \circ \varphi$ is injective: applying Lemma 3.2.2 in the same way as in the last part of the proof of Lemma 3.2.4 we obtain an injective *-homomorphism $\psi\colon A \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes B_\omega$. The reader may check that $\psi = (\mathrm{id} \otimes \pi) \circ \varphi$ on the dense subset $A\mathbb{Z} \subseteq A \rtimes_\tau \mathbb{Z}$ and that $(\mathrm{id} \otimes \pi) \circ \varphi$ therefore is injective.

□

3.2.2 The Embedding Theorem

This section, as the title suggests, is concerned with proving Kirchberg's embedding theorem. As the route towards this very deep, beautiful and highly suprising result is rather technical, we take the time here to point out the key results we will see along the way: The primary result to be used in the proof of the embedding theorem is given in Lemma 3.2.17, and the main technical difficulties in proving this lemma is solved by Proposition 3.2.9, Proposition 3.2.10 and Lemma 3.2.14. As we shall see, these results have other very interesting consequences. Indeed, Corollary 3.2.12 and Theorem 3.2.16 are rather wonderful applications of the forementioned technical results. Before we get that far, the reader may notice that we occasionally keep track of whether the isometries we choose are unitary or not. This is of no immediate consequence in this section, but will be used in the proof of Proposition 3.3.4 in the next section.

Before venturing into the proof of the first proposition we need to introduce two new definitions. A C^*-algebra A is said to be **antiliminal** if no non-zero positive element in A generates an abelian, hereditary subalgebra (see [2, Definition IV.1.1.6]) and it is said to be **prime** if I, J are ideals in A such that $IJ = 0$ then $I = 0$ or $J = 0$ (see [17, p. 158]). We need to use the following facts in the first proof: The set of pure states is weak*-dense in the set of states on an antiliminal, prime, unital C^*-algebra (see [12, lemme 11.2.4]) and the kernel of a pure state κ equals $L + L^*$ where L is the left kernel of κ, i.e. $L := \{x \in A \mid \omega(x^*x) = 0\}$ (see [19, Proposition 3.13.6]). Obviously any simple, purely infinite C^*-algebra is both antiliminal, since a simple, infinite C^*-algebra is non-abelian, and prime, since it is simple.

Proposition 3.2.6. *Let A be a unital, simple, purely infinite C^*-algebra and κ a state on A. For each finite subset $F \subseteq A$ and each $\varepsilon > 0$ there exists a non-zero projection $p \in A$ such that $\|pap - \kappa(a)p\| < \varepsilon$ for all $a \in A$.*

Proof. Since the pure states on A are weak*-dense in the set of states on A it is sufficient to prove this proposition for pure states. To see this let φ be any state on A, $F \subseteq A$ finite and $\varepsilon > 0$ be given. Choose a pure state κ such that $\|\varphi(a) - \kappa(a)\| < \varepsilon/2$ and a non-zero projection $p \in A$ such that $\|pap - \kappa(a)p\| < \varepsilon/2$ for all $a \in F$. Then for any $a \in F$

$$\begin{aligned} \|pap - \varphi(a)p\| &= \|(pap - \kappa(a)p) + \kappa(a)p - \varphi(a)p\| \\ &\le \|pap - \kappa(a)p\| + \|\kappa(a)p - \varphi(a)p\| < \varepsilon \end{aligned}$$

and hence the desired result follows.

So let κ be a pure state and let L be the left kernel of κ. Then, since κ is a state, L is a left ideal in A and hence $N := L \cap L^*$ is a hereditary subalgebra of A. Since A is purely infinte and simple and therefore has real rank zero by Theorem 2.2.6, there is an approximate unit (q_λ) for N where each q_λ is a projection according to Theorem 2.2.8. For each λ let $p_\lambda = 1 - q_\lambda$. Note that $\kappa(q_\lambda) = 0$ for all λ, hence $\kappa(p_\lambda) = 1$ implying that $p_\lambda \neq 0$ for all λ. Then for any $x \in L$ it follows that

$$\|xp_\lambda\|^2 \leq \|x^*xp_\lambda\| = \|x^*x(1-q_\lambda)\| \to 0,$$

since $x^*x \in N$ when $x \in L$, and hence $xp_\lambda \to 0$ for any $x \in L$. Let $y \in \ker\kappa$. Since κ is a pure state we may write $y = x + z^*$ for some $x, z \in L$, whence $\|p_\lambda y p_\lambda\| \leq \|xp_\lambda\| + \|p_\lambda z^*\| \to 0$. Hence we have obtained that

$$\|p_\lambda(a - \kappa(a))p_\lambda\| \to 0$$

for any $a \in A$ and the desired result follows. □

Lemma 3.2.7. *Let A be a unital, properly infinite C^*-algebra and $\rho : M_n \to A$ be a ucp. map. Then there exist a *-homomorphism $\varphi\colon M_n \to A$ and an isometry $t \in A$ such that $\rho(x) = t^*\varphi(x)t$ for all $x \in M_n$.*

Proof. Let $\{e_{ij}\}$ be a set of matrix units in M_n. The strategy of this proof is to find an element $v \in M_n \otimes M_n \otimes A$, a *-homomorphism $\psi : M_n \otimes M_n \otimes A \to A$ and an isometry $s \in A$ such that:

$$v^*(x \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \rho(x), \quad \psi(e_{11} \otimes e_{11} \otimes a) = sas^*.$$

When this has been done it follows that

$$\begin{aligned} v^*v &= v^*(1 \otimes 1 \otimes 1)v \\ &= e_{11} \otimes e_{11} \otimes \rho(1) \\ &= e_{11} \otimes e_{11} \otimes 1 \end{aligned}$$

and

$$\psi(v^*v) = \psi(e_{11} \otimes e_{11} \otimes 1) = ss^*.$$

Thus $t = \psi(v)s$ is an isometry. If we let $\varphi : M_n \to A$ be given by $\varphi(x) = \psi(x \otimes 1 \otimes 1)$ we obtain

$$\begin{aligned} t^*\varphi(x)t &= s^*\psi(v^*)\psi(x \otimes 1 \otimes 1)\psi(v)s \\ &= s^*\psi(e_{11} \otimes e_{11} \otimes \rho(x))s \\ &= s^*s\rho(x)s^*s = \rho(x) \end{aligned}$$

and this will complete the proof.

First we find v. Note that $\frac{1}{n}\sum_{i,j} e_{ij} \otimes e_{ij} \in M_n \otimes M_n$ is a projection, and therefore $\sum_{i,j} e_{ij} \otimes e_{ij}$, being a multiple of a projection, is a positive element. Hence

$$y = (\mathrm{id}_{M_n} \otimes \rho)\left(\sum_{i,j=1}^n e_{ij} \otimes e_{ij}\right) = \sum_{i,j=1}^n e_{ij} \otimes \rho(e_{ij}) \in M_n \otimes A$$

is positive. Therefore y has a positive square root in $M_n \otimes A$. Write $y^{1/2} = \sum_{i,j=1}^n e_{ij} \otimes a_{ij}$ with $a_{ij} \in A$ for each $1 \leq i, j \leq n$. Set

$$v := \sum_{i,j=1}^n e_{i1} \otimes e_{j1} \otimes a_{ji}.$$

We check that for each $1 \leq i, j \leq n$, it holds that

$$v^*(e_{ij} \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \sum_{k=1}^n a_{ki}^* a_{kj}.$$

Due to notation this is easier to do in steps:

$$\begin{aligned} v^*(e_{ij} \otimes 1 \otimes 1) &= \left(\sum_{k,l=1}^n e_{1k} \otimes e_{1l} \otimes a_{lk}^* \right) (e_{ij} \otimes 1 \otimes 1) \\ &= \sum_{l=1}^n e_{1j} \otimes e_{1l} \otimes a_{li}^* \end{aligned}$$

and therefore

$$\begin{aligned} v^*(e_{ij} \otimes 1 \otimes 1)v &= \left(\sum_{k=1}^n e_{1j} \otimes e_{1k} \otimes a_{ki}^* \right) \left(\sum_{l,m=1}^n e_{l1} \otimes e_{m1} \otimes a_{ml} \right) \\ &= \sum_{k=1}^n e_{11} \otimes e_{11} \otimes a_{ki}^* a_{kj}. \end{aligned}$$

By comparing this with

$$\begin{aligned} (y^{1/2})^* y^{1/2} &= \left(\sum_{i,j=1}^n e_{ij} \otimes a_{ji}^* \right) \left(\sum_{k,l=1}^n e_{kl} \otimes a_{kl} \right) \\ &= \sum_{i,j=1}^n \left(e_{ij} \otimes \sum_{k=1}^n a_{ki}^* a_{kj} \right) \end{aligned}$$

we see that $\sum_{k=1}^n a_{ki}^* a_{kj}$ is the (i,j)'th entry in y. Hence

$$v^*(e_{ij} \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \rho(e_{ij})$$

for all $1 \leq i, j \leq n$ and therefore

$$v^*(x \otimes 1 \otimes 1)v = e_{11} \otimes e_{11} \otimes \rho(x)$$

for all $x \in M_n$. Thus v behaves as promised in the beginning of the proof.

Next we find ψ and s; choose some system $\{f_{ij}\}_{i,j=1}^{n^2}$ of matrix units for $M_n \otimes M_n$ such that $f_{11} = e_{11} \otimes e_{11}$, and choose isometries $u_1, ..., u_{n^2}$ in A with orthogonal range projections (see Remark 2.2.1). Let $\psi : M_n \otimes M_n \otimes A \to A$ be given by $\psi(f_{ij} \otimes a) = u_i a u_j^*$ and set $s = u_1$. Obviously this choice of ψ and s yield $\psi(e_{11} \otimes e_{11} \otimes a) = sas^*$. □

Note that the isometry t is in fact not unitary. Indeed, in case $n \geq 2$ it follows that s is necessarily non-unitary and in case $n = 1$ we can simply choose a non-unitary isometry s. Then, since $ss^* = \psi(v^*v)$ and v is a partial isometry, we see that

$$tt^* = \psi(vv^*) \leq \psi(1) < 1.$$

The fact that $\psi(1) < 1$ relies on our choice of isometries, see Remark 2.2.1.

We will need the following lemma when proving Proposition 3.2.9. It is a simple application of the continuous functional calculus and will therefore not be proven (those interested may consult [15, Lemma 0.1] for a proof).

Lemma 3.2.8. *Let A be a C^*-algebra, $p \in A$ a projection and $a \in A$ satisfy $ap = a$ and $\|a^*a - p\| < 1$. Then the partial isometry $v \in A$ given by $v = a(a^*a)^{-1/2}$, where the functional calculus is applied with respect to the C^*-algebra pAp, satisfies that $v^*v = p$ and $\|a - v\| \leq \|a^*a - p\|$.*

Proposition 3.2.9. *Let A be a unital, purely infinite, simple C^*-algebra and $\rho : A \to A$ be a nuclear ucp. map. Then for all finite subsets $F \subseteq A$ and all $\varepsilon > 0$ there exists an isometry $s \in A$ such that $\|s^*as - \rho(a)\| < \varepsilon$ for all $a \in F$.*

Proof. Fix a finite subset $F \subseteq A$. Without loss of generality we may assume that $1_A \in F$. By assumption ρ is nuclear, hence there exists an $n \in \mathbb{N}$ and ucp. maps σ, η such that the diagram

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr, "\sigma"'] & & A \\ & M_n \arrow[ur, "\eta"'] & \end{tikzcd}$$

is commutative up to any given tolerance on F. Hence, without loss of generality we may assume that $\rho = \eta \circ \sigma$. Using Lemma 3.2.7 we obtain an isometry $t \in A$ and a *-homomorphism $\varphi : M_n \to A$ such that $\eta(x) = t^*\varphi(x)t$ for all $x \in M_n$, in particular $t^*\varphi(1_{M_n})t = 1_A$. The first step in the proof is finding an element $u \in A$ such that u^*au is close to $\varphi \circ \sigma(a)$ for all $a \in F$. Once this has been done we can peturb u slightly so that $u^*u = \varphi(1_{M_n})$, whence ut will become an isometry, and this will be shown to satisfy the conditions in the statement above.

Let $\varepsilon > 0$ be given, $\{e_{ij}\}$ the standard system of matrix units for M_n and $\{\xi_1, ..., \xi_n\}$ the standard basis for $\mathbb{C}^n$. Let $\kappa : M_n \otimes A \to \mathbb{C}$ be given by

$$\kappa\left(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\right) = \frac{1}{n}\sum_{i,j=1}^n \langle \sigma(a_{ij})\xi_j, \xi_i\rangle$$

Clearly κ is a linear and unital map since σ is linear and unital. Letting $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^{n^2}$ we see that

$$\kappa\left(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\right) = \frac{1}{n}\left\langle (1_{M_n} \otimes \sigma)\left(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\right)\xi, \xi\right\rangle$$

and since σ is completely positive it follows that κ is also positive, hence a state. Furthermore, it is seen from the definition of κ that

$$\sigma(a) = n\sum_{i,j=1}^n \kappa(e_{ij} \otimes a)e_{ij}.$$

Since κ is a state, there exists a non-zero projection $p \in M_n \otimes A$ such that $\|p(e_{ij} \otimes a)p - \kappa(e_{ij} \otimes a)p\| < \delta$ for all $1 \leq i, j \leq n$ and $a \in F$, where $\delta > 0$ is some small number to be determined later. Since A is purely infinite and simple, so is $M_n \otimes A$, by Proposition 2.2.9 and therefore, by Lemma 2.2.4 it contains a partial isometry v satisfying $v^*v = e_{11} \otimes \varphi(e_{11})$ and $vv^* \leq p$. If we write $v = \sum_{i,j=1}^n e_{ij} \otimes v_{ij}$, then it follows from the identity $v^*v = e_{11} \otimes \varphi(e_{11})$ that

$$\sum_{l=1}^n v_{lk}^* v_{lk} = 0$$

for $2 \leq k \leq n$. Hence, $v_{lk} = 0$ when $k \neq 1$ and we can therefore write $v = \sum_{j=1}^n e_{j1} \otimes v_j$ with $v_j \in A$. Since $vv^* \leq p$ it follows that $v^*p = v^*vv^*p = v^*$ and similarly $pv = v$. Therefore

$$e_{11} \otimes v_i^*av_j = v^*(e_{ij} \otimes a)v = v^*p(e_{ij} \otimes a)pv.$$

Hence

$$\begin{aligned}\|e_{11} \otimes v_i^* a v_j - e_{11} \otimes (\kappa(e_{ij} \otimes a)\varphi(e_{11}))\| &= \|v^* p(e_{ij} \otimes a)pv - \kappa(e_{ij} \otimes a)v^* pv\| \\ &\leq \|p(e_{ij} \otimes a)p - \kappa(e_{ij} \otimes a)p\| < \delta\end{aligned}$$

for all $a \in F$, which implies that

$$\|v_i^* a v_j - \kappa(e_{ij} \otimes a)\| < \delta.$$

Let $u := \sqrt{n} \sum_{j=1}^n v_j \varphi(e_{1j})$. Then, since $\sigma(a) = n \sum_{i,j=1}^n \kappa(e_{ij} \otimes a)e_{ij}$, it follows that

$$\begin{aligned}\|u^* a u - \varphi(\sigma(a))\| &= \|n \sum_{i,j=1}^n \varphi(e_{i1}) v_i^* a v_j \varphi(e_{j1}) - n \sum_{i,j=1}^n \kappa(e_{ij} \otimes a)\varphi(e_{ij})\| \\ &= \|n \sum_{i,j=1}^n \varphi(e_{i1}) v_i^* a v_j \varphi(e_{j1}) - n \sum_{i,j=1}^n \varphi(e_{i1})\kappa(e_{ij} \otimes a)\varphi(e_{11})\varphi(e_{1j})\| \\ &= \|n \sum_{i,j=1}^n \varphi(e_{i1})(v_i^* a v_j - \kappa(e_{ij} \otimes a)\varphi(e_{11}))\varphi(e_{1j})\| \\ &\leq n^3 \delta\end{aligned}$$

since $\varphi(e_{1j})$ and $\varphi(e_{i1})$ are partiel isometries for each $1 \leq i, j \leq n$. Since σ is unital and $1_A \in F$ we have that $u\varphi(1_{M_n}) = u \sum_{i=1}^n \varphi(e_{ii}) = u$, hence, if δ is chosen sufficiently small, Lemma 3.2.8 implies the existence of a partiel isometry $w \in A$ such that $w^* w = \varphi(1_{M_n})$ and $\|w - u\| \leq n^3 \delta$. Then $s := wt$ is an isometry in A and

$$\begin{aligned}\|s^* a s - \rho(a)\| &= \|s^* a s - t^* (\varphi \circ \sigma)(a) t\| \\ &\leq \|w^* a w - u^* a u + u^* a u - (\varphi \circ \sigma)(a)\| \\ &\leq \|w^* a(w - u) + (w - u)^* a u\| + n^3 \delta \\ &\leq (2\|a\| + 1)n^3 \delta\end{aligned}$$

for all $a \in F$. Thus, if we choose δ to satisfy

$$\delta < \frac{\varepsilon}{\max\{\|a\| + 1 \mid a \in F\} \cdot n^3},$$

we obtain the desired result. □

Once again, note that s is a non-unitary isometry. This follows since t is non-unitary (by the comments after lemma 3.2.7) and

$$ss^* = vtt^* v^* \leq \varphi(1_{M_n}) tt^* \varphi(1_{M_n}) = tt^* < 1,$$

where we have used that $tt^* \leq \varphi(1_{M_n})$.

Proposition 3.2.10. *Let E be a finite dimensional operator system in a separable, unital, exact C^*-algebra A and let $\varepsilon > 0$. There exists an $n \in \mathbb{N}$ such that given separable, unital C^*-algebras B_1, B_2 and ucp. maps $\rho_1 : E \to B_1$ and $\rho_2 : E \to B_2$ that satisfy:*

(i) *ρ_1 is injective,*

(ii) *$\|id_{M_n} \otimes \rho_1^{-1}\| \leq 1 + \varepsilon/2$ where $\rho_1^{-1} : \rho(E) \to E$,*

(iii) *B_2 is nuclear,*

then there is ucp map $\eta : B_1 \to B_2$ such that $\|\eta \circ \rho_1 - \rho_2\| < \varepsilon$, i.e. such that the diagram

$$\begin{tikzcd}
& E \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\
B_1 \arrow[rr, dashed, "\eta"'] & & B_2
\end{tikzcd}$$

commutes within ε on the unit ball of E.

Proof. The strategy of the proof is given in the following diagram:

$$\begin{tikzcd}
& E \arrow[r, "\mathrm{id}"] \arrow[dl, "\rho_1"'] \arrow[d, dashed, "\sigma_1"] & E \arrow[d, dashed, "\sigma_2"] \arrow[dr, "\rho_2"] & \\
B_1 \arrow[r, dashed, "\tau_1"'] & M_n(\mathbb{C}) \arrow[ur, dashed, "\eta_1"'] \arrow[r, dashed, "\tau_2"'] & M_r(\mathbb{C}) \arrow[r, dashed, "\eta_2"'] & B_2
\end{tikzcd} \tag{3.2}$$

where each triangle will be made to commute within some small tolerance, the maps $\sigma_1, \sigma_2, \eta_2, \tau_1$ and τ_2 are ucp. and η_1 is ucb. defined on $\sigma_1(E)$.

Consider the second triangle in the diagram above. By Corollary 1.0.22 there exists a natural number, a ucp. map $\sigma_1 : E \to M_n(\mathbb{C})$ and a unital, completely bounded map $\eta_1 : \sigma_1(E) \to E$, such that the triangle commutes, i.e. $\eta \circ \sigma_1 = \mathrm{id}$ and $\|\eta_1\|_{\mathrm{cb}} \leq 1 + \varepsilon/4$.

Now, consider the fourth triangle. Since B_2 is nuclear, so is the ucp. map $\rho_2 : E \to B_2$, which yields an $r \in \mathbb{N}$ and ucp. maps η_2, σ_2 such that $\|\rho_2 - \eta_2 \circ \sigma_2\| \leq \varepsilon/4$. Indeed, since E is finite dimensional there exists a finite number of elements $a_1, ..., a_n \in (E)_1$ such that for all $a \in (E)_1$ there exists some $1 \leq k \leq n$ such that $\|a - a_k\| < \varepsilon/16$. Choose ucp. maps η_2, σ_2 as indicated in (3.2) such that $\|(\rho_2 - \eta_2 \circ \sigma_2)(a_k)\| \leq \varepsilon/8$ for all $1 \leq k \leq n$. Then

$$\|(\rho_2 - \eta_2 \circ \sigma_2)(a_k)\| - \|(\rho_2 - \eta_2 \circ \sigma_2)(a)\| \leq \|(\rho_2 - \eta_2 \circ \sigma_2)(a_k - a)\| \leq \varepsilon/8,$$

which yields the desired estimate.

Now, the third triangle: Note that η_1, chosen above, is self-adjoint and hence $\sigma_2 \circ \eta_1$ is a self-adjoint unital completely bounded map. Let $F := \sigma_1(E) \subseteq M_n(\mathbb{C})$ and note that this is an operator system since σ_1 is ucp. Then it follows from Lemma 1.0.11 that there exists a ucp. map $\tau_2 : M_n(\mathbb{C}) \to M_r(\mathbb{C})$ such that $\|\tau_2|_F - \sigma_2 \circ \eta_1\| \leq \varepsilon/4$ ($\|\sigma_2\|_{\mathrm{cb}} \leq 1$ since σ_2 is ucp.).

Finally, consider the first triangle and the map $\sigma_1 \circ \rho_1^{-1} : \rho_1(E) \to M_n(\mathbb{C})$. Then, since the cb norm of a linear map $D \to M_n$ equals the norm of the induced map $M_n \otimes D \to M_n \otimes M_n$ (see [18, Proposition 7.9]), and $\|1_{M_n} \otimes \sigma_1\| \leq \|\sigma_1\|_{\mathrm{cb}} \leq 1$ it follows that

$$\begin{aligned}
\|\sigma_1 \circ \rho_1^{-1}\|_{\mathrm{cb}} &= \|1_{M_n} \otimes (\sigma_1 \circ \rho_1^{-1})\| \\
&\leq \|1_{M_n} \otimes \rho_1^{-1}\| \leq 1 + \varepsilon/2.
\end{aligned}$$

Thus, by Lemma 1.0.11 there exists ucp. map $\tau_1 : B_1 \to M_n(\mathbb{C})$ such that $\|\tau_1|_{\rho_1(E)} - \sigma_1 \circ \rho_1^{-1}\| \leq \varepsilon/2$, implying that $\|\tau_1 \circ \rho_1 - \sigma_1\| \leq \varepsilon/2$.

Let $\eta = \eta_2 \circ \tau_2 \circ \tau_1 : B_1 \to B_2$. Combining the estimates above, one easily obtains that $\|\eta \circ \rho_1 - \rho_2\| \leq \varepsilon$. □

Though it may not be apparent at first glance that the above lemma is indeed very useful, the next corollary at least gives an indication in this direction. First however, we need the following definition.

Definition 3.2.11 (Kirchberg Algebras)**.** A simple, separable, nuclear and purely infinite C^*-algebra is said to be a **Kirchberg algebra**.

Corollary 3.2.12. *Let A be a unital, separable, exact C^*-algebra and let B_1, B_2 be unital separable C^*-algebras with B_2 nulear.*

(i) For every pair of unital *-homomorphisms $\varphi_1 : A \to B_1$ and $\varphi_2 : A \to B_2$ where φ_1 is injective there is a sequence of ucp. maps $\eta_n : B_1 \to B_2$ such that for every $a \in A$ it holds that $(\eta_n \circ \varphi_1)(a) \to \varphi_2(a)$ in norm.

If $B_1 = B_2 = B$ is a Kirchberg algebra then there exists a sequence $(s_n) \subseteq B$ of isometries such that $s_n^* \varphi_1(a) s_n \to \varphi_2(a)$ for all $a \in A$.

(ii) Let ω be a free ultrafilter on $\mathbb{N}$ and (ρ_n) be a sequence of ucp. maps from A to B_1 such that $\rho : A \to (B_1)_\omega$ given by $\pi_\omega(\rho_1(a), \rho_2(a), ...)$ is injective. Then for each sequence (σ_n) of ucp. maps from A to B_2 there is a sequence (η_n) of ucp. maps such that

$$\lim_{\omega} \|(\eta_n \circ \rho_n)(a) - \sigma_n(a)\| = 0$$

for all $a \in A$.

If $B_1 = B_2 = B$ is a Kirchberg algebra then there exists a sequence $(s_n) \subseteq B$ of isometries such that

$$\lim_{\omega} \|s_n^* \rho_n(a) s_n - \sigma_n(a)\| = 0$$

for all $a \in A$.

Proof. (i): Let $(a_n)_{n\in\mathbb{N}} \subseteq A$ be a dense subset and for each $j \in \mathbb{N}$ let $E_j \subseteq A$ be the finite dimensional operator system given by

$$E_j := \operatorname{span}\{1_A, a_1, a_1^*, ..., a_j, a_j^*\}.$$

Obviously the restrictions $\varphi_1|_{E_j}$ and $\varphi_2|_{E_j}$ meet the requirements for an application of Proposition 3.2.10. Hence for each $n \in \mathbb{N}$ there exists $\eta_n : B_1 \to B_2$ such that $\|\eta_n \circ \varphi_1|_{E_n} - \varphi_2|_{E_n}\| \leq \frac{1}{n}$. Thus, for each $a \in \cup_{n\in\mathbb{N}} E_n$ it follows that $\|(\eta_n \circ \varphi_1)(a) - \varphi_2(a)\| \to 0$ and since $\cup_{n\in\mathbb{N}} E_n$ is dense in A the desired result follows.

The second statement follows from the first statement and Proposition 3.2.9. Namely, let (η_n) be a sequence of ucp. maps from B to itself such that $\|(\eta_n \circ \varphi_1)(a) - \varphi_2(a)\| \to 0$ for each $a \in A$. Since for each $n \in \mathbb{N}$ the map $\eta_n : B \to B$ is ucp. and B is nuclear it follows that each η_n is nuclear. Hence, let $(b_k)_{k\in\mathbb{N}} \subseteq B$ be a dense subset, and for each $n \in \mathbb{N}$ choose an isometry $s_n \in B$ such that $\|s_n^* b_k s_n - \eta_n(b_k)\| < \frac{1}{n}$ when $1 \leq k \leq n$. Then it clearly follows that $s_n^* \varphi_1(a) s_n \to \varphi_2(a)$ for all $a \in A$.

(ii): The strategy is as follows: for each finite dimensional operator system $E \subseteq A$ and each $\varepsilon > 0$ we find $X \in \omega$ and ucp. maps $\eta_k : B_1 \to B_2$ such that

$$\|(\eta_k \circ \rho_k)|_E - \sigma_k|_E\| < \varepsilon$$

for all $k \in X$. Then the argument can be completed in a fashion completely similar to what was done in (i). To do this we show that Prop 3.2.10 can be applied.

Since ρ is an injective *-homomorphism and ω is a free ultrafilter, it follows that

$$\lim_{k\to\omega} \|(1_{M_n} \otimes \rho_k)(a)\| = \|a\|$$

for all $a \in M_n \otimes A$. In particular, when a is in the unit sphere of $M_n \otimes E$ for some finite dimensional operator system E, we obtain that

$$\lim_{k\to\omega} \|(1_{M_n} \otimes \rho_k)(a)\| = 1.$$

Since E is finite dimensional, so is $M_n \otimes E$, hence the unit ball of $M_n \otimes E$ is compact. Choose $\delta > 0$ such that $(1+\varepsilon/2)^{-1} + \delta < 1$ and a finite number of elements $a_1, ..., a_m \in M_n \otimes E$ of norm 1 such that for any $a \in M_n \otimes E$ of norm 1, there exists some $1 \leq l \leq m$ for which

$$\|a - a_l\| < \delta.$$

Now we can find $X \in \omega$ such that

$$\|(1_{M_n} \otimes \rho_k)(a_l)\| > (1+\varepsilon/2)^{-1} + \delta$$

for all $1 \leq l \leq m$ and $k \in X$, hence

$$\|(1_{M_n} \otimes \rho_k)(a_l)\| - \|(1_{M_n} \otimes \rho_k)(a)\| \leq \|(1_{M_n} \otimes \rho_k)(a - a_l)\| < \delta.$$

Thus, we have obtained a set $X \in \omega$ such that

$$\|(1_{M_n} \otimes \rho_k)(a)\| \geq (1+\varepsilon/2)^{-1}$$

for all $k \in X$ and all $a \in M_n \otimes E$ of norm 1. It follows that each $\rho_k|_E$ is injective and that $\|1_{M_n} \otimes (\rho_k|_E)^{-1}\| \leq 1 + \varepsilon/2$.

As was the case in (i), the second statement in (ii) follows from the first along with Proposition 3.2.9 □

A final note on non-unitary isometries; the sequence (s_n) actually defines a non-unitary isometry in B_ω. Indeed, it follows from the comments below Proposition 3.2.9 that each s_n is a non-unitary isometry and therefore

$$\lim_\omega \|s_n s_n^* - 1\| = \lim_\omega 1 = 1,$$

implying that $\pi_\omega((s_n)_{n\in\mathbb{N}}) \in (B)_\omega$ is non-unitary.

The next two lemmas are taken from [15, Lemma 1.9, Lemma 1.10].

Lemma 3.2.13. *Let A be a unital C^*-algebra, $u \in A$ a unitary and $s \in A$ an isometry with range projection $e = ss^*$. Then*

$$\|u - (eue + (1-e)u(1-e))\| \leq \inf\{(2\|s^*us - v\|)^{1/2} \mid v \in A \text{ unitary}\}.$$

Proof. Let $v \in A$ be an arbitrary unitary. Note that $(svs^*)^*(svs^*) = e$ and that

$$\|eue - svs^*\| = \|ss^*uss^* - svs^*\| = \|s^*us - v\|$$

and hence

$$\begin{aligned}\|(eue)^*(eue) - e\| &= \|(eue)^*(eue) - (eue)^*(svs^*) + (eue)^*(svs^*) - (svs^*)^*(svs^*)\| \\ &\leq 2\|eue - svs^*\| \\ &= 2\|s^*us - v\|.\end{aligned}$$

Since

$$e = eu^*ue = (eue^*)(eue) + ((1-e)ue)^*((1-e)ue),$$

it follows that $\|(1-e)ue\| \leq \sqrt{2\|s^*us - v\|}$. Similar computations show that $\|eu(1-e)\| \leq \sqrt{2\|s^*us - v\|}$. Since e and $1-e$ are orthogonal we can use a matrix trick to see that

$$\begin{aligned}\|u - (eue + (1-e)u(1-e))\| &= \|(1-e)ue + eu(1-e)\| \\ &= \max\{\|(1-e)ue\|, \|eu(1-e)\|\} \\ &\leq \sqrt{2\|s^*us - v\|}.\end{aligned}$$

□

Lemma 3.2.14. *Let A be a unital C^*-algebra, let $s, t \in A$ be isometries and D be a unital subalgebra of A which is isomorphic to $\mathcal{O}_2$ and such that $s, t \in A \cap D'$. Then there is a unitary $z \in A$ such that whenever u, v are unitaries in $A \cap D'$, then*

$$\|z^*uz - v\| \leq 11(\max\{\|s^*us - v\|, \|t^*vt - u\})^{1/2}.$$

Proof. Let $B = A \cap D'$ and $\varphi : \mathcal{O}_2 :\to D$ be an isomorphism. Since the images of id_B and φ commute there exists a *-homomorphism $\psi : \mathcal{O}_2 \otimes B \to A$ such that $\psi(a \otimes b) = \varphi(a)b$, i.e. that takes $\mathcal{O}_2$ to D and is the identity on B. Hence, if we find a unitary $z \in \mathcal{O}_2 \otimes B$ such that

$$\|z^*(1 \otimes u)z - (1 \otimes v)\| \leq 11(\max\{\|s^*us - v\|, \|t^*vt - u\})^{1/2}$$

for all unitaries in $u, v \in B$ then $\psi(z)$ will fill the requirements of the statement. To do this, we define a variety of projections and partial isometries. Let $e_1 = ss^*$ and $f_1 = tt^*$. Set

$$e_2 = sf_1s^*, \quad f_2 = te_1t^*, \quad f_3 = te_2t^*.$$

Note that $e_1 \geq e_2$ and $f_1 \geq f_2 \geq f_3$. Set

$$p_1 = 1 - e_1, \quad p_2 = e_1 - e_2, \quad p_3 = e_2,$$

and

$$q_1 = 1 - f_1, \quad q_2 = f_1 - f_2, \quad q_3 = f_2 - f_3, \quad q_4 = f_3$$

Clearly the p_i's and q_j's form sets of mutually orthogonal projections in B summing to 1_B. Next we define a set of partiel isometries by

$$\begin{aligned} c_1 &:= p_2s = sq_1 \\ c_2 &:= t^*q_2 = p_1t^* \\ c_3 &:= t^*q_3 = p_2t^* \\ c_4 &:= t^*q_4 = p_2t^* \end{aligned}$$

and compute

$$\begin{aligned} c_1c_1^* &= p_2, \quad c_1^*c_1 = q_1 \\ c_jc_j^* &= p_{j-1}, \quad c_j^*c_j = q_j \end{aligned}$$

for $j = 2, 3, 4$. Let s_1, s_2 be the standard generators for $\mathcal{O}_2$. It easy to see that $c_i^*c_j = c_ic_j^* = 0$ when $i \neq j$ and hence it follows that

$$z = s_1 \otimes c_1 + 1 \otimes c_2 + s_2 \otimes c_3 + 1 \otimes c_4$$

is a unitary in $\mathcal{O}_2 \otimes B$. Let $u, v \in B$ be unitaries and set $\delta = \max\{\|s^*us - v\|, \|t^*vt - u\|\}$. Using that $s_1s_1^* + s_2s_2^* = 1$ and writing

$$\begin{aligned} u &= u - \sum_{i=1}^{3} p_iup_i + \sum_{i=1}^{3} p_iup_i \\ v &= v - \sum_{i=1}^{4} q_ivq_i + \sum_{i=1}^{4} q_ivq_i, \end{aligned}$$

we see that

$$\begin{aligned}\|z(1\otimes v)z^* - 1\otimes u\| \leq &\|s_1s_1^*\otimes(c_1(q_1vq_1)c_1^* - p_2up_2) + 1\otimes(c_2(q_2vq_2)c_2^* - p_1up_1)\\ &+ s_2s_2^*\otimes(c_3(q_3vq_3)c_3^* - p_2up_2) + 1\otimes(c_4(q_4vq_4)c_4^* - p_3up_3)\|\\ &+ \|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\|\\ &+ \|p_1up_1 + p_2up_2 + p_3up_3 - u\|.\end{aligned}$$

This may not look very pleasant but we can rearrange in the first term as follows

$$\begin{aligned}\|(1\otimes p_1)(1\otimes(t^*vt-u))(1\otimes p_1) + (1\otimes p_2)(s_1s_1^*\otimes(svs^*-u) + s_2s_2^*\otimes(t^*vt-u))(1\otimes p_2)\\ + (1\otimes p_3)(1\otimes t^*vt - u)(1\otimes p_3)\|\end{aligned}$$

Since $1\otimes p_1 + 1\otimes p_2 + 1\otimes p_3 = 1$ and $s_1s_1^*\otimes 1 + s_2s_2^*\otimes 1 = 1$, we can perform a matrix trick twice to obtain

$$\begin{aligned}\|z(1\otimes v)z^* - 1\otimes u\| \leq \delta + &\|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\|\\ &+ \|p_1up_1 + p_2up_2 + p_3up_3 - u\|.\end{aligned}$$

Thus, the task has been reduced to estimating the last two terms, which will be done using Lemma 3.2.13. Consulting the definitions of p_1, p_2 and p_3 leads to the realization, that we can rewrite the last term as follows

$$\begin{aligned}&\|u - (e_1ue_1 - (1-e_1)u(1-e_1)) + e_1ue_1 - (e_2ue_2 + (e_1-e_2)u(e_1-e_2))\|\\ \leq &\|u - (e_1ue_1 - (1-e_1)u(1-e_1))\| + \|u - (e_2ue_2 + (1-e_2)u(1-e_2))\|.\end{aligned}$$

Combining the above with the computation

$$\|t^*s^*ust - u\| \leq \|s^*us - v\| + \|t^*vt - u\| \leq 2\delta$$

and Lemma 3.2.13 we obtain that

$$\|p_1up_1 + p_2up_2 + p_3up_3 - u\| \leq (\sqrt{2} + \sqrt{4})\sqrt{\delta}.$$

A similar line of estimations show that

$$\|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| \leq (\sqrt{2} + \sqrt{4} + \sqrt{6})\sqrt{\delta}.$$

Combining these two estimates with the fact that since $\delta \leq 2$, then $\delta \leq \sqrt{2\delta}$, we obtain

$$\begin{aligned}\|z^*(1\otimes u)z - 1\otimes v\| &= \|z(1\otimes v)z^* - 1\otimes u\|\\ &\leq (4 + (3+\sqrt{3})\sqrt{2})\delta\\ &\leq 11\sqrt{\delta}.\end{aligned}$$

□

Although technical in nature the above lemma has some very nice applications including the following result:

Lemma 3.2.15. *Let A and B be unital C^*-algebras, D be a unital subalgebra of B which is isomorphic to $\mathcal{O}_2$, and $\varphi, \psi : A \to B \cap D'$ be unital *-homomorphisms. Suppose there are sequences (s_n) and (t_n) of isometries in $B \cap D'$ such that*

$$\|s_n^*\varphi(a)s_n - \psi(a)\| \to 0, \quad \|t_n^*\psi(a)t_n - \varphi(a)\| \to 0$$

for all $a \in A$. Then $\varphi \approx_u \psi$ in B.

Proof. This is a straight-forward application of Lemma 3.2.14 and is therefore omitted. □

Theorem 3.2.16. *Let A be a unital, separable, exact C^*-algebra.*

(i) Let B be a simple, separable, unital and nuclear C^*-algebra. Then any two unital, injective *-homomorphisms $\varphi, \psi : A \to B \otimes \mathcal{O}_2$ are approximately unitarily equivalent.

(ii) Any two unital, injective *-homomorphisms $\varphi, \psi : A \to \mathcal{O}_2$ are approximately unitarily equivalent.

Proof. (i): First note that it follows from Proposition 1.0.13 and Proposition 2.2.9 that $B \otimes \mathcal{O}_2$ is a nuclear, simple and purely infinite C^*-algebra. Furthermore since both B and $\mathcal{O}_2$ are unital and separable, so is $B \otimes \mathcal{O}_2$, i.e., it is a unital Kirchberg algebra. Thus we can apply Corollary 3.2.12 (i) to obtain sequences of isometries $(s_n), (t_n)$ in $B \otimes \mathcal{O}_2$ such that $\|s_n^* \varphi(a) s_n - \psi(a)\| \to 0$ and $\|t_n^* \psi(a) t_n - \varphi(a)\| \to 0$ for all $a \in A$.

Let $\iota : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be given by $\iota(x) = x \otimes 1$ and $\lambda : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be any isomorphism. Then $\lambda \circ \iota \approx_u \mathrm{id}_{\mathcal{O}_2}$ whence $(\mathrm{id}_B \otimes \lambda) \circ (\mathrm{id}_B \otimes \iota) \approx_u \mathrm{id}_B \otimes \mathrm{id}_{\mathcal{O}_2}$. Furthermore, it follows from Lemma 3.2.15 that $(\mathrm{id}_B \otimes \iota) \circ \varphi \approx_u (\mathrm{id}_B \otimes \iota) \circ \psi$ relative to $B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2$. Indeed, letting $(s'_n), (t'_n) \subseteq B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2$ be the sequences of isometries given by $s'_n = (\mathrm{id}_B \otimes \iota)(s_n)$ and $t'_n = (\mathrm{id}_B \otimes \iota)(t_n)$, it follows that

$$\|s_n'^* (\mathrm{id}_B \otimes \iota)(\varphi(a)) s'_n - (\mathrm{id}_B \otimes \iota)(\psi(a))\| \to 0$$
$$\|t_n'^* (\mathrm{id}_B \otimes \iota)(\psi(a)) t'_n - (\mathrm{id}_B \otimes \iota)(\varphi(a))\| \to 0$$

for all $a \in A$ and hence Lemma 3.2.15 applies.

Combining the considerations above we obtain

$$\varphi \approx_u (\mathrm{id}_B \otimes \lambda) \circ (\mathrm{id}_B \otimes \iota) \circ \varphi \approx_u (\mathrm{id}_B \otimes \lambda) \circ (\mathrm{id}_B \otimes \iota) \circ \psi \approx_u \psi.$$

(ii): Let $\iota : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be given by $\iota(x) = x \otimes 1$ and $\lambda : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be any isomorphism. Then it follows from (i) that $\iota \circ \varphi \approx_u \iota \circ \psi$ and hence,

$$\varphi \approx_u \lambda \circ (\iota \circ \varphi) \approx_u \lambda \circ (\iota \circ \psi) \approx_u \psi.$$

□

Lemma 3.2.17. *Let A be a separable, unital, exact C^*-algebra, ω a free ultrafilter and suppose that there is a unital, injective *-homomorphism $\varphi : A \to (\mathcal{O}_2)_\omega$ with a ucp. lift $\rho : A \to l^\infty(\mathcal{O}_2)$, i.e., such that the diagram*

$$\begin{tikzcd} & l^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\ A \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & (\mathcal{O}_2)_\omega \end{tikzcd}$$

commutes. Then there is a unital injective *-homomorphism from A into $\mathcal{O}_2$.

Proof. Write $\rho(a) = (\rho_1(a), \rho_2(a), \ldots)$, and let $\pi_n : l^\infty(\mathcal{O}_2) \to \mathcal{O}_2$ be the projection on to the n'th copy of $\mathcal{O}_2$. Since each π_n is a *-homomorphism each of the maps $\rho_n = \pi_n \circ \rho : A \to \mathcal{O}_2$ is ucp. By the hypothesis of the lemma we may apply Corollary 3.2.12 to obtain a sequence of isometries $(s'_n) \subseteq \mathcal{O}_2$ such that

$$\lim_\omega \|s_n'^* \rho_n(a) s'_n - \rho_{n+1}(a)\| = 0.$$

Similarly, by noting that the map given by $\pi_\omega(\rho_2(a), \rho_3(a), ...)$ also defines an injective *-homomorphism $A \to \mathcal{O}_2$, we obtain a sequence of isometries $(t'_n) \subseteq \mathcal{O}_2$ such that

$$\lim_\omega \|t'^*_n \rho_{n+1}(a) t'_n - \rho_n(a)\| = 0.$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of unitaries in A such that $E := \operatorname{span}\{u_i | i \in \mathbb{N}\} \subseteq A$ is dense and $(\delta_j)_{j=1}^\infty$ a decreasing sequence of small positive numbers (how small will be determined later). Since ω is a free filter, we can (inductively) choose a strictly increasing sequence $(k_j)_{j=1}^\infty \subseteq \mathbb{N}$ and corresponding sequences of isometries $(s_j), (t_j) \subseteq \mathcal{O}_2$ such that

$$\|(\rho_{k_j}(u_n))(\rho_{k_j}(u_n))^* - 1\| < \delta_j, \quad \|(\rho_{k_j}(u_n))^*(\rho_{k_j}(u_n)) - 1\| < \delta_j, \tag{3.3}$$

and

$$\|s_j^* \rho_{k_j}(u_n) s_j - \rho_{k_{j+1}}(u_n)\| < 2\delta_j, \quad \|t_j^* \rho_{k_{j+1}}(u_n) s_j - \rho_{k_j}(u_n)\| < 2\delta_j$$

for all $1 \le n \le j$. Now, by the equations in (3.3), we can for each $1 \le n \le j$ define unitaries

$$x_j^n = \rho_{k_j}(u_n)((\rho_{k_j}(u_n))^*(\rho_{k_j}(u_n)))^{-1/2}$$

if we ensure that $\delta_j < 1$. Then it follows from Lemma 3.2.8 that $\|x_j^n - \rho_{k_j}(u_n)\| \le \delta_j$ and hence

$$\begin{aligned}\|s_j^* x_j^n s_j - x_{j+1}^n\| &\le \delta_j + \|s_j^* \rho_{k_j}(u_n) s_j - x_{j+1}^n\| \\ &\le \delta_j + 2\delta_j + \|\rho_{k_{j+1}}(u_n) - x_{j+1}^n\| \\ &\le 4\delta_j\end{aligned}$$

and similarly

$$\|t_j^* x_{j+1}^n t_j - x_j^n\| \le 4\delta_j.$$

Then Lemma 3.2.14 implies the existence of unitaries $z_j \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|z_j(1 \otimes x_j^n) z_k^* - 1 \otimes x_{j+1}^n\| \le 2\sqrt{\delta_j}$$

from which it follows that

$$\|z_j(1 \otimes \rho_{k_j}(u_n)) z_j^* - 1 \otimes \rho_{k_{j+1}}(u_n)\| \le 2\delta_j + 2\sqrt{\delta_j}$$

when $1 \le n \le j$. Hence, if we define $y_n = z_1^* z_2^* \cdots z_{n-1}^*$ and choose the sequence (δ_j) to satisfy $2\delta_j + 2\sqrt{\delta_j} < 2^{-j}$ we see that for each $n \in \mathbb{N}$ the sequence $\{y_j(1 \otimes \rho_{k_j}(u_n)) y_j^*\}$ is Cauchy and therefore

$$\lim_{j\to\infty} y_j(1 \otimes \rho_{k_j}(u_n)) y_j^*$$

exists for all $n \in \mathbb{N}$, hence

$$\psi_0(a) = \lim_{j\to\infty} y_j(1 \otimes \rho_{k_j}(a)) y_j^*$$

is welldefined for all $a \in \bigcup_{j=1}^\infty E_j$. Since

$$\|\psi_0(a)\| = \lim_{j\to\infty} \|\rho_{k_j}(a)\| = \|\varphi(a)\| = \|a\|,$$

we can by continuity extend ψ_0 to an injective linear map $\psi : A \to \mathcal{O}_2 \otimes \mathcal{O}_2$.

It only remains to show that ψ is a *-homomorphism. By assumption

$$\lim_\omega \|\rho_{k_j}(a^*) - (\rho_{k_j}(a))^*\| = 0$$
$$\lim_\omega \|\rho_{k_j}(ab) - \rho_{k_j}(a)\rho_{k_j}(b)\| = 0$$

for all $a, b \in A$. Since A is separable and ω is free, we can arrange that this is convergence in the usual sense by switching to a suitable subsequence (this is not completely obvious, but is not very difficult to prove using a standard separability argument and properties of free filters discussed in Section 3.1). All the considerations made above are still valid for this subsequence, and we have thus obtained an embedding $\psi : A \to \mathcal{O}_2 \otimes \mathcal{O}_2$.

The desired embedding into $\mathcal{O}_2$ can now be obtained by composing ψ with any isomorphism $\lambda \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$. □

Lemma 3.2.18. *Every quasidiagonal, separable, unital and exact C^*-algebra admits a unital embedding in $\mathcal{O}_2$.*

Proof. Let ω be a free ultrafilter on $\mathbb{N}$ and for any sequence $(m_j)_{j\in\mathbb{N}}$ of integers let $\prod_{j=1}^{\infty} M_{m_j}$ denote the bounded sequences in the set-theoretic product and $\bigoplus_{j\in\mathbb{N}}^{\omega} M_{m_j}$ denote the set of sequences that converge to 0 along ω. For every quasidiagonal, separable and unital C^*-algebra A there exist a sequence of integers $(k_n)_{n\in\mathbb{N}}$, a ucp. map ρ and an injective *-homomorphism φ making the diagram

$$\begin{tikzcd} & \prod_{n=1}^{\infty} M_{k_n} \arrow[d, "\pi"] \\ A \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & \prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n\in\mathbb{N}}^{\omega} M_{k_n} \end{tikzcd}$$

commutative (see Appendix D). Since M_n embeds in $\mathcal{O}_2$ for all $n \in \mathbb{N}$, we can extend the above diagram

$$\begin{tikzcd} & \prod_{n=1}^{\infty} M_{k_n} \arrow[r, "\iota"] \arrow[d, "\pi"] & l^{\infty}(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\ A \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & \prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n\in\mathbb{N}}^{\omega} M_{k_n} \arrow[r, "\bar{\iota}"'] & (\mathcal{O}_2)_\omega \end{tikzcd}$$

where $\bar{\iota}$ is the *-homomorphism induced by $\pi_\omega \circ \iota$. Since $\iota \circ \rho$ is a ucp. map, $\bar{\iota} \circ \varphi$ is a unital embedding and A is unital, separable and exact, it follows from Lemma 3.2.17 that A admits a unital embedding into $\mathcal{O}_2$. □

Before diving into the proof of the main theorem of this section (and indeed this entire exposition), we list a few facts that will be needed. First, that for any C^*-algebra A, we have an isomorphism $C_0(\mathbb{R}) \otimes A \cong C_0(\mathbb{R}, A)$ (see [17, Theorem 6.4.17]). In particular, we see that $C_0(\mathbb{R}, A)$ is separable, when A is separable, and since $C_0(\mathbb{R})$ is nuclear, it also follows that $C_0(\mathbb{R}, A)$ is exact whenever A is exact. Second, that if a C^*-algebra contains a non-unitary isometry s, then it contains a unitary with full spectrum, namely $\exp(\frac{i\pi}{2}(s + s^*))$ (the most direct approach to this would be to show that the spectrum $\sigma(s) = [-2, 2]$, by representing faithfully on some Hilbert space $\mathcal{H}$ and choosing a sequence of unit vectors wisely).

Theorem 3.2.19 (Kirchbergs exact embedding theorem)**.** *A separable C^*-algebra A is exact if and only if there is an injective *-homomorphism $\iota : A \to \mathcal{O}_2$.*

If A is unital then ι can be chosen to be unital.

To put it another way, up to isomorphism the only separable, exact C^*-algebras are subalgebras of $\mathcal{O}_2$.

Proof. First note that the last statement follows from the first. Namely, assume that A is unital and that we have an injective *-homomorphism $\gamma : A \to \mathcal{O}_2$. Then $\gamma(1_A) \in \mathcal{O}_2$ is a non-zero projection and hence is equivalent to $1_{\mathcal{O}_2}$ (see the comments above Proposition 2.2.10). Therefore $\gamma(1_A)\mathcal{O}_2\gamma(1_A) \cong \mathcal{O}_2$ and composing γ with the latter isomorphism we obtain the desired unital embedding. Hence we concentrate on the first part of the statement. The strategy of the proof is to embed A into a certain crossed product and then apply the results from Section 3.2.1. First however we need to check that the conditions of Corollary 3.2.5 is satisfied.

Consider $C_0(\mathbb{R}, A)$ and let B be the unitization. Consider $\tau \in \mathrm{Aut}(C_0(\mathbb{R}, A))$ given by $\tau(f)(t) = f(t+1)$ and extend τ to an automorphism of B. Then, since $C_0(\mathbb{R}, A) \cong C_0(\mathbb{R}) \otimes A$, we obtain $C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z} \cong (C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A$ see Appendix B. Thus, if we can find a non-zero projection $p \in C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$ then we will have inclusions

$$A \hookrightarrow C_0(\mathbb{R}, A) \rtimes_\tau \mathbb{Z} \hookrightarrow B \rtimes_\tau \mathbb{Z}$$

where the first is given by $a \mapsto p \otimes a$ and the second is derived from crossed product theory (see Appendix B). If we let $u \in \widetilde{C_0(\mathbb{R})} \rtimes_\tau \mathbb{Z}$ be the unitary that implements τ, and $f, g \in C_0(\mathbb{R})$ be given by

$$f(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1 \\ 0, & \text{otherwise,} \end{cases} \qquad g(t) = \begin{cases} \sqrt{t - t^2}, & 0 \le t \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

then we can define $p \in C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$ to be $p = gu^* + f + ug$. This is easily seen to be a projection using the relation $uhu^* = \tau(h)$ and the definitions of f and g. Thus, we only have to show that $B \rtimes_\tau \mathbb{Z}$ can be embedded into $\mathcal{O}_2$.

Since A is exact and $C_0(\mathbb{R})$ is abelian it follows that $C_0(\mathbb{R}, A)$ is exact and therefore that both B and $B \rtimes_\tau \mathbb{Z}$ are exact. Furthermore B is quasidiagonal. This can be seen by noting that $C_0(\mathbb{R}, A) \cong C_0((0,1), A)$ (just by composing with a homeomorphism $\mathbb{R} \to (0,1)$) and the latter C^*-algebra is a subalgebra of $C_0([0,1), A)$ which is quasidiagonal (see Example D.1.1). Therefore B, being the unitization of a subalgebra of a quasidiagonal C^*-algebra, is quasidiagonal. Furthermore, B is separable, being the unitization of a separable C^*-algebra. Hence Lemma 3.2.18 shows that B can be unitally embedded into $\mathcal{O}_2$, and henceforth we consider B to be a unital subalgebra of $\mathcal{O}_2$.

Let ω be a free ultrafilter on $\mathbb{N}$ and consider now the unital, injective *-homomorphisms $\iota, \iota \circ \tau : B \to \mathcal{O}_2$. By Theorem 3.2.16 (ii) these are approximately unitarily equivalent, i.e., τ is approximately inner in $\mathcal{O}_2$. Thus Corollary 3.2.5 yields an injective *-homomorphism $\psi : B \rtimes_\tau \mathbb{Z} \to C(\mathbb{T}) \otimes (\mathcal{O}_2)_\omega$ with a ucp. lift ρ, i.e., we have a commutative diagram

$$\begin{tikzcd} & C(\mathbb{T}) \otimes l^\infty(\mathcal{O}_2) \arrow[d, "\pi"] \\ B \rtimes_\tau \mathbb{Z} \arrow[ur, "\rho"] \arrow[r, "\psi"'] & C(\mathbb{T}) \otimes (\mathcal{O}_2)_\omega. \end{tikzcd}$$

This is already beginning to shape up to an application of Lemma 3.2.17, but we are not quite ready yet.

First we note that since $\mathcal{O}_2$ contains a non-unitary isometry, for instance one of the canonical generators, we can embed $C(\mathbb{T})$ in $\mathcal{O}_2$. Furthermore, using that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and the comments made before the statement of this theorem we can find an injective *-homomorphism $\varphi : C(\mathbb{T}) \otimes (\mathcal{O}_2)_\omega \to (\mathcal{O}_2)_\omega$ by composing the embeddings

$$C(\mathbb{T}) \otimes (\mathcal{O}_2)_\omega \hookrightarrow \mathcal{O}_2 \otimes (\mathcal{O}_2)_\omega \hookrightarrow (\mathcal{O}_2 \otimes \mathcal{O}_2)_\omega \hookrightarrow (\mathcal{O}_2)_\omega.$$

Each of the above maps are chosen in the obvious way and since all C^*-algebras involved are simple we can be certain that the maps are really injective (the last one is actually an isomorphism). Similarly, using Proposition 3.2.3 we obtain embeddings

$$C(\mathbb{T}) \otimes l^\infty(\mathcal{O}_2) \hookrightarrow l^\infty(C(\mathbb{T}) \otimes \mathcal{O}_2) \hookrightarrow l^\infty(\mathcal{O}_2 \otimes \mathcal{O}_2) \hookrightarrow l^\infty(\mathcal{O}_2).$$

All that is left to check is that the following diagram is commutative

$$\begin{tikzcd} C(\mathbb{T}) \otimes l^\infty(\mathcal{O}_2) \arrow[r] \arrow[d, "\pi"'] & l^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\ C(\mathbb{T}) \otimes (\mathcal{O}_2)_\omega \arrow[r] & (\mathcal{O}_2)_\omega. \end{tikzcd}$$

But this is just matter of unravelling the maps and is therefore left for the reader. Now, Lemma 3.2.17 applies to yield an embedding $B \rtimes_\tau \mathbb{Z} \to \mathcal{O}_2$ which completes the proof.

□

3.3 Kirchberg's Tensor Product Theorems

3.3.1 On Tensor Products With $\mathcal{O}_2$

The following theorem gives a complete classification of those C^*-algebras for which $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. It is highly surprising, given how deep the statement is, that the proof is as simple as it is. Indeed it almost falls out of the embedding theorem as a corollary.

Theorem 3.3.1. *The tensor product $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ if and only if A is unital, simple, separable and nuclear.*

Proof. Assume that $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$. Since $\mathcal{O}_2$ is unital, A is isomorphic to a subalgebra of $A \otimes \mathcal{O}_2$ and therefore separable. Furthermore, if J is a non-trivial ideal in A then $J \otimes \mathcal{O}_2$ is a non-trivial ideal in $A \otimes \mathcal{O}_2$, contradicting the fact that $A \otimes \mathcal{O}_2$ is simple, hence A is also simple. Similarly, if A is not nuclear, then there exists a C^*-algebra B such that the canonical surjection $B \otimes_{\max} A \to B \otimes_{\min} A$ is not injective. But then, using that $\mathcal{O}_2$ is nuclear, we obtain that the canonical surjection

$$B \otimes_{\max} (A \otimes \mathcal{O}_2) = (B \otimes_{\max} A) \otimes \mathcal{O}_2 \to (B \otimes_{\min} A) \otimes \mathcal{O}_2 = B \otimes_{\min} (A \otimes \mathcal{O}_2)$$

is not injective, contradicting the fact that $A \otimes \mathcal{O}_2$ is nuclear.

Now we show that A must be unital. Let $\pi_1 : A \to B(\mathcal{H})$ and $\pi_2 : \mathcal{O}_2 \to B(\mathcal{K})$ be faithful and non-degenerate representations of A and $\mathcal{O}_2$, and consider $\pi_1 \otimes \pi_2 : A \otimes \mathcal{O}_2 \to B(\mathcal{H} \otimes \mathcal{K})$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for A. Then $(e_\lambda \otimes 1)_{\lambda \in \Lambda}$ is an approximate unit for $A \otimes \mathcal{O}_2$ and since π_1 and π_2 was chosen to be non-degenerate representations, this implies that $\lim_\lambda (e_\lambda \otimes 1)(\xi \otimes \eta) = \xi \otimes \eta$ for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Hence, if we let $I_0 \in B(\mathcal{H} \otimes \mathcal{K})$ denote the unit for $A \otimes \mathcal{O}_2$ we see that

$$I_0(\xi \otimes \eta) = \lim_\lambda I_0(e_\lambda \otimes 1)(\xi \otimes \eta) = \xi \otimes \eta$$

and hence, $I_0 = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$. Since $(e_\lambda \otimes 1)$ is an approximate unit for $A \otimes \mathcal{O}_2$ and π_2 was chosen to be non-degenerate it follows that $1_{\mathcal{O}_2} = I_{\mathcal{K}}$, whence

$$\lim_\lambda \|I_{\mathcal{H}} \otimes I_{\mathcal{K}} - e_\lambda \otimes 1\| = \lim_\lambda \|(I_{\mathcal{H}} - e_\lambda) \otimes I_{\mathcal{K}}\| = 0$$

which implies that $I_{\mathcal{H}} \in A$.

Assume that A is unital, simple, separable and nuclear. Then $A \otimes \mathcal{O}_2$ is also unital, simple, separable and nuclear, and therefore Kirchbergs embedding theorem implies that there exists a unital embedding $\varphi : A \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Let $\iota : \mathcal{O}_2 \to A \otimes \mathcal{O}_2$ be given by $\iota(x) = 1 \otimes x$. It follows from Theorem 3.2.16 part (i) that $\iota \circ \varphi \approx_u \mathrm{id}_{A \otimes \mathcal{O}_2}$ and from Theorem 2.2.14 that $\iota \circ \varphi \approx_u \mathrm{id}_{\mathcal{O}_2}$. The theorem now follows from Proposition 1.0.3. □

3.3.2 On Tensor Products With $\mathcal{O}_\infty$

In this section we examine under what conditions the isomorphism $A \cong A \otimes \mathcal{O}_\infty$ holds. This turns out to require more work than the corresponding result for $\mathcal{O}_2$. However, compared with the embedding section, the workload is still relatively light. Before we get into the main results though, we need a few propositions.

Proposition 3.3.2. *Let A be C^*-algebras with A exact and B simple. Then any ideal in $A \otimes_{\min} B$ is of the form $I \otimes B$ for some ideal I in A.*

Proof. This result holds in greater generality than stated here, but this will suffice for the present purposes (and it is still a quite deep result). Unfortunately we do not have time to delve into the proof, but a sketch of argument goes as follows: by Proposition 1.0.20 it follow that A/I is exact and therefore [3, Proposition 2.16] implies that any ideal J in $A \otimes_{\min} B$ will be the closure of the sum of the ideals on the form $I_1 \otimes_{\min} I_2 \subseteq J$, where $I_1 \subseteq A$ and $I_2 \subseteq B$ are ideals. Using that B is simple, we see that J must therefore be of the form $I \otimes_{\min} B$ for some ideal $I \subseteq A$. □

Lemma 3.3.3. *Let B be a unital C^*-algebra, A a unital subalgebra and $s \in B$ an isometry. If the map $a \mapsto s^*as$ for $a \in A$, is multiplicative, then ss^* commutes with all elements in A. If $s^*as = a$ for all $a \in A$, then s commutes with all elements in A.*

Proof. Let $p := ss^*$. Then for any unitary $u \in A$ the assumption that $a \mapsto s^*as$ is multiplicative implies

$$(pup)^*(pup) = s(s^*u^*s)(s^*us)s^* = ss^* = p.$$

Represent A on a Hilbert space $\mathcal{H}$ and let ξ be a unit vector in $p(\mathcal{H})$. Then

$$\begin{aligned} 1 &= \|up\xi\|^2 \\ &= \|pup\xi\|^2 + \|(1-p)up\xi\|^2 \\ &= 1 + \|(1-p)up\xi\|^2 \end{aligned}$$

and since this holds for any unit vector $\xi \in p(\mathcal{H})$ it follows that $pup = up$. By similar computations it follows that $pu^*p = u^*p$ and together these equalities imply that p commutes with all unitaries in A. Hence p commutes with all elements in A. If $s^*as = a$ then it follows from the statement made above that $as = ass^*s = ss^*as = sa$. □

Proposition 3.3.4. *Let ω be a free ultrafilter. The relative commutant $A_\omega \cap A'$ is a unital, simple, purely infinite C^*-algebra whenever A is a unital Kirchberg algebra.*

Proof. It will suffice to show that for each non-zero positive element $h \in A_\omega \cap A'$ there exists a non-unitary isometry $s \in A_\omega \cap A'$ such that $s^*hs = 1$. This will clearly imply that condition (iii) in Theorem 2.2.6 is satisfied.

Without loss of generality we may assume that $\|h\| = 1$. Let $K = \sigma(h)$. Since $f(h) \in A_\omega \cap A'$ for each $f \in C(K)$, we can define *-homomorphisms $\varphi, \psi : C(K) \otimes A \to A_\omega$ by

$$\varphi(f \otimes a) = f(h)a, \qquad \psi(f \otimes a) = f(1)a.$$

Note that $1 \in K$ since $\|h\| = 1$, hence ψ is well-defined. Furthermore, φ is injective; assume that $\ker \varphi \neq 0$. Since A is simple and nuclear, Proposition 3.3.2 implies that $\ker \varphi$ is of the form $C_0(U) \otimes A$ for some non-empty, open set $U \subseteq K$. Let $f \in C_0(U)$ be non-zero. Then $\varphi(f \otimes 1) = f(h)1 = f(h) = 0$ in contradiction with the assumption that f was non-zero.

Since $C(K)$ and A are both nuclear it follows that $C(K) \otimes A$ is also nuclear. Hence, by Theorem 1.0.16, there exist unital, completely positive lifts $\rho, \sigma : C(K) \otimes A \to l^\infty(A)$ of φ and ψ. Let $\rho(x) =$

$(\rho_1(x), \rho_2(x), ...)$ and $\sigma(x) = (\sigma_1(x), \sigma_2(x), ...)$, where each ρ_k and σ_k is a ucp. map, and use Corollary 3.2.12 part (ii) to obtain a sequence of non-unitary isometries $(s_n)_{n\in\mathbb{N}} \subseteq A$ such that

$$\lim_{\omega} \|s_n^* \rho(x) s_n - \sigma(x)\| = 0$$

for all $x \in C(K) \otimes A$. Letting $s := \pi_\omega((s_n)_{n\in\mathbb{N}})$ we see that $s \in A_\omega$ is a non-unitary isometry such that $s^*\varphi(x)s = \psi(x)$ for all $x \in C(K) \otimes A$. In particular,

$$s^*as = s^*\varphi(1 \otimes a)s = \psi(1 \otimes a)s = a$$

which implies that $s \in A_\omega \cap A'$. Furthermore; $s^*hs = s^*\varphi(\iota \otimes 1)s = \psi(\iota \otimes 1) = 1$ where $\iota \in C(K)$ is given by $\iota(t) = t$. $\square$

Proposition 3.3.5. *Let A and B be separable C^*-algebras with B unital and $\varphi : A \to B$ be an injective *-homomorphism.*

(i) Suppose there is a sequence of unitaries $(v_n) \subseteq B$ such that

$$\lim_{n\to\infty} \|v_n\varphi(a) - \varphi(a)v_n\| = 0, \quad \lim_{n\to\infty} dist(v_n^* b v_n, \varphi(A)) = 0$$

for all $a \in A$ and $b \in B$. Then $A \cong B$ and there is an isomorphism $\psi : A \to B$ such that $\psi \approx_u \varphi$.

(ii) Suppose that for some free ultrafilter ω on $\mathbb{N}$ there exists a sequence of unitaries $(v_n) \subseteq B_\omega \cap \varphi(A)'$ such that

$$\lim_{n\to\infty} dist(v_n^* b v_n, \varphi(A)_\omega) = 0$$

for all $b \in B$. Then $A \cong B$ and there is an isomorphism $\psi : A \to B$ such that $\psi \approx_u \varphi$.

Proof. (*i*): Let $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$ be countable, dense subsets. By the assumptions we can inductively choose a sequence of unitaries $(v_k) \subseteq B$ and elements $a_{j,n} \in A$ such that

$$\|v_n^* \cdots v_1^* b_j v_1 \cdots v_n - \varphi(a_{j,m})\| \leq \frac{1}{n}$$
$$\|v_n\varphi(a_j) - \varphi(a_j)v_n\| \leq 2^{-n}, \quad \|v_n\varphi(a_{j,m}) - \varphi(a_{j,m})v_n\| \leq 2^{-n}$$

for $j = 1, ..., n$ and $m = 1, ..., n-1$. It follows that the for each $j \in \mathbb{N}$ and suitably large $m, n \in \mathbb{N}$ we have

$$\begin{aligned} &\|v_1 \cdots v_n\varphi(a_j)v_n^* \cdots v_1^* - v_1 \cdots v_m\varphi(a_j)v_m^* \cdots v_1^*\| \\ &= \|v_{n+1} \cdots v_m\varphi(a_j)v_m^* \cdots v_{n+1}^*\| \\ &\leq \sum_{k=n+1}^{m} 2^{-k} \leq 2^{-n}, \end{aligned}$$

and since $\{a_j\} \subseteq A$ is dense it follows that $(v_1 \cdots v_n\varphi(a)v_n^* \cdots v_1^*)_{n=1}^\infty$ is a Cauchy sequence for all $a \in A$. Thus we can define $\psi : A \to B$ to be the *-homomorphism given by

$$\psi(a) = \lim_{n\to\infty} v_1 \cdots v_n\varphi(a)v_n^* \cdots v_1^*.$$

By construction $\psi \approx_u \varphi$ and since

$$\begin{aligned} \|\psi(a)\| &= \lim_{n\to\infty} \|v_1 \cdots v_n\varphi(a)v_n^* \cdots v_1^*\| \\ &= \lim_{n\to\infty} \|\varphi(a)\| = \|a\| \end{aligned}$$

we see that ψ is injective. Using the calculations above, we also see that

$$\|\psi(a_{j,n}) - v_1 \cdots v_n \varphi(a) v_n^* \cdots v_1^*\| \leq 2^{-n},$$

whence it follows that

$$\begin{aligned}\|b_j - \psi(a_{j,n})\| &\leq 2^{-n} + \|b_j - v_1 \cdots v_n \varphi(a_{j,n}) v_n^* \cdots v_1^*\| \\ &= 2^{-n} + \|v_n^* \cdots v_1^* b_j v_1 \cdots v_n - \varphi(a_{j,n})\| \\ &\leq 2^{-n} + \frac{1}{n}.\end{aligned}$$

Since $\{b_j\} \subseteq B$ is dense and $\psi(A)$ is a C^*-algebra, this concludes the proof.

(ii): It is sufficient to prove that the assumptions imply the assumptions in (i). Furthermore, since A and B are separable, it is sufficient to prove that, there exists a sequence of unitaries $(u_n) \subseteq B$ such that for all finite subsets $\{a_1, ..., a_N\} \subseteq A$, $\{b_1, ..., b_M\} \subseteq B$ and $\varepsilon > 0$ we can find $X \in \omega$ such that

$$\|u_k \varphi(a_i) - \varphi(a_i) u_k\| \leq \varepsilon, \quad \operatorname{dist}(u_k^* b_j u_k, \varphi(A)) \leq \varepsilon$$

for all $k \in X$, $i = 1, ..., N$ and $j = 1, ..., M$.

Let $(v_n)_{n=1}^\infty \subseteq B_\omega \cap \varphi(A)'$ be a sequence of unitaries satisfying the conditions in the hypothesis above. Let $\varepsilon > 0$ be arbitrary and choose $k \in \mathbb{N}$ and $c_1, ..., c_M \in \varphi(A)_\omega$ such that

$$\|v_k^* b_j v_k - c_j\| \leq \frac{\varepsilon}{2}.$$

Let $v := v_k$. As unitaries in B_ω lift to unitaries, we can choose a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq B$ such that $\pi_\omega((u_n)_{n=1}^\infty) = v$. Also, for each $j = 1, ..., M$ choose bounded sequences $(a_{j,n})_{n=1}^\infty \subseteq A$ such that $c_j = \pi_\omega((\varphi(a_{j,n}))_{n=1}^\infty)$. From the work done so far, we obtain that

$$\lim_\omega \|u_n \varphi(a) - \varphi(a) u_n\| = 0 \quad \text{and} \quad \lim_\omega \|u_n^* b_j u_n - \varphi(a_{j,n})\| \leq \frac{\varepsilon}{2},$$

for all $a \in A$ and $j = 1, ..., M$, i.e., there exists $X \in \omega$ such that

$$\|u_n \varphi(a) - \varphi(a) u_n\| \leq \varepsilon, \quad \|u_n^* b_j u_n - \varphi(a_{j,n})\| \leq \varepsilon$$

for all $n \in X$ and since ω is free, a standard separability argument completes the proof. □

Theorem 3.3.6. *Let A be a unital, separable C^*-algebra and B a simple, separable, unital and nuclear C^*-algebra. Then $A \cong A \otimes B$ if*

(i) B admits a unital embedding into $A_\omega \cap A'$ for some free ultrafilter ω.

(ii) The two *-homomorphisms $\alpha, \beta : B \to B \otimes B$ given by $\alpha(x) = 1 \otimes x$ and $\beta(x) = x \otimes 1$ are approximately unitarily equivalent.

Proof. Let $\varphi : A \to A \otimes B$ be the injective homomorphism given by $\varphi(a) = a \otimes 1$. The strategy of the proof is to show that φ fullfills the requirements for an application of Proposition 3.3.5 part (ii).

Let $\iota : B \to A_\omega \cap A'$ be the unital embedding from (i) and write $\iota(x) = \pi_\omega(\iota_1(x), \iota_2(x), ...)$, where $\iota_j : B \to A$ for all $j \in \mathbb{N}$. Then let $\beta : B \to \varphi(A)_\omega \cap \varphi(A)' \subseteq (A \otimes B)_\omega \cap \varphi(A)'$ be given by $\beta(b) = \pi_\omega(\varphi(\iota_1(b)), \varphi(\iota_2(b)), ...)$. Since ι is a *-homomorphism it follows that β is also a *-homomorphism. Furthermore, β is injective, since $\|\beta(b)\| = 0$ implies

$$\begin{aligned}\lim_\omega \|\varphi(\iota_n(b))\| &= \lim_\omega \|\iota_n(b)\| \\ &= \|\iota(b)\| = 0\end{aligned}$$

which, since ι was injective, implies that $b = 0$. In a similar manner, one checks that image of β indeed commutes with the image of φ. Let $\alpha : B \to (A \otimes B)_\omega$ be given by $\alpha(b) = 1 \otimes b$. It is easy to check that the images of α and β commute, and therefore, since B is nuclear and simple, it follows that the *-homomorphism $\alpha \times \beta : B \otimes B \to (A \otimes B)_\omega \cap \varphi(A)'$ given by $(\alpha \times \beta)(a \otimes b) = \alpha(a)\beta(b)$ is an isomorphism onto its image. Condition (2) implies the existence of a sequence of unitaries $(u_n) \subseteq B \otimes B$ such that $\lim_{n\to\infty} \alpha \times \beta(u_n^*(b \otimes 1)u_n) = \alpha \times \beta(1 \otimes b)$. Letting $v_n = (\alpha \times \beta)(u_n)$ we therefore obtain a sequence of unitaries in $(A \otimes B)_\omega \cap \varphi(A)'$ such that

$$\begin{aligned}\lim_{n\to\infty} v_n^* \alpha(b) v_n &= \lim_{n\to\infty} v_n^*(\alpha \times \beta)(b \otimes 1) v_n \\ &= (\alpha \times \beta)(1 \otimes b) = \beta(b).\end{aligned}$$

Hence,

$$\lim_{n\to\infty} v_n^*(a \otimes b)v_n = \lim_{n\to\infty} v_n^* \varphi(a)\alpha(b) v_n = \varphi(a)\beta(b) \in \varphi(A)_\omega$$

for all $a \in A$, $b \in B$. Clearly, this implies that

$$\lim_{n\to\infty} \operatorname{dist}(v_n^* c v_n, \varphi(A)_\omega) = 0$$

for all $c \in A \otimes B$ and therefore Proposition 3.3.5 yields the desired result. □

The final step before we get to the main result is the following uniqueness proposition for $\mathcal{O}_\infty$. During the proof we use the fact that inclusions of corners $\iota : pAp \to A$ gives rise to an injective group homomorphism $K_0(\iota) : K_0(pAp) \to K_0(A)$ for any unital C^*-algebra A and projection $p \in A$. This is not difficult to do using the standard picture of K_0 see [21, Proposition 3.1.7].

Proposition 3.3.7 (Uniqueness for $\mathcal{O}_\infty$)**.** *Any two unital *-homomorphisms φ, ψ from $\mathcal{O}_\infty$ into a unital, purely infinite, simple and nuclear C^*-algebra A are approximately unitarily equivalent.*

Proof. First, we prove the statement in the case where $[1_A]_0 = 0$ in $K_0(A)$. Let $s_1, s_2...$ be the standard generators for $\mathcal{O}_\infty$. Note that for each $j \in \mathbb{N}$ the projections $\varphi(s_j s_j^*)$ and $\psi(s_j s_j^*)$ are equivalent with 1_A and hence $[\varphi(s_j s_j^*)]_0 = [\psi(s_j s_j^*)]_0 = 0$ in $K_0(A)$ for all $j \in \mathbb{N}$. Furthermore, since $\varphi(s_j s_j^*)$ is orthogonal to $\varphi(s_i s_i^*)$ whenever $i \neq j$, and similarly for ψ, we obtain that

$$\left[\sum_{i=1}^n \varphi(s_i s_i^*)\right]_0 = \left[\sum_{i=0}^n \psi(s_i s_i^*)\right]_0 = 0$$

in $K_0(A)$ for each integer $n \in \mathbb{N}$. Using this we see that

$$0 = [1_A]_0 = \left[1_A - \sum_{i=1}^n \varphi(s_i s_i^*)\right]_0 + \left[\sum_{i=1}^n \varphi(s_i s_i^*)\right]_0 = \left[1_A - \sum_{i=1}^n \varphi(s_i s_i^*)\right]_0$$

and similarly for ψ. By Proposition 2.2.10 it follows that for each integer n the non-zero projections

$$1_A - \sum_{i=1}^{n-1} \varphi(s_i s_i^*), \quad 1_A - \sum_{i=1}^{n-1} \psi(s_i s_i^*), \quad 1_A$$

are equivalent. Hence, we may choose isometries $t_n, r_n \in A$ such that

$$\sum_{i=1}^{n-1} \varphi(s_i s_i^*) + t_n t_n^* = 1_A = \sum_{i=1}^{n-1} \psi(s_i s_i^*) + r_n r_n^*.$$

We aim to finish this part of the proof by applying Theorem 2.2.15 to the unital *-homomorphisms $\varphi_n, \psi_n : \mathcal{O}_n \to A$ given by mapping the canonical generators to $\varphi(s_1), \varphi(s_2), ..., \varphi(s_n), t_n$ respectively $\psi(s_1), \psi(s_2), ..., \psi(s_n), r_n$. First, we need to choose n to be even (which is easily done), and we need to show that the K_1 class of the unitary element

$$\sum_{i=1}^{n-1} \varphi(s_i)\psi(s_i)^* + t_n r_n^*$$

belongs to $(n-1)K_1(A)$. Although this is not necessarily true at the outset it may always be arranged by replacing t_n with wt_n for some suitable unitary w in $(t_n t_n^*)A(t_n t_n^*)$ (This is a K-theoretical exercise and therefore we don't give the argument. The interested reader may go though exercise 8.9 in [21]). Therefore there exists a sequence of unitaries $u_n \in A$ such that $u_n^* \varphi(s_j) u_n \to \psi(s_j)$ when $j = 1, ..., n-1$ and since this trick can be done for each even n, this completes the first part of the proof.

Now for the general case; let ω be a free ultrafilter on $\mathbb{N}$. By Proposition 3.3.4 the C^*-algebra $A_\omega \cap A'$ is simple and purely infinite, in particular there exists projections $p, q \in A_\omega \cap A'$ such that $1 \sim p \sim q$ and $p + q \leq 1$. Hence, we can find projections $p, q, r \in A_\omega \cap A'$ such that $1 \sim p \sim q$ and $1 = p + q + r$. From this we deduce that $[p+r]_0 = [q+r]_0 = 0$ in $K_0(A_\omega \cap A')$, and therefore the comments made before the statement of this proposition implies that $[p+r]_0 = 0$ in $K_0((p+r)A_\omega(p+r))$ and $[q+r]_0 = 0$ in $K_0((q+r)A_\omega(q+r))$. We aim to apply the first part of the proof to $(p+r)A_\omega(p+r)$ and $(q+r)A\omega(q+r)$, but first a little notation; for each projection $e \in A_\omega \cap A'$ and unital *-homomorphism $\rho : \mathcal{O}_\infty \to A$ let $\rho_e : \mathcal{O}_\infty \to eA_\omega e$ be given by $\rho_e(x) = \rho(x)e$. Note that if $e, e' \in A_\omega \cap A'$ are orthogonal projections then $\rho_e + \sigma_{e'}$ also defines a *-homomorphism for each pair of *-homomorphisms $\rho, \sigma : \mathcal{O}_\infty \to A$.

Since $[p+r]_0 = 0$ in $K_0((p+r)A_\omega(p+r))$ the first part of the proof implies that we can find a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq (p+r)A_\omega(p+r)$ such that $u_n^* \varphi_{p+r}(x) u_n \to \psi_{p+r}(x)$ for all $x \in \mathcal{O}_\infty$. Letting $v_n = u_n + q$ we see that v_n is an unitary in A_ω and that

$$v_n^*(\varphi_{p+r}(x) + \varphi_q(x))v_n = u_n^* \varphi_{p+r}(x) u_n + \varphi_q(x) \to \psi_{p+r}(x) + \varphi_q(x),$$

i.e. that $\varphi_{p+r} + \varphi_q \approx_u \psi_{p+r} + \varphi_q$ in A_ω. Similar considerations show that $(\varphi_q + \psi_r) + \psi_p \approx_u \psi_{q+r} + \psi_p$. Putting this together we see that

$$\begin{aligned} \varphi &= \varphi_{p+r} + \varphi_q \\ &\approx_u \psi_{p+r} + \varphi_q \\ &= (\varphi_q + \psi_r) + \psi_p \\ &\approx_u \psi_{q+r} + \psi_p \\ &= \psi \end{aligned}$$

in A_ω and by Lemma 3.1.5 this completes the proof. □

Finally we are ready to prove the last tensor product theorem.

Theorem 3.3.8. *Let A be a unital, simple, separable and nuclear C^*-algebra. Then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite.*

Proof. If A is simple and separable then $A \otimes \mathcal{O}_\infty$ is simple and purely infinite by Proposition 2.2.9, and thus, if $A \cong A \otimes \mathcal{O}_\infty$ then A is purely infinite.

Suppose conversely that A is purely infinite (and unital, simple, separable and nuclear). Then, by Proposition 3.3.4, $A_\omega \cap A'$ is simple and purely infinite, whence there is a unital embedding $\mathcal{O}_\infty \to A_\omega \cap A'$. By Proposition 3.3.7 the two unital *-homomorphisms $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ given by $x \mapsto 1 \otimes x$ and $x \mapsto x \otimes 1$ are approximately unitarily equivalent and we may therefore apply Theorem 3.3.6 to get an isomorphism $A \cong A \otimes \mathcal{O}_\infty$. □

APPENDIX A

Filters

The principal goal of this Appendix is to introduce filters to the reader who is not familiar with these. We cover only the most basic facts here (for more see [6, Appendix J]).

A.1 Filters

For any set X we let $\mathcal{P}(X)$ denote the powerset of X. A filter on X is a non-empty subset of X that is stable under inclusions and finite intersections and does not contain the empty set. To be more precise:

Definition A.1.1. Let X be a set and F a non-empty subset $F \subseteq \mathcal{P}(X)$. Then F is said to be a **filter** on X if it satisfies the following properties:

(i) $\emptyset \notin F$.

(ii) If $A \in F$ and $A \subseteq B \subseteq X$ then $B \in F$.

(iii) If $A, B \in F$ then $A \cap B \in F$.

The following is an immediate consequence of this definition.

Lemma A.1.2. *Let X be a set and F any filter on X. Then it holds that*

(i) For any $B \in \mathcal{P}(X)$ it does not happen that $B \in F$ and $(X \backslash B) \in F$.

(ii) For any finite system of sets $A_1, ..., A_n \in F$ it holds that $\bigcap_{i=1}^{n} A_i \in F$.

(iii) $X \in F$.

In some sense, when choosing a filter on a set X we decide which sets we consider to be 'large'. This is of course a picture with some deficiencies as will be clear from the following examples, but it is nonetheless a helpful picture to keep in mind. The claims in the examples to follow are not justified as they are quite easy to check using the definition. The curious reader is encouraged to do them as (very) easy excercises.

Example A.1.1. Let X be a set and $\Omega \subseteq \mathcal{P}(X)$ such that $\emptyset \notin \Omega$ and for each $B_1, B_2 \in \Omega$ there exists $B \in \Omega$ such that $B \subseteq B_1 \cap B_2$. Then

$$F(\Omega) := \{A \in \mathcal{P}(X) \mid \text{There exists } A' \in \Omega \text{ such that } A' \subseteq A\}$$

is a filter on X and $\Omega \subseteq F(\Omega)$. This filter is called the filter *generated by* Ω.

Example A.1.2. Let $(I, \leq)$ be a directed set and $I_\alpha := \{\beta \in I \mid \alpha \leq \beta\}$. Clearly the set $\Omega = \{I_\alpha \mid \alpha \in I\}$ satisfy the conditions required in the preceding example and therefore $F(\Omega)$ is a filter on I called the **residual filter** on I.

Example A.1.3. Let X be a topological space and $x \in X$. Then the set F_x of all neighbourhoods of x in X is a filter on X. Here, neighbourhoods means a set, not necessarily open, containing x and an open neighboorhood of x.

Example A.1.4. If $A_0 \subseteq X$ is a non-empty subset of X then $\{A \in \mathcal{P}(X) \mid A_0 \subseteq A\}$ is a filter on X. A filter F on X is said to be **principal** if there exists some non-empty subset $A_0 \subseteq X$ such that $F = \{A \in \mathcal{P}(X) | A_0 \subseteq A\}$. In this case F is said to be the *principal filter based on* A_0.

The definition of filters turns out to be too inclusive in most applications and therefore one usually restricts ones attention to so-called free filters, much in the same way that the only topological spaces that are considered in practice are the Hausdorff spaces. The definition is as follows:

Definition A.1.3. Let X be a set and F a filter on X. Then F is said to be **free** if $\bigcap_{X \in F} X = \emptyset$.

Another interesting concept is that of ultrafilters:

Definition A.1.4. Let X be a set and F a filter on X. Then F is said to be an **ultrafilter** on X if it is maximal, i.e., if F' is a filter on X such that $F \subseteq F'$, then $F = F'$.

That these concepts are of interest will become clear once we discuss convergence along filters. Before we get to that however we list a few results about ultrafilters. The proofs can be found in most litterature on the subject of filters for instance [6], and as a result they are omitted here.

Theorem A.1.5. *Let X be a non-empty set and F_0 a filter on X. Then there exists an ultrafilter ω on X containing F_0.*

Proof. Consider the set Φ of all filters on X containing F_0 ordered under inclusion. This is a partially ordered set, and a straightforward application of Zorn's Lemma yields a maximal element $\omega \in \Phi$, which is the desired ultrafilter. $\square$

Note that if F_0 is a free filter then the ultrafilter F in the above proposition will also be free.

Proposition A.1.6. *Let X be a set and F a filter on X. Then the following are equivalent:*

(i) F is an ultrafilter

(ii) For each $A \subseteq X$ either $A \in F$ or $(X \backslash A) \in F$.

Proof. The proof will only be outlined, but the reader should not experience any difficulties in filling out the details. The proof relies on the following fact: Given a filter F on X and $A_0 \subseteq X$ such that $A_0 \notin F$ and $(X \backslash A_0) \notin F$, there exists a filter F' on X such that $A_0 \in F'$ and $F \subseteq F'$. The filter F' is given by:

$$F' := \{B \in \mathcal{P}(X) \mid \text{there exists } A \in F, A' \in F_0 \text{ such that } A \cap A' \subseteq B\},$$

where F_0 is the principal filter based on A_0. It is left as an easy exercise for the reader to check that this is indeed a filter.

Now to prove the statement. Assume that F is ultrafilter and that there exist some set $A_0 \subseteq X$ such that $A_0, (X \backslash A_0) \notin F$. Then it follows that there exists some filter F' such that $A_0 \in F'$ and $F \subseteq F'$, contradicting maximality of F. On the other hand, assume that F satisfies (ii) and that F' is a filter on X such that $F \subset F'$. Choose some set $A_0 \in F' \backslash F$. By assumption $(X \backslash A_0) \in F \subset F'$ and thus $(X \backslash A_0) \cap A = \emptyset \in F'$, contradicting the assumption that F' is a filter. ☐

A.2 Convergence Along Filters

As promised earlier, now comes a discussion of convergence along filters.

Definition A.2.1. Let X be a topological space, I a set and F a filter on I. A subset $(x_i)_{i \in I} \subseteq X$ is said to **converge along** F to some $x \in X$, in symbols $x_i \xrightarrow{i \to F} x$ or $\lim_F x_i = x$, if

$$\{i \in I \mid x_i \in U\} \in F$$

for all neighbourhoods U of x.

Note that one could replace neighbourhoods with open neighboorhoods in this definition.

Remark A.2.1. In the applications to follow this definition will often be replaced with an equivalent statement, namely that for all (open) neighbourhoods U of $x \in X$ there exists some $A \in F$ such that $x_i \in U$ for all $i \in A$. This is clearly equivalent with Definition A.2.1 and has the advantage of being slightly more intuitive (i.e., x_i is close to x for a 'large' number of i's in I) and closer to the definition of convergence of nets and sequences.

It should now be clear why we usually require filters to be free. If X is a topological space, I is an index set and F is the filter on I generated by $\{i_0\}$ for some $i_0 \in I$, then $x_i \xrightarrow{i \to F} x_{i_0}$ for any $(x_i)_{i \in I} \subseteq X$. Hence, when we insist that filters should be free we really instist that all boring instances of converging sets are excluded.

Proposition A.2.2. *In the setup of Definition A.2.1 and Remark A.2.1 the limit x is unique if X is Hausdorff.*

Proof. Let $y \in X$ and $y \neq x$. Since X is Hausdorff there exists disjoint open neighboorhoods U_x and U_y of x and y respectively. Thus the sets

$$\{i \in I \mid x_i \in U_x\} \quad \text{and} \quad \{i \in I \mid x_i \; U_y\}$$

are disjoint. The former of these sets is in F and hence, by property (i) and (ii) in definition A.1.1, the latter is not. ☐

Theorem A.2.3. *Let X be a topological space. The following conditions are equivalent:*

(i) X is compact

(ii) For any ultrafilter F on X there exists some point $x_0 \in X$ such that F contains all neighbourhoods of x_0.

Proof. Assume (i) and let ω be any ultrafilter on X. Let $A_1, ..., A_n \in \omega$. Since

$$\bigcap_{i=1}^{n} \overline{A_i} \supseteq \bigcap_{i=1}^{n} A_i \neq \emptyset$$

it follows that the collection $(\overline{A})_{A\in\omega}$ has the finite intersection property and hence, by compactness of X, it has non-empty intersection. Let $x \in \bigcap_{A\in\omega} \overline{A}$. For every $V \in F_x$ it follows that $x \in V \cap A$ and hence the collection

$$\Omega := \{A \cap V \mid A \in \Omega, V \in F_x\}$$

has the finite intersection property. Let $F(\Omega)$ be the filter generated by Ω. By construction $\omega \subseteq F(\Omega)$ (since X is a neighboorhood of x), and therefore $\omega = F(\Omega)$ by Corollary A.1.6. Since $F_x \subseteq F(\Omega) = \omega$, the desired conclusion follows.

Assume (ii) and let $(C_i)_{i\in I}$ be a collection of closed subsets of X with the finite intersection property. Then there exists an ultrafilter ω on X such that $(C_i)_{i\in I} \subseteq F$. By assumption there exists some $x \in X$ such that $F_x \subseteq \omega$ and hence for each C_i and $V \in F_x$ it follows that $C_i \cap V \neq \emptyset$. Since each C_i is closed this implies that $x \in C_i$ and hence $x \in \bigcap_{i\in I} C_i \neq \emptyset$. Therefore the desired result follows. □

Corollary A.2.4. *Let X be a compact, topological space, $(x_i)_{i\in I} \subseteq X$ and ω an ultrafilter on I. Then there exists $x_0 \in X$ such that $x \xrightarrow{i\to\omega} x_0$. Furthermore, if X is Hausdorff this limit is unique.*

Proof. The last statement follows from Proposition A.2.2. Hence only the first part of the statement will be proven.

Consider the collection of sets $\{x_i \mid i \in A\}$ for all $A \in \omega$. This collection has the finite intersection property since

$$\{x_i \mid i \in A\} \cap \{x_i \mid i \in B\} \supseteq \{x_i \mid i \in A \cap B\} \neq \emptyset$$

and hence there exists some ultrafilter ω' on X containing this collection. By the previous theorem there exists some $x_0 \in X$ such that $F_{x_0} \subseteq \omega'$ and this x_0 will be shown to be the limit of $(x_i)_{i\in I}$ along the ultrafilter ω.

Let V be some neighbourhood of x_0 and assume that $\{i \in I \mid x_i \in V\} \notin \omega$. Then, by Definition A.1.4, $U := \{i \in I \mid x_i \notin V\} \in F$. By construction $\{x_i \mid i \in U\} \in \omega'$ and $V \in \omega'$. But since

$$\{x_i \mid i \in U\} \cap V = \emptyset$$

it follows that $\emptyset \in \omega'$ in contradiction with Definition A.1.1 part (i). Hence, for every neighbourhood V of x_0 it holds that $\{i \in I \mid x_i \in V\} \in \omega$, i.e., $x_i \xrightarrow{i\to\omega} x_0$. □

Proposition A.2.5. *Let $f : X \to Y$ be a map between topological spaces X and Y. Then f is continuous if and only if for each index set I, each filter $\mathcal{F}$ on I and each indexed subset $(x_i)_{i\in I} \subseteq X$ converging to some $x \in X$ along $\mathcal{F}$ it holds that $(f(x_i))_{i\in I}$ converges to $f(x)$ in Y along $\mathcal{F}$.*

Proof. We only prove the 'only if' direction, since this is all we will need. The other direction is difficult to do using the techniques introduced here.

Assume that f is continuous and that $(x_i)_{i\in I}$ converges to $x \in X$ along $\mathcal{F}$. Let U be a neighbourhood of $f(x)$ in Y. Then $V := f^{-1}(U)$ is a neighbourhood of x in X and hence there exists some $F \in \mathcal{F}$ such that $x_i \in U$ for all $i \in F$. This, of course, implies that $f(x_i) \in U$ for all $i \in F$ completing the proof. □

APPENDIX B

Crossed Products

In this Appendix we seek to give a (very) basic introduction to the subject of crossed products. We do this for arbitrary discrete, at least to start with and then, towards the end we prove a few selected results which will be needed when we embed exact C^*-algebras in $\mathcal{O}_2$. However there are numerous results which are simply stated, not proven, in fact everything apart from the last proposition (see [11, Chapter VIII] for proofs).

B.1 Crossed Products

Let A be a C^*-algebra and Γ a discrete, countable group with unit e. A C^*-dynamical system is a triple (A, Γ, α) where $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an action of Γ on A, i.e. a group homomorphism. We write α_s instead of $\alpha(s)$ to ease the notation a bit. Given such a triple we let $A\Gamma$ denote the set of finite sums

$$A\Gamma := \left\{ \sum_{s\in\Gamma} a_s s \mid a_s \in A \right\}.$$

Introducing the formal rules $sas^{-1} = \alpha_s(a)e$ and $(as)^* = s^{-1}a$ for all $s \in \Gamma$ and $a \in A$, we obtain a *-algebra by setting

$$\left(\sum_{s\in\Gamma} a_s s\right) + \left(\sum_{s\in\Gamma} b_s s\right) = \sum_{s\in\Gamma} (a_s + b_s)s$$

$$\left(\sum_{s\in\Gamma} a_s s\right) \cdot \left(\sum_{t\in\Gamma} b_t t\right) = \sum_{t\in\Gamma} \left(\sum_{s\in\Gamma} a_s \alpha_s(b_{s^{-1}t})\right) t$$

$$\left(\sum_{s\in\Gamma} a_s s\right)^* = \sum_{s\in\Gamma} \alpha_s(a_{s^{-1}})s.$$

These operations are obtained by defining them as one would expect and then playing around with the sums using the rule $sas^{-1} = \alpha_s(a)$. Note that $A\Gamma$ contains a copy of A, namely Ae.

Now we proceed to define the notion of a covariant homomorphism. The terminology is not standard, but have been applied since it is the opinion of the author that the term covariant representation, which is standard terminology, should be reserved for the case $B(\mathcal{H})$.

Definition B.1.1. Let (A, Γ, α) be a C^*-dynamical system. A covariant homomorphism (φ, u) of (A, Γ, α) into a unital C^*-algebra B, consists of a *-homomorphism $\varphi : A \to B$ and a group homomorphism $u : \Gamma \to \mathcal{U}(B)$ such that

$$\varphi(\alpha_s(a)) = u_s \varphi(a) u_s^*$$

for all $a \in A$ and $s \in \Gamma$. If $B = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then (φ, u) is called a covariant representation.

In the above we have adopted the notation $u(s) = u_s$.

Proposition B.1.2. *If (φ, u) is a covariant homomorphism from (A, Γ, α) into B, then*

$$(\varphi \times u)\left(\sum_{s\in\Gamma} a_s s\right) := \sum_{s\in\Gamma} \varphi(a_s) u_s$$

defines a *-homomorphism from $A\Gamma$ into B.

The proof is omitted, but the interested reader should do it. Linearity is clear and the fact that $(\varphi\times u)$ is multiplicative and preserves adjoints follows easily by using that (φ, u) is a covariant homomorphism.

We now turn to representing $A\Gamma$ on a Hilbert space. Let G be some set. Then we get a Hilbert space $l^2(G)$ by setting

$$l^2(G) := \left\{ x : G \to \mathbb{C} \mid \sum_{g\in G} |x(g)|^2 < \infty \right\}.$$

We equip $l^2(G)$ with the obvious vectorspace structure and the inner product $\langle x, y\rangle = \sum_{g\in G}\langle x(g), y(g)\rangle$. Note that any sum $\sum_{g\in G} |x(g)|$ is convergent if and only if $|x(g)| \neq 0$ for only countably many g. Using this it is not difficult to see that $l^2(G)$ is a Hilbert space for any set G, and that the set $(\delta_g)_{g\in G}$ where $\delta_g(h) = 1$ if $h = g$ and $\delta_g(h) = 0$ otherwise, constitutes an orthonormal basis for $l^2(G)$.

In our case we consider $l^2(\Gamma) \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$ and then we have a canonical unitary representation $\lambda : \Gamma \to \mathcal{U}(B(l^2(\Gamma) \otimes \mathcal{H}))$ given by $\lambda_s(\delta_t \otimes \xi) = \delta_{st} \otimes \xi$. It is easy to see that λ_s is indeed an unitary by checking that $\lambda_s^*(\delta_t \otimes \xi) = \delta_{s^{-1}t} \otimes \xi$. It is not uncommon in the litterature to refer to this representation as the left regular representation of Γ

Lemma B.1.3. *If π is a representation of A on $\mathcal{H}$, then $\hat{\pi} : A \to B(l^2(\Gamma) \otimes \mathcal{H})$ given by*

$$\hat{\pi}(a)(\delta_s \otimes \xi) = \pi(\alpha_{s^{-1}}(a))(\xi)$$

is a representation on $l^2(\Gamma, \mathcal{H})$ and $(\hat{\pi}, (\lambda \otimes I_{\mathcal{H}})$ is a covariant representation of $A\Gamma$ on $l^2(\Gamma) \otimes \mathcal{H}$. Furthermore if π is faithful so is $\hat{\pi}$.

Now we are ready to define the crossed product and the reduced crossed product:

Definition B.1.4. If (A, Γ, α) is a C^*-dynamical system we let the crossed product, denoted $A \rtimes_\alpha \Gamma$, be the completion of $A\Gamma$ with respect to the norm

$$\|S\| := \sup\{\|(\pi \times u)(S)\| \mid (\pi, u) \text{ is a covariant } {}^*\text{-representation of } A\Gamma\}.$$

We define the reduced crossed product, denoted $A \rtimes_{\alpha,r} \Gamma$, to be the completion of $A\Gamma$ in the norm

$$\|S\|_r = \|(\hat{\sigma} \times \lambda)(S)\|$$

where σ is the universal representation of A.

It is not completely obvious that $\|\cdot\|$ is well-defined, but it is not difficult to see that

$$\|(\pi \times u)\left(\sum_{s\in\Gamma} a_s s\right)\| \leq \sum_{s\in\Gamma} \|a_s\|,$$

and hence that $\|S\| < \infty$ for all $S \in A\Gamma$. Before moving on to $\mathbb{Z}$-actions we summarize the most useful properties of the crossed product. They follow quite easily from the results stated so far.

Proposition B.1.5. *Let (A, Γ, α) be a C^*-dynamical system.*

(i) If (φ, u) is a covariant homomorphism of (A, Γ, α) into B then there is a *-homomorphism $(\varphi \rtimes u) : A \rtimes_\alpha \Gamma \to B$ such that

$$(\varphi \rtimes u)\left(\sum_{s\in\Gamma} a_s s\right) = \sum_{s\in\Gamma} \varphi(a_s) u_s.$$

(ii) The map $\iota : A \to A \rtimes_\alpha \Gamma$ given by $\iota(a) = ae$ defines an embedding.

(iii) If A is unital then $u_s := 1_A s \in A \rtimes_\alpha \Gamma$ defines a unitary satisfying $u_s \iota(a) u_s^* = \iota(\alpha_s(a))$ for all $a \in A$ and $s \in \Gamma$ and

$$\sum_{s\in\Gamma} a_s s = \sum_{s\in\Gamma} a_s u_s,$$

for all $\sum_{s\in\Gamma} a_s s \in A\Gamma$. We say that u_s implements α_s.

(iv) If A is unital, then $s \mapsto 1_A s$ defines an injective group homomorphism.

Since A can be embedded in $A \rtimes_\alpha \Gamma$ we will simply identify A with the appropriate subalgebra and omit any reference to ι. Similarly, in case A is unital, we will simply consider Γ to be a subgroup of $\mathcal{U}(A \rtimes_\alpha \Gamma)$.

One may wonder what can be said about $A \rtimes_\alpha \Gamma$ in case A is not unital. Given an action $\alpha : \Gamma \to A$ of Γ on a non-unital C^*-algebra A we can easily extend the action to $\tilde{\alpha} : \Gamma \to \tilde{A}$ by letting $\tilde{\alpha}(s) = \tilde{\alpha_s}$. The following proposition reduces virtually every problem concerning unital C^*-algebras to the unital case.

Proposition B.1.6. *Let (A, Γ, α) be a non-unital C^* dynamical system, $U_s = 1s \in \tilde{A} \rtimes_{\tilde{\alpha}} \Gamma$ and $V_s = 1s \in \mathbb{C} \rtimes_{id} \Gamma$. Then the sequence:*

$$0 \longrightarrow A \rtimes_\alpha \Gamma \xrightarrow{\iota \times U} \tilde{A} \rtimes_{\tilde{\alpha}} \Gamma \xrightarrow{\pi \times V} \mathbb{C} \rtimes_{id} \Gamma \longrightarrow 0$$

is exact.

In particular, it follows that we can always find a unitary that implements α_s, although we may have to go to $\tilde{A} \rtimes_{\tilde{\alpha}} \Gamma$ to find it.

We now turn our attention to crossed products with $\mathbb{Z}$ since these are the ones that will be used in the main body of the exposition. First we note that, since $\mathbb{Z}$ is a cyclic group generated by 1, any action $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$ is unquely determined by its action in $1_\mathbb{Z}$, meaning that if we set $\tau := \alpha_1$, then for all $n \in \mathbb{Z}$ it holds that $\alpha_n = \tau^n$. Thus, when dealing with $\mathbb{Z}$-actions we only specify the single automorphism that generates the action, usually denoted τ. Similarly, if B is a unital C^*-algebra then any unitary $v \in B$ gives rise to a group homomorphism $v : \mathbb{Z} \to \mathcal{U}(B)$ by setting $v_n = v^n$. Hence a covariant homomorphism (φ, v) of (A, τ) into B consists simply of a homomorphism $\varphi : A \to B$ and a unitary $v \in B$ satisfying

$$v\varphi(a)v^* = \varphi(\tau(a)).$$

Suppose that A is unital. If we let $u \in A \rtimes_\alpha \mathbb{Z}$ be the unitary that implements τ, i.e., $1_A 1_\mathbb{Z}$, we see that

$$\sum_{n\in\mathbb{Z}} a_n n = a_n u^n$$

for all $\sum_{n\in\mathbb{Z}} a_n n \in A\mathbb{Z}$. Since $A \rtimes_\alpha \mathbb{Z}$ is the completion of $A\mathbb{Z}$ with respect to a certain norm we conclude that

$$\left\{\sum_{n\in\mathbb{Z}} a_n u^n \mid a_n \in A, n \in \mathbb{Z}\right\}$$

is a dense subset of $A \rtimes_\alpha \mathbb{Z}$. In other words the C^*-algebra $A \rtimes_\alpha \mathbb{Z}$ is generated by A and u. Crossed products with $\mathbb{Z}$ enjoys the following useful property:

Proposition B.1.7. *If $\alpha \in Aut(A)$ then $A \rtimes_\alpha \mathbb{Z} \cong A \rtimes_{\alpha,r} \mathbb{Z}$.*

If we consider the C^*-algebra $\mathbb{C}$, and a discrete group Γ we can now define $\mathbb{C} \rtimes_\alpha \Gamma$. Since there is precisely one automorphism of $\mathbb{C}$, namely the identity, this crossed product must be $\mathbb{C} \rtimes_{\mathrm{id}} \Gamma$.

Definition B.1.8. For any discrete group Γ we define the **full group C^*-algebra** of Γ, denoted $C^*(\Gamma)$, to be $\mathbb{C} \rtimes_{\mathrm{id}} \Gamma$.

Actually this is sort of cheating, since the group C^*-algebra is a concept which can be defined without any reference to crossed products, but it is true that this C^*-algebra is isomorphic to the one defined above and thus no serious harm is done.

Proposition B.1.9. *Let $\mathbb{T} \subseteq \mathbb{C}$ be the unit circle. Then $C(\mathbb{T}) \cong C^*(\mathbb{Z})$.*

Proof. See [5]. $\square$

Now to a final result on crossed products that will be needed in the main body of this exposition. It also serves as a nice exercise to practice playing around with crossed products.

Proposition B.1.10. *For any pair of C^*-algebras A, C and any automorphism $\tau \in Aut(A)$ it holds that $(A \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \cong (A \otimes_{\min} C) \rtimes_{\tau\otimes id} \mathbb{Z}$.*

Proof. Let π_1 be a faithful representation of A on $\mathcal{H}_1$, π_2 be a faithful representation of B on $\mathcal{H}_2$ and $\lambda : \mathbb{Z} \to \mathcal{U}(B(l^2(\mathbb{Z})))$ be the standard group homomorphism. Then, using Proposition B.1.3 and the definition of the spatial tensor product, we obtain faithful representations

$$\begin{aligned}(\hat{\pi}_1 \rtimes \lambda) \otimes \pi_2 &: (A \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \to B(l^2(\mathbb{Z}) \otimes \mathcal{H}_1 \otimes \mathcal{H}_2)\\ (\widehat{\pi_1 \otimes \pi_2}) \rtimes \lambda &: (A \otimes_{\min} C) \rtimes_{\tau\otimes\mathrm{id}} \mathbb{Z} \to B(l^2(\mathbb{Z}) \otimes \mathcal{H}_1 \otimes \mathcal{H}_2).\end{aligned}$$

Noting that $A\mathbb{Z} \odot C \subseteq (A \rtimes_\tau \mathbb{Z}) \otimes_{\min} C$ and $(A \odot C)\mathbb{Z} \subseteq (A \otimes_{\min} C) \rtimes_{\tau\otimes\mathrm{id}} \mathbb{Z}$ are dense subsets, we only need to show that the representations agree on these sets. This boils down to unravelling the maps involved. Let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}_1$, $(e_j)_{j\in J}$ a basis for $\mathcal{H}_2$ and $(\delta_n)_{n\in\mathbb{Z}}$ the standard basis for $l^2(\mathbb{Z})$. Then

$$(\hat{\pi}_1 \rtimes \lambda) \otimes \pi_2(an \otimes c)(\delta_k \otimes e_i \otimes e_j) = \delta_{k+n} \otimes \pi_1(\tau^{-n-k}(a)(e_i) \otimes \pi_2(c)(e_j)$$

and

$$\begin{aligned}((\widehat{\pi_1 \otimes \pi_2}) \rtimes \lambda)((a \otimes c)n)(\delta_k \otimes e_i \otimes e_j) &= \delta_{k+n} \otimes (\pi_1 \otimes \pi_2)((\tau \otimes \mathrm{id})^{-n-k}(a \otimes c))(e_i \otimes e_j)\\ &= \delta_{k+n} \otimes \tau^{-k-n}(a)(e_i) \otimes \pi_2(c)(e_j)\end{aligned}$$

from which it follows that $(A \rtimes_\tau \mathbb{Z}) \otimes_{\min} C \cong (A \otimes_{\min} C) \rtimes_{\tau\otimes\mathrm{id}} \mathbb{Z}$ (if we identify both algebras with their image under the faithful representations above, we actually get equality). $\square$

APPENDIX C

Hereditary Subalgebras

In this Appendix we seek to indtroduce the reader to the most basic facts concerning hereditary subalgebras. They will be used throughout the main body of the exposition without reference. The results and proofs are taken from [17, Section 3.2].

C.1 Hereditary Subalgebras

Definition C.1.1. Let A be a C^*-algebra and $B \subseteq A$ a subalgebra. Then B is said to be a **hereditary subalgebra** if $b \in B_+$ and $0 \leq a \leq b$ implies that $a \in B$.

Obviously both 0 and A are hereditary subalgebras of $\mathcal{A}$ and so is any intersection of hereditary subalgebras. Thus the notion of a smallest hereditary containing a given set $S \subseteq A$, namely the intersection of all hereditary subalgebras containing S, makes sense, and this subalgebra is said to be *generated by* S. The following example is quite easy, but it will be important later on.

Example C.1.1. Let A be a C^*-algebra and $p \in A$ a projection. Then pAp is a hereditary subalgebra of A. It is easy to convince oneself of the equality

$$pAp = \{a \in A \mid pa = ap = a\}.$$

Hence pAp is always unital, even when A is not, with unit p. Furthermore the above description of pAp easily shows that it is closed, and obviously it is a *-subalgebra, hence a subalgebra of A. To see that it is hereditary, let $b \in A_+$, $pap \in (pAp)_+$ and assume $b \leq pap$. Then $0 \leq (1-p)b(1-p) \leq (1-p)pap(1-p) = 0$ which implies that $\|b^{1/2}(1-p)\| = 0$ which again implies that $b(1-p) = 0$. Hence $bp = pb = b$ and therefore $b \in pAp$.

It turns out that there is a close relation between hereditary subalgebras and the closed left-ideals of a C^*-algebra, as described in the following theorem.

Theorem C.1.2. *Let A be a C^*-algebra.*

(i) Let $L \subseteq A$ be a left ideal. Then $L \cap L^*$ is a hereditary subalgebra of A. Furthermore the map $L \mapsto L \cap L^*$ is a bijection between the set of left ideals and the hereditary subalgebras of A.

(ii) Let L_1, L_2 be left ideals in A. Then $L_1 \subseteq L_2$ if and only if $(L_1 \cap L_1^*) \subseteq (L_2 \cap L_2^*)$.

(iii) If $B \subseteq A$ is a hereditary subalgebra then

$$L(B) := \{a \in A \mid a^*a \in B\}$$

is the unique left ideal corresponding to B.

Proof. (i) The last part of the statement will have been proven once (*ii*) and (*iii*) have been proven, so for the moment we contend ourselves to proving the first part.

Let $L \subseteq A$ be a left ideal in A. Clearly $L \cap L^*$ is a subalgebra of $\mathcal{A}$. To see that it is hereditary let $b \in (L \cap L^*)_+$, $a \in A_+$ and assume $a \leq b$. Since L is a left ideal there is an approximative unit $(e_\lambda) \subseteq L$. Since $a \leq b$ it follows that $(1-u_\lambda)a(1-u_\lambda) \leq (1-u_\lambda)b(1-u_\lambda)$ and hence

$$\|a^{1/2}(1-u_\lambda)\|^2 \leq \|(1-u_\lambda)b(1-u_\lambda)\| \leq \|b(1-u_\lambda)\|.$$

The last inequality can be seen by noting that $\|(1-u_\lambda)\| = \sup_{\varphi \in S} |\varphi(1) - \varphi(u_\lambda)| \leq 1$. It follows that $a^{1/2} \in L$ since $a^{1/2}u_\lambda \in L$, and therefore $a \in L \cap L^*$.

(ii) Since $L_1 \subseteq L_2$ implies $L_1^* \subseteq L_2^*$, one implication is clear. Let $(u_\lambda) \subseteq L \cap L^*$ be an approximative unit and $a \in L$. Then

$$\|a - u_\lambda\|^2 = \|(1-u_\lambda)a^*a(1-u_\lambda)\| \leq \|a^*a(1-u_\lambda)\|,$$

hence $\lim_\lambda au_\lambda = a$, since $a^*a \in L \cap L^*$. Furthermore, $(u_\lambda) \subseteq L_2$ and therefore $a \in L_2$.

(iii) First it needs to be shown that $L(B) := L$ is an ideal. Let $a, b \in L$. Then

$$(a+b)^*(a+b) \leq (a+b)^*(a+b) + (a-b)^*(a-b) = 2a^*a + 2b^*b \in B,$$

and hence $a + b \in B$. Similarly

$$(ab)^*(ab) = b^*a^*ab \leq \|a\|^2 b^*b \in B.$$

In a completely similar fashion one can show that L is also stable under scalar multiplication.

If $b \in B$ then $b^*b \in B$ and $bb^* \in B$ hence $b \in L \cap L^*$. On the other hand if $0 \leq b \in L \cap L^*$ then $b^2 \in B$, hence $b \in B$. Since the positive elements span a C^*-algebra it follows that $L \cap L^* = B$. ☐

As a first application of this proposition we prove the following.

Proposition C.1.3. *Let A be a C^*-algebra and $B \subseteq A$ a subalgebra. Then B is hereditary if and only if $bab' \in B$ for each $b, b' \in B$ and $a \in A$.*

Proof. Assume that B is hereditary. Then there exists some left-ideal L such that $B = L \cap L^*$. Hence $(ba)b' \in L$ and $(b'^*a^*)b^* \in L$ for each $b, b' \in B$ and $a \in A$, implying that $bab' \in B$.

Suppose that $\mathcal{B}$ has the other mentioned property and let $(u_\lambda) \subseteq \mathcal{B}$ be an approximative unit, and assume that $0 \leq a \leq b$ for some $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then $0 \leq \|(1-u_\lambda)a(1-u_\lambda)\| \leq \|(1-u_\lambda)b(1-u_\lambda)\|$ and therefore $a^{1/2} = \lim a^{1/2}u_\lambda$, which implies that $a = \lim u_\lambda a u_\lambda \in \mathcal{B}$ yielding the desired result. ☐

A few obvious consequences of this result deserves to be mentioned.

Corollary C.1.4. *Every ideal of a C^*-algebra is a hereditary subalgebra.*

Corollary C.1.5. *Let $a \in A_+$. Then $\overline{aAa}$ is the hereditary subalgebra generated by a.*

Proof. It follows easily from Proposition C.1.3 that this is actually a hereditary subalgebra, and proving that $a \in \overline{aAa}$ is a straightforward consequence of the existence of approximate units. □

Given the close relation between hereditary subalgebras of A and ideals in A the following result is perhaps not all that surprising. Before going into the proof we introduce the following notation; if A, B, C are subalgebras of the C^*-algebra D we let $S_{A,C}(B) := \overline{\mathrm{span}\{abc \mid a \in A, b \in B, c \in D\}}$, and we let $S_A(B) := S_{A,A}(B)$. In particular, if B is a subalgebra of A and I is an ideal in B then $S_A(I)$ is the ideal in A generated by I.

Proposition C.1.6. *Let B be a hereditary subalgebra of a C^*-algebra A, and J an ideal in B. Then there exists an ideal I in A, such that $J = B \cap I$.*

Proof. Let $I = S_A(I)$. Then I is an ideal in A, and since J is a C^*-algebra and $B \subseteq A$ is hereditary we have that $J^3 = J$ and $B \cap I = BIB$ (use approximate units to see this). Then, because B is hereditary and J is an ideal in B we see that

$$B \cap I = BIB = B(S_A(J^3))B \subseteq S_{B,J}(A)JS_{J,B}(A) \subseteq BJB.$$

Thus $B \cap I \subseteq BJB = J$ and the reverse inclusion is obvious, hence $B \cap I = J$. □

Corollary C.1.7. *Let B be a hereditary subalgebra of a simple C^*-algebra A. Then B is simple.*

It turns out that hereditary subalgebras are very well-behaved in general, not only in connection with simple C^*-algebras. They also behave nicely when trying to extend states and in other cases. In the present context however these results are sufficient.

APPENDIX D

Quasidiagonal C^*-algebras

This Appendix is not meant to be a general introduction to quasidiagonal C^*-algebras. Instead it presents the straigtest possible route towards Lemma D.1.6, which is the only statement we will need for the results of this exposition. For a general overview of the theory of quasidiagonal C^*-algebras see [4]. As the notion of quasidiagonality is simpler in the separable setting, and this is the only setting we really need, we will throughout this section assume that all Hilbert spaces and C^*-algebras are separable

D.1 Quasidiagonal C^*-algebras

Definition D.1.1. If A is a concrete C^*-algebra, i.e. $A \subseteq B(\mathcal{H})$, then A is said to be **quasidiagonal** if there exists a sequence of finite rank projections $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to 1_{B(\mathcal{H})}$ in the strong operator topology and $\|[a, p_n]\| \to 0$ for all $a \in A$.

This definition is only really valid in the separable setting. To do this properly, one should define quasidiagonality by a local property and then prove, that in the separable setting, the above definition and the local property agree.

Definition D.1.2. Let A be a C^*-algebra and $\pi : A \to B(\mathcal{H})$ a representation. Then π is said to be a **quasidiagonal representation** if $\pi(A) \subseteq B(\mathcal{H})$ is quasidiagonal. A is said to be a **quasidiagonal C^*-algebra** if A admits a faithful, quasidiagonal representation.

Remark D.1.1. It is not uncommon in the litterature to refer to C^*-algebras that satisfy this condition as **weakly quasidiagonal** C^*-algebras and reserve the term quasidiagonal for C^*-algebras for which all representations are quasidiagonal. There is good reason for this distinction, as the two notions do not agree, but once again we are only interested in the above situation in the present exposition.

One may notice that we do not require the representation of A in Definition D.1.2 to be non-degenerate. However this can always be arranged, see for example [4, Lemma 3.10]. It is also evident from the definition that if A is quasidiagonal so is the unitization of A as well as any subalgebra of A.

Proposition D.1.3. *Let A be a separable C^*-algebra and $\pi : A \to B(\mathcal{H})$ be a faithful, quasidiagonal representation of A. Then there exist a sequence $(q_n)_{n\in\mathbb{N}}$ of pairwise orthogonal, finite rank projections such that:*

(i) The sum $\sum_{i=1}^{\infty} q_i$ converges to $I_{\mathcal{H}}$ in the strong operator topology.

(ii) For all $a \in A$ it holds that $\pi(a) - \sum_{i=1}^{\infty} q_i \pi(a) q_i \in \mathcal{K}(\mathcal{H})$.

(iii) For all $a \in A$ the sequence $\|[\pi(a), q_i]\|$ is convergent.

Furthermore, the map $\rho : A \to B(\mathcal{H})$ given by $\rho(a) = \sum_{i=1}^{\infty} q_i \pi(a) q_i$ is essentially a *-homomorphism, i.e., it is a *-homomorphism modulo the compact operators.

Proof. We only sketch the proof here, and leave the details for the reader.

We can choose a subsequence (p_j) of (p_i) such that if we define $q_j = p_j - p_{j-1}$ then $\|[q_j, \pi(a)]\| < 2^{-j}$. Obviously, this sequence (q_j) satisfy (i) and (ii). It is not difficult to see that $\rho(a)$ exists as a SOT-limit and that β is a cp. map hence norm-continuous. Proving the last statements is a matter of finding $\pi(a) - \rho(a)$ as a norm-limit of finite rank operators, more specifically $\sum_{i=1}^{\infty} [q_i, \pi(a)] q_i$ and then writing $\rho(ab) - \rho(a)\rho(b)$ as a sum of compact operators. $\square$

Definition D.1.4. An operator $T \in B(\mathcal{H})$ is called block-diagonal if there exists a sequence (q_i) of pairwise orthogonal, finite rank projections such that $[q_i, T] = 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} q_i = 1$.

Using Proposition D.1.3 it is easy to see that a C^*-algebra $A \subseteq B(\mathcal{H})$ is quasidiagonal if and only if all $a \in A$ can be decomposed as $a = T + K$ where T is block-diagonal and K is compact. Now we are almost ready for the result that we are interested in. As you will recall, given faithful representation $\pi : A \to B(\mathcal{H})$, we can obtain a new faithful representation $\hat{\pi} : A \to B(l^2(\mathbb{N}) \otimes \mathcal{H})$ simply by embedding $B(\mathcal{H})$ in $l^\infty(B(\mathcal{H}))$ which can be considered as a subset of $B(l^2(\mathbb{N}) \otimes \mathcal{H})$ in a natural way, i.e., by letting $(b_n)_{n\in\mathbb{N}}(\delta_k \otimes \xi) = b_k(\xi)$.

Proposition D.1.5. *Let A be a quasidiagonal C^*-algebra and $\pi : A \to B(\mathcal{H})$ be a faithful and quasidiagonal representation. Then the induced representation $\hat{\pi} : A \to B(l^2(\mathbb{N}) \otimes \mathcal{H})$ is also faithful and quasidiagonal.*

Proof. Obviously $\hat{\pi}$ is also faithful. To see that $\hat{\pi}$ is quasidiagonal let $(p_i)_{i\in\mathbb{N}} \subseteq B(\mathcal{H})$ be a sequence of projections as in the definition of quasidiagonality. Let $\tilde{p}_i = (p_i, ..., p_i, 0, ...)$, where the p_i's occupy the first i positions. Then

$$\|[\tilde{p}_i, \hat{\pi}(a)]\| = \|[p_i, \pi(a)]\|$$

hence $\|[\tilde{p}_i, \hat{\pi}(a)]\| \to 0$. Furthermore, for any $\xi = (\xi_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}) \otimes \mathcal{H}$ we have that

$$\|\tilde{p}_i(\xi) - \xi\|\| = \sum_{k=1}^{i} \|p_i(\xi_k) - \xi_k\|^2 + \sum_{k=i+1}^{\infty} \|\xi_k\|^2$$

and then by standard arguments we see that $(\tilde{p}_i)_{i\in\mathbb{N}}$ converges to 1 in the strong operator topology. $\square$

Lemma D.1.6. *Let A be a unital, quasidiagonal C^*-algebra and ω a free ultrafilter on $\mathbb{N}$. Then there exists a sequence of integers $(k_n)_{n\in\mathbb{N}}$ and maps ρ, φ making the following diagram commutative*

$$\begin{tikzcd} & \prod_{n=1}^{\infty} M_{k_n} \arrow[d, "\pi_\omega"] \\ A \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & \prod_{n=1}^{\infty} M_{k_n} / \bigoplus_{n=1}^{\infty} M_{k_n} \end{tikzcd}$$

where φ is an injective *-homomorphism and ρ is ucp.

Proof. First, let $\pi : A \to B(\mathcal{H})$ be a faithful, quasidiagonal representation of A on $\mathcal{H}$, and let $\hat{\pi} : A \to B(l^2(\mathbb{N}) \otimes \mathcal{H}) := \mathbb{B}$ be the induced representation, as indicated in the paragraf preceding the statement of this lemma. Let $(p_i) \subseteq B(\mathcal{H})$ be given as in the definition of quasidiagonality and $(\tilde{p}_i) \subseteq \mathbb{B}$ be the corresponding sequence of projections. Next, we choose $(\tilde{q}_i)_{i\in\mathbb{N}} \subseteq \mathbb{B}$ as in Proposition D.1.3, i.e., such that $\sum_{i=1}^{\infty} \tilde{q}_i = 1$ and $\hat{\rho} : \mathbb{B} \to \mathbb{B}$ given by $\hat{\rho}(a) = \sum_{i=1}^{\infty} \tilde{q}_i a \tilde{q}_i$ is essentially a *-homomorphism. Since each $\tilde{q}_i$ is a finite-rank projection we may identify $\tilde{q}_i \mathbb{B} \tilde{q}_i$ with M_{k_i}, where $k_i = \operatorname{rank}(\tilde{q}_i)$, and hence $\hat{\rho}(\mathbb{B})$ with $\prod_{i=1}^{\infty} M_{k_i}$. Thus, since each of the maps $a \mapsto \tilde{q}_i a \tilde{q}_i$ is ucp. and $\hat{\pi}$ is a representation, we obtain a ucp. map $\hat{\rho} \circ \hat{\pi} : A \to \prod_{i=1}^{\infty} M_{k_i}$.

We let $\bigoplus_{i=1}^{\infty} M_{k_i}$ denote the sequences in $\prod_{i=1}^{\infty} M_{k_i}$ that converge to 0 along ω, let $\rho = \hat{\rho} \circ \hat{\pi}$ and

$$\pi_\omega : \prod_{i=1}^{\infty} M_{k_i} \to \prod_{i=1}^{\infty} M_{k_i} / \bigoplus_{i=1}^{\infty} M_{k_i}$$

be the quotient map. We have to show that $\pi_\omega \circ \rho$ is an injective *-homomorphism.

First we prove injectivity. Assume that $\pi_\omega \circ \rho(a) = 0$ for some $a \in A$, i.e., that

$$\lim_{i\to\omega} \|\tilde{q}_i \hat{\pi}(a) \tilde{q}_i\| = 0.$$

By Proposition D.1.3 we can write $\hat{\pi}(a) = \hat{\rho}(a) + K$, where $K \in \mathbb{B}$ is a compact operator. Noting that $(\tilde{p}_i)$ is an approximate unit for the compact operators in $\mathbb{B}$ we infer that $\lim_{i\to\omega} \|\tilde{q}_i K \tilde{q}_i\| = 0$. From this we deduce that

$$\begin{aligned}\lim_{i\to\omega} \|\tilde{q}_i \hat{\pi}(a) \tilde{q}_i\| &= \lim_{i\to\omega} \|\tilde{q}_i \hat{\rho}(a) \tilde{q}_i + \tilde{q}_i K \tilde{q}_i\| \\ &= \lim_{i\to\omega} \|\rho(a)(i)\| = 0\end{aligned}$$

Using a matrix trick we see that

$$\|\tilde{q}_i \hat{\pi}(a) \tilde{q}_i\| \geq \|p_i \pi(a) p_i\|$$

and hence

$$\lim_{i\to\omega} \|p_i \pi(a) p_i\| = 0.$$

Since ω is a free filter there exists a subsequence of $(p_i \pi(a) p_i)$ that converges strongly to 0. But $p_i \to 1$ and multiplication is SOT-continuous on bounded sets, implying that $p_i \pi(a) p_i \to \pi(a)$. Hence $\pi(a) = 0$.

Now we have to establish that $\varphi := \pi_\omega \circ \rho$ is a *-homomorphism. The only part of this that is not wholly trivial is proven multiplicativity, hence that is the only part we will show here. To that end let $a, b \in A$ be given. Proposition D.1.3 yields that ρ is essentially a *-homomorphism, hence $\rho(ab) - \rho(a)\rho(b)$ is a compact operator in $\mathbb{B}$. Since $(\tilde{p}_i)$ is an approximate unit for the compacts, we see that

$$\|\tilde{q}_i(\beta(ab) - \beta(a)\beta(b))\tilde{q}_i\| \leq \|(\beta(ab) - \beta(a)\beta(b)) - (\beta(ab) - \beta(a)\beta(b))\tilde{p}_{i-1}\|,$$

i.e., that $\pi_\omega(\beta(ab) - \beta(a)\beta(b)) = 0$.

□

In conclusion we state a theorem of Voiculescu without proof, which we will need in the proof of Kirchberg's embedding theorem. But first a quick reminder concerning homotopy is in order.

Definition D.1.7. Let A, B be C^*-algebras and $\varphi, \psi : A \to B$ be *-homomorphisms. Then φ and ψ are said to be **homotopic** if there exists a family of *-homomorphisms φ_t, $t \in [0, 1]$ such that for each $a \in A$ the map $t \mapsto \varphi_t(a)$ is continuous, $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.

The C^*-algebras A, B are said to be **homotopy equivalent** if there exists *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi\psi$ is homotopic to id_B and $\psi\varphi$ is homotopic to id_A.

This definition should be well-known to all those who have dealt with K-theory.

Theorem D.1.8 (Voiculescu). *Let A and B be C^*-algebras, with B quasidiagonal. If there exists *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi\varphi$ is homotopic to id_A then A is quasidiagonal.*

Example D.1.1. We can use Voiculescu's theorem to show that $C([0,1), A)$ is quasidiagonal, where

$$C([0,1), A) = \{f : [0,1) \to A \mid f \text{ is continuous and } f(0) = 0\}.$$

Namely, if we can show that $C([0,1), A)$ is homotopy equivalent to 0, then Voiculescu's theorem yields the desired conclusion. To prove this we only need to show that the identity on $C([0,1), A)$ is homotopic to the zero map.

For each $t \in [0,1]$ let $\varphi_t : C([0,1), A) \to C([0,1), A)$ be given by $\varphi_t(f)(s) = f(ts)$. Clearly this is a continuous family of *-homomorphism. Furthermore, $\varphi_1 = \mathrm{id}_{C([0,1),A)}$ and for any $f \in C([0,1), A)$ and $s \in [0,1)$ we see that $\varphi_0(f)(s) = f(0 \cdot s) = 0$ and hence φ_0 is the zero map.

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