On Quasidiagonal C*-Algebras
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Abstract

The project conveys a survey of quasidiagonality. Quasidiagonality originates from block-diagonality of bounded operators acting on Hilbert spaces, a concept introduced by Halmos. Quasidiagonality extends to the realm of C^*-algebras expressed in terms of completely positive maps and has proven itself to be an adamant invariant of C^*-algebras. The first aim of the project is to establish the notion of quasidiagonal C^*-algebras properly in its various disguises, following closely the footsteps of Voiculescu's approach in establishing the abstract shape of quasidiagonality alongside its concrete ones.

Studying quasidiagonal C^*-algebras eventually forces one to consider group theoretic, probably approximation natured, properties translating into quasidiagonality of the reduced group C^*-algebra. Indeed, Jonathan Rosenberg succeeded in proving that quasidiagonality of the reduced group C^*-algebra associated to a discrete group G implies amenability of the group and conjectured the converse to be true. Prior until recently, the validity of the converse remained unanswered with a partial converse being confirmed by Ozawa, Rørdam and Sato. The full converse of Rosenberg's theorem was answered in the affirmative by a recent paper due to Tikuisis, White and Winter invoking various disciplines in the operator algebraic setting including, but not limited to, order zero maps, KK-theory and lifting theorems of separable nuclear - and/or quasidiagonal C^*-algebras. In this project, the proofs of the aforementioned partial converse together with investigations of ingredients used during the derivation of the full version are carried out in detail.

Contents

Prologue

The project carries an exposition of quasidiagonality. Quasidiagonality has roots reaching block-diagonal operators, a concept introduced by Halmos that generalized the notion of diagonalizable matrices to the infinite-dimensional case. Being block-diagonal is quite restrictive, so one modifies block-diagonality by weakening the condition and thus creates an approximation natured version. Since C*-algebras arguably form generalizations of subalgebras occurring in bounded operators acting on Hilbert spaces, it is natural to extend the quasidiagonality to this context. A number of participants, especially Voiculescu, succeeded in achieving a version of quasidiagonal C*-algebras including a characterization based on completely positive maps.

Quasidiagonality is a rigid structure to impose on a C*-algebra and has been extensively studied partly to achieve classification results concerning nuclear separable C*-algebras. A significant example is the reduced group C*-algebra associated to a discrete group in which recent progress has been made by providing an affirmative answer to Rosenberg's conjecture. Rosenberg's conjecture states the converse of Rosenberg's theorem; that quasidiagonality of the reduced group C*-algebra implies amenability of the group in play. The project essentially attempts to convey a survey of quasidiagonlity in general and understand the aforementioned converse of Rosenberg's conjecture.

In the first chapter, a broad treatment of various preliminaries are given with the notable theory occurring being unique properties of UHF-algebras and ultraproducts. The chapter includes background theory regarding UHF-algebras, AF-algebras, crossed products, group C*-algebras, the prerequisites, inductive limits and ultraproducts. It is based on knowledge gathered from texts such as [2,], [5], [17], [12] and [10].

The second chapter tears through the seventh chapter in [2] and attempts to present quasidiagonality in detail with the overall aim of understanding the numerous characterizations and examples. Permanence properties are proven as well, having an added emphasis on the heralded homotopy invariance theorem due to Voiculescu's.

The third chapter gives a crash course on KK-theory, the UCT-class and groupoid C*-algebras with the sole purpose of comprehending, to some extend, these abstract and indispensable beasts when understanding quasidiagonality. For the record, the project solely considers Cuntz picture of KK-theory, sadly, and the theory hereof is extracted from his original article [4].

The fourth chapter pursues the proof of a partial converse to Rosenberg's conjecture, proven in the article [9] by Ozawa, Rørdam and Sato. The theorem is the deepest one emerging in the project and a whole chapter is devoted to it for this particular reason. We even present the results of Chou/Osin from the articles [3] and [8], which the partial answer exploits.

The fifth and final chapter brings an exposition of certain considerations concerning quasidiagonality that Tikuisis, White and Winter use to prove a full-fledged answer to Rosenberg's conjecture, in the affirmative. Although the project initially sought to understand, to whichever extend possible, this particular answer, the writer had to settle with discussing various aspects in the article [13]. The chapter includes mentioning of quasidiagonal traces, embedding/lifting criteria for quasidiagonality and order zero maps, all based on [15] and [13].

Acknowledgment. I genuinely appreciate Mikael Rørdam's guidance during the project. It is a pleasure learning from you and the subject had both challenging and intriguing features. I cannot deny having learned a tremendous amount over the past half year. Sincerely, thank you.

Chapter 1

Preliminaries

We begin our journey by introducing the notation and preliminary results necessary to understand the topics emerging in this project. The chapter is quite long, however, most of the results have general shapes and the writer deemed isolating the statements herein, as opposed to continuously deriving these as lemmas whenever needed, to be advantageous. Thus, anyone familiar with the preliminary results may safely skip the entire chapter and start dwelling into the world of quasidiagonality instead, perhaps returning later on.

The project assumes a solid knowledge of general operator algebra theory including von Neumann algebraic aspects such as the existence of the enveloping von Neumann algebra associated to every C*-algebra, fundamentals concerning contractive completely positive maps, K-theory, the classification of UHF - and AF algebras and the reader must be fluent in approximation properties with an added emphasis on nuclearity. Furthermore, we demand that category theoretic terms sound quaint to the audience. Of course, we shall exhibit the main theory required, partly to settle the notation once and for all.

1.1 The Basics

Here we present the notation together with basic constructions arising in the group theoretic and C*-algebraic frameworks. We shall begin the discussion by addressing groups at first. For the record, there will be close to zero proofs occurring herein save some considerations of the GNS-construction in the separable case, which we exploit repetitively in the project.

Prerequisites concerning groups. Groups are typically denoted by G, H and N; never assumed to be discrete unless specified otherwise. The neutral element of a group G is denoted by 1_G or occasionally merely 1 should the group in question be understood. In fact, we assume these to be topological, although we rarely work outside the setting of discrete groups.

Group actions on some set M, with M having an algebraic structure or not, are commonly denoted by α or β. We implicitly assume that the automorphism groups of any such set M refers to automorphism in the corresponding category. As such any action $\alpha\colon G \longrightarrow \mathrm{Aut}(M)$ will act by homeomorphisms provided M is a topological space, by automorphisms of groups provided M is a group and so on. To avoid an overwhelming amount of paranthesis, we denote $\alpha(g)$ by α_g instead.

A group G is *free* provided no relations occur on G eg. the integers $\mathbb{Z}$. Since free groups are uniquely determined by their generating sets, one often represents these by F_S having S being the generating set. The free group F_S is known to fulfill the following universal property: given any set function $f\colon S \longrightarrow G$ there exists a unique group homomorphism $\varphi\colon F_s \longrightarrow G$ such that $\varphi_{|S} = f_{|S}$.

Given abelian groups G and H, let $\mathrm{Hom}_{\mathbb{Z}}(G, H)$ be the abelian group of group homomorphism $G \longrightarrow H$ equipped with pointwise operations.

Prerequisites concerning C*-Basics. For any normed space X, the ball of radius $r > 0$ centered around the origin will be denoted $(X)_1$ and the dual of X by X^*. Hilbert spaces are typically symbolically represented as $\mathcal{H}$ and $\mathcal{K}$. Within any Hilbert space, the orthogonal complement of a subspace M is written as $M^\perp$. For every bounded operator $T\colon \mathcal{H} \longrightarrow \mathcal{K}$, we let $\operatorname{Ran} T$ symbolize the norm-closure of the image $T(\mathcal{H})$ in $\mathcal{K}$.

C*-algebras are denoted by A, B and C throughout the entire project. These are never assumed to be unital nor separable unless stated otherwise. We denote by 1_A the unit of A in the presence of one. In the following section, A and B will be fixed C*-algebras whereas $\mathcal{H}$ and $\mathcal{K}$ will be some pair of fixed Hilbert spaces.

- The Banach algebra consisting of bounded operators $\mathcal{H} \longrightarrow \mathcal{K}$ between Hilbert spaces is represented as $\mathrm{B}(\mathcal{H}, \mathcal{K})$, abbreviating $\mathrm{B}(\mathcal{H})$ in the case where $\mathcal{H} = \mathcal{K}$ and $\mathbb{M}_n$ whenever $\mathcal{H} = \mathbb{C}^n$. For every Hilbert space $\mathcal{H}$, we denote the finite rank operators hereon by $\mathrm{F}(\mathcal{H})$ and the compact operators by $\mathrm{K}(\mathcal{H})$. We shorten the notation into $\mathbb{F}$ and $\mathbb{K}$, respectively, whenever $\mathcal{H}$ is separable.

- We write C^*-Alg to symbolize te category of C*-algebras having $*$-homomorphism[1] as morphisms. The subcategory of separable C*-algebras is denoted C^*_s-Alg. We adopt the convention of calling a two-sided ideal in C^*-Alg a *$*$-ideal* for brevity. Thus, C*-algebras are called *simple* if they only contain trivial $*$-ideals.
  - Every $*$-homomorphism is automatically contractive, hence continuous. Moreover, their images are closed, so they form C*-algebras themselves.
  - An injective $*$-homomorphism is called a *$*$-monomorphism.* Every $*$-homomorphism π is injective if and only if π is an isometry. A surjective $*$-homomorphism is called a *$*$-epimorphism.*
  - $\operatorname{Hom}(A, B)$ will represent the biadditive [2] abelian group of $*$-homomorphism $A \longrightarrow B$.

- The *unitalization* A^+ of A is the unital C*-algebra containing A as an ideal. As a complex vector space, it may be identified with $A \oplus \mathbb{C}$ having as $*$-algebraic operations the ones defined as

$$(a + \lambda 1_{A^+})(b + \mu 1_{A^+}) = (ab + \mu a + \lambda b) + (\lambda\mu)1_{A^+} \quad \text{and} \quad (a + \lambda 1_{A^+})^* = a^* + \overline{\lambda} 1_{A^+}.$$

  It is a well-established fact that $A^+ \cong A \oplus \mathbb{C}$ whenever A admits a unit. The assignment $A \mapsto A^+$ is functorial in the sense that any bounded linear map $\pi\colon A \longrightarrow B$ (resp. $*$-homomorphism) extends to a bounded linear map $\pi_+\colon A^+ \longrightarrow B^+$ (resp. $*$-homomorphism) via

$$\pi_+(a + \lambda 1_{A^+}) = \pi(a) + \lambda 1_{B^+}.$$

- The collection of positive elements in A is denoted by A_+, the collection of self-adjoints by A_{sa}, the set of projections by $\operatorname{Proj}(\mathcal{H})$ and the group of unitaries by $\mathcal{U}(A)$. We simply write $\mathcal{U}(\mathcal{H})$ and $\operatorname{Proj}(\mathcal{H})$ whenever $A = \mathrm{B}(\mathcal{H})$. Every self-adjoint element is the sum of two positive elements, hence every element in A is the sum of at most four positive elements in A.
  - We shall stipulate that $a \geq 0$ whenever a belongs to A_+. The set A_{sa} will implicitly be endowed with the order $\leq$ defined by declaring that $a \leq b$ if and only if $b - a \geq 0$. The positive elements A_+ form a positive cone in A when equipped with the relation $\leq$.
  - For two projections p and q in $\operatorname{Proj}(\mathcal{H})$, the reader is assumed to be familiar with the relations,

$$p \leq q \quad \Longleftrightarrow \quad \operatorname{Ran} p \subseteq \operatorname{Ran} q \quad \Longleftrightarrow \quad pq = p = qp.$$

- The spectrum $\sigma(a)$ associated to an element a inside a unital C*-algebra A consists of all complex numbers λ for which $\lambda 1_A - a$ cannot be invertible in A. We define the spectrum of an element a belonging to a non-unital C*-algebra to be the spectrum of a regarded as an element in the unitalization. The spectrum $\sigma(a)$ defines a nonempty norm-compact subspace of $(A)_{\|a\|}$.

[1] a prior, $*$-homomorphism are non-unital in this project.

[2] Additivity being with respect to finite direct sums of C*-algebras and ℓ^∞-sums in the general case.

- **Gelfand, Naimark.** Every commutative C*-algebra A is $*$-isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω. Moreover, A is unital if and only if Ω is compact.

- For any normal element a inside A, there exists a $*$-isomorphism $C(\sigma(a)) \longrightarrow C^*(a)$ given by $f \mapsto f(a)$, known as *the continuous functional calculus*, such that for every continuous function $f\colon \sigma(a) \longrightarrow \mathbb{C}$ one has $\sigma(f(a)) = f(\sigma(a))$[3].

- An *approximate unit* of some ideal $I \triangleleft A$ is an increasing net $\{e_\alpha\}_{\alpha\in\Lambda}$ consisting of contractions $e_\alpha \geq 0$ acting as a unit in the norm limit on A, i.e. $\lim_{\alpha\in\Lambda} e_\alpha a = \lim_{\alpha\in\Lambda} a e_\alpha = a$ for each $a \in A$. The C*-algebra A is called *σ-unital* should it admit a countable approximate unit. Additionally,

$$\|a + I\| = \lim_{\alpha\in\Lambda} \|a - e_\alpha a\|, \quad a \in A.$$

  The approximate $\{e_\alpha\}_{\alpha\in\Lambda}$ unit is called *quasicentral* provided that $\lim_{\alpha\in\Lambda}(e_\alpha a - a e_\alpha) = 0$ for each $a \in A$ as well. Every $*$-ideal of some C*-algebra admits an approximate unit $\{e_\alpha\}_{\alpha\in I}$ and quasicentral one may be extracted from the convex hull of an approximate unit [4].

- $\mathrm{M}_n(A)$ denotes the C*-algebra consisting of $n \times n$-matrices with values in A whose operations are multiplication of matrices and involution being the operation of taking the conjugate transpose. The assignment $A \mapsto \mathrm{M}_n(A)$ is functorial in the sense that every linear map $\pi\colon A \longrightarrow B$ induces a linear map $\varphi_n\colon \mathrm{M}_n(A) \longrightarrow \mathrm{M}_n(B)$ given by

$$\varphi_n([a_{ij}]_{i,j}) = [\varphi(a_{ij})]_{i,j}$$

  for every $[a_{ij}]$ inside $\mathrm{M}_n(A)$, frequently called the n'th *amplification* of φ. The n'th amplification of a $*$-homomorphism remains a $*$-homomorphism.

- A bounded linear map $\varphi\colon A \longrightarrow B$ is said to be *positive* whenever $\varphi(A_+) \subseteq B_+$. A positive bounded linear functional $\omega\colon A \longrightarrow \mathbb{C}$ is called a *state* should $\|\omega\| = 1$, which is equivalent to $\omega(1_A) = \|\omega\| = 1$ in the event of A being unital. Given any positive linear functional φ on A, one has the following Cauchy-Schwarz esque inequality

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b), \qquad a, b \in A. \tag{1.1}$$

  - Any bounded linear map $\varphi\colon A \longrightarrow B$ is called *faithful* if $\varphi(a^*a) > 0$ for all nonzero a in A. The corresponding ideal measuring the deficiency of faithfulness is $\mathcal{L}_\varphi = \{a \in A : \varphi(a^*a) = 0\}$.
  - The state space $S(A)$ is the subspace of A^* consisting of all states ω on A. In the unital case, $S(A)$ defines a weak*-compact convex subspace of A^*. The state space is large enough to separate points in A, that is, for every $a \in A$ there exists a state $\omega\colon A \longrightarrow \mathbb{C}$ such that $\omega(a) \neq 0$. Therefore, one has $a \leq b$ in A if and only if $\omega(a) \leq \omega(b)$ for all $\omega \in S(A)$.

- A positive linear functional $\tau\colon A \longrightarrow \mathbb{C}$ is called a *trace* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. The trace is called a *tracial state* if it is a state. The C*-algebra A is said to be *monotracial* should it admit a unique tracial state. The unique faithful tracial state $\mathbb{M}_n$, denoted τ_n, is the $\mathbb{C}$-linear extension of $e_{ii} \mapsto 1$, where e_{ii} is the (i,i)'th matrix unit. Moreover, the following are equivalent in the presence of a unit on A:

  - τ is a trace on A.
  - $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.
  - $\tau(u^*au) = \tau(a)$ for all $a \in A$ and every $u \in \mathcal{U}(A)$.

[3]The latter statement perhaps known as *the spectral mapping theorem to some.*
[4]This is difficult to prove. A proof may be found in [5, Theorem I.9.16].

- A *partial isometry* in A is an element v therein such that v^*v defines a projection in A. Furthermore, v satisfies the following constantly exploited characteristic properties:
  - v is a partial isometry in A if and only if v^* is.
  - $v = vv^*v$ together with $v^*vv^* = v^*$ hold.
  - In $\mathrm{B}(\mathcal{H})$, an operator ν is a partial isometry if and only if it restricts to an isometry on $\ker \nu^{\perp}$.
- A $*$-homomorphism of the form $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ is called a representation of A. Furthermore, we adopt the following conventions concerning representations.
  - π is called *separable* if $\mathcal{H}$ is separable.
  - π is called *non-degenerate* if $\pi(A)\mathcal{H}$ is norm-dense in $\mathcal{H}$.
  - A vector ξ in $\mathcal{H}$ is called *cyclic* with respect to π provided that $\pi(A)\xi$ is norm-dense in $\mathcal{H}$.
- For every positive linear functional φ on an involutive algebra A, there exist a Hilbert space $\mathcal{H}_\varphi$, a representation $\pi_\varphi\colon A \longrightarrow \mathrm{B}(\mathcal{H}_\varphi)$ together with a cyclic unit vector ξ_φ. The Hilbert space $\mathcal{H}_\varphi$ is the quotient $A/\mathcal{L}_\varphi$ endowed with the inner product $\langle\cdot,\cdot\rangle\colon \mathcal{H}_\varphi \times \mathcal{H}_\varphi \longrightarrow \mathbb{C}$ given by

  $$\langle [a],[b]\rangle = \varphi(b^*a)$$

  for all $a, b \in A$ and is commonly denoted $L^2(A,\varphi)$. The pair $(\xi_\varphi, \pi_\varphi)$ recover φ via

  $$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle \tag{1.2}$$

  for all a belonging to A. The obtained triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is called *the GNS-triple associated to* φ. For every C*-algebra A, let $\mathcal{H}_u = \bigoplus_{\omega\in S(A)} \mathcal{H}_\omega$. The *universal representation* associated to A is the induced faithful non-degenerate representation $\pi_u\colon A \longrightarrow \mathrm{B}(\mathcal{H}_u)$. As such any C*-algebra canonically embeds into $\mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ non-degenerately.

We will primarily deal with separable C*-algebras, so we derive a separable version of the GNS construction that we shall invoke implicitly whenever separability is is assumed. Recall that for every self-adjoint element $a \in A$ there exists some ω in $S(A)$ fulfilling $|\omega(a)| = \|a\|$.

Proposition 1.1.1. *Every separable C*-algebra admits a canonical faithful non-degenerate separable representation.*

Proof. Suppose A is a separable C*-algebra having $\{a_n\}_{n\geq 1}$ as its dense subset. Upon rescaling in conjunction with splitting each a_n into the their real and imaginary parts, i.e., writing $a_n = a_n^r + ia_n^i$ with $a_n^r, a_n^i \in A$ being self-adjoint, we may assume that each a_n must be self-adjoint unit vectors. Choose for every positive integer n some state ω_n acting on A subject to $|\omega_n(a_n)| = \|a_n\| = 1$. Define from the sequence $\{\omega_n\}_{n\geq 1}$ of states, a positive linear functional $\omega\colon A \longrightarrow \mathbb{C}$ by

$$\omega(\cdot) = \sum_{n=1}^{\infty} \frac{1}{2^n}\omega_n(\cdot).$$

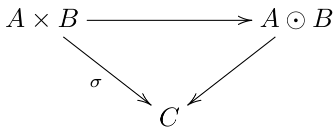
The functional ω becomes bounded due to each ω_n being contractive. By density, any unit vector $a \geq 0$ in A lies within a distance strictly smaller than 1 of some a_n. Therefore one has $|1 - \omega_n(a)| = |\omega(a_n - a)| < 1$ whereof $\omega_n(a) > 0$. The normalization $a' = a\|a\|^{-1}$ of some $a \geq 0$ inside A thus must satisfy $\omega(a') \geq \omega_n(a') > 0$ as a consequence, hence $\omega(a) > 0$ for every $a \in A_+$, verifying that ω has to be faithful. The associated Hilbert space $L^2(A,\omega)$ may hereby be identified with A and the equation (1.2) applied to $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ yields

$$\|\pi_\omega(a)\|^2 = \langle \pi_\omega(a^*a)\xi_\omega, \xi_\omega\rangle = \omega(a^*a)$$

for all $a, b \in A$. Faithfulness of ω therefore entails faithfulness of π_ω so that A represents faithfully and non-degenerately on $\mathcal{H}_\omega \cong A$, which is separable. This completes the proof. □

Tensor Products of C*-Algebras. Tensor products appear at almost every single page after exceeding page 12, and we recall various facts thereof, at least the most commonly used ones. However, the reader is assumed to be fluent in spatial tensor products of C*-algebras.

Algebraically, the tensor product $A \odot B$ of algebras is the unique algebra such that for every algebra C and bilinear map $\sigma\colon A \times B \longrightarrow C$ there exists a unique homomorphism $A \odot B \longrightarrow C$ making the diagram



commute. The algebraic tensor product $A \odot B$ of involutive algebras may be endowed with factor-wise multiplication and involution to form an involutive algebra, meaning $(a \otimes b)(a_0 \otimes b_0) = aa_0 \otimes bb_0$ and $(a \otimes b)^* = a^* \otimes b^*$ on elementary tensors, itself compatible with $*$-representations. The aforementioned compatibility being that given a pair of $*$-representations $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ and $\varrho\colon B \longrightarrow \mathrm{B}(\mathcal{K})$, the map $\pi \otimes \varrho\colon A \odot B \longrightarrow \mathrm{B}(\mathcal{H} \otimes \mathcal{K})$ defined via the formula $(\pi \otimes \varrho)(a \otimes b) = \pi(a) \otimes \varrho(b)$ defines a $*$-representation. Here $\mathcal{H} \otimes \mathcal{K}$ is endowed with the inner product

$$\langle \xi \otimes \eta, \xi_0 \otimes \eta_0 \rangle = \langle \xi, \xi_0 \rangle \cdot \langle \eta, \eta_0 \rangle,$$

which is easily seen to satisfy $\|\xi \otimes \eta\| = \|\xi\| \cdot \|\eta\|$. We define the *spatial tensor product*, denoted $A \otimes B$, to be the norm closure of $A \odot B$ under the spatial tensor norm defined thus: Choose once and for all (choice being irrelevant, although this is non-trivial) a pair of faithful representations $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ and $\varrho\colon B \longrightarrow \mathrm{B}(\mathcal{K})$, then declare that

$$\Big\| \sum_{k=1}^{n} a_k \otimes b_k \Big\| = \Big\| \sum_{k=1}^{n} \pi(a_k) \otimes \varrho(b_k) \Big\|.$$

The spatial tensor product is an additive, in both variables, associative bifunctor being covariant in both variables. Functoriality extends to the following property: Every pair of $*$-homomorphism $\pi\colon A \longrightarrow C$ and $\varrho\colon B \longrightarrow C'$ induce a $*$-homomorphism $A \otimes B \longrightarrow C \otimes C'$ via $a \otimes b \mapsto \pi(a) \otimes \varrho(b)$.

Prerequisites concerning completely positive maps. Completely positive maps are the heart of the project and are, without a shadow of a doubt, our favorite morphisms next to $*$-homomorphism. Numerous facts regarding completely positive maps will be exploited constantly throughout the project. The most essential being presented here.

A positive linear map $\varphi\colon A \longrightarrow B$ between C*-algebras is *completely positive*, abbreviated c.p, if the n'th amplification $\varphi_n\colon M_n(A) \longrightarrow M_n(B)$ is positive for every $n \in \mathbb{N}$. We call a contractive c.p map φ c.c.p and a unital c.p map u.c.p. Every c.c.p map fulfills $\|\varphi(1_A)\| = \|\varphi\|$ in the unital case, hence u.c.p maps are c.c.p. One easily verifies that positive linear maps are involutive, so c.p maps are as well.

- Let B be a C*-subalgebra of A. A *conditional expectation* $E\colon A \longrightarrow B$ is a contractive completely positive map such that $E_{|B} = \mathrm{id}_B$ and $E(bab_0) = bE(a)b_0$ for all $a \in A$ and $b, b_0 \in B$.

- A linear map $\varphi\colon A \longrightarrow B$ between C*-algebras is *nuclear* if there are nets $(\varphi_\alpha)_{\alpha \in J}$ and $(\psi_\alpha)_{\alpha \in J}$ consisting of c.c.p maps $\varphi_\alpha\colon A \longrightarrow \mathrm{M}_{n(\alpha)}$ together with $\psi_\alpha\colon \mathrm{M}_{n(\alpha)} \longrightarrow A$ fulfilling $\psi_\alpha \circ \varphi_\alpha \to \varphi$ in the point-norm topology. For a separable C*-algebra we require such sequences to exist and one may always choose the c.c.p maps in play to be unital should φ be unital. We call A *nuclear* whenever the identity hereon is nuclear.

- **Stinespring's dilation theorem.** For every completely positive map $\varphi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ there exists a triple $(\sigma, V, \mathcal{K})$, called the *Stinespring dilation of* φ, consisting of a $*$-homomorphism $\sigma\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ and a bounded operator $V\colon \mathcal{H} \longrightarrow \mathcal{K}$ witnessing φ via $\varphi(\cdot) = V^*\sigma(\cdot)V$.

- **Arveson's extension theorem.** $\mathrm{B}(\mathcal{H})$ is an injective object in the category of C*-algebras with c.c.p maps as morphisms. That is, every c.c.p map $\varphi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$, with A being C*-subalgebra of B, extends to a c.c.p map $\psi\colon B \longrightarrow \mathrm{B}(\mathcal{H})$.

- **The correspondence theorem.** There is a one-to-one correspondence of sets between completely positive maps $\mathbb{M}_n \longrightarrow A$ and positive elements in $\mathrm{M}_n(A)$ determined by $\varphi \mapsto [\varphi(e_{ij})]_{i,j}$ where e_{ij} denotes the (i,j)'th unit matrix in $\mathbb{M}_n$.

Prerequisites concerning von Neumann algebras. We will barely address von Neumann algebras in this project, however, we require some theory regarding the enveloping von Neumann algebra associated to every C*-algebra, so we establish the concepts required hereof. Understanding von Neumann algebras forces one to understand, at minimum, one locally convex topology on $\mathrm{B}(\mathcal{H})$.

- The *strong-operator topology* on $\mathrm{B}(\mathcal{H})$ is defined to be the locally convex Hausdorff topology induced by the family of seminorms $a \mapsto \|a\xi\|$ indexed over all $\xi \in \mathcal{H}$. We write $\text{sot-}\lim_{i\in\Lambda} a_i$ to denote the strong-operator limit of a net $(a_i)_{i\in\Lambda}$ should it exist.

- Every increasing net $(a_i)_{i\in\Lambda}$ consisting of self-adjoint operators, with respect to the ordering on self-adjoint elements, admits a strong-operator limit.

- According to von Neumann's *double commutant theorem*, a C*-algebra $\mathscr{M} \subseteq \mathrm{B}(\mathcal{H})$ defines a von Neumann algebra if and only if one, hence both, of the following conditions hold:

$$\overline{\mathscr{M}}^{\mathrm{sot}} = \mathscr{M} \quad \text{or} \quad \mathscr{M}'' = \mathscr{M}$$

  where $\mathscr{M}'$ denotes the commutant of $\mathscr{M}$, i.e,. $\mathscr{M}' = \{b \in \mathrm{B}(\mathcal{H}) : ab = ba, \text{ for all } a \in \mathscr{M}\}$, and $\mathscr{M}'' = (\mathscr{M}')'$. A von Neumann algebra, should it appear, will be denoted by $\mathscr{M}$.

- The double dual A^{**} of any C*-algebra becomes a von Neumann algebra via the isomorphism $A^{**} \cong \pi_u(A)''$, called the *enveloping von Neumann algebra.*

Prerequisites concerning ordinals. We review some facts about ordinals, excluding proofs entirely. For a rigorous introduction to ordinals along with transfinite induction, the reader is urged to consult the book [6]. By definition, an ordinal α is a non-empty set endowed with the relation $\in$ while obeying the rules:

$$x \in \alpha \;\wedge\; y \in x \;\Rightarrow\; y \in \alpha \quad \text{and} \quad x \in y \;\wedge\; y \in x \;\Rightarrow\; x = y.$$

- Let $\mathcal{O}$ be the collection[5] of all ordinals. We define an order relation $<$ on $\mathcal{O}$ by stipulating that $\alpha < \beta$ if and only if $\alpha \in \beta$. An ordinal α is called a *successor* provided it attains the form $\alpha = \beta \cup \{\beta\}$ for some ordinal β differing from α, which evidently fulfills $\beta \in \alpha$.

- We define the *successor* of an ordinal α by $s(\alpha) = \alpha \cup \{\alpha\}$. In accordance with this terminology, an ordinal α is called a *successor* whenever $\alpha = s(\beta)$ for some ordinal β. An ordinal α which is not a successor is called *a limit ordinal* and must be of the form $\alpha = \bigcup_{\beta<\alpha} \beta$. It is well-known that every ordinal is either a successor or a limit ordinal, but never both.

- One commonly defines an additive structure on $\mathcal{O}$ by $\alpha + 0 = \alpha$, then inductively defining the operations via $\alpha + s(\beta) := s(\alpha + \beta)$ and $\alpha + \lambda = \bigcup_{\beta<\lambda} s(\alpha + \beta)$ for each limit ordinal λ. In particular, $\alpha + 1 = s(\alpha)$. Thus, $\sum_{i\in I} \alpha_i$ is merely the recursively defined sum of the ordinals α_i over some indexing set I. It evidently contains all the ordinals α_i.

- **The principle of transfinite induction.** Suppose $P(\alpha)$ is a property defined on all ordinals α. If $P(\beta)$ is true for every ordinal $\beta < \alpha$, then $P(\alpha)$ must be true.

[5] Strictly speaking this is not a set, but we still treat it as one.

A K-theoretric toolkit. We exhibit the knowledge of K-theory that are needed to read the notes fluently, certain result being stated in the future. K-theory of C*-algebra arises from studying homotopic projections and unitaries upon passing to higher dimensional matrix algebras, so we must address the homotopy matter, especially due to the notion bearing independent interest itself. Throughout the section, I will denote the closed interval $[0, 1]$ in $\mathbb{R}$.

Definition. Suppose A and B are C*-algebras. Then,

- any pair of unitaries $u, v \in A$, in the presence of a unit, are called *homotopic* if there exists a continuous map $\theta\colon I \longrightarrow \mathcal{U}(A)$ fulfilling $\theta(0) = u$ and $\theta(1) = v$;
- any pair of projections $p, q \in A$ are called *Murray - von Neumann equivalent* if there exists a partial isometry v in A satisfying $p = v^*v$ and $q = vv^*$. The obtained Murray - von Neumann relation $\sim$ defines an equivalence relation on A;
- any pair of $*$-homomorphisms $\pi, \varrho\colon A \longrightarrow B$ are said to be *homotopic* if there exists a family $\{\sigma_t\}_{t\in I}$ of $*$-homomorphisms $\sigma_t\colon A \longrightarrow B$ fulfilling $\sigma_0 = \pi$ and $\sigma_1 = \varrho$ with $t \mapsto \sigma_t$ being continuous. The obtained relation $\sim_h$ defines an equivalence relation on $\mathrm{Hom}(A, B)$.
- We refer to A as *homotopically dominating* B if there exist $*$-homomorphisms $\pi\colon A \longrightarrow B$ and $\varrho\colon B \longrightarrow A$ subject to the relation $\pi\varrho \sim_H \mathrm{id}_B$. In the event of A and B homotopically dominating one another through the same pair of $*$-homomorphisms, meaning $\pi\varrho \sim_h \mathrm{id}_B$ and $\varrho\pi \sim_h \mathrm{id}_A$, we call A and B *homotopy equivalent* while symbolically writing $A \cong_h B$.

Our sole purpose of applying K-theory concerns unital C*-algebras, so we confine ourselves to this special case. Fix some unital C*-algebra A and define

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathrm{Proj}(\mathrm{M}_n(A)) \quad \text{together with} \quad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}(\mathrm{M}_n(A)).$$

Alternatively, one may regard the latter as being an inductive limit in the category of groups, which we introduce in the upcoming section. As such the reader may find it valuable to return to this section afterwards. We define associative operations $\oplus$ on these entities by mapping a pair $a \in \mathrm{M}_n(A)$ and $b \in \mathrm{M}_n(B)$ into the matrix having a in the first diagonal entry and b in the second, or in shorthand the assignment $(a, b) \mapsto \mathrm{diag}(a, b)$.

Define an equivalence relation on $\mathcal{P}_\infty(A)$ by declaring that $p \sim_0 q$, with $p \in \mathrm{M}_n(A)$ and $q \in \mathrm{M}_m(A)$, if and only if there exists an element $v \in \mathrm{M}_{n,m}(A)$ such that $p = v^*v$ together with $q = vv^*$ hold. Define hereafter another equivalence relation on $\mathcal{U}_\infty(A)$ by stipulating that $u \sim_1 v$, where $u \in \mathrm{M}_n(A)$ and $v \in \mathrm{M}_m(A)$, if and only if one may find some integer $k \geq n, m$ such that $u \oplus 1_{k-n}$ becomes homotopic to $v \oplus 1_{k-m}$. The obtained quotients are the K-groups.

Definition. For every given unital C*-algebra A, we define the K-groups to be the abelian groups $\mathrm{K}_0(A) = \mathcal{P}_\infty(A)/\sim_0$ and $\mathrm{K}_1(A) = \mathcal{U}_\infty(A)/\sim_1$ endowed with the composition $\oplus$.

Theorem 1.1.2. *The abelian groups $\mathrm{K}_n(A)$ for $n = 0, 1$ associated to any C*-algebra A produce additive covariant functors $\mathrm{K}_n(\cdot)\colon$ C*-Alg $\longrightarrow$ Ab assigning to each $*$-homomorphism $\pi\colon A \longrightarrow B$ the induced group homomorphism $\mathrm{K}_n(\pi)\colon \mathrm{K}_n(A) \longrightarrow \mathrm{K}_n(B)$ defined as $\mathrm{K}_*(\pi)[a] = [\pi_k(a)]$ for each $a \in \mathrm{M}_k(A)$. Furthermore, these satisfy the following properties:*

- K_0 *is stable, meaning* $\mathrm{K}_0(\mathrm{M}_n(A)) \cong \mathrm{K}_0(A)$.
- K_n *is homotopy invariant, meaning* $\pi \sim_h \varrho$ *entails that* $\mathrm{K}_n(\pi) = \mathrm{K}_n(\varrho)$.
- K_n *is split-exact, meaning it maps split-exact sequences to split-exact sequences.*
- K_n *is half-exact, meaning it maps exact sequences to sequences which are exact in the middle.*

1.2 Ultrafilters and Products of C*-Algebras

Towards the end of project, we will be exploiting unique convergence properties of ultrafilters. The current section provides an introduction to ultrafilters and general products of C*-algebra in extreme brevity having most proofs omitted. We begin the exposition of ultrafilters by addressing the basics.

Definition. Let S be some set. A family $\mathcal{F}$ consisting of subsets in S is said to

- *be nontrivial* if $\emptyset \notin \mathcal{F}$;
- *be direct* if $A \subseteq B \subseteq S$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$;
- *have the finite intersection property* if $A, B \in \mathcal{F}$ entails $A \cap B \in \mathcal{F}$;
- *be maximal* if for every $A \subseteq S$ one must have that either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$.

We call $\mathcal{F}$ a *filter* provided it is nontrivial, direct and has the finite intersection property. A maximal filter $\mathcal{F}$ is called an *ultrafilter*. Note that the maximality property of an ultrafilter forces every subset $A \subseteq S$ or its complement to lie therein, but never both.

Let M be some subcollection of subsets in S without $\emptyset$ and with the finite intersection property. We define the *filter generated by* M, symbolically denoted by $\mathcal{F}(M)$, to be the family consisting of all $V \subseteq S$ such that there exist members $J_1, \ldots, J_n \in M$ whose intersection belong to V. The finite intersection property together with directness are easily verified and for nontriviality suppose the empty set did lie in $\mathcal{F}(M)$. Then there are n-many members $J_1, \ldots, J_n$ in M whose intersection is empty, so $\emptyset = J_k \cap J_l \in M$ for some indices $1 \leq k, l \leq n$, a contradiction.

Another construction of a filter may be given as follows. Suppose $M \subseteq S$ is nonempty and define $\mathcal{F}$ to be the family of $B \subseteq S$ containing M. One easily checks that $\mathcal{F}$ defines a filter on S. We call $\mathcal{F}$ *the principal filter associated to* S. An ultrafilter $\mathcal{F}$ is said to be *free* if it is non-principal. Here is one important existence result concerning ultrafilters.

Proposition 1.2.1. *Every filter is contained in an ultrafilter. In particular, for any nonempty set S and nonempty family M of subsets therein, there exists a free ultrafilter $\mathcal{F}$ on S containing M.*

Proof. A rigorous proof of the first claim may be recovered in [2, theorem A.5]. For the latter, let $\mathcal{F} = \mathcal{F}(M)$ be the filter generated by M to obtain a containment $M \in \mathcal{F}$. Apply hereafter the first part to extend $\mathcal{F}$ to an ultrafilter. □

Our main interest in ultrafilters concerns convergence along these, a notion we emphasize on.

Definition. Let X be a topological space and $(x_s)_{s\in S}$ some net over a directed set S that admits a filter $\mathcal{F}$. The net $(x_s)_{s\in S}$ is said to *converge to x along* $\mathcal{F}$ if, for every open neighbourhood U around x the corresponding set $S(U) = \{s \in S : x_s \in U\}$ belongs to $\mathcal{F}$. Equivalently, if for every open neighbourhood U around x, there exists some $A \in \mathcal{F}$ fulfilling $\mathcal{N}_A = \{x_s \in X : s \in A\} \subseteq U$. The resulting limit point x is denoted $\lim_{s\to\mathcal{F}} x_s$.

Plainly, every continuous map $f\colon X \longrightarrow Y$ satisfies $f(\lim_{i\to\mathcal{F}}(x_i)) = \lim_{i\to\mathcal{F}} f(x_i)$ for any filter $\mathcal{F}$ on some set I and net $(x_i)_{i\in I}$ converging along $\mathcal{F}$. We now derive the absolute reason to consider ultrafilters in this project: convergence is easy to guarantee.

Theorem 1.2.2. *Suppose X denotes a Hausdorff topological space and let S be some nonempty directed set admitting an ultrafilter $\mathcal{F}$. Under these premises, any net $(x_s)_{s\in S}$ converging along $\mathcal{F}$ has a unique limit point. Moreover, any net converges $\mathcal{F}$ provided X is compact.*

Proof. Suppose that x and y are two distinct limit points of $(x_s)_{s\in S}$ along $\mathcal{F}$. Choose two disjoint open neighbourhoods U and V around these, respectively. The corresponding sets $S(U)$ and $S(V)$ obviously become disjoint. The finite intersection property combined with nontriviality of $\mathcal{F}$ ensures that both cannot lie in $\mathcal{F}$ simultaneously, a contradiction.

To ease the notation, let $\mathcal{N}_A^c$ be the closure of $\mathcal{N}_A$ in X, $\mathcal{F}$ be an ultrafilter on S and write $\mathcal{N} = \bigcap_{A\in\mathcal{F}} \mathcal{N}_A^c$. Any limit point x of $(x_s)_{s\in S}$ along $\mathcal{F}$ must belong to $\mathcal{N}$. Otherwise x would belong to the open set $X \setminus \mathcal{N}_A^c$ for some $A \in \mathcal{F}$ and by convergence we may find some $B \in \mathcal{F}$ such that $\mathcal{N}_B \subseteq X \setminus \mathcal{N}_A^c$ from which $\mathcal{N}_A \cap \mathcal{N}_B = \emptyset$, contradicting $A \cap B \neq \emptyset$.

Conversely, every element in $\mathcal{N}$ determines a limit point of $(x_s)_{s\in S}$. Indeed, by maximality of $\mathcal{F}$ we have $S(X \setminus U) = X \setminus S(U) \in \mathcal{F}$ if $S(U)$ lies outside $\mathcal{F}$ some open neighbourhood U of x. Therefore we must have $x \in \mathcal{N}^c_{S(X\setminus U)} \subseteq X \setminus U$ since $X \setminus U$ is closed, a contradiction.

The two previous observation combine into the following, invoking uniqueness of limits along $\mathcal{F}$: every net $(x_s)_{s\in S}$ converges along $\mathcal{F}$ if and only if $\mathcal{N}$ is nonempty. However, due to the collection of the closed sets $\mathcal{N}_A^c$ indexed over $\mathcal{F}$ having the finite intersection property, compactness of X ensures that the intersection of all its members, i.e. $\mathcal{N}$, must be nonempty as desired. Voila. □

Notable Remark. Given an inclusion $\mathcal{F} \subseteq \mathcal{G}$ of filters, convergence along $\mathcal{F}$ evidently implies convergence along $\mathcal{G}$. Moreover, one may check that ordinary convergence of sequences is equivalent to convergence along the Frechét filter, which is defined to be the family all subsets having a finite complement. Since every free ultrafilter contains the associated Frechét filter in the infinite case, ordinary convergence of sequences implies convergence along a free ultrafilter on $\mathbb{N}$.

Definition. Suppose $(A_i)_{i\in I}$ denotes a collection of C*-algebras over some indexing set I. We define the ℓ^∞-sum of the sequence $(A_i)_{i\in I}$ as the C*-algebra defined by

$$\ell^\infty(A_i, I) = \left\{ (a_i)_{i\in I} : a_i \in A_i,\ \sup_{i\in I} \|a_i\| < \infty \right\}.$$

endowed with ordinary coordinatewise operations and supremum norm. Furthermore, we define the c_0-sum $c_0(A_i, I)$ of $(A_i)_{i\in I}$ to be the closure of the subset consisting of tuples $(a_i)_{i\in I}$ in $\ell^\infty(A_i, I)$ such that $\|a_i\| = 0$ for all except finitely many indices $i \in I$. The subspace $c_0(A_i, I)$ becomes a $*$-ideal inside $\ell^\infty(A_i, I)$. The resulting quotient

$$\ell(A_i, I) = \frac{\ell^\infty(A_i, I)}{c_0(A_i, I)}$$

becomes a C*-algebra. Given a free ultrafilter ω on some indexing set I, we construct the ultraproduct of $(A_i)_{i\in I}$ as follows. Let $c_0^\omega(A_i, I)$ be the collection of elements $(a_i)_{i\in I}$ inside $\ell^\infty(A_i, I)$ satisfying $\lim_{i\to\omega} \|a_i\| = 0$. No mischief appears here: by boundedness of $(a_i)_{i\in I}$, the corresponding real-valued sequence $(\|a_i\|)_{i\in I}$ lies in some bounded precompact set, whereupon *theorem* 1.3.2 applies to assure the existence of the above limit. One may show that $c_0^\omega(A_i, I)$ defines a $*$-ideal inside $\ell^\infty(A_i, I)$. We hereof define the *ω-ultra product of* $(A_i)_{i\in I}$ by

$$\ell_\omega(A_i, I) = \frac{\ell^\infty(A_i, I)}{c_0^\omega(A_i, I)}$$

and endow it with the C*-norm[6]

$$\|\varrho((a_i)_{i\in I})\| = \lim_{i\to\omega} \|a_i\|, \tag{1.3}$$

where $\varrho\colon \ell^\infty(A_{i,I}) \longrightarrow \ell_\omega(A_i, I)$ is the canonical quotient map. The formula is meaningful on the merits of *theorem* 1.3.2 once more. There is an alternative configuration of computing the norm in $\mathcal{Q}(A_n)$ in a fashion mirroring the ultraproduct norm, see for instance [10, lemma 6.1.3] for a proof:

$$\|\varrho(a)\| = \limsup_{n\to\infty} \|a_n\|. \tag{1.4}$$

[6]here the $(a_i)_{i\in I}$ is implicitly a chosen lift of its image under ϱ, the choice posing no obstruction, see ....

1.3 Inductive Limits and Infinite Tensor Products

The current section carries an exposition of inductive limits and in particular infinite tensor product C*-algebras, starting with general inductive limits including the group theoretic version. A large portion of these facts are considered well-known in this project, however, the required results are stated including references to make the project more self-contained. Inductive limits behave well within both the C*-algebraic - and group algebraic framework. Perhaps the reader will swiftly assert how do adopt the inductive limit notion to define infinite tensor products at first glimpse.

Definition. Let $\mathscr{A}$ be a category. An *inductive system* in $\mathscr{A}$ consists of a family of pairs $(A_i, \varphi_i)_{i\in\mathcal{I}}$, where $\mathcal{I}$ is some directed nonempty set, A_i denotes an object in $\mathscr{A}$ for every $i \in \mathcal{I}$ and $\varphi_{ij} \colon A_i \longrightarrow A_j$ is a morphism for every pair of indices $i \leq j$ in $\mathcal{I}$ fulfilling

$$\varphi_{ii} = \mathrm{id}_{A_i} \quad \text{together with} \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \ \text{ for } \ i \leq j \leq k.$$

The morphisms φ_{ij} are commonly referred to as the *connecting morphisms*. In the event of $\mathcal{I}$ being countable, we frequently forego the mentioning of a system and instead present the inductive system in terms of a sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

Such a sequence is called an *inductive sequence*, wherein for $n \leq m$ we write

$$\varphi_{n,m} = \varphi_{m-1} \circ \varphi_{m-2} \circ \ldots \circ \varphi_n \colon A_n \longrightarrow A_m.$$

An inductive limit of an inductive system $(A_i, \varphi_i)_{i\in\mathcal{I}}$ in $\mathscr{A}$ is a pairing $(A, \{\varphi_i^\infty\}_{i\in\mathcal{I}})$, with A denoting an object $\mathscr{A}$ and $\varphi_i^\infty \colon A_n \longrightarrow A$ being a morphism, such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\quad\varphi_{ij}\quad} & A_j \\ & {\scriptstyle\varphi_i^\infty}\searrow \quad \swarrow{\scriptstyle\varphi_j^\infty} & \\ & A & \end{array}$$

commutes for every pair of indices $i \leq j$ in $\mathcal{I}$. The pairing is required to fulfill the following *universal property*: for any additional pairing $(B, \{\lambda_i^\infty\}_{i\in\mathcal{I}})$, there exists a unique morphism $\lambda \colon A \longrightarrow B$ fulfilling $\lambda \circ \varphi_i^\infty = \lambda_i^\infty$ for each index $i \in \mathcal{I}$. Applying the universal property twice easily entails that the inductive limit, should it exist, is unique up to isomorphism in the category $\mathscr{A}$. Hence we may freely speak of *the inductive limit* A of an inductive system $(A_i, \varphi_i)_{i\in\mathcal{I}}$ and we shall denote it by $A = \varinjlim(A_i, \varphi_{ij})$. The above diagram translates into commutativity with $i = n$ and $j = n + 1$ whenever the indexing set $\mathcal{I}$ is countable.

In the category of C*-algebras with $*$-homomorphisms as morphisms, one must address the issue of whether inductive limits even exist. Fortunately, they always do herein and quite a fair amount of added structure may be uncovered. Since we shall primarily work with inductive sequences of C*-algebras, we avoid dwelling too deep into the general setting of C*-algebras. For completeness, the group theoretic version will be supplied to the reader with the construction added.

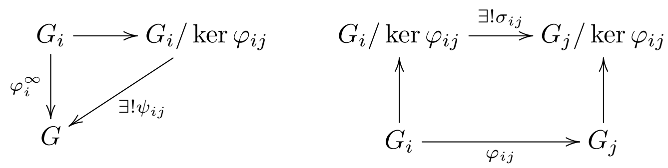
Proposition 1.3.1. *In the category of groups with group homomorphisms as morphisms, inductive limits of inductive systems exist. In fact, the inductive limit of an inductive system may be explicitly described in the following manner. Suppose $(G_i, \varphi_{ij})_{i\in\mathcal{I}}$ denotes an inductive system and let G be the quotient of the disjoint union $\bigcup_{i\in\mathcal{I}} G_i$ with respect to the equivalence relation*

$$g_i \sim g_j \ \text{ in } G \quad \overset{def.}{\iff} \quad \exists k \in \mathcal{I} \colon \varphi_{ik}(g_i) = \varphi_{jk}(g_j).$$

Then there exists an isomorphisms $\varinjlim(G_i, \varphi_{ij}) \cong G$ of groups. Furthermore, inductive limits arising from inductive sequences of abelian groups exist and the same statement remains true in the category of ordered abelian groups.

Proof. The proof is straightforward. One merely verifies that G must satisfy the universal property of inductive limits whereupon existence together with uniqueness implies the desired. For the latter two assertions, we refer to [10, 6.2.5, 6.2.6] for rigorous proofs. □

Remark. There is a precedence to regard an inductive limit G of an inductive system $(G_i, \varphi_{ij})_{i\in\mathcal{I}}$ as a union $G = \bigcup_{i\in\mathcal{I}} G_i$. This is quite justifiable as the inductive limit, by construction, for most practical purposes behaves in this sense. The main idea may be sketched thus: replacing the i'th stage G_i with the quotient $G_i/\ker\varphi_{ij}$ for all $i \leq j$ assures the existence of the following commutative diagrams; upon invoking the universal property of quotient maps to factor each $\varphi_i^\infty : G_i \longrightarrow G$ and composition $G_i \longrightarrow G_j \longrightarrow G_j/\ker\varphi_{ij}$ through the quotient $G_i/\ker\varphi_{ij}$, for all $i \leq j \leq k$ in $\mathcal{I}$.



Exploiting commutativity will reveal that the pairing $(G, \{\psi_{ij}\})_{i,j\in\mathcal{I}}$ determines an isomorphic model of G associated to the directed system of quotients $G_i/\ker\varphi_{ij}$, whose connecting morphisms σ_{ij} become injections by construction. The benefits of such a point of view is, amongst several, preimages under group homomorphisms preserve inductive limits, meaning if $\varphi\colon G \longrightarrow H$ is a group homomorphism having G be a directed limit viewed as the union $\bigcup_{i\in G_i} G_i$, then

$$\varphi^{-1}(G) = \varphi^{-1}\left(\bigcup_{i\in\mathcal{I}} G_i\right) = \bigcup_{i\in\mathcal{I}} \varphi^{-1}(G_i) \cong \varinjlim \varphi^{-1}(G_i). \tag{1.5}$$

Observations such as these are frequently hidden between the lines in other literature. We shall neither refer to these observations and consider the previous discussion sufficient[7].

Proposition 1.3.2. *Suppose $(A_i, \pi_{ij})_{i\in\mathcal{I}}$ denotes an inductive system in the category C^*-Alg. Then the system $(A_i, \pi_{ij})_{i\in I}$ admits an inductive limit $(A, \{\pi_i^\infty\}_{i\in\mathcal{I}})$ and satisfies the following properties.*

- *The union $\bigcup_{i\in\mathcal{I}} \pi_i^\infty(A_i)$ determines a dense subset of A. One may replace each summand by A_i should each π_{ij} be a $*$-monomorphism.*
- *For $\mathcal{I} = \mathbb{N}$: $\|\pi_n^\infty(a)\| = \lim_{m\to\infty} \|\pi_{n,m}(a)\|$ for all $a \in A$ and integer $n \geq 1$.*
- *For $\mathcal{I} = \mathbb{N}$: The kernel of π_n^∞ equals the set of all $a \in A_n$ for which $\|\pi_{n,m}(a)\| \to 0$ as $m \to \infty$.*

Proof. See [10, 6.2.4]. □

Our main application of inductive limits will be to properly introduce infinite tensor algebras. Inductive limits in conjunction with the preceding theorem, more specifically the first property thereof, provide a method to construct infinite tensor products by iteratively embed n-tensor algebras into one another and then consider its inductive limit. More formally, suppose $A_1, A_2, \ldots, A_m$ denotes a finite collection of unital C*-algebras. We define the iterated tensor product of these as

$$\bigotimes_{n=1}^{m} A_n = \left(\bigotimes_{n=1}^{m-1} A_n\right) \otimes A_m$$

[7]Hopefully the arguments granted will convince the reader of the validity, for going into further details supplies no greater epiphany according to the writer.

Due to tensor products obeying an associative law, that is, $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ canonically in the C*-algebraic sense, the ordinary rules of tensor calculus extend naturally to the iterated version. As such one foregos placing the paranthesis overall. Elementary tensor elements become those of the form $\bigotimes_{m=1}^{n} a_n$ with a_n belonging to A_n for each $1 \leq n \leq m$. Furthermore, since the spatial tensor norm determines a cross-norm, a straightforward induction argument reveals that

$$\left\| \bigotimes_{n=1}^{m} a_n \right\| = \prod_{n=1}^{m} \|a_n\|.$$

for any elementary tensor. It is apparent that $\bigotimes_{n=1}^{m} 1_{A_n}$ serves as a unit for the algebra. A special case of iterated tensor products naturally appear in the form of finite dimensional ones: Suppose $n_1, n_2, \dots n_m$ are integers and let N denote their product. By successively applying the identity $\mathbb{M}_k \otimes \mathbb{M}_\ell \cong \mathbb{M}_{k\ell}$ one may deduce that $\bigotimes_{k=1}^{m} \mathbb{M}_{n_k} \cong \mathbb{M}_N$. Another reoccurring version is the notion of F-folds of tensor products; suppose F denotes a finite set, A denotes some unital C*-algebra and define in accordance with the above, modulo an appropriate choice of enumeration of F, *the spatial F-fold tensor algebra* by

$$\bigotimes_F A := A \otimes A \ldots \otimes A \qquad (|F|\text{-times}).$$

Notice that whenever $n \leq m$ are positive integers and $A_1, A_2, \dots, A_m$ is a collection of unital C*-algebras, one has a canonical $*$-monomorphism $\pi \colon \bigotimes_{k=1}^{n} A_k \longrightarrow \bigotimes_{k=1}^{m} A_k$ given by the assignment

$$\pi\left(\bigotimes_{k=1}^{n} a_k \right) = \bigotimes_{k=1}^{n} a_k \otimes \bigotimes_{k=n+1}^{m} 1_{A_k}.$$

Similarly, if $M \subseteq N$ is an inclusion of finite sets one has a natural embedding $\bigotimes_M A \hookrightarrow \bigotimes_N A$ given by adding the unit to the remaining $|N \setminus M|$-many tensor-factors onto elements in the former algebra provided A is unital. These observations permit us to extend our notion to the infinite case.

Definition. Suppose $(A_n)_{n\geq 1}$ denotes some sequence of unital C*-algebras and let π_n be the canonical $*$-monomorphism discussed previously. We define the infinite tensor product associated to $(A_n)_{n\geq 1}$ to be the inductive limit

$$\bigotimes_{k=1}^{\infty} A_k = \varinjlim \left(\bigotimes_{k=1}^{n} A_k, \pi_k \right).$$

On the virtues of proposition 1.3.2, it contains the union $\bigcup_{n=1}^{\infty} (\bigotimes_{k=1}^{n} A_k)$ as a dense subspace. Moreover, if A denotes a unital C*-algebra and N some infinite set, let $\mathcal{F}$ denote the collection of finite sets contained in N and let $\pi_F \colon \bigotimes_F A \longrightarrow \bigotimes_G A$ be the $*$-monomorphism induced via the inclusion $F \subseteq G$ occurring in $\mathcal{F}$. Using this notation, we define the spatial N-fold tensor product of A to be the inductive limit

$$\bigotimes_N A = \varinjlim \left(\bigotimes_F A, \pi_F \right),$$

which contains the union $\bigcup_{F \in \mathcal{F}} (\bigotimes_F A)$ as a dense subset.

Remark. Since the tensor products have the above dense subsets, elementary tensors inside the tensor-algebra $\bigotimes_{n=1}^{\infty} A_n$ of unital C*-algebras are those of the form $a = \bigotimes_{n=1}^{\infty} a_n$ where a_n differs from 1_{A_n} for only finitely many indices n. Naturally, an analogue version is given for infinite N-folds $\bigotimes_N A$ of some fixed C*-algebra A. Lastly, for such an elementary tensor a one has

$$\|a\| = \prod_{n \in V} \|a_n\|, \tag{1.6}$$

where V denotes the collection of $n \in \mathbb{N}$ for which $a_k \neq 1_{A_k}$. This generalizes the above formula.

1.4 A word on AF - and UHF Algebras

Approximately finite dimensional algebras, abbreviated into AF-algebras, are interesting examples of C*-algebras arising from inductive limits while being classified. In fact, George Elliot classified AF-algebras solely in terms of their ordered K-theory. A special case of AF-algebras is the class consisting of UHF-algebras, a shorthand for *uniformly hyperfinite*, whose classification was established by Glimm without K-theory. Glimm's work included the use of supernatural numbers, which is a notion suitable for describing UHF-algebras in terms of infinite tensor products. Since we shall exploit properties of UHF-algebras during the survey of elementary amenable groups and their connections to quasidiagonality, a brief treatment of these has been deemed necessary.

Definition. An *AF-algebra* is an inductive limit of finite-dimensional C*-algebras. The subclass of *UHF-algebras* consists of C*-algebras that are $*$-isomorphic to some inductive limit of matrix-algebras $\mathbb{M}_{n(k)}$ with unital connecting $*$-monomorphisms. To distinguish these from one another, it is customary to keep track of the indices $n(k)$ and collect these into a sequence $\{n(k)\}_{k\geq 1}$ commonly referred to as the *type* of the algebra $\varinjlim \mathbb{M}_{n(k)}$.

Proposition 1.4.1. *An inductive limit A of simple C*-algebras A_n containing a common unit and having unital $*$-monomorphisms $\varphi_n\colon A_n \longrightarrow A_{n+1}$ as connected morphisms is simple and unital.*

Proof. By hypothesis, the connecting morphisms are embeddings, which permit us to assume that $\bigcup_{n=1}^{\infty} A_n$ defines a norm-dense subspace of A. Suppose I is $*$-ideal inside A differing from A itself. In particular, I cannot contain the identity on A and it admits a nontrivial quotient morphism $\pi\colon A \longrightarrow A/I$ whose restriction π_n to A_n defines an ideal $\ker \pi_n \trianglelefteq A_n$. Upon each A_n being simple, the kernel of π_n has to equal either $\{0\}$ or A_n with the latter failing due to $0 \neq \pi(1_A) = \pi_n(1_A)$. The map π_n must therefore be isometric for each $n \in \mathbb{N}$, whence

$$\bigcup_{n=1}^{\infty} A_n \subseteq \{a \in A : \|\pi(a)\| = \|a\|\}.$$

As the right-hand side is norm-closed in A, it coincides with A by density. It follows that π must be a $*$-monomorphism, forcing I to be the zero ideal as desired. □

Lemma 1.4.2. *Suppose $A_1 \subseteq A_2 \subseteq \ldots$ is an increasing sequence of C*-algebras whose union is dense inside a unital C*-algebra A. If A_n contains the unit of A and admit a trace for every integer $n \geq 1$, then A admits a tracial state. In particular, inductive limits of C*-algebras admitting tracial states with unital $*$-monomorphisms as connecting morphisms admit tracial states.*

Proof. The proof revolves around a clever application of the Hahn-Banach extension theorem combined with the finite intersection property. Let T_n denote the collection of states on A whose restriction to A_n defines a tracial state. By hypothesis, A_n admits a trace τ_0. On the merits of Hahn-Banach's extension theorem, τ_0 extends to a state τ on A whose restriction must be a traicial state on A_n. The set T_n must be non-empty hereof, and closed (hence compact) because

$$T_n = \bigcap_{a,b\in A_n} \{\varphi \in S(A) : \varphi(ab - ba) = 0\}.$$

Due to the collection $(A_n)_{n\geq 1}$ being increasing, one may infer that $(T_n)_{n\geq 1}$ must be decreasing, so the finite intersection property yields $T = \bigcap_{n\geq 1} T_n \neq \emptyset$. The element therein determines a state on the union $\bigcup_{n=1}^{\infty} A_n$, hence extends to A by density in conjunction with continuity, proving existence of a trace on A. The final statement is immediate from the first. □

Proposition 1.4.3. *Any UHF-algebra is faithfully monotracial, unital and simple.*

Proof. Having established the previous lemma, the proof is quite dull. Recall that $\mathbb{M}_n$ is monotracial, unital and simple for every integer $n \geq 1$. Since every UHF-algebra A satisfies the hypotheses of the previous lemma on the merits of $\mathbb{M}_n$ being unital monotracial and simple for every positive integer n, it admits a tracial state τ. By simplicity of A, the ideal $\mathcal{L}_\tau = \{a \in A : \tau(a^*a) = 0\}$ must be the zero ideal for otherwise $\tau = 0$. This implies that τ is faithful.

Concerning uniqueness of the trace: any additional tracial state ϱ on A must agree with τ when restricted to $\mathbb{M}_{n(k)}$, whereupon uniqueness of traces on matrix-algebras entails that $\varrho_{|A_0} = \tau_{|A_0}$ with $A_0 = \bigcup_{n=1}^\infty \mathbb{M}_{n(k)}$. Since the latter is dense in A, continuity guarantees that $\varrho = \tau$. Therefore A becomes monotracial while the remaining two properties follow from the preceding proposition. □

The previous rather accessible invariants of UHF-algebras, although fruitful, may be improved upon tremendously. In fact, Glimm completely classified UHF-algebras via the notion of supernatural numbers even without using K-theory. Supernatural numbers permit us to rephrase UHF-algebras more conveniently, hence a brief survey of these is carried out.

Definition. A *supernatural number* N is an increasing sequence of nonnegative integers $(n_k)_{k\geq 1}$ including the possibility of $n_k = \infty$ for any $k \in \mathbb{N}$. Formally, we regard the supernatural number N as being the formal infinite prime factorization

$$N = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots$$

where n_k may be any nonnegative integer or ∞ with the p_k factors being primes listed in order. The product of two supernatural numbers $N = (n_k)_{k\geq 1}$ and $M = (m_k)_{k\geq 1}$ is $NM = (n_k + m_k)_{k\geq 1}$. We write $N|M$ whenever $n_k \leq m_k$ for all indices k and refer to this as N *dividing* M.

Adopting the notation of Glimm, given a sequence of natural numbers $(n_k)_{k\geq 1}$ fulfilling $n_k | n_{k+1}$ for all $k \in \mathbb{N}$ there is an associated supernatural number $\delta[(n_k)] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ defined as follows. Formally, decompose $n_k = p_1^{\alpha_{1,k}} p_2^{\alpha_{2,k}} \cdots$, i.e. regard each of the integers n_k as a supernatural number, then define hereby

$$\alpha_m = \sup\{\alpha_{mk} : k \in \mathbb{N}\}.$$

An UHF-algebra of type $(n_k)_{k\geq 1}$ thus admits a supernatural number $\delta[(n_k)]$. Hence we denote the UHF-algebra having N as "the supernatural number" N associated to it by $\mathbb{M}_N$, calling $\mathbb{M}_N$ to be the UHF-algebra of *type* N.

Given a sequence $(n_k)_{k\geq 1}$ consisting of natural numbers, it is apparent that $n(k)$ divides $\delta[(n_k)]$ for every integer $k \geq 1$ as supernatural numbers, so in some sense $\delta[(n_k)]$ is the "supremum" of the integers n_k occurring in the sequence. We illustrate the adamant use of both K-theory and supernatural numbers by stating the most fundamental result in the classification of UHF-algebras.

Theorem 1.4.4 (Glimm, Elliot)**.** *Let A and B be UHF-algebras of type N and M, respectively. The following four statements are equivalent:*

- $A \cong B$.
- $N = M$.
- $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$.

In other words, UHF-algebras are completely classified via supernatural numbers and K-theory.

Proof. See [10, theorem 7.4.5] for a rigorous proof. □

The K-theory of UHF - and AF-algebras is wealthy: there are intriguing features concerning morphisms of these C*-algebras and their K-theoretic kin. One may often recover $*$-homomorphisms from morphisms between the associated K-groups, which we shall exploit momentarily. For the sake of reference and adjusting the statements accordingly, we exhibit these.

Lemma 1.4.5. *Suppose (A_n, φ_n) is an inductive sequence in the category of C*-algebras with limit A and let B be any UHF - or finite dimensional C*-algebra. Assume that one has two positive group homomorphisms $\alpha\colon K_0(A_1) \longrightarrow K_0(B)$ and $\beta\colon K_0(B) \longrightarrow K_0(A)$ satisfying $\beta \circ \alpha = K_0(\varphi_1^\infty)$. Under these premises, there exists a positive group homomorphism $\omega\colon K_0(B) \longrightarrow K_0(A_n)$ for some positive integer n, which is unital provided each φ_n and α are unit preserving morphisms.*

Proof. We omit proving it in detail and simply assert that the proof of [10, proposition 7.3.3] may be adjusted to the UHF case. An inspection of the proof will reveal that finite-dimensionality is strictly speaking not required and only continuity of the K_0 functor is being applied. □

For the upcoming proposition, recall that a C*-algebra A has the *cancellation property* if for every pair of projections $p, q \in \mathcal{P}_\infty(A)$ one has $[p]_0 = [q]_0$ if and only if $p \sim_0 q$. Examples of C*-algebras having the cancellation property include finite-dimensional ones and AF-algebras.

Proposition 1.4.6. *Let A be a finite dimensional C*-algebra and B some unital C*-algebra having the cancellation property. If so, any two $*$-homomorphisms $\varphi, \psi\colon A \longrightarrow B$ are equal in K_0 if and only if they are unitarily equivalent. Moreover, any positive unit-preserving group homomorphism $\alpha\colon \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(B)$ stems from a unital $*$-homomorphisms $\varphi\colon A \longrightarrow B$, i.e., $\mathrm{K}_0(\varphi) = \alpha$.*

Proof. See [10, proposition 7.3.2] for a proof. □

Supernatural numbers are paramount and permit us to characterize UHF-algebras as infinite tensor products. Let A be the UHF-algebra of type $(n_k)_{k\geq 1}$, meaning A equals the inductive limit of wherein n_k divides n_{k+1} for every index k. For each index k, write $n_k d_{k+1} = n_{k+1}$. Then the canonical $*$-isomorphism $\mathbb{M}_n \otimes \mathbb{M}_m \cong \mathbb{M}_{nm}$ ensures that $\mathbb{M}_{n_k} = \mathbb{M}_{n_{k-1}} \otimes \mathbb{M}_{d_k}$ for every $k \in \mathbb{N}$. Applying these identifications successively, the inductive sequence transforms into

$$\mathbb{M}_{n_1} \longrightarrow \mathbb{M}_{n_1} \otimes \mathbb{M}_{d_2} \longrightarrow \mathbb{M}_{n_1} \otimes \mathbb{M}_{d_2} \otimes \mathbb{M}_{d_3} \longrightarrow \cdots$$

Stipulating that $n_1 = d_1$ hereof yields $A = \bigotimes_{n=1}^\infty \mathbb{M}_{d_n}$. Any repetition of the divisors d_k may be collected into the factors of the form $\mathbb{M}_{d_n^{\alpha_n}}$ with α_n potentially being infinite. The associated sequence $(\alpha_n)_{n\geq 1}$ is precisely the powers arising in the supernatural number associated to A. As such, if N and N' are the supernatural numbers of $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$, respectively, then

$$\mathbb{M}_N \otimes \mathbb{M}_{N'} = \left(\bigotimes_{n=1}^\infty \mathbb{M}_{p_n^{\alpha_n}}\right) \otimes \left(\bigotimes_{n=1}^\infty \mathbb{M}_{p_n^{\beta_n}}\right) \cong \bigotimes_{n=1}^\infty \left(\mathbb{M}_{p_n^{\alpha_n}} \otimes \mathbb{M}_{p_n^{\beta_n}}\right) \cong \bigotimes_{n=1}^\infty \mathbb{M}_{p_n^{\alpha_n+\beta_n}}.$$

The second isomorphism is the assignment $(\bigotimes_{n=1}^\infty a_n) \otimes (\bigotimes_{n=1}^\infty b_n) \mapsto \bigotimes_{n=1}^\infty (a_n \otimes b_n)$ defined on elementary tensors and then extended via continuity. The latter C*-algebra is precisely the UHF-algebra having NN' as supernatural number, whereupon we infer that $\mathbb{M}_N \otimes \mathbb{M}_{N'} \cong \mathbb{M}_{NN'}$ must be valid. Using a simple induction argument, given UHF-algebras $\mathbb{M}_{N_1}, \mathbb{M}_{N_2}, \ldots, \mathbb{M}_{N_m}$ one obtains

$$\bigotimes_{n=1}^m \mathbb{M}_{N_n} \cong \mathbb{M}_{N_1 N_2 \cdots N_m}. \tag{1.7}$$

For the infinite case, one needs to tread more carefully, so the proof has been separated into several parts. As a matter of fact, our strategy will be to investigate supernatural numbers associated to direct limits of UHF-algebra, initiating this with some intermediate observations.

Lemma 1.4.7. *Suppose A denotes the limit of the inductive sequence (A_k, φ_k) with each A_k being UHF, B is some finite dimensional C*-algebra and let some integer $m \in \mathbb{N}$ be fixed. Under these premises, there exists a unital $*$-homomorphism $\pi\colon B \longrightarrow A$ if and only if there exists some $k \in \mathbb{N}$ together with a unital $*$-homomorphism $\varrho\colon B \longrightarrow A_k$.*

Proof. The if part trivially holds, since A_k naturally embeds unitally into the limit A. For the converse, we apply lemma 1.4.5 replacing A_1 by $\mathbb{C}$ without loss of generality[8]. By hypothesis, the assumed unital $*$-homomorphism $\pi\colon A \longrightarrow B$ induces a unit-preserving positive group homomorphisms $\alpha\colon \mathrm{K}_0(B) \longrightarrow \mathrm{K}_0(A)$ while the canonical embedding $\mathbb{C} = A_1 \hookrightarrow B$ induces a unit-preserving positive group homomorphism $\beta\colon \mathrm{K}_0(A_1) \longrightarrow \mathrm{K}_0(B)$. These morphisms evidently satisfy $\alpha\beta = \mathrm{K}_0(\pi_1^\infty)$, so we may invoke lemma 1.4.5 to determine a positive unit-preserving group homomorphism $\gamma\colon \mathrm{K}_0(B) \longrightarrow \mathrm{K}_0(A_k)$. Due to UHF-algebras having the cancellation property, proposition 1.4.6 applies to imply that γ is induced via a unital $*$-homomorphisms $B \hookrightarrow A_k$. □

Lemma 1.4.8. *Suppose A denotes the limit of an inductive sequence $(\mathbb{M}_{n_k}, \varphi_k)$ of matrix-algebras. Let $n = \delta[(n_k)]$ be the corresponding supernatural number occurring in the inductive sequence. Then there exists a unital $*$-homomorphism $\mathbb{M}_m \longrightarrow A$ if and only if $m|n$.*

Proof. This is merely a restatement of a previous remark: m must divide n_k whenever there exists a unital $*$-homomorphism $\mathbb{M}_m \longrightarrow \mathbb{M}_{n_k}$ for some positive integer $k \neq 1$, so m divides n as n_k divides n for every such k. Hence m divides n provided a unital $*$-homomorphism $\mathbb{M}_m \longrightarrow B$ exists.

Conversely, if m divides n then it must divide n_k for some $k \in \mathbb{N}$, whereupon the existence of some unital $*$-homomorphism $\mathbb{M}_m \longrightarrow \mathbb{M}_{n_k}$ is guaranteed as asserted. □

Proposition 1.4.9. *Let $(A_k, \varphi_k)_{k\geq 1}$ be an inductive sequence of UHF-algebras with limit A wherein the connecting morphisms $\varphi_k\colon A_k \longrightarrow A_{k+1}$ are unital. Suppose in addition A_k has*

$$N(k) = \prod_{i=1}^{\infty} p_i^{n_k(i)}$$

as its associated supernatural number for every $k \in \mathbb{N}$. Under these premises, the tensor product $A_1 \otimes \ldots \otimes A_n$ is canonically isomorphic to the UHF-algebra of type $N_1 N_2 \cdots N_n$ and A becomes an UHF-algebra of type

$$N = \prod_{i=1}^{\infty} p_i^{n(i)}; \quad n(i) = \sup\{n_k(i) : k \in \mathbb{N}\}.$$

In particular, the C-algebra $\bigotimes_{\mathbb{N}} \mathbb{M}_{N(k)}$ determines an UHF-algebra of type $N' = \prod_{k=1}^{\infty} N(k)$.*

Proof. The former assertion was already established. For the remaining ones, notice that the supernatural number N described in the statement is precisely the supernatural number subject to the following relation: For every natural number m one has that $m|N$ if and only if there exists some index k for which $m|N(k)$. Verifying this is almost accomplished instantly when considering $m \in \mathbb{N}$ as a supernatural number. Combining the two previous lemmas entails

$$m|N \iff \mathbb{M}_m \hookrightarrow A \iff \exists k : \mathbb{M}_m \hookrightarrow A_k \iff \exists k : m|N(k).$$

This immediately implies the sought conclusion concerning the supernatural number N and A. For the latter claim, simply consider the obtained identifications

$$\bigotimes_{k=1}^{\infty} \mathbb{M}_{N(k)} = \varinjlim \Big(\bigotimes_{k=1}^{n} \mathbb{M}_{N(k)} \Big) \overset{(1.7)}{\cong} \varinjlim (\mathbb{M}_{N(1)\cdots N(n)}) \cong \mathbb{M}_{\delta[(N_1 \cdots N_n)_n]} = \mathbb{M}_{N'}$$

to conclude the desired. □

[8] A contains a copy of $\mathbb{C}$ regardless, so passing to this new sequence leaves the limit unaltered.

Note that the final statement ensures that $\bigotimes_I \mathbb{M}_N \cong \mathbb{M}_N$ whenever $N = p^\infty$ for some prime p and I is countable. We finalize the section with an observation concerning UHF-algebras, namely that they all contain a rich family of conditional expectations preserving the norm sufficiently well.

Proposition 1.4.10. *Let $\mathbb{M}_N$ be an UHF-algebra of type $N = (n_1, n_2, \dots)$ and $\mathbb{M}_N = \bigotimes_{k=1}^m \mathbb{M}_{n(k)}$. There exists a conditional expectation $E_m \colon \mathbb{M}_N \longrightarrow \mathbb{M}_{n(m)}$ defined by*

$$\bigotimes_{k=1}^{\infty} a_k \longmapsto \bigotimes_{k=1}^{n(m)} a_k$$

where $n(m) = n_1 n_2 \cdots n_m$. Moreover, the composed map $\mathbb{M}_n \longrightarrow E_m(\mathbb{M}_N) \overset{\iota_m}{\hookrightarrow} \mathbb{M}_N$ is an isometry.

Proof. The map E_m is obviously a conditional expectation on elementary tensors, hence on all of $\mathbb{M}_N$ by continuity. Finally, note that for every elementary tensor $a = \bigotimes_{k=1}^\infty a_k$ in $\mathbb{M}_N$ one has

$$\|\iota_m E_m(a)\| = \left\| \bigotimes_{k=1}^{n} a_k \otimes \bigotimes_{k=n+1}^{\infty} 1_k \right\| \overset{(1.6)}{=} \prod_{k=1}^{n} \|a_k\| = \|a\|$$

By linearity and continuity of the maps involved, we are done. □

1.5 From Groups to C*-Algebras

It was conjectured by Rosenberg that quasidiagonality was equivalent to the discrete group in question being amenable after he proved one implication. In essence, the recent discoveries of Tikusis, Winter and White provide the converse, so a small note concerning group C*-algebras is supplied. The construction is assumed to be well-known in these notes, however, it was deemed necessary to recall certain parts. The reader is referred to [2, section 4.1] for proofs.

We adopt the following conventions. Given a discrete group G, the point image $u(s)$ of a unitary representation $u \colon G \longrightarrow \mathcal{U}(\mathcal{H})$ is written as u_s to dodge potential confusion. The concrete one encoding the left-translation on G is called the *left regular representation* and we denote it λ or λ^G for emphasis on G, that is, the unitary representation $\lambda \colon G \longrightarrow \mathcal{U}(\ell^2(G))$ given by

$$\lambda_s \xi(t) = \xi(s^{-1}t)$$

for all $s, t \in G$. The collection $\{\delta_s\}_{s\in G}$ consisting of one-point masses $\delta_s \colon G \longrightarrow \{0, 1\}$, meaning $\delta_s(g) = 1$ if $g = s$ and zero otherwise, determines an orthonormal basis of $\ell^2(G)$ whereof λ rewrites into $\lambda_s \delta_t = \delta_{st}$ for all $s, t \in G$. As a remaining ingredient, given a discrete group G we define the *complex group ring* $\mathbb{C}[G]$ associated to G as the $\mathbb{C}$-algebra $\mathbb{C}[G] = C_c(G, \mathbb{C})$ or alternatively

$$\mathbb{C}[G] = \left\{ \sum_{s\in G} a_s s : a_s \in \mathbb{C} \text{ only finitely many } a_s \text{ are nonzero} \right\}.$$

Here s abbreviates δ_s, such that $a_s \neq 0$ for all except finitely many elements s in G. $\mathbb{C}[G]$ is endowed with the natural product and involution

$$\left(\sum_{s\in G} a_s s\right) \cdot \left(\sum_{t\in G} b_t t\right) = \sum_{s,t\in G} a_s b_t st \quad \text{and} \quad \left(\sum_{s\in G} a_s s\right)^* = \sum_{s\in G} \overline{a_s} s^{-1}.$$

The image of $\mathbb{C}[G]$ under the left-regular representation λ hereby defines an involutive subalgebra of $\mathrm{B}(\ell^2(G))$ in the sense that λ extends linearly to $\mathbb{C}[G]$ by setting $\lambda\left(\sum_{s\in G} a_s s\right) = \sum_{s\in G} a_s \lambda_s$. Hence the image $\lambda(\mathbb{C}[G])$ contains the collection $\{\lambda_s : s \in G\}$ as a basis and a straightforward computation reveals that λ in fact becomes faithful, hence the following notion becomes meaningful.

Definition. The *reduced group C*-algebra* associated to a discrete group G is the norm closure of $\lambda(\mathbb{C}[G])$ regarded as an involutive subalgebra in $\mathrm{B}(\ell^2(G))$. The resulting C*-algebra is symbolically represented by $C^*_\lambda(G)$ and the assignment $G \mapsto C^*_\lambda(G)$ determines a functor $\mathbf{Grp} \longrightarrow \mathbf{C^*\text{-}Alg}$.

Observation. Suppose G denotes a discrete group. The reduced group C*-algebra admits a canonical faithful tracial state[9] τ given by

$$\tau(a) = \langle a\delta_{1_G}, \delta_{1_G} \rangle$$

For the sake of completeness, we state Rosenberg's theorem. We shall not formally define amenable groups, since the author could not decide between the almost uncountable number of equivalent characterizations. Regardless, here is Rosenberg's theorem.

Theorem 1.5.1 (Rosenberg)**.** *Under the premise of the reduced group C*-algebra associated to a discrete group being quasidiagonal, the group in question must be amenable.*

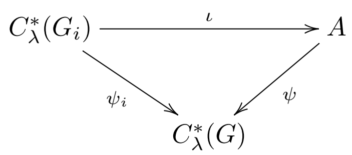
Before proceeding, a functorial property of the assignment $G \mapsto C^*_\lambda(G)$ will be taken into account.

Proposition 1.5.2. *Suppose $(G_i)_{i\in I}$ is some directed system of discrete groups with respect to inclusions $G_i \hookrightarrow G_j$ for all indices $i \leq j$. Let G be the corresponding direct limit. Under these premises, one has a $*$-isomorphism of the reduced group C*-algebras*

$$C^*_\lambda(\varinjlim G_i) \cong \varinjlim C^*_\lambda(G_i).$$

In other words, $C^*_\lambda(\cdot)$ is a continuous functor [10]*.*

Proof. According to [2, Corollary 2.5.12], any pair of indices $i \leq j$ induce an embedding of unital C*-algebras $\psi_{ij} \colon C^*_\lambda(G_i) \hookrightarrow C^*_\lambda(G_j)$, hence forms a directed system of C*-algebras whose corresponding limit we shall denote by A for simplicity. Upon G_i embedding into G for any $i \in I$, we may deduce that there exists a net $(\psi_i)_{i\in I}$ consisting of $*$-monomorphism $C^*_\lambda(G_i) \hookrightarrow C^*_\lambda(G)$ fulfilling $\psi_i = \psi_{ij} \circ \psi_j$ whenever $i \leq j$ in I. The universal property of direct limits thus produces a unique $*$-homomorphism $\psi \colon A \longrightarrow C^*_\lambda(G)$ such that the diagram below commutes for all $i \in I$.



We assert that ψ must necessarily be a $*$-isomorphism. We may identify A with the norm-closure of the union $\bigcup_{i\in I} C^*_\lambda(G_i)$. Assume $a = \lim_{n\to\infty} a_{i(n)}$ wherein $a_{i(n)}$ belongs to $C^*_\lambda(G_i)$ for some index $i \in I$. Since $*$-homomorphisms are injections precisely when they are isometric, it suffices to verify that $\|\psi(a)\| = \|a\|$. However, due to ψ_j being an isometry by the very same reasoning, we may exploit norm continuity of each ψ_j and commutativity of the diagram to write

$$\|\psi(a)\| = \|\lim_{n\to\infty} \psi(a_{i(n)})\| = \lim_{n\to\infty} \|\psi_i(a_{i(n)})\| = \lim_{n\to\infty} \|a_{i(n)}\| = \|a\|$$

proving ψ to be an embedding. In order to show that ψ must be a $*$-epimorphism, observe that each algebra $\lambda(\mathbb{C}[G_i])$ embeds into A. By density of the involving algebras, every element in $C^*_\lambda(G)$ is some norm-limit of linear combinations of elements in the involutive algebra $\lambda(\mathbb{C}[G_i])$. Commutativity of the diagram ensures that ψ maps onto any such element by continuity combined with linearity. This finalizes the proof. □

[9] However, this tracial state is rarely unique! In fact, uniqueness is connected to simplicitly of $C^*_\lambda(G)$.
[10] A functor F is continuous if it preserves direct limits,

1.6 Crossed Products

The functor $G \mapsto C^*_\lambda(G)$ encodes the group in the sense that G under the image of λ sits faithfully therein. However, the construction lacks the merits of remembering group actions $G \curvearrowright A$ on C*-algebras. The crossed product is a C*-algebraic object seeking to mend this particular flaw, copying the ordinary construction of semidirect products with the group being replaced by a unital C*-algebra. The crossed product thus needs to contain the following data: a copy of both the group and C*-algebra in question and an encoded version of the action $G \curvearrowright A$. The majority of results together with proofs are contained in [2, section 4.1].

Definition. A *C*-dynamical system* is a triple (A, α, G) consisting of a topological group G acting continuously on a C*-algebra A via $*$-automorphisms $\alpha\colon G \longrightarrow \mathrm{Aut}(A)$. We will denote an automorphism $\alpha(s)$ stemming from α applied to an element s in G by α_s instead to avoid confusion.

Suppose (A, α, G) denotes a dynamical system, wherein A is unital and G is discrete. Such a system will frequently be referred to as being *unital and discrete* for obvious reasons. Let $C_c(G, A)$ denote the linear space of finitely supported maps $\xi\colon G \longrightarrow A$ or, equivalently,

$$C_c(G, A) = \left\{ \sum_{s \in G} a_s s : a_s \in A \text{ only finitely many } a_s \text{ are nonzero} \right\}.$$

Once more the s is an abbreviation for the one-point mass $s \mapsto \delta_s$. Due to A being unital, the group G admits a group homomorphism $u\colon G \longrightarrow \mathcal{U}(C_c(G, A))$ given by $s \mapsto 1_A s$. We endow $C_c(G, A)$ with the following *twisted convolution* as product and *twisted involution* as involution;

$$\left(\sum_{s \in G} a_s s\right) \cdot \left(\sum_{t \in G} b_t t\right) = \sum_{s,t \in G} a_s \alpha_s(b_t) st \quad \text{, respectively,} \quad \left(\sum_{s \in G} a_s s\right)^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

These maps turns $C_c(G, A)$ into an involutive algebra generated by A and the unitaries $\{u_s\}_{s \in G}$ such that $u\colon G \longrightarrow C_c(G, A)$ implements the action of α in the sense that $\alpha_s(\cdot) = u_s(\cdot)u_s^*$ for every s belonging to G. Thus, the construction determines an involutive algebra with canonical generating elements, which contains an isomorphic copy of A via the assignment $a \mapsto a u_e$ while containing a copy of G via $s \mapsto u_s$. The remaining ingredient is the C* norm. As always, the easiest way to form one is to steal one.

Definition. A *covariant representation* of a unital discrete C*-dynamical system (A, α, G) is a triple $(\pi, \varphi, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$, a representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ together with some unitary representation $\varphi\colon G \longrightarrow \mathcal{U}(\mathcal{H})$ implementing the action α, meaning $(\pi \circ \alpha_s)(\cdot) = \varphi_s(\cdot)\varphi_s^*$ for all s in G. The *integrated form* of $(\pi, \varphi, \mathcal{H})$ is the induced representation $\pi_\alpha \times \varphi\colon C_c(G, A) \longrightarrow \mathrm{B}(\mathcal{H})$ defined on generating elements by

$$(\pi_\alpha \times \varphi)(a u_s) = \pi(a)\varphi_s.$$

The covariant representation $(\pi, \varphi, \mathcal{H})$ is said to be *faithful* (respectively *non-degenerate*) provided π is faithful (respectively non-degenerate) as a representation.

Definition. The *universal or full crossed product* $A \rtimes_\alpha G$ associated to a unital discrete C*-dynamical system (A, α, G) is the norm closure of the involutive algebra $C_c(A, G)$ with respect to the universal norm, meaning

$$\|x\|_u := \big\{ \|\pi(x)\|_{\mathcal{H}} : (\pi, \varphi, \mathcal{H}) \text{ non-degenerate covariant representation of } (A, \alpha, G) \big\}.$$

Moreover, $A \rtimes_\alpha G$ fulfills the following universal property: every covariant representation $(\pi, \varphi, \mathcal{H})$ hereof induces a $*$-homomorphism $\pi_\alpha \times \varphi\colon A \rtimes_\alpha G \longrightarrow \mathrm{B}(\mathcal{H})$.

Much alike the reduced group C*-algebra, we attempt to borrow a norm in $\mathrm{B}(\mathcal{H})$ for some suitable Hilbert space $\mathcal{H}$. Covariant representations allow this in the faithful case. However, what canonical representation will always exist? The answer is the naive one: use the left-regular representation and tensor it with whichever faithful representation on the C*-algebra you prefer. Elaborating upon this further, suppose (A, α, G) denotes unital discrete C*-dynamical system, let $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ be any faithful representation and let $\lambda\colon G \longrightarrow \mathcal{U}(\ell^2(G))$ be the usual left-regular representation on G. Define a representation $\pi_\alpha\colon A \longrightarrow \mathrm{B}(\mathcal{H}) \otimes \ell^2(G)$ by

$$\pi_\alpha(a)(\xi \otimes \delta_s) = \pi(\alpha_{s^{-1}}(a))\xi \otimes \delta_s,$$

for every $s \in G$, $a \in A$ and $\xi \in \mathcal{H}$. One may verify that the integrated form $\pi_\alpha \times (1 \otimes \lambda)$, commonly abbreviated into $\pi_\alpha \times \lambda$, becomes faithful as well. We may hereof finish our construction. A priori, one may expect the choice of π to have an impact, which fortunately is revealed to be false, cf. [2, Proposition 4.1.5] for an argument of the matter.

Definition. The *reduced crossed product* $A \rtimes_{\alpha,r} G$ associated to a unital discrete C*-dynamical system (A, α, G) is the norm-closure of $C_c(G, A)$ under the image of the faithful representation $\pi_\alpha \times \lambda$, the choice of π being irrelevant. Symbolically written, we define

$$A \rtimes_{\alpha,r} G = \overline{(\pi_\alpha \times \lambda)(C_c(G, A))}^{\|\cdot\|} \hookrightarrow \mathrm{B}(\mathcal{H} \otimes \ell^2(G)).$$

In particular, $A \rtimes_\alpha G$ is the C*-algebra generated by the set $\{\pi_\alpha(a), \lambda_s : a \in A, s \in G\}$.

The reader might ponder why no mentioning of functoriality concerning the reduced crossed product has appeared. The reason is simply that we have not introduced the morphisms properly, for ordinary $*$-homomorphism are insufficient due to the twisted convolution imposed on $C_c(G, A)$ demanding interactions with the action. Therefore, it seems adequate to address these.

Definition. Suppose (A, α, G) and (B, β, G) denote two discrete unital C*-dynamical systems. A unital $*$-homomorphism $\varphi\colon A \longrightarrow B$ is said to be *G-equivariant* whenever the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_s} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\beta_s]{} & B \end{array}$$

commutes for all $s \in G$. In the event of $\varphi\colon A \longrightarrow B$ being a G-equivariant morphism, it induces a $*$-homomorphism $\varphi_c\colon C_c(G, A) \longrightarrow C_c(G, B)$ via the formula $\varphi_c(au_s) = \varphi(a)u_s$ on generating elements. The corresponding $*$-homomorphism $A \rtimes_{\alpha,r} G \longrightarrow B \rtimes_{\beta,r} G$ extending φ_c is denoted φ_r, whereas the induced $*$-homomorphism $A \rtimes_\alpha G \longrightarrow B \rtimes_\beta G$ is abbreviated into φ_u.

Here are some frequently exploited facts concerning equivariant morphisms. These are applied constantly in the literature with a minimal amount of reference.

Proposition 1.6.1. *The assignments $(A, \alpha, G) \mapsto A \rtimes_{\alpha,r} G$ and $(A, \alpha, G) \mapsto A \rtimes_\alpha G$ from the category of C*-dynamical systems with G-equivariant maps determines a functor assigning to each G-equivariant morphism φ the extensions φ_r and φ_u, respectively.*

We finalize the section with two permanence properties that will be needed later on. We derive these here, since they bear independent interest on their own. The pivotal one is an associativity-esque one: the reduced crossed product commutes with the semidirect product. This has at minimum two benefits; first of all one may often pass some problem into studying a semidirect product instead of successive crossed products, and secondly it is quite pleasing to the eyes.

Proposition 1.6.2. *Suppose (A, α, G) denotes a discrete unital C*-dynamical system. Define the tensor product action $\alpha \otimes \alpha \colon G \times G \curvearrowright A \otimes A$ by letting $(\alpha \otimes \alpha)_{(s,t)} = \alpha_s \otimes \alpha_t$ for all $s, t \in G$. Then one has an isomorphism of C*-algebras*

$$(A \otimes A) \rtimes_{\alpha \otimes \alpha} G^2 \cong (A \rtimes_{\alpha,r} G) \otimes (A \rtimes_{\alpha,r} G).$$

Proof. Fix a faithful representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$. The map $\bar{\pi} = \pi \otimes \pi : A \otimes A \longrightarrow \mathrm{B}(\mathcal{H} \otimes \mathcal{H})$ becomes faithful, hence the integrated form $\bar{\pi}_{\alpha\otimes\alpha,r} \times \bar{\lambda}$ determines $M = (A \otimes A) \rtimes_{\alpha\otimes\alpha,r} G^2$ with $\bar{\lambda}$ denoting the left-regular representation associated to the direct product G^2. On the other hand, the spatial tensor product $N = (A \rtimes_{\alpha,r} G) \otimes (A \rtimes_{\alpha,r} G)$ is fully determined via the representation $(\pi_\alpha \times \lambda) \otimes (\pi_\alpha \times \lambda)$. To ease the notation, we shall identify A with its isomorphic image in $\mathrm{B}(\mathcal{H})$. The dense involutive subalgebra of M is generated by elementary tensors $a \otimes b$ and the canonical unitaries $\{u_{(s,t)}\}_{(s,t)\in G\times G}$ implementing the action $\alpha \otimes \alpha$. Furthermore, the identity

$$\begin{aligned}(\bar{\pi}_\alpha \times \bar{\lambda})((a \otimes b)u_{(s,t)})(\xi \otimes \eta \otimes \delta_{(g,h)}) &= [(\alpha_g \otimes \alpha_h)^*(a \otimes b) \otimes \lambda_{(s,t)}](\xi \otimes \eta \otimes \delta_{(g,h)}) \\ &= \alpha_g^*(a)\xi \otimes \alpha_h^*(b)\eta \otimes \delta_{(sg,th)}\end{aligned}$$

must be valid for all $a, b \in A$, $s, t, g, h \in G$ and $\xi, \eta \in \mathcal{H}$. One hereafter observes that the latter expression belongs to the algebra $\mathrm{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(G \times G))$. Keeping this in mind, consider the unitary operator $v\colon \ell^2(G^2) \longrightarrow \ell^2(G) \otimes \ell^2(G)$ defined by letting $v\delta_{(s,t)} = \delta_s \otimes \delta_t$. The operator evidently fulfills $v^*(\delta_s \otimes \delta_t) = \delta_{(s,t)}$ for every $s, t \in G$, whereupon $\ell^2(G^2)$ may be identified with $\ell^2(G) \otimes \ell^2(G)$. The switch operator, which merely swaps the tensor coordinates, is another unitary, so the latter expression above may be identified with

$$\begin{aligned}(\bar{\pi}_\alpha \times \bar{\lambda})((a \otimes b)u_{(s,t)})(\xi \otimes \eta \otimes \delta_{(g,h)}) &\overset{\cong}{\mapsto} (\alpha_g^*(a)\xi \otimes \delta_{sg}) \otimes (\alpha_h^*(b)\eta \otimes \delta_{th}) \\ &= (\pi_\alpha \times \lambda)(au_s)(\xi \otimes \delta_g) \otimes (\pi_\alpha \times \lambda)(bu_t)(\xi \otimes \delta_h).\end{aligned}$$

The dense subalgebra of N is generated by the set $\{(\pi \times \lambda)(au_s) \otimes (\pi \times \lambda)(bu_t) : a, b \in A,\ s, t \in G\}$ and the right-hand side above must belongs to N, whence any generating element in M corresponds uniquely to a generating element in N, proving the sought isomorphism $M \cong N$ by continuity combined with linearity of the involved maps. ☐

Proposition 1.6.3. *Let (A, α, G) be a discrete unital C*-dynamical system and suppose H is some discrete group acting on G by automorphisms $\varphi\colon H \longrightarrow \mathrm{Aut}(G)$. The action $\beta\colon G \rtimes H \curvearrowright A$ defined by setting $\beta_{(s,t)}(\cdot) = \alpha_{\varphi_t(s)}(\cdot)$ for all $(s,t) \in G \rtimes H$ extends the action α*[11] *and*

(i) there exists an action $\tau\colon H \curvearrowright A \rtimes_\alpha G$ such that $(A \rtimes_\alpha G) \rtimes_\tau H \cong A \rtimes_\beta (G \rtimes H)$;

(ii) there exists an action $\tau\colon H \curvearrowright A \rtimes_{\alpha,r} G$ such that $(A \rtimes_{\alpha,r} G) \rtimes_{\tau,r} H \cong A \rtimes_{\beta,r} (G \rtimes H)$.

Proof. Define a map $\tau\colon H \longrightarrow \mathrm{Aut}(A \rtimes_\alpha G)$ by letting $\tau_t(au_s) = \beta_{(1_G,t)}(a)u_{\varphi_t(s)}$ for every generating element au_s belonging to $A \rtimes_\alpha G$ and every t in H, where the collection $\{u_t\}_{t\in G}$ denotes the generating unitaries in $C_c(G, A)$. Since the assignments $t \mapsto u_t$, $t \mapsto \varphi_t$ and $t \mapsto \beta_{(1_G,t)}$ are homomorphisms, τ is readily shown to be an action provided it attains values $\mathrm{Aut}(A \rtimes_\alpha G)$. We only prove the multiplicativity property of the image point τ_t for any $t \in H$, the involutive part being proven similarly. Let $a, a_0 \in A$ and $s, s_0 \in G$ be given. Then

$$\begin{aligned}\tau_t(au_s a_0 u_{s_0}) &= \tau_t(a\alpha_s(a_0)u_{ss_0}) \\ &= \beta_{(1_G,t)}(a\alpha_s(a_0))u_{\varphi_t(ss_0)} \\ &= \beta_{(1_G,t)}(a\beta_{(s,1_H)}(a_0))u_{\varphi_t(ss_0)} \\ &= \beta_{(1_G,t)}(a)\alpha_{\varphi_t(s)}(a_0)u_{\varphi_t(s)}u_{\varphi_t(s_0)} \\ &= \beta_{(1_G,t)}(a)u_{\varphi_t(s)}\beta_{(1_G,t)}(a_0)u_{\varphi_t(s_0)} \\ &= \tau_t(au_s)\tau_t(a_0u_{s_0}).\end{aligned}$$

[11]In the sense that $\beta_{|G} = \alpha$.

Here the fifth equality is based on $\alpha_{\varphi_t(s)}(\cdot) = u_{\varphi_t(s)}(\cdot)u^*_{\varphi_t(s)}$ for all $s \in G$ and $t \in H$. Therefore the triple $(A \rtimes_\alpha G, \tau, H)$ becomes a discrete unital C*-dynamical system. It remains to be proven that this C*-dynamical systems indeed produces the two isomorphisms in the statements (i)-(ii).

(i) For the sake of simplicity, let $M = (A \rtimes_\alpha G) \rtimes_\tau H$ and $N = A \rtimes_\beta (G \rtimes H)$. Adopting this notation, we shall identify the corresponding C*-algebras in play by their dense subspaces, namely

$$A \rtimes_\alpha G = \overline{\text{span}}\{au_s : a \in A, s \in G\},$$
$$N = \overline{\text{span}}\{aw_g : a \in A, g \in G \rtimes H\},$$
$$M = \overline{\text{span}}\{au_s v_t : a \in A, t \in H, s \in G\},$$

with the collections $\{v_t\}_{t\in H}$ and $\{w_g\}_{g\in G\rtimes H}$ consisting of the canonical generating unitaries implementing the respective actions τ and β. Observe that $\beta_{(1_G,t)}(a) = \tau_t(a) = v_t a v_t^*$ for all a belonging to A whereas $u_{\varphi_t(s)} = \tau_t(u_s) = v_t u_s v_t^*$. Due to the action φ on G translating into $\varphi_t(s) = tst^{-1}$ in $G \rtimes H$ for all s in G and t in H, one may deduce that $w_t w_s w_t^* = w_{\varphi_t(s)}$ holds. Define $\sigma\colon C_c(H, A \rtimes_\alpha G) \longrightarrow C_c(G \rtimes H, A)$ by the assignment $au_s v_t \mapsto aw_{st}$. Now, fasten your seatbelt and notice that combining all this yields

$$\begin{aligned}
\sigma(au_s v_t \cdot a_0 u_{s_0} v_{t_0}) &= \sigma(au_s \tau_t(a_0 u_{s_0}) v_{tt_0}) \\
&= \sigma([au_s \beta_{(1_G,t)}(a_0) u_{\varphi_t(s_0)}] v_{tt_0}) \\
&= \sigma([a\beta_{(s,t)}(a_0)] u_{s\varphi_t(s_0)} v_{tt_0}) \\
&= a\beta_{(s,t)}(a_0) w_s (w_t w_{s_0} w_t^*) w_t w_{t_0} \\
&= a\beta_{(s,t)}(a_0) w_{(s,t)} w_{(s_0,t_0)} \\
&= \sigma(au_s v_t)\sigma(a_0 u_{s_0} v_{t_0}).
\end{aligned}$$

for all $s, s_0 \in G$, $t, t_0 \in H$ and $a, a_0 \in A$. During the third equality, $u_s \beta_{(1_G,t)}(\cdot) = \alpha_s(\cdot)u_s$ being valid for $s \in G$ and $t \in H$ is exploited. Another cumbersome computation shows that σ preserves the involution, hence σ constitutes a $*$-homomorphism, which is clearly an isomorphism. The universal property of the full crossed product entails the sought isomorphism, proving that $N \cong M$.

(ii) Let us maintain the notation in (i) except with the crossed products being reduced. Upon choosing some faithful representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$, we may infer the inclusion $A \subseteq \mathrm{B}(\mathcal{H})$ without loss of generality. Suppose λ^G is the left-regular representation of G, λ^H the one for H and $\bar{\lambda}$ is the one associated to $G \rtimes H$. For every $s, g \in G$, $t, h \in H$, $\xi \in \mathcal{H}$ and every $a \in A$,

$$\begin{aligned}
((\pi_\alpha \times \lambda^G)_\tau \times \lambda^H)(au_s v_t)(\xi \otimes \delta_g \otimes \delta_h) &= [(\pi_\alpha \times \lambda^G)(\tau_{h^{-1}}(au_s))](\xi \otimes \delta_g) \otimes \lambda^H_t \delta_h \\
&= [(\pi_\alpha \times \lambda^G)(\beta_{(1,h^{-1})}(a) u_{\varphi_{h^{-1}}(s)})](\xi \otimes \delta_g) \otimes \delta_{th} \\
&= \alpha^*_g \beta^*_{(1,h)}(a)\xi \otimes \lambda^G_{\varphi_{h^{-1}}(s)} \delta_g \otimes \delta_{th} \\
&= \beta^*_{(g,h)}(a)\xi \otimes \delta_{\varphi_{h^{-1}}(s)g} \otimes \delta_{th}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\pi_\beta \times \bar{\lambda})(aw_{(s,t)})(\xi \otimes \delta_{(g,h)}) &= (\beta^*_{(g,h)}(a) \otimes \bar{\lambda}_{(s,t)})(\xi \otimes \delta_{(g,h)}) \\
&= \beta^*_{(g,h)}(a)\xi \otimes \delta_{(s\varphi_t(g),th)}
\end{aligned}$$

and the unitary $\delta_s \otimes \delta_t \mapsto \delta_{(\varphi_t(s),t)}$ is readily seen to provide the sought identification. □

Proposition 1.6.4. *Suppose (A, α, G) is some unital discrete C*-dynamical system. The associated linear map $E\colon C_c(G, A) \longrightarrow A$ defined on generating elements by $E(\sum_{s\in G} a_s u_s) = a_e$, where u_s denotes the s'th unitary implementing the action α, defines a contractive linear map extending to a faithful conditional expectation $A \rtimes_{\alpha,r} G \longrightarrow A$.*

Proof. See [2, Proposition 4.1.9] for a proof. □

Chapter 2

Quasidiagonality

Quasidiagonality originates from a concept introduced by Halmos in the 1970's, known as block-diagonality of operators acting on Hilbert spaces. This particular notion generalizes the ability to diagonalize ordinary $n \times n$-matrices by capturing essential properties of diagonal matrices. The first chapter seeks to slowly present quasidiagonality in its many formulations, initiating the exposition by discussing block-diagonality vividly and then proceed to addressing the C*-algebraic version. Afterwards, several core equivalent formulations of quasidiagonal C*-algebras will be derived together with permanence propertiers.

2.1 From Block-Diagonality to Quasidiagonal C*-Algebras

The current section attempts to provide some intuition behind block-diagonal operators to the reader prior to rigorously presenting the notion. We shall exhibit the principle of block-diagonality, then attempt to capture the essence by an example. For the record, we adopt the canonical commutator bracket notation, meaning $[a, b] = ab - ba$ for elements in general algebras.

Definition. A bounded operator a acting on a Hilbert space $\mathcal{H}$ is said to be *block-diagonal* if there exists an increasing net $(p_\alpha)_{\alpha\in\Lambda}$ of finite rank projections on $\mathcal{H}$ converging to the identity $1_\mathcal{H}$ in the strong-operator topology such that $\|[a, p_\alpha]\| = 0$ for every index α in Λ. In the event of $\mathcal{H}$ being separable one implicitly demands that a sequence of such projections may be chosen.

In order to obtain some intuition, consider the case where $\mathcal{H} = \ell^2(\mathbb{N})$ endowed with the usual orthonormal basis $\{\delta_n\}_{n\geq 1}$. The family $\{p_n\}_{n\geq 1}$ consisting of orthogonal projections p_n onto the closed subspace spanned by $\{\delta_1, \dots, \delta_n\}$ fulfills $p_n \to 1_\mathcal{H}$ in the strong-operator sense. Let a be a block-diagonal operator acting on $\mathcal{H}$ with respect to $(p_n)_{n\geq 1}$. The triangle inequality effortlessly entails $\|[a, p_n - p_{n-1}]\| = 0$ while another computations reveals that $p_{n+1} - p_n \perp p_n - p_{n-1}$ for every positive integer n, hence we obtain a decomposition $\mathcal{H} = \bigoplus_n (p_n - p_{n-1})\mathcal{H}$ wherein $p_0 = 0$. This yields the following matrix picture of a:

$$a = \begin{pmatrix} ap_1 & & & \\ & a(p_2 - p_1) & & \\ & & a(p_3 - p_2) & \\ & & & \ddots \end{pmatrix}.$$

Inspired by this, we interpret T as being “block-diagonal”. In general one cannot expect operators to be block-diagonal and therefore one modifies the notion by transforming into quasidiagonality.

Definition. An element a inside $\mathrm{B}(\mathcal{H})$ is said to be *quasidiagonal* provided there exists an increasing net $(p_\alpha)_{\alpha\in\Lambda}$ consisting of finite rank projections converging to the identity $1_{\mathcal{H}}$ in the strong operator topology and fulfilling $\|[a,p_\alpha]\| \to 0$. A collection Ω of elements in $\mathrm{B}(\mathcal{H})$ is said to be *quasidiagonal* if such an increasing net $(p_\alpha)_{\alpha\in\Lambda}$ of finite rank projections satisfying $\|[a,p_\alpha]\| \to 0$ for all $a \in \Omega$ exists. We implicitly require a sequence of such projections to exist provided that $\mathcal{H}$ is separable.

Our first objective will be to "localize" the notion of quasidiagonality, thus revealing the local nature of quasidiagonality as opposed to its arguably more global appearance. For the record, we shall temporarily refer to the following property as being locally quasidiagonal for emphasis for now, in spite of this becoming redundant in due time.

Definition. A collection Ω of elements in $\mathrm{B}(\mathcal{H})$ is referred to as being *locally quasidiagonal* if for all $\varepsilon > 0$, every finite $\mathcal{F} \subseteq \Omega$ and every finite $\mathcal{N} \subseteq \mathcal{H}$ there exists a finite rank projection p acting on $\mathcal{H}$ subject to $\|[a,p]\| < \varepsilon$ together with $\|p\xi - \xi\| < \varepsilon$ for all $\xi \in \mathcal{N}$ and $a \in \mathcal{F}$.

Obviously, the ordinary definition of quasidiagonality must imply the local one. However, proving equivalence is slightly technical, so we shall isolate a perturbation trick at first. Using the trick we may derive that quasidiagonality and its local cousin are the same. However, we shall only prove the separable version, leaving changing various passages into the net-lingo to you.

Lemma 2.1.1. *Suppose A denotes a unital C*-algebra containing two projections p and q.*

(i) If $\|p-q\| < 1$, then there exist partial isometries ν and μ inside A such that $u = \nu+\mu$ defines a unitary in A satisfying $uqu^* = p$ and $\|1_A - u\| \leq 4\|p-q\|$.

(ii) If $\|pq - q\| < 1/4$, then there exist partial isometries ν and μ inside A such that $u = \nu + \mu$ defines a unitary in A satisfying $uqu^* \leq p$ and $\|1_A - u\| \leq 10\|p-q\|$.

Proof. **(i)** Let $a = pq$ throughout the entire proof and notice that $\|a^*a - q\| = \|q(p-q)q\| < 1$ by hypothesis. The involutive algebra qAq has q as the unit, whereof a^*a must be invertible in qAq[1]. The corresponding positive square root $|a|_q^{-1}$ therefore becomes an invertible positive element in qAq. Setting $\nu = a|a|_q^{-1}$ yields the polar decomposition $a = \nu|a|_q$ within qAq. Furthermore,

$$\nu^*\nu = |a|_q^{-1}(a^*a)|a|_q^{-1} = |a|_q^{-1}|a|^2|a|_q^{-1} = q \quad \text{and} \quad \nu\nu^* = a|a|_q^{-2}a^* = p(q|a|_q^{-2}q)p \in pAp$$

Through these relations we infer that $\nu^*\nu = q$ and $\nu\nu^* \leq p$. In an analogue fashion, aa^* becomes invertible in pAp because of the estimate $\|aa^* - p\| = \|p(q-p)p\| < 1$ while being a subject to $\nu\nu^* \geq p$, whence $\nu\nu^* = p$ follows. We proceed to estimating ν; observe that

$$\||a| - q\| \leq \|a^*a - q\| = \|qpq - q^3\| \leq \|p - q\|.$$

In conjunction with $\nu q = \nu$, the above yields the estimate

$$\|q - \nu\| \leq \|q - a\| + \|\nu|a| - \nu\| = \|(q-p)q\| + \|\nu(|a| - q)\| < 2\|p - q\| \tag{2.1}$$

Defining $b = qp$ and applying the same argument to the orthogonal projections, we may determine a partial isometry μ for which $q^\perp = \mu^*\mu$ and $p^\perp = \mu\mu^*$. Moreover, the very same approach as above easily implies that $\|q^\perp - \mu\| \leq 2\|p-q\|$. Routine calculations reveal that $u = \nu + \mu$ must be a unitary in A, so the assertion stems from (2.1) granting $\|1_A - u\| \leq \|q-\nu\| + \|q^\perp - \mu\| \leq 4\|p-q\|$ combined with the following calculation. Consider the expansion

$$\begin{aligned} uqu^* = (\nu+\mu)q(\nu^*+\mu^*) &= \nu q\nu^* + \nu q\mu^* + \mu q\nu^* + \mu q\mu^* \\ &= \nu\nu^* + \nu\mu^* + \mu\nu^* + \mu\nu^*\nu\mu \\ &= p + \nu\mu^* + \mu\nu^* + \mu\nu^*\nu\mu. \end{aligned}$$

[1] In a unital C*-algebras, any element a of distance strictly smaller than 1 from the identity turns $1_A - a$ invertible.

According to the rearrangement, verifying that $\mu\nu^* = \mu^*\nu = 0$ will grant us $uqu^* = p$. However, due to μ, ν being partial isometries combined with $q = \nu^*\nu$, $q^\perp = \mu^*\mu$, we may hereof deduce that $\mu\nu^* = \mu q^\perp \nu^* = \mu\nu^* - \mu q\nu^* = 0$. Similarly, $\mu^*\nu = 0$, completing the proof of part (i).

(ii) Abbreviate $\varepsilon = \|q - pq\| < 1/4$ and let $b = pq$. Due to $\|q - b^*b\| = \|q^2 - qpq\| = \varepsilon < 1$, the positive element b^*b admits an invertible positive square root $|b|_q^{-1}$ in qAq such that $\nu := b|b|_q^{-1}$ defines a partial isometry fulfilling $b = \nu|b|$, $q = \nu^*\nu$ and $p_0 = \nu\nu^* \leq p$. Moreover, $\|q - \nu\| \leq 2\varepsilon$ via an argument completely similar to the one in (i), thereby entailing

$$\begin{aligned}\|q - \nu\nu^*\| \leq \|q - \nu\| + \|\nu - \nu\nu^*\| &\leq \|q - \nu\| + \|\nu\nu^*\nu - \nu\nu^*\| \\ &\leq \|q - \nu\| + \|q - \nu\| \\ &\leq 4\varepsilon.\end{aligned}$$

In particular, we obtain $\|q^\perp - p_0^\perp\| \leq 4\varepsilon < 1$. Part (i) thus applies to produce a partial isometry μ fulfilling $q^\perp = \mu^*\mu$ together with $p_0^\perp = \mu\mu^*$. This must satisfy $\|q^\perp - \mu\| \leq 2\|q^\perp - p_0^\perp\| \leq 8\varepsilon$ according to the exact same estimate as in part (i). Finally, letting $u = \nu + \mu$ provides us with a unitary in A subject to $uqu^* \leq p$ and $\|1_A - u\| \leq \|q - \nu\| + \|q^\perp - \mu\| \leq 10\varepsilon$. □

Proposition 2.1.2. *A separable $\Omega \subseteq \mathrm{B}(\mathcal{H})$ is quasidiagonal if and only if Ω is locally quasidiagonal.*

Proof. Quasidiagonality implying localy quasidiagonality is obviously true whereas the converse is far more troublesome to handle. Let $\mathcal{N} \subseteq \mathcal{H}$, $F \subseteq \Omega$ be fixed finite subsets and let some $\delta > 0$ be given (to be specified later on). The overall idea revolves around considering the orthogonal projection onto the closed linear space K spanned by $\mathcal{N}$ and extract a suitable projection from its unit ball, then fix whatever gap may arise.

The unit ball of K must be compact by finiteness of $\mathcal{N}$, so the open cover consisting of the open balls $B(\xi, \delta)$ with $\xi \in (K)_1$ admits a finite subcover. As such there exists a finite collection $F' = \{\xi_1, \ldots, \xi_m\} \subseteq (K)_1$ such that the corresponding collection of open balls $B(\xi_k, \delta)$ for $1 \leq k \leq m$ cover $(K)_1$. In particular, whenever p is a finite rank projection satisfying $\|p\eta - \eta\| < \delta$ for vectors $\eta \in F'$, then for every unit vector $\xi \in K$ we have

$$\|pq\xi - q\xi\| \leq \|q\xi - \xi'\| + \|\xi - \xi'\| + \|pq\xi - \xi\| \leq 3\delta,$$

where q is the orthogonal projection onto K and ξ' is some vector in F' fulfilling $\|q\xi - \xi'\| < \delta$ together with $\|\xi - \xi'\| < \delta$, whose existence stems from the covering of $(K)_1$. Upon Ω being locally quasidiagonal, such a projection p exists including the approximation property $\|[p, a]\| < \delta$ for every $a \in F$. If we specify δ slightly by demanding $\delta < 1/4$, we may invoke lemma 2.1.1 to find a unitary u in $\mathrm{B}(\mathcal{H})$ subject to $q \leq upu^*$ and $\|1 - u\| \leq 30\delta$. Thus,

$$\|p - upu^*\| = \|pu^* + p - pu^* - upu^*\| \leq \|(1 - u)pu^*\| + \|p(1 - u^*)\| \leq 60\delta. \tag{2.2}$$

Define hereof a projection q by setting $e := upu^*$. We assert that e is the sought projection. To see this, one continues to compute; for each element a inside F one has

$$\|upu^*a - aupu^*\| \leq \|upu^* - p\| \cdot \|a\| + \|a\| \cdot \|upu^* - p\| + \|[a, p]\| \overset{(2.2)}{<} \delta(120\|a\| + 1).$$

For the latter condition, suppose ξ belongs to $\mathcal{N}$ and let ξ' be an element of F' making $\|q\xi - \xi'\| < \delta$ together with $\|\xi - \xi'\| < \delta$ valid. Under these premises, using that $q \leq p$ one obtains

$$\|e\xi - \xi\| \leq \|q\xi - \xi\| \leq \|q\xi - \xi'\| + \|\xi - \xi'\| \leq 2\delta.$$

Since δ was not specified further than being strictly smaller than $1/4$, we can make it sufficiently small to force the two preceding estimates strictly smaller than ε. One could for instance choose $\delta > 0$ small enough to force $\delta < \varepsilon/(4\delta(120M + 1))$ where $M = \max\{\|a\| : a \in F\}$. Voila. □

Having established the local appearance of quasidiagonality, we proceed towards the central concept in this project: quasidiagonal C*-algebras. The definition is quite natural in the sense that every C*-algebra corresponds to an isomorphic of a subalgebra in $\mathrm{B}(\mathcal{H})$ due to the GNS-construction, wherein quasidiagonality becomes meaningful.

Definition. A representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ of some C*-algebra is *quasidiagonal* if its image $\pi(A)$ defines a quasidiagonal subcollection of $\mathrm{B}(\mathcal{H})$. The C*-algebra A is referred to as being *quasidiagonal* if a quasidiagonal faithful representation of A exists.

As usual, it is convenient to adapt invariants of C*-algebras to approximation-esque versions. The characterization is due to Voiculescu and we aim towards deriving one of his key results concerning quasidiagonality: the aforementioned notion coincides with the approximation natured one. Proving this, however, is quite involved and we require the aid of another involved theorem of Voiculescu. Its statement alone is quite difficult to understand, so we introduce or perhaps recall some concepts regarding representations.

Definition. Two representations $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ and $\varrho\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ of a C*-algebra A are said to

- *be approximately unitarily equivalent* if there exists a sequence $(u_n)_{n\geq 1}$ of unitaries $u_n\colon \mathcal{H} \longrightarrow \mathcal{K}$ fulfilling that for every a belonging to A one has $\|\varrho(a) - u_n\pi(a)u_n^*\| \to 0$ as $n \to \infty$;
- *be approximately unitarily relative to the compacts* if π and ϱ are approximately unitarily equivalent via unitaries $u_n\colon \mathcal{H} \longrightarrow \mathcal{K}$ such that $\varrho(a) - u_n\pi(a)u_n^*$ is compact for each a in A.

Furthermore, a representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ is called *essential* provided that its image in $\mathrm{B}(\mathcal{H})$ contains no nonzero compact operators, that is, $\pi(A) \cap \mathrm{K}(\mathcal{H}) = \{0\}$.

Remark. Every representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ of a C*-algebra A gives rise to an essential representation by the infinite inflation meaning the $*$-homomorphism $\pi_\infty\colon A \longrightarrow \bigoplus_n \mathrm{B}(\mathcal{H}) \cong \mathrm{B}(\bigoplus_n \mathcal{H})$ defined by $A \mapsto (\pi(a))_{n\geq 1}$. This is quite clear from the outset, since, loosely speaking, an element of the form $(\pi(a))_{n\geq 1}$ cannot occur as a norm-limit of finite rank operators.

The deep results due to Voiculescu are the following three. For the record, we omit proving the two first and confine ourselves with proving the third, being the variation we shall deploy.

Theorem 2.1.3 (Voiculescu). *Suppose $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and suppose A denotes a separable C* subalgebra in $\mathrm{B}(\mathcal{H})$ having $1_{\mathcal{H}}$ as a unit. If $\iota\colon A \hookrightarrow \mathrm{B}(\mathcal{H})$ denotes the inclusion and $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ is any unital representation whose restriction $\pi_{|A\cap K(\mathcal{H})}$ is trivial, then ι must be approximately unitarily equivalent to $\iota \oplus \pi\colon A \longrightarrow \mathrm{B}(\mathcal{H}) \oplus \mathrm{B}(\mathcal{K})$ relative to the compacts.*

Corollary 2.1.4. *Let A be a unital C*-algebra admitting a pair of faithful essential representations $\pi_1, \pi_2\colon A \longrightarrow \mathrm{B}(\mathcal{H})$. Under these premises, π_1 and π_2 become approximately unitarily equivalent relative to the compacts.*

As promised, Voiculescu's theorems tend to have difficult statements. The one we shall invoke later on is no exception. In fact, we must introduce another technical term regarding contractive completely positive maps into $\mathrm{B}(\mathcal{H})$ whose sole obstruction towards becoming full-fledged $*$-homomorphisms is $\mathrm{K}(\mathcal{H})$. For the record, $\mathrm{Q}(\mathcal{H})$ denotes the Calkin algebra associated to $\mathcal{H}$.

Definition. Let $q\colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathrm{Q}(\mathcal{H})$ be the quotient map and A some unital C*-algebra. A unital completely positive map $\varphi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ is referred to as being *a representation modulo the compacts* provided that $q \circ \varphi\colon A \longrightarrow \mathrm{Q}(\mathcal{H})$ becomes a $*$-homomorphism. We call φ a *faithful representation modulo the compacts* if $q \circ \varphi$ is faithful.

Theorem 2.1.5 (Voiculescu)**.** *Suppose A denotes a unital separable C^*-algebra admitting a faithful separable representation $\varphi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ modulo the compacts. Let $\eta_\varphi\colon A \longrightarrow \mathbb{R}^+$ be the map*

$$\eta_\varphi(a) = 2\max\{\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2}\}.$$

If so, any faithful unital separable essential representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ admits a sequence $(u_n)_{n\geq 1}$ consisting of unitaries $u_n\colon \mathcal{H} \longrightarrow \mathcal{K}$ subject to

$$\limsup_{n\to\infty} \|\pi(a) - u_n\varphi(a)u_n^*\| \leq \eta_\varphi(a)$$

for every element a belonging to A.

Proof. The reader is kindly asked to humor the following train of thought. Suppose we had established the existence of a representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ together with unitaries $u_n\colon \mathcal{H} \longrightarrow \mathcal{K}$ satisfying the conditions of the theorem. Any additional faithful essential representation $\varrho\colon A \longrightarrow \mathrm{B}(\mathcal{H}')$ will be approximately unitarily equivalent to π relative to the compacts according to *corollary* 2.1.4, say via unitaries $v_n\colon \mathcal{K} \longrightarrow \mathcal{H}'$. The sequence $(w_n)_{n\geq 1}$ of unitaries $w_n = v_n u_n\colon \mathcal{H} \longrightarrow \mathcal{H}'$ hereof satisfies the sought bound

$$\limsup_{n\to\infty} \|\varrho(a) - w_n\varphi(a)w_n^*\| \leq \limsup_{n\to\infty}(\|\varrho(a) - v_n\pi(a)v_n^*\| + \|v_n\pi(a)v_n^* - w_n\varphi(a)w_n^*\|) \leq \eta_\varphi(a)$$

Thus the proof amounts to verifying the existence of such a representation π. To achieve existence, suppose $(\sigma, V, \mathcal{L})$ denotes the unital Stinespring dilation corresponding to the u.c.p map φ, that is, the map $\sigma\colon A \longrightarrow \mathrm{B}(\mathcal{L})$ is a representation and $v\colon \mathcal{H} \longrightarrow \mathcal{L}$ is the bounded operator witnessing φ by $\varphi(\cdot) = v^*\sigma(\cdot)v$ for all a inside A. Write $p = vv^*$ to obtain a projection and observe that

$$\begin{aligned}(p^\perp\sigma(a)p)^*(p^\perp\sigma(a)p) &= p\sigma(a^*)p^\perp\sigma(a)p\\ &= vv^*\sigma(a^*a)vv^* - vv^*\sigma(a^*)vv^*\sigma(a)vv^*\\ &= v\varphi(a^*a)v^* - v\varphi(a^*)\varphi(a)v^*.\end{aligned}$$

for all $a \in A$. This grants us the estimate

$$\|p^\perp\sigma(a)p\| \leq \|\varphi(a^*a) - \varphi(a)\varphi(a^*)\|^{1/2} \leq 2^{-1}\eta_\varphi(a). \tag{2.3}$$

Decompose $\mathcal{L}$ into the direct sum $\mathcal{L} = p\mathcal{H} \oplus p^\perp\mathcal{H}$. The bounded operator $\sigma(a)$ acting on $\mathcal{L}$, when viewed as a bounded operator acting on the above decomposition, has the matrix form

$$M_\sigma(a) := \begin{pmatrix} p\sigma(a) & p\sigma(a)p^\perp \\ p^\perp\sigma(a)p & p^\perp\sigma(a)\end{pmatrix}.$$

This may be readily verified by a straightforward computation. Let $N_\sigma(a)$ denote the associated matrix arising from $M_\sigma(a)$ by deleting the diagonal parts of $M_\sigma(a)$. One evidently has

$$\|N_\sigma(a)\| = \max\{\|p^\perp\sigma(a)p\|, \|(p^\perp\sigma(a)p)^*\|\} \overset{(2.3)}{\leq} 2^{-1}\eta_\varphi(a). \tag{2.4}$$

We have arrived at the vital part of the proof. The main idea is arguably to construct matrices whose difference becomes a matrix appearing as a block matrix having reoccurring copies of N_σ along the diagonal. Due to demanding essential representations, we ought to make infinite copies of the already established representation, upon which the remark on page 27 arrives to our aid. To this end, consider the Hilbert space rearrangement[2]

$$p\mathcal{L} \oplus \left(\bigoplus_{n=1}^\infty p^\perp\mathcal{L} \oplus p\mathcal{L}\right) = \bigoplus_{n=1}^\infty (p\mathcal{L} \oplus p^\perp\mathcal{L}) = \bigoplus_{n=1}^\infty \mathcal{L}.$$

[2]the "identification" simply being the choice of where the parentheses start.

Keeping the preceding in mind, define $\varrho_\infty := \bigoplus_n \varrho$ and $\sigma_\infty = \bigoplus_n \sigma$ where $\varrho(a)$ is the bounded operator having the following matrix form for all $a \in A$:

$$M_\varrho(a) = \begin{pmatrix} \sigma(a)_{22} & \sigma(a)_{21} \\ \sigma(a)_{12} & \sigma(a)_{11} \end{pmatrix}.$$

Herein $\sigma(a)_{ij}$ is the (i,j)'th entry occurring in the matrix $M_\sigma(a)$. The operator $\sigma(a)_{11} \oplus \varrho_\infty(a)$ may be regarded as the matrix

$$T = \begin{pmatrix} \sigma(a)_{11} & 0 & 0 & \cdots \\ 0 & M_\varrho(a) & 0 & \cdots \\ 0 & 0 & M_\varrho(a) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

whereas $\sigma_\infty(a)$ has as associated matrix the one with copies of $M_\sigma(a)$ along the diagonal. Under the matrix picture, the corresponding matrix of $\theta(\cdot) = \sigma(\cdot)_{11} \oplus \varrho_\infty(\cdot) - \sigma_\infty(\cdot)$ becomes the block-matrix having copies of $N_\sigma(a)$ along the diagonal with alternating signs, starting with the negative alteration, i.e., $-N_\sigma(a)$. More vividly described:

$$M_\theta = \begin{pmatrix} 0 & -\sigma(a)_{21} & 0 & 0 & \cdots \\ -\sigma(a)_{12} & 0 & \sigma(a)_{21} & 0 & \cdots \\ 0 & \sigma(a)_{12} & 0 & -\sigma(a)_{21} & \cdots \\ 0 & 0 & -\sigma(a)_{12} & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Splitting M_θ into the sum of two diagonal parts and applying the triangle inequality ensures that

$$\|\sigma(a)_{11} \oplus \varrho_\infty(a) - \sigma_\infty(a)\| \le 2\|N_\sigma(a)\| \overset{(2.2)}{\le} \eta_\varphi(a) \tag{2.5}$$

for each element a in A. This matrix trick has given us a potential candidate and the assumptions imposed on φ will reveal themselves to reach our end goal. Let us abbreviate $B = \varphi(A) + \mathrm{K}(\mathcal{H})$ and note that due to φ being a faithful separable representation modulo the compacts, B becomes a unital separable C*-algebra contained in $\mathrm{B}(\mathcal{H})$. If q denotes the canonical quotient map $\mathrm{B}(\mathcal{H}) \longrightarrow \mathrm{Q}(\mathcal{H})$ of the Calkin algebra restricted to B, then one has an isomorphism $q(B) \cong A$. According to theorem 2.1.3, the inclusion $\iota\colon B \hookrightarrow \mathrm{B}(\mathcal{H})$ is unitarily equivalent to the representation $\iota \oplus (\varrho_\infty \circ q)$ relative to the compacts. So there are unitaries $w_n\colon \mathcal{H} \longrightarrow \mathcal{H} \oplus (\bigoplus_n p^\perp \mathcal{L} \oplus p\mathcal{L})$ fulfilling

$$\|\iota(b) \oplus (\varrho_\infty \circ q)(b) - w_n \iota(b) w_n^*\| \to 0.$$

for every $b \in B$. The identifications $B \cong \iota(B)$ and $q(B) \cong A$ permit us to replace $\iota(b)$ above with $\varphi(a)$ while viewing the image of q as elements in A, meaning we may replace $q(b)$ with an element in A above. This in turn allows us to assume, without loss of generality, that

$$\|\varphi(a) \oplus \varrho_\infty(a) - w_n \varphi(a) w_n^*\| \to 0 \tag{2.6}$$

for every element a inside A. Our candidate for a representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ and sequence $(u_n)_{n\ge1}$ of unitaries $u_n\colon \mathcal{H} \longrightarrow \bigoplus_n \mathcal{L}$ will be the representation $\pi = \sigma_\infty$, the separable Hilbert space $\mathcal{K} = \bigoplus_n \mathcal{L}$ and as unitary $u_n = (v \oplus 1)w_n$ for all $n \in \mathbb{N}$. The remainder of the proof amounts to making some computations. Indeed the expression $v^*\sigma(\cdot)v = \varphi(\cdot)$ rearranges into $p\sigma(\cdot) = v\varphi(\cdot)v^*$ when conjugating by v, whereof

$$(v \oplus 1)[\varphi(a) \oplus \varrho_\infty(a)](v \oplus 1)^* = \sigma(a)_{11} \oplus \varrho_\infty(a)$$

must be true for all $a \in A$, thereby entailing

$$\begin{aligned}\limsup_{n\to\infty} \|\pi(a) - u_n\varphi(a)u_n^*\| &= \limsup_{n\to\infty} \|\sigma_\infty(a) - (v \oplus 1)w_n\varphi(a)w_n^*(v \oplus 1)^*\| \\ &\le \limsup_{n\to\infty} (\|\sigma_\infty(a) - \sigma(a)_{11} \oplus \varrho_\infty(a)\| + \|(\varphi(a) \oplus \varrho_\infty(a)) - w_n\varphi(a)w_n^*\|\end{aligned}$$

which is bounded by $\eta_\varphi(a)$ on the merits of (2.4) combined with (2.5). This completes the proof. □

We conclude our smaller detour and return to quasidiagonality. Firstly, the abstract approximation characterization of quasidiagonal C*-algebras. We temporarily invent our own name for the notion until the equivalence of characterizations have been verified, although Voiculescu apparently did not make one himself. As a remark, the proof works in the non-separable case. However, the proof is essentially the same modulo technical variations with nets. We settle with the separable case, since this will be sufficient for our purposes.

Definition. A C*-algebra A is said to be *abstractly quasidiagonal* should there exist a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ consisting of contractive completely positive maps $\varphi_\alpha \colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ which is *asymptotically multiplicative* meaning $\|\varphi_\alpha(ab) \to \varphi_\alpha(a)\varphi_\alpha(b)\|$ and *asymptotically isometric* meaning $\|\varphi_\alpha(a)\| \to \|a\|$ for all $a, b \in A$. In the separable case, one requires the net be a sequence instead.

At long last, we may truly determine the various approximation characterizations of quasidiagonality. Furthermore, we shall simply refer to quasidiagonality as being either of the equivalent versions after the proof, perhaps specifying later on if deemed necessary. The proof is a collaboration of several participants although the credit is commonly granted to Voiculescu. Without further ado:

Theorem 2.1.6. *Let A be a separable unital C*-algebra. Under this hypothesis, the following conditions are equivalent.*

(i) A is quasidiagonal.

(ii) A is abstractly quasidiagonal.

(iii) A is abstractly quasidiagonal via unital completely positive maps.

(iv) Every faithful unital essential separable representation of A must be quasidiagonal.

Proof. The implications (iv) ⇒ (i) and (iii) ⇒ (ii) are trivial, so we need only verify the train of implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) to complete the proof.

(i) ⇒ (ii): The proof of this implication does not revolve around how to cook up the desired c.c.p maps, but why the only choice one can make works. Suppose $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$ denotes the quasidiagonal faithful separable quasidiagonal representation. Let $(p_n)_{n\ge1}$ be a sequence of finite rank operators converging to the identity in the strong operator sense such that $\|[p_n, \pi(a)]\| \to 0$ as $n \to \infty$ for all a in A. The naive approach is to simply compresses the image of $\pi(a)$ via the existing projections. This yields c.c.p maps $\psi_n \colon A \longrightarrow p_n\mathrm{B}(\mathcal{H})p_n \cong \mathbb{M}_{k(n)}$ defined in terms of the assignments $a \mapsto p_n\pi(a)p_n$ for all positive integers n. The desired result hereby stems from

$$\begin{aligned}\|p_n\pi(a)\pi(b)p_n - p_n\pi(a)p_n\pi(b)p_n\| &= \|p_n(p_n\pi(a) - \pi(a)p_n)\pi(b)p_n\| \\ &\le \|[p_n, \pi(a)]\| \cdot \|b\| \to 0.\end{aligned}$$

being valid for all $a, b \in A$ in conjunction with π being faithful entailing that, for all $a \in A$, one has

$$\|p_n\pi(a)p_n - \pi(a)\| \le \|p_n\pi(a)(p_n - 1_\mathcal{H})\| + \|(p_n - 1_\mathcal{H})\pi(a)\| \to 0.$$

(ii) ⇒ (iii). This is possibly the least interesting, although important, of the implications while having a tedious amount of technical joggling using the continuous functional calculus. Ergo, the level of details might seem low. Let $\psi_n \colon A \longrightarrow \mathbb{M}_{k(n)}$ be the sequence of asymptotically isometric - and multiplicative c.c.p maps associated to A. Plainly, one may assume that

$$\|\psi_n(1_A)^2 - \psi_n(1_A)\| < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{2.7}$$

According to the continuous functional calculus, one may deduce that the spectrum of $\psi_n(1_A)$ must be contained in $[0,1]$ since ψ_n is positive and contractive[3]. The spectral elements in fact satisfy a more restrictive condition: due to the containment

$$\sigma(\psi_n(1_A)^2 - \psi_n(1_A)) \subseteq \sigma(\psi_n(1_A))^2 - \sigma(\psi_n(1_A))$$

combined with (2.7), any spectral element λ of $\psi_n(1_A)$ must satisfy $|\lambda^2 - \lambda| < 1/n$ for all positive integers n. Hence the spectrum of $\psi_n(1_A)$ must belong to $[0,1/n) \cup (1-1/n,1]$. Given any bounded interval $I \subseteq \mathbb{R}$, let χ_I denote the indicator map supported on I. If $f \colon \sigma(\psi_n(1_A)) \longrightarrow \{0,1\}$ is the mapping $\chi_{[1/2,1]}$ restricted to the closed set $I_n := \sigma(\psi_n(1_A))$, then for every $t \in I_n$ one has

$$|t - f(t)| = |t-1|\chi_{[1/2,1]} + |t|\chi_{[0,1/2)} < \frac{\chi_{I_n}}{n} < \frac{1}{n}.$$

The isometric property of the continuous functional calculus thus yields $||\psi_n(1_A) - f(\psi_n(1_A))|| < 1/n$ for all positive integers n. Letting $p_n := f(\psi_n(1_A))$ provides a projection such that

$$\|\psi_n(1_A)p_n - p_n\| \leq \|p_n\| \cdot \|\psi_n(1_A) - p_n\| \to 0.$$

The element $\psi_n(1_A)p_n$ thus becomes invertible in the matrix algebra $p_n\mathbb{M}_{k(n)}p_n$, hence its positive square root admits an inverse herein and another routine calculations will reveal that its distance from p_n tends to zero as n tends to infinity. The obtained sequence of unital completely positive $\varphi_n \colon A \longrightarrow \mathbb{M}_{k(n)}$ given by the map

$$\varphi_n(\cdot) = (\psi_n(1_A)p_n)^{1/2}\psi_n(\cdot)(\psi_n(1_A)p_n)^{1/2}$$

therefore works. We omit presenting the latter two computations and leave them to the reader.

(iii) ⇒ (iv) We have reached the critical point of the proof, wherein we must invoke Voiculescu's theorem. Suppose $(\varphi_n)_{n\geq 1}$ is a sequence consisting of asymptotically isometric - and multiplicative unital c.p maps $\varphi_n \colon A \longrightarrow \mathbb{M}_{k(n)}$. Define accordingly a c.p map $\pi \colon A \longrightarrow \ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N})$ by identifying the codomain by $\mathrm{B}(\mathcal{H})$ for $\mathcal{H} = \bigoplus_n \mathbb{C}^{k(n)}$ and taking the infinite inflation map. Let q_n denote the orthogonal projection of $\mathcal{H}$ onto its n'th component. By construction, the projection q_n witnesses φ_n in the sense that $\varphi_n(\cdot) = q_n\pi(\cdot)q_n$ for all $n \in \mathbb{N}$. Recall that an operator a acting on the product $\ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N})$ is compact if and only if $\|q_naq_n\| \to 0$ as n tends to infinity. As such, if $\pi(a)$ is compact for some element a in A, then $\|\varphi_n(a)\| \to 0$ and thereof $\|a\| = 0$ due to $(\varphi_n)_{n\geq 1}$ being asymptotically isometric. Furthermore,

$$\|q_n(\pi(ab) - \pi(a)\pi(b))q_n\| = \|q_n\pi(ab)q_n - q_n\pi(a)\pi(b)q_n\| = \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$$

is true for all elements $a, b \in A$. As such $\pi(ab) - \pi(a)\pi(b)$ must be compact for all $a, b \in A$, which in combination with the preceding observations imply that π must be a faithful representation modulo the compacts. The main idea behind π revolves around its rather explicit nature while allowing us to invoke Voiculescu's theorem, thereby permitting us to perturb its corresponding projection onto the n'th components via unitaries in the following manner. Suppose $\varrho \colon A \longrightarrow \mathrm{B}(\mathcal{K})$ is some faithful essential separable representation and let $\varepsilon > 0$ together with finite subsets $F \subseteq A$ and $\mathcal{N} \subseteq \mathcal{K}$ be

[3] More precisely, we apply the spectral mapping theorem here.

fixed. Upon $q_n \perp q_{n+1}$ for every positive integer n, the sum $p_n = \sum_{i=1}^n q_n$ becomes a projection readily verified to fulfill that $p_n \to 1_{\mathcal{H}}$ in the strong operator topology while satisfying $p_n\pi(a) = \pi(a)p_n$ for all $a \in A$ and $n \in \mathbb{N}$. These projections are the natural candidates to force the norm $\|[\varrho(a), p_n]\|$ to become small. This particular choice tends to fail, so we must perturb our representation ϱ slightly and this is where Voiculescu's theorem enters the scene: Choose a positive integer N large enough to ensure

$$\|\varphi_n(aa^*) - \varphi_n(a)\varphi(a^*)\|^{1/2} < \varepsilon/2 \text{ and } \|\varphi_n(a^*a) - \varphi_n(a^*)\varphi_n(a)\|^{1/2} < \varepsilon/2. \tag{2.8}$$

for every n exceeding N. Voiculescu's theorem, meaning theorem 2.1.5, permits us to determine some unitary $v\colon \mathcal{H} \longrightarrow \mathcal{K}$ satisfying

$$\|\varrho(a) - v\pi(a)v^*\| \leq \eta_\pi(a) \overset{(2.8)}{<} \varepsilon \tag{2.9}$$

The remainder of the proof is simply to perturb the projections p_n by setting $e_n := vp_nv^*$, for then the aforementioned relations between the family of projections p_n and point images $\pi(a)$ combined with (2.9) permit us to infer that

$$\begin{aligned}\|e_n\varrho(a) - \varrho(a)e_n\| &= \|vp_nv^*\varrho(a) - \varrho(a)vp_nv^*\| \\ &\leq \|vp_nv^*\varrho(a) - vp_n\pi(a)v^*\| + \|v\pi(a)p_nv^* - \varrho(a)vp_nv^*\| \\ &\leq \|v^*\varrho(a) - \pi(a)v^*\| + \|v\pi(a) - \varrho(a)v\| \\ &= \|v^*\varrho(a) - v^*v\pi(a)v^*\| + \|v\pi(a)v^*v - \varrho(a)v\| \\ &< \varepsilon\end{aligned}$$

whenever n is sufficiently large and every $a \in F$. Lastly, we have

$$\|e_n\xi - \xi\| = \|vp_nv^* - vv^*\xi\| = \|(p_n - 1_{\mathcal{H}})(v^*\xi)\| \to 0$$

for all $\xi \in \mathcal{N}$, completing the proof. □

2.2 Properties, Examples and Homotopy Invariance

Having established the notion of quasidiagonal C*-algebras in various disguises, we continue the survey by deriving a few basic permanence properties. None of these properties are particularly powerful except for the homotopy invariance, which is an indispensable tool in the quasidiagonality arsenal. However, we shall begin with justifying that assuming a C*-algebra to be unital when considering quasidiagonality is relatively harmless.

Proposition 2.2.1. *The unitalization of any quasidiagonal C*-algebra is quasidiagonal.*

Proof. Suppose $(\varphi_\alpha)_{\alpha\in\Lambda}$ is the assumed net asymptotically isometric - and multiplicative contractive completely positive maps $\varphi_\alpha\colon A \longrightarrow \mathbb{M}_{n(\alpha)}$. Due to $\mathbb{M}_{n(\alpha)}$ being unital, we may extend each φ_α to a unital completely positive map $\psi_\alpha\colon A^+ \longrightarrow \mathbb{M}_{n(\alpha)}$ by setting $\psi_\alpha(a + \lambda 1_{A^+}) = \varphi_\alpha(a) + \lambda 1_n$; consult for instance [2, proposition 2.2.1] for a proof of this fact. Due to

$$\begin{aligned}&\|\psi_\alpha(a + \lambda 1_{A^+})\psi_\alpha(b + \mu 1_{A^+}) - \psi(ab + \lambda b + \mu a + \lambda\mu 1_{A^+})\| \\ &= \|(\varphi_\alpha(a) + \lambda 1_{A^+})(\varphi_\alpha(b) + \mu 1_{A^+}) - \varphi_\alpha(ab) - \lambda\varphi_\alpha(b) - \mu\varphi_\alpha(A) + \lambda\mu 1_{A^+}\| \\ &= \|\varphi_\alpha(a)\varphi_\alpha(b) - \varphi_\alpha(ab)\| \to 0\end{aligned}$$

being valid for all $a, b \in A$ and scalars $\lambda, \mu \in \mathbb{C}$, we deduce that $(\psi_\alpha)_{\alpha\in\Lambda}$ becomes an asymptotically multiplicative net of unital completely positive maps. The asymptotically isometric property may be proven through a similar and easiar computation. This verifies the claim on the merits of theorem 2.1.6 and Voiculescu's abstract characterization of quasidiagonality. □

The next lemma contains two semi-obvious permanence properties regarding hereditary aspects of quasidiagonality. The former one will be needed during the proof of homotopy invariance, however, the proof merely mimics previous techniques, adapting them ever slightly, whereas restriction to subalgebras retains quasidiagonality is obvious. As such, proofs have been omitted as no reward arises from dwelling further into the matter.

Proposition 2.2.2. *Suppose Ω denotes a quasidiagonal set of operators acting on a Hilbert space $\mathcal{H}$. Under this premise, the C*-algebra generated by Ω becomes quasidiagonal. Furthermore, quasidiagonality passes to subalgebras.*

Now, for some of the more intriguing permanence properties, namely that quasidiagonality is preserved under ℓ^∞-sums, hence c_0-sums due to the previous proposition, and tensoring with respect to the spatial norm. The proofs are surprisingly easy in spite of their benefactors; you merely make the naive choice.

Proposition 2.2.3. *Let $(A_n)_{n\geq 1}$ denote a sequence of C*-algebras. Then each A_n is quasidiagonal if and only if $\ell^\infty(A_n, \mathbb{N})$ is quasidiagonal.*

Proof. The if part is trivial since $A_n \hookrightarrow \ell^\infty(A_n, \mathbb{N})$ isometrically. In order to prove the converse, let $\varphi^n_{\alpha_n} : A_n \longrightarrow \mathbb{M}_{k(\alpha_n)}$ denote the corresponding asymptotically multiplicative - and isometric nets of contractive completely positive maps indexed over sets Λ_n for every positive integer n. Define accordingly a net of bounded linear maps $\varphi_{(\alpha_n)_{n\geq 1}} : \ell^\infty(A_n, \mathbb{N}) \longrightarrow \ell^\infty(\mathbb{M}_{k(\alpha_n)}, \mathbb{N})$, indexed over the directed product $\Lambda_1 \times \Lambda_2 \times \ldots$, by

$$(b_n)_{n\geq 1} \longmapsto (\varphi^n_{\alpha_n}(b_n))_{n\geq 1}.$$

By hypothesis, each coordinate component of $\varphi_{(\alpha_n)_{n\geq 1}}$ is asymptotically multiplicative - and isometric. The assertion follows immediately from the definition of the ℓ^∞-sum norm. □

Proposition 2.2.4. *The spatial tensor product of two quasidiagonal C*-algebras is again quasidiagonal. In particular, the spatial tensor product $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ of unital C*-algebras is quasidiagonal if and only if each factor is quasidiagonal.*

Proof. Suppose A and B are two quasidiagonal C*-algebras admitting quasidiagonal faithful representations $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ and $\varrho\colon B \longrightarrow \mathrm{B}(\mathcal{K})$. The tensor map $\pi \otimes \varrho$ is well-known to produce a faithful representation $A \otimes B \longrightarrow \mathrm{B}(\mathcal{H} \otimes \mathcal{K})$ allowing us to identify the tensor product $A \otimes B$ with the C*-algebra generated by elementary tensors on the form $\pi(a) \otimes \varrho(b)$[4] with $a \in A$ and $b \in B$.

Therefore, we it suffices prove that for every pair of finite sets $F_1 \subseteq A$, $F_2 \subseteq B$, every pair of finite sets $\mathcal{N}_1 \subseteq \mathcal{H}$, $\mathcal{N}_2 \subseteq \mathcal{K}$ and $\varepsilon > 0$ there exists a finite rank projection p acting on $\mathcal{H} \otimes \mathcal{K}$ such that one has

$$\|[p, \pi(a) \otimes \varrho(b)]\| < \varepsilon \quad \text{together with} \quad \|p(\xi \otimes \eta) - \xi \otimes \eta\| < \varepsilon$$

whenever $a \otimes b \in F_1 \otimes F_2$ and $\xi \otimes \eta \in \mathcal{N}_1 \otimes \mathcal{N}_2$ are elementary tensors. Let such elements be given, then exploit quasidiagonality to choose finite rank projections p and q fulfilling

$$\|[p, \pi(a)]\| < \frac{\varepsilon}{2\max_{a\in F_1} \|q\varrho(b)\|}, \quad \|p\xi - \xi\| < \varepsilon^{1/2},$$

$$\|[q, \varrho(b)]\| < \frac{\varepsilon}{2\max_{b\in F_2} \|\pi(a)p\|}, \quad \|q\eta - \eta\| < \varepsilon^{1/2}.$$

[4] This is based on the choice of faithful representations having no impact on the spatial tensor product.

The projection $p_0 = p \otimes q$ admits a range of dimension equaling the product $\dim p = \dim(p)\dim(q)$ meaning it must be of finite rank itself so that the estimates

$$\begin{aligned}\|[p \otimes q, \pi(a) \otimes \varrho(b)]\| &= \|p\pi(a) \otimes q\varrho(b) - \pi(a)p \otimes \varrho(b)q\| \\ &= \|(p\pi(a) - \pi(a)p) \otimes q\varrho(b) + \pi(a)p \otimes (q\varrho(b) - q\varrho(b))\| \\ &\leq \|[p, \pi(a)]\| \cdot \|q\varrho(b)\| + \|\pi(a)p\| \cdot \|[q, \varrho(b)]\| \\ &< \varepsilon\end{aligned}$$

and

$$\|(p \otimes q)(\xi \otimes \eta) - \xi \otimes \eta\| = \|p\xi - \xi\| \cdot \|q\eta - \eta\| < \varepsilon$$

yield the sought conclusion via *lemma* 2.2.3, proving the first assertion. The "if" part of the remaining statement follows from repeated application of the first whereas the "only if" part stems from the canonical mappings $A_k \hookrightarrow A_1 \otimes \ldots \otimes A_n$ being $*$-monomorphisms combined with quasidiagonality passing to subalgebras. □

To achieve a proof circumventing a notational nightmare, we isolate a preliminary result conveying a local formulation of Voiculescu's abstract characterization of quasidiagonality. The proof is merely a standard trick passing from finite subsets to nets using inclusions.

Lemma 2.2.5. *A C*-algebra is quasidiagonal if and only if for every finite subsets $F \subseteq A$ and every $\varepsilon > 0$ there exists a c.c.p map $\varphi\colon A \longrightarrow \mathbb{M}_n$ fulfilling*

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon \quad \textit{together with} \quad \|\varphi(a)\| > \|a\| - \varepsilon$$

for any pair of elements $a, b \in F$.

Proof. The "only if" part is immediate, so we settle with the reverse implication. Suppose that given any pairing (F, ε) consisting of a finite subsets $F \subseteq A$ and $\varepsilon > 0$, the map $\varphi_{F,\varepsilon}\colon A \longrightarrow \mathbb{M}_{n(F,\varepsilon)}$ denotes the assumed c.c.p map subject to the estimates described in the assertion. The collection $\mathcal{F}$ of pairs (F, ε) endowed with the order $\leq$ defined by stipulating that $(F, \varepsilon) \leq (F', \varepsilon')$ if and only if $F \subseteq F'$ while simultaneously $\varepsilon' < \varepsilon$ is easily seen to produce a directed set, whereupon $(\varphi_\alpha)_{\alpha \in \mathcal{F}}$ defines a net of c.c.p maps $A \longrightarrow \mathbb{M}_{n(\alpha)}$. Indeed the net satisfies the sought estimates by construction, proving the claim. □

Proposition 2.2.6. *Let $(A_n, \varphi_n)_{n \in \mathbb{N}}$ be an inductive sequence consisting of quasidiagonal C*-algebras having $*$-monomorphisms as connecting morphisms. Then the associated inductive limit becomes a quasidiagonal C*-algebra.*

Proof. According to the previous lemma, let $F \subseteq A$ be finite and choose your favorite $\varepsilon > 0$. Without loss of generality, we may assume that A is the norm closure of $\bigcup_n A_n$. Ergo, for every pair $a, a_0 \in F$ may be norm approximated, within any tolerance $\delta > 0$, via another pair $b, b_0 \in A_n$ for some sufficiently large positive integer n.

Keeping this in mind, the hypothesis combined with the preceding lemma entails that A_n admits a c.c.p map $\varphi\colon A_n \longrightarrow \mathbb{M}_n$ within an $\varepsilon/3$ tolerance of being multiplicative and isometric. Arveson's extension permits us to extend φ to a c.c.p map $\psi\colon A \longrightarrow \mathbb{M}_n$. Moreover, choosing δ sufficiently small, then exploiting continuity of the involved c.c.p maps forces the estimates $\|a - b\| < \delta$ and $\|a_0 - b_0\| < \delta$ to imply the bounds $\|\psi(bb_0) - \psi(aa_0)\| < \varepsilon/3$ and $\|\psi(a)\psi(a_0) - \psi(b)\psi(b_0)\| < \varepsilon/3$, whereof one obtains the sought bound

$$\|\psi(a)\psi(a_0) - \psi(aa_0)\| \leq \|\psi(a)\psi(a_0) - \psi(b)\psi(b_0)\| + \|\psi(b)\psi(b_0) - \psi(bb_0)\| + \|\psi(bb_0) - \psi(aa_0)\| < \varepsilon.$$

Of course, an entirely analogous argument supplies the estimate $\|\psi(a)\| > \|a\| - \varepsilon$ for all $a \in F$, so we consider our work finished. □

Examples (a).

- Finite-dimensional C*-algebras. Indeed, the classification of finite-dimensional C*-algebras in conjunction with *proposition* 2.2.4 evidently guarantee this.
- AF-algebras are quasidiagonal being inductive limits of finite-dimensional C*-algebras by definition. This provides us with numerous examples despite of general inductive limits typically failing to be quasidiagonal, including UHF-algebras of whichever type you prefer.
- Every AF-embeddable C*-algebra becomes quasidiagonal due to the previous example combined with quasidiagonality passing to subalgebras.
- The compact operators $\mathrm{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ are quasidiagonal. In particular, the stabilization $A \otimes \mathbb{K}$ of any quasidiagonal C*-algebra remains quasdiagonal.

Another lovely feature of quasidiagonality is the existence of faithful tracial states.

Proposition 2.2.7. *Quasidiagonal unital C*-algebras admit a tracial states. In particular, simple quasidiagonal unital C*-algebras admit a faithful tracial state.*

Proof. By hypothesis, A admits u.c.p asymptotically multiplicative maps $\varphi_\alpha \colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ indexed over a directed set Λ. Let hereof $\tau_\alpha \colon \mathbb{M}_{n(\alpha)} \longrightarrow \mathbb{C}$ be the canonical tracial state for every index α and define accordingly $\tau \colon A \longrightarrow \mathbb{C}$ by setting τ to be the weak* cluster-point of the net consisting of states $\tau_\alpha \circ \varphi_\alpha$ acting on A, existence assured via weak*-compactness of the state space. Due to weak*-continuity of addition, one has

$$\tau(ab) - \tau(ba) = (\lim_\alpha \tau_\alpha[\varphi_\alpha(ab) - \varphi_\alpha(ba)]) = \lim_\alpha \tau_\alpha[\varphi(a)\varphi(b) - \varphi(b)\varphi(b)] = 0$$

with the second equality stemming from φ_α being asymptotically multiplicative. For the latter assertion, recall that the subset $\mathcal{L}_\tau$ consisting of elements a in A fulfilling $\tau(a^*a) = 0$ determines a two-sided $*$-ideal in A, hence must coincide with A or be the zero ideal by simplicity. Since $\tau \neq 0$ the former cannot occur, so τ becomes faithful, completing the proof. □

Unfortunately, investigating whether the maximal tensor products of quasidiagonal C*-algebras remains quasidiagonal is still undergoing whereas general inductive limits of quasidiagonal ones may fail to become quasidiagonal. Another important class of C*-algebras contained in the quasidiagonal class is the so-called residually finite dimensional ones. We shall present a rather thorough survey of residually finite dimensional C*-algebras to emphasize on a few techniques arising hereby.

Definition. A C*-algebra A is said to be *residually finite dimensional*, abbreviated into RFD, if there exists a net $(\pi_\alpha)_{\alpha\in\Lambda}$ consisting of $*$-homomorphism $\pi_\alpha \colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ such that the corresponding inflation $*$-homomorphism $\pi \colon A \longrightarrow \ell^\infty(\mathbb{M}_{n(\alpha)}, \Lambda)$ becomes faithful. Equivalently, A is RFD if for every nonzero element $a \in A$ there exists a $*$-homomorphism $\pi_\alpha \colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ such that $\pi_\alpha(a) \neq 0$. As usual, a sequence of such homomorphisms is demanded whenever A is separable.

Example. Abelian C*-algebras are automatically RFD-algebras. Indeed, for any locally compact Hausdorff space Ω and bounded continuous map $f \colon \Omega \longrightarrow \mathbb{C}$ differing from zero, one has $f(z) \neq 0$ for some element z belonging to Ω so that one may define $\psi_z \colon C_0(\Omega) \longrightarrow \mathbb{C}$ by

$$\psi_z(f) = f(z).$$

By construction $\psi_z(f) \neq 0$, proving that $C_0(\Omega)$ must be residually finite dimensional, whereupon the assertions follows from Gelfand-Neimark's theorem.

Proposition 2.2.8. *Residually finite dimensional C*-algebras are quasidiagonal.*

Proof. Suppose A is an RFD-algebra and suppose $\pi_{\alpha,a}\colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ is the $*$-homomorphism mapping a corresponding nonzero element a in A to some nonzero element. To simplify the notation, set $A_0 = A \setminus \{0\}$ and define thus $\pi\colon A \longrightarrow \ell^\infty(\mathbb{M}_{n(a)}, A_0)$ as the infinite inflation indexed over A_0 of these existing $*$-homomorphisms, that is, the assignment $b \mapsto (\pi_{\alpha,a}(b))_{a\in A_0}$. Identify the codomain of π side with $\mathrm{B}(\bigoplus_{a\in A_0} \mathbb{C}^{n(\alpha)})$ via the $*$-isomorphism $(T_a)_{a\in A_0} \mapsto T$ where T is the bounded operator on the latter algebra given by

$$T(\xi_a)_{k=1}^{n(a)} = (T_a\xi_a)_{k=1}^{n(\alpha)}.$$

It is clear that π must be faithful due to each factor being so, hence it remains to be shown that its image must be quasidiagonal. Write $\mathcal{H} = \bigoplus_{a\neq 0} \mathbb{C}^{n(\alpha)}$ to ease the notation. For every nonzero element $a \in A$, let $q_a\colon \mathcal{H} \longrightarrow \mathbb{C}_{n(\alpha)}$ be the canonical orthogonal projection. The strong-operator sum $p_a := \sum_{b\leq a} q_b$ defines a projection acting on $\mathcal{H}$ converging in the strong-operator topology towards the identity $1_{\mathcal{H}}$. The rank of p_a must equal the sum of ranks of each q_b appearing in its sum, hence it has finite rank. Furthermore,

$$\begin{aligned}
\|\pi(a)p_a\xi - p_a\pi(a)\xi\| &= \|(\pi_a(a))_{a\neq 0}p_a\xi - p_a(\pi_a(a))_{a\neq 0}\xi\| \\
&= \|(\pi_a(a)\xi_b)_{b\leq a} - p_a(\pi_a(a)\xi_a)_{a\neq 0}\| \\
&= \|(\pi_a(a)\xi_b)_{b\leq a} - (\pi_a(a)\xi_b)_{b\leq a}\| \\
&= 0
\end{aligned}$$

for all $a \in A$ and $\xi = (\xi_b)_{b\neq 0}$ in $\mathcal{H}$. We conclude that $\pi(A)$ must be block-diagonal. □

2.3 Homotopy Invariance

As a grand finale to the chapter, we derive the heralded result due to Voiculescu. It asserts that quasidiagonality is a homotopy invariant in the category C*-Alg. The merits to such a permanence properties are aplenty. For instance, it provides additional examples of quasidiagonal C*-algebras, namely the suspension and cone associated to any C*-algebra. The proof calls for meticulous care, so it has been separated into several parts with an emphasis on the actual strategy. Despite of not being adequately equipped to prove Voiculescu's theorem, we present a portion of it and thereof verify quasidiagonality of the aforementioned algebras, partly to encapsulate the main idea in the proof. The reader is strongly urged to consult the page 7 before venturing further.

Theorem 2.3.1 (Voiculescu)**.** *Any C*-algebra homotopically dominated by a quasidiagonal C*-algebra must be quasidiagonal. In particular, quasidiagonality is a homotopy invariant in the category of C*-algebras having $*$-homomorphisms as morphisms.*

Proof. Suppose $\pi\colon A \longrightarrow B$ and $\varrho\colon B \longrightarrow A$ are $*$-homomorphisms such that $\pi\varrho \cong_H \mathrm{id}_B$ with A being quasidiagonal. Define accordingly $\vartheta\colon B \longrightarrow B \oplus \varrho(B)$ by the assignment $b \mapsto \pi\varrho(b) \oplus \varrho(b)$. There exists by hypothesis some $*$-homotopy continuously transforming $\pi\varrho$ into the identity on B, say $\sigma\colon B \times I \longrightarrow B$ satisfies $\sigma(b, 0) = \pi\varrho(b)$ and $\sigma(b, 1) = b$ for every element b in B. The obtained pairing (σ, ϑ) defines a map $\sigma_\vartheta\colon (B \oplus \varrho(B)) \times I \longrightarrow B \oplus \varrho(B)$ given by

$$\sigma_\vartheta(b, \varrho(b_0), t) = \sigma(b, t) \oplus \varrho(b_0)$$

remains continuous in each variable, while the restriction $(b, \varrho(b_0)) \mapsto \sigma_\vartheta(b, \varrho(b_0), t)$ automatically determines a $*$-homomorphism for all parameters $0 \leq t \leq 1$, which transforms $\pi\varrho \oplus \varrho$ into $\mathrm{id}_B \oplus \varrho$ by construction. Here is the crucial trick involved in the proof: suppose that given any pair of homotopic $*$-homomorphisms $\mu, \nu\colon A \longrightarrow B$ with ν being a $*$-monomorphism and $\mu(A)$ quasidiagonal, one may deduce quasidiagonality of A. If this were true, then due to $\mathrm{id}_B \oplus \varrho$ clearly being faithful in conjunction with the inclusion

$$\vartheta(B) = \{(\pi\varrho(b), \varrho(b)) \colon b \in B\} \cong \varrho(B) \hookrightarrow A$$

guaranteeing quasidiagonality of $\vartheta(B)$, one obtains quasidiagonality of B. Here the identifications is merely the assignment from $\varrho(B)$ onto $\vartheta(B)$ given by $\varrho(b) \mapsto (\pi\varrho(b), \varrho(b))$, which is readily checked to determine an isomorphism of C*-algebras based on faithfulness granting

$$\|\vartheta(b)\| = \max\{\|\pi\varrho(b)\|, \|\varrho(b)\|\} = \|\varrho(b)\|$$

for every element $b \in B$. This completes the proof, modulo the lemma 2.3.6 below. □

Recall that given a C*-algebra, there exists two C*-algebras arising hereby n a functorial manner, namely the *suspension and cone*. The cone of a C*-algebra, denoted by $C(A)$, consists of all continuous functions $f \colon I \longrightarrow A$ such that $f(0) = 0$ with $I = [0, 1]$ and the suspension, denoted by $S(A)$, is the $*$-ideal herein consisting of all continuous functions $f \colon I \longrightarrow A$ such that $f(0) = f(1) = 0$. The cone and suspension of a C*-algebra are paramount in K-theory and we shall derive quasidiagonality of these.

Corollary 2.3.2. *The functors $A \mapsto C_A$ and $A \mapsto S_A$ from the category of C*-algebras maps into the subcategory of quasidiagonal C*-algebras. Described less abstractly, the suspension and cone of a C*-algebra is always quasidiagonal.*

Proof. Due to quasidiagonality passing to subalgebras, it suffices to show that C_A must be quasidiagonal for any C*-algebra A. However, the cone is always contractible, that is, homotopic to the trivial algebra $\{0\}$ via the homotopy $\sigma \colon C_A \times I \longrightarrow C_A$ given by $\sigma(f, t)(s) = f(ts)$ for all $0 \leq s \leq 1$ and continuous map $f \colon I \longrightarrow A$ for which $f(0) = 0$: one has $\sigma(f, 1)(s) = f(s)$ together with $\sigma(f, 0) = f(0) = 0$. Therefore we may deduce that C_A is homotopy equivalent to a single point with the latter trivially being quasidiagonal. Thus C_A becomes quasidiagonal because of the homotopy invariance theorem, proving the claim. □

As discussed, the proof of Voiculescu's theorem relies on lemma 2.3.6. The preceding proof, in which the lemma was assumed valid, unfolds the general strategy in Voiculescu's approach. The proof exploits various trickery, so to maintain sanity these parts are tackled individually. First of and foremost, we shall use the following criterion for quasidiagonality. Its content provides a local sufficient condition for a C*-algebra to be quasidiagonal in terms of families consisting of representations. The reader should be warned of the next couple of results, details have generally been reduced into a bare minimum to stay on track.

Lemma 2.3.3. *A C*-algebra A fulfilling the following condition must be quasidiagonal: for every $\varepsilon > 0$ and finite subset $F \subseteq A$ there exist a representation $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$ together with a finite rank projection p acting on $\mathcal{H}$ subject to the estimates*

$$\|[p, \pi(a)]\| < \varepsilon \quad \text{and} \quad \|p\pi(a)p\| \geq \|a\| - \varepsilon \tag{2.10}$$

for all a belonging to F.

Proof. Let $F \subseteq A$ be finite, let $\varepsilon > 0$ be fixed and set $M = \max\{\|a\| : a \in F\}$. Suppose (π, p) denotes the pairing fulfilling the estimates (2.10) in the statement with respect to the pair $(F, \varepsilon/M)$. Compression with bounded operators always yields a c.c.p map. Hence letting $a \mapsto p\pi(a)p$ be denoted by φ leaves a c.c.p map subject to

$$\|\varphi(a)\varphi(b) - \varphi(ab)\| = \|p\pi(a)(p\pi(b) - \pi(b)p)p\| \leq M \cdot \|\pi(b)p - \pi(b)p\| < \varepsilon$$

In a similar manner one obtains $\|\varphi(a)\| > \|a\| - \varepsilon$, so the desired stems from lemma 2.2.5. □

Proceeding further, some preparations concerning quasicentral approximate units.

Proposition 2.3.4. *Suppose I denotes a σ-unital two-sided $*$-ideal in some C^*-algebra A. Then there exists a quasicentral approximate unit $\{q_n\}_{n\geq 1}$ in I fulfilling $q_n q_{n-1} = q_{n-1}$ for all $n \in \mathbb{N}$.*

Proof. Let $z = \sum_{n=1}^{\infty} 2^{-n} e_n$ having $\{e_n\}_{n\geq 1}$ denote the sequence acting as an approximate unit. Then z is strictly positive according to proposition A.3.1. Construct any continuous function $\psi_n \colon (0,1] \longrightarrow [0,1]$ having support outside the open interval $(0, 2^{-n})$, attaining the value 1 on the closed bounded interval $[2^{-(n-1)}, 1]$ and linear in between the gap of the intervals. We have

$$\psi_n(t)\psi_{n-1}(t) = \begin{cases} 1, & \text{if } 0 < t < \frac{1}{2^n}, \\ 0, & \text{if } \frac{1}{2^{n-1}} \leq t \leq 1, \\ \psi_{n-1}(t), & \text{if } z \text{ is in between the above.} \end{cases}$$

simply because $[2^{-n}, 1] \subseteq [2^{-(n+1)}, 1]$ for all positive integers n. According to the proposition A.3.1, constructing a sequence of increasing positive elements $\{\psi_n(z)\}_{n\geq 1}$ determining an approximate unit of I suffices. To achieve this, note that the multiplicative property of the continuous functional calculus implies $\psi_n(z)\psi_{n-1}(z) = \psi_{n-1}(z)$ for all $n \in \mathbb{N}$, and if ω is a state on I we have

$$\|\omega(\psi_n(z))\| \leq \|\psi_n(z)\| = \|\psi_n\|_\infty \to 1,$$

since $[2^{-n}, 1] \to (0, 1]$. Strict positivity of z easily entails $\lim_{n\to\infty} \psi_n(z)a = \lim_{n\to\infty} a\psi_n(z) = a$. The sequence $\{\psi_n(z)\}_{n\geq 1}$ hereof becomes an approximate unit in I, from which a quasicentral approximate unit $\{q_n\}_{n\geq 1}$ may be extracted. We leave the verifying the relation $q_n q_{n-1} = q_{n-1}$ to the reader; one requires the proof of existence of quasicentral approximate units for this. □

The intriguing feature of quasicentral approximate units is its property of asymptotically commuting with elements, a property heavily resembling one of the quasidiagonal conditions. The preceding result permits us to arrange the relation $q_n q_{n-1} = q_{n-1}$ which becomes essential when applying a matrix trick. Another minor observation we shall invoke is a continuity-esque result.

Lemma 2.3.5. *For every $\varepsilon > 0$ and element f in $C_0(0, 1]$, there exists a $\delta > 0$ such that for every C^*-algebra A and unit vectors $e, a \in A$ with $\|[e, a]\| < \delta$ one has $\|[f(e), a]\| < \varepsilon$.*

Proof. Through an application of the Stone-Weierstrass approximation theorem, we may approximate f via polynomials within an $\varepsilon/2$ tolerance, thus permitting us to assume $f = \lambda_1 x + \ldots + \lambda_n x^n$ with each λ_k belonging to $\mathbb{C}$. To ease the notation, denote by $D_a(b)$ the derivation $[b, a] = ba - ab$ for any pair of elements $a, b \in A$. In the event of b being of unit length, we consider the action of D_a on powers, whereof we obtain the rearrangement

$$D_a(b^{k+1}) = (b^{k+1}a - bab^{k+1}) + (bab^{k+1} - ab^{k+1}) = bD_a(b^k) + D_a(b)b^k.$$

combined with an induction argument the bounds $\|D_a(b^{k+1})\| \leq \|D_a(b^k)\| + \|D_a(b)\|$ for all $k \leq n$. Thus $\|D_a(b^k)\| \leq k\|D_a(b)\|$ for every k'th power not exceeding n. Ergo, choosing $\delta = \varepsilon/M$, where the constant is $M = \sum_{k=1}^{n} k|\lambda_k|$, and assuming $\|D_a(e)\| = \|[e, a]\| < \delta$ entails according to the continuous functional calculus the estimate

$$\|[f(e), a]\| = \left\| \sum_{k=1}^{n} \lambda_k D_a(e^k) \right\| \leq \sum_{k=1}^{n} k|\lambda_k| \, \|D_a(b)\| < \varepsilon,$$

proving the claim. □

Lemma 2.3.6. *Let $\varphi, \psi \colon A \longrightarrow B$ be $*$-homotopic $*$-homomorphisms with φ a $*$-monomorphism and $\psi(B)$ determining a quasidiagonal C*-algebra. Under these premises, A becomes quasidiagonal.*

Proof. **Step 1.** Let $\varphi, \psi \colon A \longrightarrow B$ be $*$-homotopic $*$-homomorphisms with φ being a monomorphism and the image of ψ being quasidiagonal. Suppose $\pi \colon B \longrightarrow \mathrm{B}(\mathcal{K})$ denotes a faithful separable essential representation of B and let $\sigma \colon I \times A \longrightarrow B$ be the $*$-homotopy transforming φ into ψ, i.e., $\sigma(0,a) = \varphi(a)$ and $\sigma(1,a) = \psi(a)$ for all a in A. Upon applying lemma 2.3.3, it will suffice to show that given an $\varepsilon > 0$ and finite subset $F \subseteq A$ there exist a representation $\varrho : A \longrightarrow \mathrm{B}(\mathcal{H})$ and a finite rank projection p acting on $\mathcal{H}$ such that

$$\|[p, \varrho(a)]\| < \varepsilon \quad \text{and} \quad \|p\varrho(a)p\| \geq \|a\| - \varepsilon.$$

Let $\delta > 0$ be an arbitrary parameter (to be specified in the future after some setup). We may filter I with a finite collection of intervals on the form $V_n^k = [k/n, (k+1)/n]$ for $0 \leq k \leq n-1$ by compactness of the unit interval. Norm continuity of the map $t \mapsto \sigma(t,a)$ combined with the convergence $V_n^k \to \{0\}$ for $n \to \infty$ implies the existence of some positive integer N_δ for which

$$\|\sigma(k/n, a) - \sigma((k+1)/n, a)\| \leq \delta. \tag{2.11}$$

holds for all $0 \leq k \leq n-1$, $n \geq N_\delta$ and $a \in A$. Due to F being finite, the orthogonal projection q onto the image of the linear space spanned by F under φ has finite rank and satisfies the constraint

$$\|q\pi(\varphi(a))q\| \geq \|a\| - \varepsilon, \quad a \in F, \tag{2.12}$$

due to φ being an isometry, or with some slight abuse of notation $\|q\varphi(a)q\| \geq \|a\| - \varepsilon$ under the identification $B \cong \pi(B)$. Next up, the quasicentral approximate unit trick: Since π is separable, the corresponding ideal of compact operators on $\mathcal{H}$ becomes separable, hence it admits a quasicentral approximate unit $\{e_n\}_{n\geq 1}$ satisfying $e_n e_{n-1} = e_{n-1}$ for any $n \in \mathbb{N}$ according to proposition 2.3.4. The finite rank operators on $\mathcal{K}$ are dense in the compact ones, so there exists for every positive integer k some finite rank operator f_k^n acting on $\mathcal{K}$ within $1/k$ distance of e_n. The quasicentral property ensures that the right-hand side of

$$\|[a, f_n^k]\| \leq \|af_n^k - ae_n\| + \|ae_n - e_n a\| + \|e_n a - f_n^k a\|$$

can be made arbitrarily small for each $a \in F$. Thus, upon replacing each e_k with f_k, we may assume that e_k has finite rank in addition to being quasicentral, positive, of unit length, subject to $e_k e_{k-1} = e_{k-1}$ for all $k \in \mathbb{N}$, while $q \leq e_1 \leq e_2 \leq \ldots \leq e_n \leq 1$ and on the merits of $\{e_n\}_{n\geq 1}$ being quasicentral have the property

$$\|[e_k, \sigma(k/n, a)]\| \leq \delta, \quad 0 \leq k \leq n-1, \;\; a \in F. \tag{2.13}$$

We arrive at a pivotal point. Since $\psi(A)$ is quasidiagonal and $\pi_{|\psi(A)}$ essential, theorem 2.1.6 permits us to find a finite rank projection p such that $\|[p, \psi(F)]\|$ and $\|p\xi - \xi\|$ become arbitrarily small for any basis vector of the finite-dimensional linear space Ran e_n. The approach presented during the last half in the proof of proposition 2.1.2 allows one to determine a unitary perturbing p in a manner that makes it dominate q[5]. The perturbed version of p remains a projection, which still asymptotically commutes with $\psi(F)$. hence it will asymptotically commute with elements of the form $\sigma(k/n, a)$ for all $0 \leq k \leq n-1$ and $a \in A$.

We will not dwell into the technical estimates, for they are mere repetitions of previous ones and therefore not particularly rewarding. The crucial point to be stressed is: We may ensure that e_n becomes a finite rank projection such that (2.13) remains valid for F_n.

[5] indeed the $\delta > 0$ prescribed during the proof has no constraints attached to it, so we merely have to choose our current $\delta > 0$ sufficiently small for our purposes.

Step 2. We proceed to the second vital step of the proof. The main idea revolves around "encoding" the projection e_n into an *almost* diagonal matrix that almost commutes with the representation $a \mapsto \bigoplus_{k=0}^{n} \sigma(k/n, a)$. To accomplish this, define a bounded operator $v\colon \mathcal{K} \longrightarrow \mathcal{K}^{n+1}$ by

$$v\xi = \bigoplus_{k=0}^{n} x_k^{1/2}\xi$$

where $x_k = e_k - e_{k-1}$ and $x_0 = e_0$ for all indices $k \leq n$. It ought to be apparent that each x_k defines a positive operator. Suppose ξ and η belong to $\mathcal{K}$. From the computations

$$\langle v^*v\xi, \eta\rangle_{\mathcal{K}} = \Big\langle \bigoplus_{k=0}^{n} x_k^{1/2}\xi, \bigoplus_{k=0}^{n} x_k^{1/2}\eta \Big\rangle_{\mathcal{K}^{n+1}} = \sum_{k=0}^{n} \langle x_k\xi, \eta\rangle_{\mathcal{K}} = \langle e_n\xi, \eta\rangle$$

one may deduce that v must be a partial isometry, whereupon $p = vv^*$ becomes a projection. This projection will be our candidate and to verify this we identify its associated $(n+1)\times(n+1)$ matrix. To ease the computational burden, we observe that for two indices i and j of at bare minimum distance 2 apart, say with $i > j$ without loss of generality, one has

$$\begin{aligned} x_i^{1/2}x_j^{1/2} = (e_i - e_{i-1})(e_j - e_{j-1}) &= e_ie_j - e_{i-1}e_j - e_ie_{j-1} + e_{i-1}e_{j-1} \\ &= e_j - e_j - e_{j-1} + e_{j-1} \\ &= 0 \end{aligned}$$

due to the property $e_ke_{k-1} = e_{k-1}$ being valid for every index $0 \leq k \leq n$ entailing that $e_ie_j = e_j$ along with $e_{i-1}e_j = e_j$ (using the hypothesis $i - 1 > j$). Keeping this in mind, notice that due to

$$\Big\langle \xi, \sum_{k=0}^{n} x_k^{1/2}\eta_k \Big\rangle_{\mathcal{K}} = \sum_{k=0}^{n} \langle x_k^{1/2}\xi_k, \eta_k\rangle_{\mathcal{K}} = \Big\langle \bigoplus_{k=0}^{n} x_k^{1/2}\xi, \eta \Big\rangle_{\mathcal{K}^{n+1}} = \langle v\xi, \eta\rangle_{\mathcal{K}^{n+1}},$$

where $\eta = (\eta_o, \ldots, \eta_n)$ together with $\xi = (\xi_0, \ldots, \xi_n)$, the adjoint of v must be operator $\sum_{k=0}^{n} T_k^{1/2}$. Combining this with our preceding relation entails

$$vv^* = \sum_{k=0}^{n} vx_k^{1/2} = \bigoplus_{\ell=0}^{n}\sum_{k=0}^{n} x_\ell^{1/2}x_k^{1/2} = \big[x_0 + x_0^{1/2}x_1^{1/2}, x_1^{1/2}x_0^{1/2} + x_1 + x_1^{1/2}x_2^{1/2}, \ldots, x_n\big].$$

The operator on the right-hand side may be more conveniently regarded as the matrix

$$M(p) = \begin{pmatrix} x_0 & x_0^{1/2}x_1^{1/2} & 0 & 0 & \cdots \\ x_1^{1/2}x_0^{1/2} & x_1 & x_1^{1/2}x_2^{1/2} & 0 & \cdots \\ 0 & x_2^{1/2}x_1^{1/2} & x_2 & x_2^{1/2}x_1^{1/2} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix}$$

Define lastly $\varrho\colon A \longrightarrow \mathrm{B}(\mathcal{K}^{n+1})$ via the assignment $a \mapsto \bigoplus_{k=0}^{n} \sigma(k/n, a)$. By construction of our well-designed semimagical collection $\{e_k\}_{k=0}^{n}$ in conjunction with the relation (2), we have

$$\|p\varrho(a)p\| = \max_{0\leq k\leq n} \|e_k\sigma(k/n, a)e_k\| \geq \|q\sigma(k/n, a)q\| \geq \|a\| - \varepsilon.$$

for every element $a \in A$. One ought to be careful during the final inequality, since we strictly speaking only verified it on $\varphi(A)$. However, choosing a sufficiently large positive integer n forces $\sigma(k/n, \cdot)$ to approach $\varphi(\cdot)$ within any tolerance, whereof we may apply (2.12) accordingly. The remainder of the proof amounts to verifying that p asymptotically commutes with $\varrho(A)$.

Final Step. Split $M(p)$ up into a sum of three diagonal matrices $M(p) = U + D + L$ with U denoting the upper diagonal matrix, L the lower one and D the diagonal part. Proving that each of these asymptotically commutes with $\varrho(a)$ for any fixed unit vector a in A will suffice. The diagonal part is straightforward. Indeed, for some index $0 \le k \le n-1$ we have according to the relations (2.11)-(2.13), the bound

$$\begin{aligned}
&= \|x_k \sigma(k/n, a) - \sigma(k/n, a) x_k\| \\
&\le \||e_{k-1}\| \cdot \|\sigma((k-1)/n, a) - \sigma(k/n, a)\| + \|[e_k, \sigma(k/n, a)]\| \\
&+ \|[e_{k-1}\sigma((k-1)/n, a) - \sigma(k/n, a) e_{k-1}\| \\
&\le \delta + \delta + \||e_{k-1}\| \cdot \|\sigma((k-1)/n, a) - \sigma(k/n, a)\| + \|[e_{k-1}, \sigma(k/n, a)]\|
\end{aligned}$$

which becomes smaller than 4δ due to (2.11). This settles the diagonal part and for the remaining ones we require the aid of our preceding lemma. Certainly, according to lemma 2.3.5 we may arrange $\delta > 0$ in a manner such that

$$\|[\sigma(\ell/n, a), x_k^{1/2}]\| \le \varepsilon \quad \text{whenever} \quad |\ell - k| \le 1. \tag{2.14}$$

Here the constraint $|\ell - k| \le 1$ is sufficient on the merits of (3). Now, due to

$$[L, \varrho(a)] = \left(x_{k+1}^{1/2} x_k^{1/2} \sigma(k/n, a) - \sigma(k/n, a) x_{k+1}^{1/2} x_k^{1/2} \right)_{k=0}^{n},$$

another triangle-inequality trick stemming from adding and subtracting

$$x_k^{1/2} \sigma(k/n, a) x_k^{1/2} \quad \text{together with} \quad \sigma((k+1)/n, a) x_{k+1}^{1/2} x_k^{1/2}$$

to the norm $\|[L, \varrho(a)]\|$, then exploiting (4) will provide us with $\|[L, \varrho(a)]\| \le 3\varepsilon$. A similar computations grants the same estimate for U, whence $\|[M(p), \varrho(a)]\|$ may be made smaller than $6\varepsilon + 4\delta$, which ought to be apparent because of the symmetry occurring in $M(p)$. Choosing δ sufficiently small will force $\|[M(p), \varrho(a)]\|$ smaller than ε, completing the proof. □

Chapter 3

KK-Theory, Hilbert C*-modules and Groupoid C*-algebras

During the previous chapter, the notion of quasidiagonality was established for abstract C*-algebras together with a plethora of permanence properties and examples. In spite of Rosenberg's conjecture having an answer in the affirmative, the required machinery to even partially answering this is vast. This chapter seeks to collect a humble fraction of these concepts including a crash course in KK-theory, Hilbert C*-modules and groupoid C*-algebras. We urge the reader to endure the lack of rigor in certain aspects emerging in the KK-theory sections; the missing ingredients ought to be minor difficulties or, at bare minimum, completely standard arguments. Regardless, the notions are independently very general and exotic tools in the operator theoretic arsenal.

3.1 A Crash Course in KK-Theory: The KK-Groups

KK-theory was developed by G.G. Kasparov using Fredholm-modules to build a bifunctor from the category of separable C*-algebras into the category of abelian groups. The resulting theory succeeded in encapsulating both the ordinary K-theory in one variable while containing information concerning the K-homology in the second variable.

Joachim Cuntz managed to redefine KK-theory through the notion of quasihomomorphisms and we adopt his picture of KK-theory. The approach of Cuntz arguably appears more abstract, however, certain properties that otherwise are troublesome become easier to tackle. For the record, the author apologizes for the generality and attempts to justify it by emphasizing on the key components exploited during the constructions. Let us begin with the basics.

Definition. Let A and B be C*-algebras. A *prequasihomomorphism* from A into B is a diagram

$$A \overset{\varphi_+}{\underset{\varphi_-}{\rightrightarrows}} M \rhd I \xrightarrow{\ \varrho\ } B$$

wherein M denotes a C*-algebra, $I \subseteq M$ defines a $*$-ideal and $\varphi_\pm$ are $*$-homomorphisms fulfilling the relation $\varphi_+(a) - \varphi_-(a) \in I$ for all elements $a \in A$.

It is apparent that any $*$-homomorphism $\varphi\colon A \longrightarrow B$ gives rise to a prequasihomomorphism with $M = I$, $\varphi_\pm = \varphi$ and $\varrho = \mathrm{id}_B$, so indeed we have generalized the former condition. The prefix "pre" leaves a slightly undesired generality for our purposes, hence we shall impose additional constraints to our morphisms. The ideas behind the additional axioms will be unraveled momentarily.

Definition. Suppose A and B are C*-algebras. A *quasihomomorphism* is a prequasihomomorphism

$$A \underset{\varphi_-}{\overset{\varphi_+}{\rightrightarrows}} M \rhd I \subseteq B$$

such that, as the diagram suggests, the connecting map $\varrho\colon I \longrightarrow B$ is a $*$-monomorphism and:

- M is generated by the images $\varphi_+(A)$ and $\varphi_-(A)$. (Q1)
- The ideal I is generated by the set $\{\varphi_+(a) - \varphi_-(a) : a \in A\}$. (Q2)
- The composed $*$-homomorphism $A \xrightarrow{\varphi_\pm} M \longrightarrow M/I$ is an isometry. (Q3)

We denote a quasihomomorphisms by a pair (φ_+, φ_-), occasionally symbolically representing it by writing $(\varphi_+, \varphi_-)\colon A \rightrightarrows B$ and refer it the pairing as *passing through* $I \lhd M$ for emphasis on the entities involved. Note that the first two axioms automatically ensure that $A \longrightarrow M \longrightarrow M/I$ automatically a $*$-epimorphism, hence a $*$-isomorphism according to (Q3).

Quasihomomorphisms may strike the reader as being mysterious entities, so we shall attempt to unravel the idea by exhibiting a prototypical setting wherein quasihomomorphisms appear naturally (in fact, some take this as the definition). Let A and B be C*-algebras admitting a pair of $*$-homomorphisms $\varphi_+, \varphi_-\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$, where $\mathbb{K}$ denotes the compact operators on some separable Hilbert space. Since the multiplier algebra $\mathcal{M}(A)$ contains A as an essential ideal,

$$A \underset{\varphi_-}{\overset{\varphi_+}{\rightrightarrows}} \mathcal{M}(A \otimes \mathbb{K}) \rhd A = A$$

defines a quasihomomorphism from A into $\mathcal{M}(A \otimes \mathbb{K})$. The reasoning behind the added tensor product with the compacts will be revealed shortly. The main principle behind quasihomomorphisms is that they define bona fide $*$-homomorphisms when we enlarge the domain using free products.

Definition. Let R be a ring. We define Q_R to be the ring generated by abstract symbols $q(a)$ and a, for all elements a inside R, satisfying the relation

$$q(ab) = aq(b) + bq(a) - q(a)q(b) \quad \text{and} \quad q(a+b) = q(a) + q(b) \tag{3.1}$$

for all $a, b \in R$.

The ring Q_R comes equipped with canonical ring monomorphisms $\iota, \bar{\iota}\colon R \hookrightarrow Q_R$, defined for all elements a in R via $\iota(a) = a$ together with $\bar{\iota}(a) = a - q(a)$. Hence their images produce two copies of R that together generate Q_R, the verifications being routine calculations based on (3.1). One commonly regards Q_R as being generated by these images, in which case the corresponding ideal q_R generated by the symbols $q(a)$ becomes the ideal generated by $\iota(\cdot) - \bar{\iota}(\cdot)$.

The above construction is perhaps more familiar to the reader in the language of free products. Indeed Q_R is the free product $R * R$ of the ring with itself, hence may be regarded as the unique ring such that whenever there exist two homomorphisms from R into some ring S, then there exists a unique ring-homomorphism $\alpha\colon R * R \longrightarrow S$ making the diagram below commute:

$$\begin{array}{ccccc} R & \xrightarrow{\bar{\iota}} & R * R & \xleftarrow{\iota} & R \\ & \searrow & \downarrow{\scriptstyle \alpha} & \swarrow & \\ & & S & & \end{array}$$

Returning to the C*-algebraic context, notice that given a quasihomomorphism $(\varphi_+, \varphi_-)\colon A \rightrightarrows B$ the associated map $q(\cdot) = \varphi_+(\cdot) - \varphi_-(\cdot)$ defines a linear map, which is readily checked to fulfill

- $q(ab) = \varphi_+(a)q(b) + q(b)\varphi_-(a) = \varphi_-(a)q(b) + q(a)\varphi_+(b)$ for all $a, b \in A$;
- $q(ab) + q(a)q(b) = \varphi_+(a)q(b) + q(a)\varphi(b)$ for all $a, b \in A$.

These relations, with an increased attentions towards the latter, ought to be compared to Q_A. Hereof, one expects some connection between QA and quasihomomorphisms, which is precisely the content of the following discussion. We may naturally endow Q_A with an involution by declaring q to preserve the involution of A meaning $q(a)^* := q(a^*)$ and we may evidently regard Q_A as a complex vector space by extending each involving maps, such as ι and $\bar{\iota}$, linearly. This gives rise to the following construction.

Definition. For every C*-algebra A, we define Q_A to be the completion of the underlying involutive algebra under the universal norm $\|\cdot\|_q \colon Q_A \longrightarrow \mathbb{R}^+$ defined by

$$\|x\|_q = \sup\left\{\|\pi(x)\| : \pi\colon Q_A \longrightarrow \mathrm{B}(\mathcal{H}) \text{ non-degenerate representation}\right\}.$$

We denote the completion of Q_A and the ideal q_A by Q_A and q_A again, respectively.

Remark. If you read the previous construction with skeptical eyes, you are completely on track: there is a priori absolutely no reason for free products to exist. We omit the proof of the existence concerning free products of C*-algebras together with the fact that the universal norm in fact is a full-fledged norm as opposed to a seminorm.

Observation 1: Induced morphisms. Suppose A and B are C*-algebra. Let $\varphi, \psi\colon A \longrightarrow B$ be $*$-homomorphisms. The induced map $Q(\varphi, \psi)\colon Q_A \longrightarrow B$ defined on generating elements by

$$\iota(\cdot) \mapsto \varphi(\cdot) \quad \text{together with} \quad \bar{\iota}(\cdot) \mapsto \psi(\cdot)$$

becomes continuous, hence may be turned into a $*$-homomorphism by extending $\mathbb{C}$-linearly and via continuity. Conversely, every $*$-homomorphism $Q_A \longrightarrow B$ arises in this particular fashion, since $*$-homomorphisms on Q_A are uniquely determined by their action on the generating elements. We define the *universal quasihomomorphism associated to* A in accordance with the diagram below.

$$A \overset{\iota}{\underset{\bar{\iota}}{\rightrightarrows}} Q_A \rhd q_A \subseteq Q_A$$

Conversely, any prequasihomomorphism

$$A \overset{\varphi_+}{\underset{\varphi_-}{\rightrightarrows}} M \rhd I \overset{\varrho}{\longrightarrow} B$$

gives a $*$-homomorphism $Q(\varphi_+, \varphi_-)\colon Q_A \longrightarrow B$ whose restriction to the ideal q_A is an additional $*$-homomorphism $q(\varphi_+, \varphi_-)\colon q_A \longrightarrow B$ having values in I, for elements in q_A are the of the form $\iota(\cdot) - \bar{\iota}(\cdot)$, which are mapped into $\varphi_+(\cdot) - \varphi_-(\cdot)$, whereof (Q2) applies. The induced $*$-homomorphism $q(\varphi, \psi)$ associated to a pair $\varphi, \psi\colon A \longrightarrow B$ of $*$-homomorphisms is generally the assignment $q(\cdot) \mapsto \varphi(\cdot) - \psi(\cdot)$ due to q_A being generated by the differences $\iota(\cdot) - \bar{\iota}(\cdot)$ of the canonical embeddings. In particular, the zero morphism $q_A \longrightarrow I$ is precisely $q(\varphi, \varphi)$.

The two observations combine into a complete characterization of quasihomomorphisms and prequasihomomorphisms. We prove the characterization for the sake of completeness and to comfort ourselves with the aforementioned concepts. However, we require some notions.

Definition. Two prequasihomomorphisms

$$A \rightrightarrows M_1 \triangleright I_1 \longrightarrow B \qquad\qquad A \rightrightarrows M_2 \triangleright I_2 \longrightarrow B$$

are referred to as being *isomorphic* if $M_1 \cong M_2$ and $I_1 \cong I_2$ as C*-algebras in a compatible manner. The compatibility condition being commutativity of the following diagram:

$$\begin{array}{ccccccc}
A & \rightrightarrows & M_1 & \hookleftarrow & I_1 & \longrightarrow & B \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
A & \rightrightarrows & M_2 & \hookleftarrow & I_2 & \longrightarrow & B
\end{array}$$

On the merits of the compatibility in conjunction with the identifications, isomorphic prequasihomomorphisms determine the same morphisms in the appropriate sense.

Proposition 3.1.1. *For any pair of C*-algebras A and B, there exists a one-to-one correspondence between quasihomomorphisms $A \rightrightarrows B$ and the set $\operatorname{Hom}(q_A, B)$. The correspondence is given by*

$$A \overset{\varphi_+}{\underset{\varphi_-}{\rightrightarrows}} M \triangleright I \longrightarrow B \qquad \overset{\Phi}{\longmapsto} \qquad q(\varphi_+, \varphi_-) = \varphi_+(\cdot) - \varphi_-(\cdot),$$

$$\varphi\colon q_A \longrightarrow B \qquad \overset{\Psi}{\longmapsto} \qquad A \overset{p\iota}{\underset{p\bar{\iota}}{\rightrightarrows}} Q_A/\ker\varphi \triangleright q_A/\ker\varphi \subseteq B,$$

where p denotes the canonical quotient map.

Proof. The proof is based on the previous observations combined with certain identifications arising hereby. In fact, the tricky part of the proof is understanding why the map Ψ works; they will automatically become mutual inverses afterwards. Let $(\varphi_+, \varphi_-)\colon A \rightrightarrows B$ be any prequasihomomorphism passing through $M \triangleright I$. The image of $Q(\varphi_+, \varphi_-)$ is the C*-algebra generated by the images $\varphi_\pm(A)$, which isomorphic to M due to (Q1). Due to the property (Q3), the $*$-homomorphism

$$\mu: \quad A \cong \iota(A) \xrightarrow{Q(\varphi_+,\varphi_-)} M \longrightarrow M/I$$

must be a $*$-monomorphism. We proceed to investigating the role of q_A in terms of the maps involved. Consider the induced map $\sigma = Q(\mathrm{id}_A, \mathrm{id}_A)$. The restrictions of σ onto $\iota(A)$ and $\bar{\iota}(A)$ are obviously injections whereas $\sigma(q_A) = \{0\}$ by construction. It follows that $q_A \ker\sigma$, whereupon

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_A & \longrightarrow & q_A & \xrightarrow{\sigma} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q(\varphi_+,\varphi_-)} & & \downarrow{\scriptstyle Q(\varphi_+,\varphi_-)} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & I & \longrightarrow & M & \longrightarrow & M/I & \longrightarrow & 0
\end{array}$$

becomes commutative with exact rows and the upper one being split. Some diagram chasing exploiting that injectivity of μ will reveal that the kernel of $Q(\varphi_+, \varphi_-)$ is the kernel of $q(\varphi_+, \varphi_-)$, both determining ideals in q_A hereby. We denote the coinciding kernel by N for simplicity. Due to the vertical maps, save μ, being $*$-epimorphisms, the first isomorphism theorem implies that $Q_A/N \cong M$ together with $q_A/N \cong I$. Ergo, the quasihomomorphisms

$$A \rightrightarrows Q_A/N \hookleftarrow q_A/N \qquad\qquad A \rightrightarrows M \hookleftarrow I$$

must be isomorphic, implying $\Psi\Phi = \mathrm{id}$. Conversely, given a $*$-homomorphism $\varphi\colon q_A \longrightarrow B$, the quasihomomorphism $\Psi(\varphi)$ has the induced $*$-homomorphism $q(p\iota, p\bar{\iota})$ assigned to it via Φ, where p denotes the quotient map. However, one has $q(p\iota, p\bar{\iota})(q(a)) = p(q(a))$ for all $a \in A$. Since the map $q(a) \mapsto p(q(a))$ is merely φ in disguise, we deduce that $\Phi\Psi = \mathrm{id}$, completing the proof. □

Observation 2: Functoriality and projection maps. Let A and B be C*-algebras.

- We unravel properties of q_A by extending it to a genuine functor. Suppose $\varphi\colon A \longrightarrow B$ denotes a $*$-homomorphism and define $q(\varphi)\colon q_A \longrightarrow q_B$ via the actions $\iota(\cdot) \mapsto \iota(\varphi(\cdot))$ and $\bar{\iota}(\cdot) \mapsto \bar{\iota}(\varphi(\cdot))$. There are two distinguished $*$-homomorphisms $\pi_0^A, \pi_1^A\colon q_A \longrightarrow A$ associated A, namely

$$\pi_0^A = q(\mathrm{id}_A, 0) \quad \text{and} \quad \pi_1^A = q(0, \mathrm{id}_A).$$

- We define the *transposition morphism* on q_A to be the $*$-automorphism β^A acting hereon, defined by interchanging the point-images $\iota(a)$ and $\bar{\iota}(a)$ for every $a \in A$. Notably,

$$\pi_0^A \circ \beta^A = \pi_1^A \quad \text{and} \quad (q(\varphi,\psi) \circ \beta^A)(q(a)) = \psi(a) - \varphi(a), \quad a \in A. \tag{3.2}$$

  The latter one stems from $q(\varphi,\psi)$ mapping any generating element $\iota(a) - \bar{\iota}(a) = q(a)$ into the difference $\varphi(a)-\psi(a)$, however, β^A interchanged these two terms beforehand, granting the asserted. The significance of β^A is paramount, for its induced map in KK-theory becomes the inversion map. We will omit the superscripts when addressing a single C*-algebra for brevity.

Modulo the slightly obvious notion of homotopic quasihomomorphisms, KK-theory is nearly within our grasp. However, the tricky part has risen: the actual group operation requires some meticulous care. As a final preparation, we take a minor detour towards stable C*-algebras.

Definition. For every C*-algebra A, we define $d_i\colon A \longrightarrow \mathrm{M}_2(A)$ to be the canonical i'th diagonal embedding for each index $i = 1, 2$. Using this notation, we refer to A as being *stable* if there exists an isomorphism $\kappa\colon \mathrm{M}_2(A) \longrightarrow A$ such that κd_1 and κd_2 are homotopic to the identity on A.

Lemma 3.1.2 (Stabilization)**.** *For every C*-algebra A, the spatial tensor product $A \otimes \mathbb{K}$ is stable.*

Proof. Recall that $\varinjlim \mathrm{M}_n(A)$, with the canonical diagonal embeddings as connecting morphisms, may be identified with the norm-closure of the union $\bigcup_{k=1}^{\infty} \mathrm{M}_k(A)$. Moreover, the involutive algebra $A \odot F \subseteq A \otimes \mathbb{K}$ is norm-dense where $F = \bigcup_{k=1}^{\infty} \mathbb{M}_k$, because F lies densely within the compact operators on $\ell^2(\mathbb{N})$. To deduce that $\varinjlim \mathrm{M}_n(A) \cong A \otimes \mathbb{K}$ it suffices to show that $A \odot F \cong \bigcup_{k=1}^{\infty} \mathrm{M}_k(A)$ as involutive algebras by density in conjunction with uniqueness of $*$-norms. It is apparent that

$$A \odot F \cong A \odot \left(\bigcup_{k=1}^{\infty} \mathbb{M}_k \right) = \bigcup_{k=1}^{\infty} A \odot \mathbb{M}_k \cong \bigcup_{k=1}^{\infty} \mathrm{M}_k(A),$$

where the latter isomorphism stems from summing the isomorphisms $A \otimes \mathbb{M}_k \cong \mathrm{M}_k(A)$ over all indices $k \in \mathbb{N}$. Exploiting this yields the sought identification due to

$$\mathrm{M}_2(A \otimes \mathbb{K}) = \varinjlim \mathrm{M}_2(\mathrm{M}_n(A)) \cong \varinjlim \mathrm{M}_{2n}(A) \cong A \otimes \mathbb{K}$$

being valid as inductive limits remain unchanged when passing to subsequences. The last two conditions are omitted and we stop here to stay on track. ☐

Frequently, the procedure prescribed in the lemma is referred to as *stabilization*. Having settled the stability technicalities, the KK-groups are within our grasp. Without further ado: the KK-theory group will be defined.

Definition. We declare that two quasihomomorphisms $(\varphi_+, \varphi_-)\colon A \rightrightarrows B$ and $(\psi_+, \psi_-)\colon A \rightrightarrows B$ are *homotopic* provided there exists a family $(\sigma_+^t, \sigma_-^t)\colon A \rightrightarrows B$ consisting of quasihomomorphisms indexed continuously (in norm) over the unit interval $[0,1] \subseteq \mathbb{R}$ and such that one has $\sigma_\pm^0 = \varphi_\pm$ together with $\sigma_\pm^1 = \psi_\pm$. The equivalence relation is denoted by $\sim_q$.

Remark. One may readily verify that $\sim_q$ and the ordinary homotopy equivalence relation coincide under the identification prescribed in proposition 3.1.1. We will therefore not distinguish between the two points of view, however, having both has several beneficial factors; frequently quasihomomorphisms are properly suited to tackle one situation and vice versa.

Definition. For every pair of C*-algebras A and B, we define the KK-*theory* or KK-*group* as

$$\mathrm{KK}(A,B) = \mathrm{Hom}(q_A, B \otimes \mathbb{K})/\sim_h$$

with the relation being the ordinary one of $*$-homomorphisms. Equivalently, we define $\mathrm{KK}(A,B)$ to be the set consisting of all quasihomomorphisms modulo $\sim_q$.

Proposition 3.1.3. *Let B be any stable C*-algebra and define on $\mathrm{Hom}(q_A,B)/\sim_h$ a composition $''+''$ hereon by setting*

$$[\varphi] + [\psi] = [\kappa(\varphi \oplus \psi)]$$

for any pair of quasihomomorphisms $\varphi, \psi\colon A \longrightarrow B$. Here κ denotes the existing $*$-isomorphism $\mathrm{M}_2(A) \cong A$ and we abbreviate $\varphi\oplus\psi = d_1\varphi + d_2\psi$. Under these premises, $[q_A,B] = \mathrm{Hom}(q_A,B)/\sim h$ defines an abelian group having $+$ as the additive composition, $[0]$ as the neutral element and with the inversion map being $[\varphi] \mapsto [\varphi\beta^A]$. In particular, $\mathrm{KK}(\cdot,\cdot)$ defines an abelian group.

Proof. We confine ourselves with tackling inverse elements and the neutral element, for the remaining axioms are trivially true because of lemma 3.1.2. Suppose $\varphi\colon q_A \longrightarrow B$ is a $*$-homomorphisms and note that the identity $[\varphi] + [0] = [\kappa(d_1\varphi)] = [\varphi]$ holds due to $\kappa d_1 \cong_H \mathrm{id}_A$ being true by stability, proving $[0]$ to be an additive right-sided neutral elements whereas the left-sided part is proven similarly by exploiting $\kappa d_2 \cong_H \mathrm{id}_A$. According to proposition 3.1.1. proving $[\varphi] + [\varphi\beta^A] = [0]$ amounts to showing that for every quasihomomorphism $(\varphi_+, \varphi_-)\colon A \rightrightarrows B$ one has

$$\left(\begin{pmatrix}\varphi_+ & 0\\ 0 & \varphi_-\end{pmatrix}, \begin{pmatrix}\varphi_- & 0\\ 0 & \varphi_+\end{pmatrix}\right) \sim_q \left(\begin{pmatrix}\varphi_+ & 0\\ 0 & \varphi_-\end{pmatrix}, \begin{pmatrix}\varphi_+ & 0\\ 0 & \varphi_-\end{pmatrix}\right),$$

since $q(\psi,\psi) = 0$ for any $*$-homomorphism $\psi\colon q_A \longrightarrow B$. The sought homotopy family is merely the one fixing the first entry and rotating the second accordingly, all of which are continuous maps. □

3.2 Properties of KK-Theory and the UCT-class

We continue our journey into the wonders of KK-theory and address unique properties associated to the functor(s). Unfortunately, there will be a lack of proofs during this section, for deriving either demands effort. Alas, we must settle with understanding the majority of statements. Afterwards, we discuss the UCT-class in brevity. This class of C*-algebras has staggering properties suitable for classification of nuclear C*-algebras. The first unique aspect of KK-theory that distinguishes itself from ordinary K-theory and K-homology: the Kasparov product. However, proving the existence of the Kasparov product requires a technical intermediate result.

Successive Free Products. Suppose A is some C*-algebra. For every positive integer n we inductively define the *n-fold free product* of A to be $q_A^n = qq_A^{n-1}$ and $q_A^1 = q_A$. The assignment $A \mapsto q_A^n$ thus defines a functor for every such integer n having $q^n(\varphi)\colon q_A^n \longrightarrow q_B^n$ as induced morphism with respect to some $*$-homomorphism $\varphi\colon A \longrightarrow B$.

Lemma 3.2.1. *For every separable C*-algebra A, the associated C*-algebras q_A and q_A^2 become homotopy equivalent up to stabilizing by, meaning spatially tensoring with $\mathrm{M}_2(\cdot)$. Specified further in detail, there exists a $*$-homomorphism $\sigma_A \colon q_A \longrightarrow \mathrm{M}_2(q_A^2)$ satisfying*

$$\pi_0\sigma_A \cong_H d_1 \quad \text{together with} \quad \sigma_A\pi_0' \cong_H d_1',$$

where the maps d_1 and d_1' represent the canonical upper-left diagonal embeddings $q_A \hookrightarrow \mathrm{M}_2(q_A)$ and $q_A^2 \hookrightarrow \mathrm{M}_2(q_A^2)$, respectively, while π_0 and π_0' denote the induced projection maps $\mathrm{M}_2(q_A^2) \longrightarrow \mathrm{M}_2(q_A)$ and $q_A^2 \longrightarrow q_A$ from the second observation, respectively.

Proof. See [4, theorem 1.6] for a rigorous proof of the statement. □

We omit the proof of the following theorem due to Cuntz and instead focus on the pivotal application whereof the Kasparov product arises. The derivation revolves around using the lemma to ensure that the identifications are compatible, thereby retaining an associative bilinear product.

Theorem 3.2.2 (Cuntz)**.** *Suppose A, B and C are C*-algebras.*

(i) There exists for any pair $n, m \in \mathbb{N}$ an associative product

$$[q_A^n, B] \times [q_B^m, C] \longrightarrow [q_A^{n+m}, C]; \quad ([\varphi], [\psi]) \longmapsto [\psi q^m(\varphi)].$$

(ii) Whenever A is separable and B is stable, the morphism $\pi_0^* \colon [q_A^n, B] \longrightarrow [q_A^{n+1}, B]$ given by

$$\pi_0^*([\varphi]) = [\varphi\pi_0],$$

where π_0^* is the aforementioned projection map $q_A^{n+1} \longrightarrow q_A^n$, defines an isomorphism of abelian groups. In particular, standard induction yields $[q_A^n, B] \cong [q_A, B]$.

(iii) Whenever A is separable and B together with C are stable, there exists a $\mathbb{Z}$-bilinear product

$$[q_A, B] \times [q_B, C] \longrightarrow [q_A, C]; \quad [\varphi] \cdot [\psi] = [\psi_2 q(\varphi)_2 \sigma_A].$$

Here σ_A is the $*$-homomorphism in lemma 3.2.1, ψ_2 is the 2-amplification $\mathrm{M}_2(qB) \longrightarrow \mathrm{M}_2(C)$ of the $*$-homomorphism $\psi \colon q_B \longrightarrow C$ and $q(\varphi)_2$ is the 2-amplification associated to $q(\varphi)$. We naturally identify $\mathrm{M}_2(C)$ with C in the KK-groups to obtain the designated codomain.

In order to fully appreciate the Kasparov product, we throw some properties into the mix. It turns out that the Kasparov product succeeds in describing the product of induced maps on KK-theory and the composition of the quasihomomorphism from the ambient spaces. For the sake of convenience, given a $*$-homomorphism $\varphi \colon A \longrightarrow B$, let $\mathrm{KK}(\varphi)$ to be the class in KK induced by the composition $q_A \xrightarrow{\pi_0} A \xrightarrow{\varphi} B$.

Theorem 3.2.3 (Kasparov Product)**.** *Let A, B and C be C*-algebras with A separable. Under these premises, there exists a $\mathbb{Z}$-bilinear associative product $\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \longrightarrow \mathrm{KK}(A, C)$. In addition, for every pair of $*$-homomorphisms $\varphi \colon A \longrightarrow B$ and $\psi \colon B \longrightarrow C$ of C*-algebras, the Kasparov product satisfies the following compatibility properties:*

- $\mathrm{KK}(\varphi) \cdot \mathrm{KK}(\psi) = \mathrm{KK}(\varphi \circ \psi)$;
- *If φ is an isomorphism, then $\mathrm{KK}(\varphi)$ is an isomorphism;*
- *The associated abelian group $\mathrm{KK}(A, A)$ becomes a ring having the Kasparov product as multiplicative composition and $1_A = \mathrm{KK}(\mathrm{id}_A) = [\pi_0^A]$ as the multiplicative identity.*

Proof. Upon tensoring with the identity on the compact operators $\mathbb{K}$ acting on a separable Hilbert space, we obtain a homomorphism $\tau\colon [q_B, C \otimes \mathbb{K}] \longrightarrow [q_{B\otimes\mathbb{K}}, C \otimes \mathbb{K}]$. Due to tensoring with $\mathbb{K}$ producing a stable C*-algebra, theorem 3.2.2 produces an associative $\mathbb{Z}$-linear "·" product

$$[q_A, B \otimes \mathbb{K}] \times [q_B, C \otimes \mathbb{K}] \xrightarrow{\mathrm{id}\times\tau} [q_A, B \otimes \mathbb{K}] \times [q_{B\otimes\mathbb{K}}, C \otimes \mathbb{K}] \stackrel{\cdot}{\longrightarrow} [q_A, C \otimes \mathbb{K}]$$

This establishes existence. To achieve the compatibility properties, observe that given any triple (φ, ψ, ϕ) of $*$-homomorphism for which the codomain of ϕ coincides with the domain of the remaining, two one has the relation $q(\varphi, \psi) \circ q(\phi) = q(\varphi\phi, \psi\phi)$. As such for every pair of classes $[\alpha] \in [q_A, B]$ and $\beta \in [q_B, C]$ one may infer that

$$\begin{aligned}
[\alpha] \cdot [\psi\pi_0^A] &= [(\psi\pi_0^A)_2 q(\alpha)_2 \sigma_A] \\
&= [\psi_2(q(\mathrm{id}_A, 0) \circ q(\alpha))_2 \sigma_A] \\
&= [\psi_2(\alpha\pi_0^A)_2 \sigma_A] \\
&= [\psi_2 \alpha_2 d_1] \\
&= [\psi\alpha],
\end{aligned}$$

due to $\pi_0\sigma_A$ being homotopic to the diagonal embedding map $d_1\colon q_A \longrightarrow \mathrm{M}_2(q_A)$ on the merits of lemma 3.2.1. Here we naturally regard the resulting 2-amplifications as merely being the ordinary maps when composing with d_1. In a similar fashion, one may deduce that $[\varphi\pi_0^A] \cdot [\beta] = [\beta q(\varphi)]$. Applying these considerations grants us the rearrangement

$$\mathrm{KK}(\varphi) \cdot \mathrm{KK}(\psi) = [\varphi\pi_0^A] \cdot [\psi\pi_0^A] = [\psi\varphi\pi_0^A] = \mathrm{KK}(\psi \circ \varphi)$$

for ever pair of $*$-homomorphisms $\varphi\colon A \longrightarrow B$ and $\psi\colon B \longrightarrow C$, verifying the first property whereas the two remaining ones are immediate consequences. This finalizes the proof. ☐

Having build the KK-groups together with the corresponding Kasparov product, we proceed towards understanding the axioms of KK-theory as an abstract functor. In fact, Higson proved that the functor $\mathrm{KK}(A, \cdot)$ from the category of separable C*-algebras into the category of abelian groups is the unique stable homotopy invariant and split-exact functor up to natural transformation. We will not dwell further into this, however, we shall describe KK-theory abstractly in categorical terms.

Definition. We define **KK** to be the category having separable C*-algebras as objects and elements in the corresponding KK-groups as morphisms. The category **KK** gives rise to a functor from the category of separable C*-algebras into **KK** mapping a morphism $\varphi\colon A \longrightarrow B$ into $\mathrm{KK}(\varphi)$.

The abstract framework has several beneficial factors, including the characterization due to Higson that provides a certain degree of uniqueness. For instance, having the KK-groups be the morphisms in **KK** supplies us with a versatile notion of KK-equivalence described next. For the record, we shall omit writing class $[\cdot]$ for the elements in **KK** unless confusion may occur. Afterwards we discuss functorial properties.

Definition. Let A and B be objects in **KK**. A morphism α in $\mathrm{KK}(A, B)$ is said to be *invertible* provided there exists another morphism in $\mathrm{KK}(B, A)$, suggestively denoted α^{-1}, fulfilling the relations $\alpha \cdot \alpha^{-1} = 1_A$ and $\alpha^{-1} \cdot \alpha = 1_B$. The collection of invertible morphisms in $\mathrm{KK}(A, B)$ is commonly denoted by $\mathrm{KK}(A, B)^{-1}$. Two separable C*-algebras are referred to as being *KK-equivalent*, abbreviated symbolically into $A \cong_{\mathrm{KK}} B$, if the corresponding set $\mathrm{KK}(A, B)^{-1}$ is nonempty or, equivalently, if they are isomorphic objects in **KK**.

Proposition 3.2.4. *Suppose A and B are separable C*-algebras. Any isomorphism $\varphi : A \longrightarrow B$ induces an invertible morphism $\mathrm{KK}(\varphi)$ in $\mathrm{KK}(A, B)$, i.e., an invertible element in* ***KK***.

Proof. An obvious consequences of the rule $\mathrm{KK}(\varphi) \cdot \mathrm{KK}(\psi) = \mathrm{KK}(\varphi \circ \psi)$ from theorem 3.2.3. □

Theorem 3.2.5. *For every pair of separable C*-algebra A and B, the associated KK-group determines a bifunctor given by the assignments $A \mapsto \mathrm{KK}(A, \cdot)$ and $B \mapsto \mathrm{KK}(\cdot, B)$ such that the first is contravariant while the second is covariant. Furthermore, the bifunctor fulfills the following.*

- $\mathrm{KK}(\cdot,\cdot)$ *is split-exact in both variables meaning for every split-exact sequence in the category C^*_s-Alg, the sequence obtained by applying $\mathrm{KK}(\cdot,\cdot)$ in either variable remains split-exact.*
- $\mathrm{KK}(A,\cdot)$ *is homotopy invariant meaning any two homotopic morphisms $\varphi, \psi \colon B_1 \longrightarrow B_2$ in C^*_s-Alg induce the same morphism $\mathrm{KK}(A, B_1) \longrightarrow \mathrm{KK}(A, B_2)$.*
- $\mathrm{KK}(A,\cdot)$ *is stable meaning the $*$-homomorphism $B \longrightarrow B \otimes \mathbb{K}$ defined by $b \mapsto b \otimes p$, where p is some rank-one projection in $\mathbb{K}$, induces an isomorphism $\mathrm{KK}(A, B) \longrightarrow \mathrm{KK}(A, B \otimes \mathbb{K})$.*

Proof. We restrict ourselves to proving split-exactness. Choose some split-exact sequence

$$0 \longrightarrow I \xrightarrow{\ \psi\ } A \underset{\pi}{\overset{s}{\rightleftarrows}} B \longrightarrow 0$$

in the C*-algebraic setting. In general, any exact sequence in either the category of C*-algebras or abelian groups on the above form splits if and only if one has a decomposition $I \oplus B \cong A$. The strategy will be to obtain such a decomposition, and upon tensoring with $\mathfrak{K}$ we may assume that A and B are stable C*-algebras. At first we shall reduce the problem into constructing invertible morphisms in $\mathrm{KK}(A, B \oplus I)$ and $\mathrm{KK}(B \oplus I, A)$. Consider any pair of C* algebras A_0 and B_0, separable if you will, then consider the free product $q(A_0 \oplus B_0)$. The direct sum $q_{B_0} \oplus q_{B_0}$ evidently admits two $*$-monomorphisms $q_{B_0}, q_{A_0} \hookrightarrow q_{A_0} \oplus q_{B_0}$. The universal property of free products thus produces a commutative diagram

$$\begin{array}{ccccc} q_{A_0} & \longrightarrow & q_{A_0 \oplus B_0} & \longleftarrow & q_{B_0} \\ & \searrow & \downarrow{\scriptstyle \alpha} & \swarrow & \\ & & q_{A_0} \oplus q_{B_0} & & \end{array}$$

wherein each non-vertical morphism is a $*$-monomorphism. By commutativity of the diagram, α must be an isomorphism. Therefore, the functor $q(\cdot)$ is additive. Additivity of the spatial tensor product, meaning $(A_0 \oplus B_0) \otimes \mathbb{K} \cong (A_0 \otimes \mathbb{K}) \oplus (B_0 \otimes \mathbb{K})$, in conjunction with additivity of the bifunctor $\mathrm{Hom}(\cdot,\cdot)$ entail the trail of identifications of abelian groups

$$\mathrm{KK}(A_0 \oplus B_0, B) \cong [q_{A_0} \oplus q_{B_0}, B \otimes \mathbb{K}] \cong [q_{A_0}, B \otimes \mathbb{K}] \oplus [q_{B_0}, B \otimes \mathbb{K}] = \mathrm{KK}(A_0, B) \oplus \mathrm{KK}(B_0, B),$$

$$\mathrm{KK}(A, A_0 \oplus B_0) \cong [q_A, (A_0 \otimes \mathbb{K}) \oplus (B_0 \otimes \mathbb{K})] \cong [q_A, A_0 \otimes \mathbb{K}] \oplus [q_A, B_0 \otimes \mathbb{K}] = \mathrm{KK}(A, A_0) \oplus \mathrm{KK}(A, B_0).$$

Formulated more concisely, the KK-bifunctor is biadditive. If there were invertible morphisms in $\mathrm{KK}(A, B \oplus I)$, then $\mathrm{KK}(\cdot, A) \cong \mathrm{KK}(\cdot, B \otimes I) \cong \mathrm{KK}(\cdot, B) \oplus \mathrm{KK}(\cdot, I)$ and similarly for the second variable, proving split-exactness. Ergo, our objective will be to determine two morphisms that are multiplicatively inverses to one another.

To construct these two invertible morphisms, let throughout e_{ij} be the (i,j)'th unit matrix in the full separable matrix algebra $\mathbb{M}_2$. Define α as the unique element in $\mathrm{KK}(A, B \oplus I)$ induced by the quasihomomorphism $(\pi \oplus \mathrm{id}_I, 0 \oplus s\pi) \colon \mathrm{M}_2(A) \rightrightarrows B \oplus I$. Let further $\beta = \mathrm{KK}(\varphi)$, where

$\varphi\colon B \oplus I \longrightarrow \mathrm{M}_2(A)$ is the $*$-homomorphism defined via $(b, a) \mapsto s(b) \otimes e_{11} + \psi(a) \otimes e_{22}$. Suppose $x = b \oplus a$ belongs to $B \oplus I$ and notice that when regarding α and β as being $*$-homomorphisms,

$$\begin{aligned} \alpha\beta(\iota(x) - \bar{\iota}(x)) &= \alpha q(\varphi, 0)(\iota(x) - \bar{\iota}(x)) \\ &= \alpha(s(b) \otimes e_{11} + \psi(a) \otimes e_{22}) \\ &= q(\pi \oplus \mathrm{id}, 0 \oplus s\pi)(s(b) \otimes e_{11} + \psi(a) \otimes e_{22}) \\ &= \pi s(b) \otimes e_{11} + \psi(a) \otimes e_{22} \\ &= b \otimes e_{11} + \psi(a) \otimes e_{22} \end{aligned}$$

whereupon $\alpha\beta$, through an application of the first rule in theorem 3.2.5, may be identified with the element arising from the $*$-homomorphism $\varrho\colon B \oplus I \longrightarrow B \oplus I$ given by the 2×2-matrix $(b, a) \mapsto b \otimes e_{11} + a \otimes e_{22}$. This matrix clearly represents the identity on $B \oplus I$. The reader is encouraged to verify that $\beta\alpha$ may in a resembling manner be identified with a $*$-homomorphism, which is $*$-homotopic to the one written in matrix form as $b \mapsto b \otimes e_{11}$. Since this merely represents the identity on B, we may infer that $\varphi = \psi^{-1}$. Voila. □

We arrive at the finale of KK-theory required to fully understand the remainder of the project: the universal coefficient theorem. The theorem is due to the work of Rosenberg and Schoebet. Formally, it describes a sufficient condition of a separable C*-algebra to satisfy a K-theoretic version of the ordinary universal coefficient theorem from homological algebra by using KK-equivalence. However, to truly understand the origin the writer deemed a slightly detour more insightful.

Observation 3. Let A and B be separable C*-algebras. One may prove that KK-theory encodes both K-groups by establishing (far easier spoken than achieved) the existence of two isomorphisms

$$\mathrm{K}_0(B) \cong \mathrm{KK}(\mathbb{C}, B) \quad \text{and} \quad \mathrm{K}_1(B) \cong \mathrm{KK}(C_0(\mathbb{R}), B).$$

Exploiting these identifications permits us to define group homomorphisms

$$\begin{aligned} \kappa_0\colon \mathrm{KK}(A, B) &\longrightarrow \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)); \qquad \kappa_0([\varphi])([p]) = [p] \cdot [\varphi], \\ \kappa_1\colon \mathrm{KK}(A, B) &\longrightarrow \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B)); \qquad \kappa_1([\varphi])([u]) = [u] \cdot [\varphi]. \end{aligned}$$

These satisfy that $\kappa_n(\mathrm{KK}(\varphi))[a] = \mathrm{K}(\varphi)([a])$ for $n = 0, 1$ and a being either a projection or unitary, depending on the n, for every $*$-homomorphism $\varphi\colon A \longrightarrow B$. Moreover, they are compatible with the Kasparov product in the sense that for all quasihomomorphisms $\varphi, \psi\colon q_A \longrightarrow B \otimes \mathbb{K}$.

$$\kappa_n([\varphi] \cdot [\psi]) = \kappa_n([\varphi]) \circ \kappa_n([\psi]), \quad n = 0, 1. \tag{3.3}$$

An interesting consequence of (3.3) is that κ_* maps invertible elements into isomorphisms between the corresponding K-groups. The converse, however, may be false. Much alike the ordinary universal coefficient theorem, one seeks to implement a group measuring the failure of a converse. This leads to the vital UCT-class and the universal coefficient theorem, proven in [11].

Definition. A C*-algebra A is called K-abelian whenever $A \cong_{\mathrm{KK}} C_0(\Omega)$ for some locally compact Hausdorff space Ω, meaning A is KK-equivalent to some abelian C*-algebra. We define the UCT-class $\mathcal{U}$ to be class consisting of separable K-abelian C*-algebras and let $\mathcal{U}_N$ be the subclass wherein each member is nuclear.

Theorem 3.2.6 (Rosenberg, Schoechet)**.** *Suppose A is some σ-unital C*-algebra in the* UCT *and let B be any separable σ-unital C*-algebra. Then there exists a split-exact sequence in C^*_s-Alg:*

$$0 \longrightarrow \mathrm{Ext}(\mathrm{K}_n(A), \mathrm{K}_{n+1}(B)) \longrightarrow \mathrm{KK}(A, B) \xrightarrow{\kappa_0 \oplus \kappa_1} \mathrm{Hom}(\mathrm{K}_n(A), \mathrm{K}_n(B)) \longrightarrow 0$$

As a special case, any morphism α in $\mathrm{KK}(A, B)$ must be invertible if and only if both $\kappa_0(\alpha)$ and $\kappa_1(\alpha)$ determine isomorphisms of abelian groups.

3.3 Hilbert C*-Modules

Hilbert C*-modules form a natural generalization of Hilbert spaces by permitting the inner product to attain values in any C*-algebra. The notion was originally introduced by Kardison, although he allowed only values in $C(\Omega)$ to be taken. Hilbert C*-modules are significant in the construction of groupoid C*-algebras and Kasparov's picture of KK-theory. We solely consider the former application and commence with presenting basic notions, thereafter addressing towards completions.

Definition. Suppose A denotes a C*-algebra and let $\mathcal{H}$ be a $\mathbb{C}$-vector space equipped with a left A-module structure. An *A-valued semi-inner product* on $\mathcal{H}$ is a map $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \longrightarrow A$ subject to

- $\langle \xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$;
- $\langle \cdot, \cdot \rangle$ is linear in the second variable;
- $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for all vectors $\xi, \eta \in \mathcal{H}$;
- $\langle \xi, a\eta \rangle = \langle \xi, \eta \rangle a$ for all vectors $\xi, \eta \in \mathcal{H}$ and $a \in A$.

We call $\langle \cdot, \cdot \rangle$ an *A-valued inner product* whenever $\langle \xi, \xi \rangle = 0$ occurs if and only if $\xi = 0$. In the event of $\mathcal{H}$ having an A-valued inner product, we define a norm $\| \cdot \|_{\mathcal{H}} \colon \mathcal{H} \longrightarrow \mathbb{R}^+$ by $\|\xi\|_{\mathcal{H}}^2 = \|\langle \xi, \xi \rangle\|_A$, thereof mimicking the idea of ordinary Hilbert spaces.

It is apparent that Hilbert spaces admit $\mathbb{C}$-valued inner products. Furthermore, there is an obvious notion analogue to the above wherein the linear space $\mathcal{H}$ is assumed be a right A-module instead. Before finalizing the notion of Hilbert C*-modules, we derive a Cauchy-Schwarz esque inequality for left A-module endowed with an A-valued semi-inner product.

Lemma 3.3.1 (á la Cauchy-Schwarz)**.** *Let A be a C*-algebra and $\mathcal{H}$ some left (resp. right) A-module equipped with an A-valued semi-inner product. Then one has*

$$\|\langle \xi, \eta \rangle\| \leq \|\xi\| \cdot \|\eta\| \quad \textit{together with} \quad \|a.\xi\| \leq \|\xi\| \cdot \|a\|$$

for every pair of vectors $\xi, \eta \in \mathcal{H}$ and $a \in A$.

Proof. Suppose ξ and η were two nonzero vectors inside $\mathcal{H}$. A straightforward application of the well-known inequality $a^*ba \leq \|b\| a^*a$ for elements in the C*-algebra A, where $b \geq 0$, in conjunction with the substitution $a = \langle \xi, \eta \rangle$ permits us to deduce the inequality

$$\langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \geq \langle \xi, \eta \rangle^* \frac{\langle \xi, \xi \rangle}{\|\xi\|^2} \langle \xi, \eta \rangle. \tag{3.4}$$

To provide the reader with some notational overview, we abbreviate $x = \xi \|\xi\|^{-1}$, $a = \langle \xi, \eta \rangle$ and $y = \|\xi\| \eta$. Collecting these substitutions, keeping (3.4) in mind and highly invoking the axioms of an A-valued semi-inner product yields the estimate

$$\begin{aligned} 0 \leq \langle ax - y, ax - y \rangle &= a^* \langle x, x \rangle a - a^* \langle x, y \rangle - \langle y, x \rangle a + \langle y, y \rangle \\ &= \langle \xi, \eta \rangle^* \frac{\langle \xi, \xi \rangle}{\|\xi\|^2} \langle \xi, \eta \rangle - a^* \langle x, y \rangle - \langle y, x \rangle a + \|\xi\|^2 \langle \eta, \eta \rangle \\ &\leq \langle \xi, \eta \rangle^* \langle \xi, \eta \rangle + \|\xi\|^2 \langle \eta, \eta \rangle. \end{aligned}$$

Applying the norm of A on the above grants the sought inequality. The remaining assertion merely stems from $\|a\xi\|^2 = \|a^* \langle \xi, \xi \rangle a\| \leq \|\xi\|^2 \|a\|^2$ becoming valid from the first part. $\square$

Definition. Suppose A denotes a C*-algebra. A left A-module $\mathcal{H}$ endowed with an A-valued inner product such that the associated norm on $\mathcal{H}$ is complete is called *a Hilbert C*-module over A*.

Examples.

- Any Hilbert space is a Hilbert C*-module over $\mathbb{C}$.
- Every C* -algebra A is a Hilbert C*-module over itself having as A-valued inner product, the mapping $\langle \cdot, \cdot \rangle_A$ defined by $\langle a, b\rangle_A = a^*b$ for all $a, b \in A$.
- Let Ω be a compact Hausdorff space, $\mathcal{H}$ be any Hilbert space and suppose $\{\mathcal{H}_x\}_{x\in\Omega}$ is a collection of subspaces in $\mathcal{H}$. Let E be the subspace of $C(\Omega, \mathcal{H})$ whose elements $\xi : \Omega \longrightarrow \mathcal{H}$ fulfill $\xi(x) \in \mathcal{H}_x$ for all x in Ω. Then E may be endowed with a left $C(\Omega)$-action and $C(\Omega)$-valued inner product each given by
$$(g.\xi)(x) = \xi(x)g(x) \quad \text{respectively,} \quad \langle \xi, \eta\rangle(x) = \langle \xi(x), \eta(x)\rangle_{\mathcal{H}}$$
for all $g \in C(\Omega)$, $\xi \in E$ and $x \in \Omega$.

Completion. Suppose $\mathcal{H}_0$ denotes a left (or right) A-module having an A-valued semi-inner product, where A is some C*-algebra. Inspired by the typical separation and completion approach, there is a procedure turning $\mathcal{H}_0$ into a full-fledged Hilbert A-module, executed as follows. According to the first inequality appearing in lemma 3.3.1, the subset $\mathcal{L}$ in $\mathcal{H}_0$ consisting of all vectors ξ fulfilling $\langle \xi, \xi\rangle = 0$ becomes a closed involutive subspace therein. The second inequality of the lemma entails that $\mathcal{L}$ determines a submodule of $\mathcal{H}_0$, so the pairing
$$\mathcal{H} = \overline{\mathcal{H}/\mathcal{L}} \quad \text{and} \quad \langle \xi + \mathcal{L}, \eta + \mathcal{L}\rangle_{\mathcal{H}} = \langle \xi, \eta\rangle_{\mathcal{H}_0}$$
naturally forms a Hilbert C*-module. Now, because of the mirror imaging features of Hilbert C*-modules, one may naively (and rightfully) address the concept of bounded operators acting on these entities. Such a notion is indeed achievable and desired, however, one must tread carefully in order to fully capture essentials from Hilbert space theory including the existence of an adjoint.

Definition. Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert C*-modules defined over some common C*-algebra A. Any linear map $T : \mathcal{H} \longrightarrow \mathcal{K}$ for which there exists an additional linear map $T^* : \mathcal{K} \longrightarrow \mathcal{H}$, called *the adjoint of* T, satisfying the relation $\langle T\xi, \eta\rangle_{\mathcal{K}} = \langle \xi, T^*\eta\rangle_{\mathcal{H}}$ for all $\xi, \eta \in \mathcal{H}$ is called an adjointable operator.

We define $\mathrm{B}(\mathcal{H}, \mathcal{K})$ to be the collection of all adjointable operators equipped with pointwise scalar multiplication and addition, adopting the convention $\mathrm{B}(\mathcal{H}, \mathcal{H}) = \mathrm{B}(\mathcal{H})$ wherein a multiplicative structure defined via ordinary composition of maps is implicitly imposed.

A priori, the notion may appear flawed, for no boundedness conditions are imposed. The reason behind this is somewhat elusive at first glance. Luckily, the good old magical theorem due to Banach-Steinhaus comes to our aid in the following manner. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert C*-modules over a common C*-algebra A. Let T be some adjointable operator, $\xi, \eta \in \mathcal{H}$ and fix $a \in A$. Then
$$\langle \eta, a(T\xi) - T(a\xi)\rangle = \langle \eta, T\xi\rangle a - \langle T^*\eta, \xi\rangle a = 0$$
proving A-linearity. Concerning boundedness, let $\sigma_\xi(\cdot) = \langle T\xi, \cdot\rangle$ for every vector ξ in the unit ball of $\mathcal{H}$. The norm of $\sigma_\xi(\cdot)$ is bounded by $||T^* \cdot ||$ on the merits of the Cauchy-Schwarz inequality, hence the Banach-Steinhaus theorem applies to show that the set $\{||\sigma_\xi(\cdot)|| : \xi \in (\mathcal{H})_1\}$ must be bounded. This clearly shows boundedness of any adjointable operator. Obviously, our main interest lies in $\mathrm{B}(\mathcal{H})$ for this indeed becomes a C*-algebra in a fashion almost identical to the ordinary case.

Unfortunately we will not dwell much further into the Hilbert C*-module theory. The remaining aspect we shall shed light upon is the module $\mathrm{B}(\mathcal{H})$ arising from adjointable operators acting on a Hilbert C*-module and elaborate further on the second example above.

Let $\mathcal{H}$ be a Hilbert C*-module over some C*-algebra A. In a fashion entirely analogue to the algebra of bounded operators on Hilbert spaces, $B(\mathcal{H})$ defines an involutive normed algebra. Since the A-valued inner product on $\mathcal{H}$ must be contractive in both variables due to lemma 3.3.1, $B(\mathcal{H})$ determines a closed subspace of the Banach algebra consisting of bounded operators $\mathcal{H} \longrightarrow \mathcal{H}$. Lastly, the C*-identity is easily deduced from lemma 3.3.1 so that $B(\mathcal{H})$ becomes a C*-algebra in a way compatible with the module structure as described below.

Lemma 3.3.2. *Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert C*-modules over a common C*-algebra A. Let T be an adjointable operator $\mathcal{H} \longrightarrow \mathcal{K}$. If so, one has $\|T\xi\| \leq \|T\| \cdot \|\xi\|$ for every vector ξ inside $\mathcal{H}$.*

Proof. Let ω be some state on A. The induced map map $\omega(\langle\cdot,\cdot\rangle)\colon \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}^+$ defines a semi-inner product (linear in the second variable) on $\mathcal{H}$ in the ordinary sense. This permits us to apply the usual Cauchy-Schwarz inequality of inner product spaces repetitively to obtain

$$\begin{aligned}
\omega(\langle T^*T\xi,\xi\rangle) &\leq \omega(\langle T^*T\xi, T^*T\xi\rangle)^{1/2} \cdot \omega(\langle\xi,\xi\rangle)^{1/2} \\
&= \omega(\langle T^*T)^2\xi,\xi\rangle)^{1/2} \cdot \omega(\|\xi\|^2)^{1/2} \\
&\vdots \\
&\leq \omega(\langle (T^*T)^{2n}\xi,\xi\rangle)^{2^{-n}} \cdot \omega(\|\xi\|^2)^{1/2+1/4+\dots} \\
&= \omega(\langle (T^*T)^{2n}\xi,\xi\rangle)^{2^{-n}} \cdot \omega(\|\xi\|^2)^{1-2^{-n}} \\
&\leq \|\xi\|^{2^{-n}} \cdot \|T\|^2 \cdot \omega(\|\xi\|^2)^{1-2^{-n}}.
\end{aligned}$$

The latter converges to $\|T\|^2 \cdot \omega(\|\xi\|^2)$ whenever $n \to \infty$. Due to the choice of state being arbitrary and the state space separating points in A, the sought inequality follows. ☐

Before ending the section, we ought to describe morphisms preserving the underlying structure of Hilbert C*-modules. Indeed we are poised to take both topological and algebraic aspects into account for the matter.

Definition. Suppose $\mathcal{H}$ and $\mathcal{K}$ denote two left Hilbert C*-module over A and B, respectively. A *morphism* $\pi\colon \mathcal{H} \longrightarrow \mathcal{K}$ is an A-linear bounded map, meaning

- $\pi(a.\xi) = a.\pi(\xi)$ for all $a \in A$ and $\xi \in \mathcal{H}$;
- $\pi(\lambda\xi + \mu\eta) = \lambda\pi(\xi) + \mu\pi(\eta)$ for all $\lambda, \mu \in \mathbb{C}$ and $\xi, \eta \in \mathcal{H}$.

A similar notion concerns right Hilbert C*-modules. If π maps surjectively onto $\mathcal{K}$, we call π an *epimorphism* of Hilbert C*-modules, whereas we refer to π as being a *monomorphism* provided the relation $\langle\pi(\xi),\pi(\eta)\rangle_{\mathcal{K}} = \langle\xi,\eta\rangle_{\mathcal{H}}$ is valid for any pair of vectors $\xi, \eta \in \mathcal{H}$. We call π an *isomorphism*, if π is both monic and epic, whereof $\mathcal{H}$ and $\mathcal{K}$ are called isomorphic, symbolically writing $\mathcal{H} \cong \mathcal{K}$ or perhaps $\mathcal{H} \cong_A \mathcal{K}$ for emphasis on the ambient module structure.

Evidently, isomorphic Hilbert C*-modules may be identified with one another, since the property of being a monomorphism forcing the monomorphism in question to be an isometry with respect to the norms induced via the C*-valued inner products. This concludes the section, so we proceed towards applying the Hilbert C*-module framework into usage by constructing a whole new family of C*-algebras arising from groupoids.

3.4 Groupoid C*-Algebras

Groupoids generalize groups by great lengths, partly by relaxing certain conditions such as uniqueness of a neutral element. The definition of a groupoid varies between the literature despite of these being equivalent and in this project we adopt one primarily suitable for the C*-algebraic framework. Obstacles arise during the procedure of forming a C*-algebra arguably due to the general nature of groupoids. However, the Hilbert C*-module theory established prior until now arrives to our rescue, as we shall witness. We initiate the construction by understanding groupoids.

Definition. A *groupoid* is a small category in which every morphisms is invertible. Elaborating further, a groupoid G consists of a distinguished set G_0 of elements referred to as the *units of* G, a set of morphisms between objects in G_0, commonly denoted by G as well, and maps $s, r\colon G \longrightarrow G_0$, called the *source* and *range*, respectively. The involved maps must fulfill the following axioms.

- The set G_2 of pairs $g, h \in G$ fulfilling $s(g) = r(h)$ admits an associative composition $G_2 \longrightarrow G$ written multiplicatively. A pair $g, h \in G$ determining an element in G_2 is called *composable*.
- For every $u \in G_0$, there is an identity morphism 1_u meaning $1_u g = g$ and $g1_u = g$ for all $g \in G$.
- Every morphism g admits an inverse morphism g^{-1} in G, that is, $g^{-1}g = s(g)$ and $gg^{-1} = r(g)$.
- One has $gs(g) = g = r(g)g$ and $s(u) = u = r(u)$ for all $g \in G$ and $u \in G_0$.
- One has $s(gh) = s(h)$ and $r(gh) = r(g)$ for all $(g, h) \in G_2$.

The third and fourth conditions may appear meaningless at first sight. However, the sheer existence of an identity morphisms 1_u for each unit u yields an injection $G_0 \hookrightarrow G$ of sets merely via the assignment $u \mapsto 1_u$, hence it is customary to regard G_0 as an actual subset of G. To preemptively purge computational confusion, we derive some minor rules.

Lemma 3.4.1. *Suppose G denotes a groupoid. One has:*

- *For any two decomposable elements $g, h \in G$, one has $(gh)^{-1} = h^{-1}g^{-1}$;*
- *for every morphisms $g \in G$, one has $s(g) = r(g^{-1})$ together with $r(g) = s(g^{-1})$;*
- *for every unit u in G, one has $u^{-1} = u$;*
- *for any decomposable pair $g, g_0 \in G$ with $gg_0 = h$ for some suitable morphism h, one has $g = hg_0^{-1}$ together with $g_0 = g^{-1}h$.*

Proof. Each of these properties are borderline trivial, so we settle with proving the the second and fourth rule. Let g be a morphism belonging to G. The axioms of the structure maps provide the identity $s(g^{-1}) = s(gg^{-1}) = s(r(g)) = r(g)$, and similarly one deduces the relation $r(g^{-1}) = s(g)$. To verify the fourth rule, one simply notes that

$$g_0 = r(g_0)g_0 = s(g)g_0 = (g^{-1}g)g_0 = g^{-1}h$$

due to $s(g) = r(g_0)$ by hypothesis. Voila. □

Definition. A *groupoid homomorphism* is a map $\varphi\colon G \longrightarrow H$ between groupoids such that the inclusion $(\varphi \times \varphi)(G_2) \subseteq H_2$ holds while being multiplicative, meaning

$$\varphi(gg_0) = \varphi(g)\varphi(g_0)$$

for every composable pair $g, g_0 \in G$. Note that $\varphi(g^{-1}) = \varphi(g)^{-1}$ and $\varphi(G_0) \subseteq H_0$ automatically.

Since we never require a groupoid to be discrete, we ought to impose some topological conditions. For this purpose, recall that a *local homeomorphisms* $\sigma\colon X \longrightarrow Y$ of topological spaces is a continuous map such that given any point x in X there exists an open neighbourhood U around x for which the image $\sigma(U)$ is open in Y and the restriction $\sigma_{|U}$ is a homeomorphism onto its image.

Definition. A *topological groupoid* is a groupoid G endowed with a Hausdorff topology turning the associated structure maps

$$s, r : G \longrightarrow G_0, \quad g \mapsto g^{-1} \text{ and } G_2 \longrightarrow G$$

continuous, where G_2 inherits the induced subspace topology from the product $G \times G$ and G_0 inherits the subspace topology from G. We call a topological group G *etalé* provided that the range and source maps are local homeomorphisms on G. We abbreviate *locally compact second countable* into LCSC for brevity.

Examples.

- Any group G having $G_0 = \{e\}$ and $G = G$ becomes a groupoid. The required map $G \times G \longrightarrow G$ translates into the usual composition in G. If G is discrete, then G is an etalé LCSC groupoid.
- Any topological space X having $X = G_0$ and $G = \{\mathrm{id}_x : x \in X\}$ becomes a topoloigcal groupoid having $s(x) = x = r(x)$ for all x belonging to X.
- Suppose M is some set having an equivalence relation $\sim$ associated to it. Then $G_0 = M$ and we declare that there exists a morphism $n \longrightarrow m$ if and only if $m \sim n$.

Proposition 3.4.2. *The set of units G_0 in an etalé LCSH groupoid G is open and closed.*

Proof. The diagonal associated to any Hausdorff space is closed, hence to prove that G_0 is closed we seek to recognize G_0 as the preimage thereof under some continuous map. To accomplish this, define $\sigma\colon G \longrightarrow G \times G$ by $g \mapsto (s(g), r(g))$, which is obviously continuous. Since s and r restrict to the identity on G_0, we deduce that G_0 is the preimage of the diagonal on G under σ as desired.

To prove that G_0 must be open, let u be some element therein and choose $V \subseteq G$ to be an open neighbourhood around u turning $s_{|V}$ into a homeomorphism. The intersection $G_0 \cap V \subseteq G_0$ is open within G whereupon $W = V \cap s^{-1}(G_0 \cap V)$ becomes a neighbourhood around u. The objective will be to verify that $W \subseteq G_0$. However, any element g in W is mapped into an element in $V \cap G_0$. Hence we may deduce that $s(s(g)) = s(g)$ in V meaning $g = s(g) \in G_0$ as s defines a homeomorphism on V, proving the claim. □

Notation. Let G be an etalé locally compact groupoid. We hereon define the *source and range fibers* associated to a unit u belonging to G by setting

$$G_u = \{g \in G : s(g) = u\} \quad \text{respectively,} \quad G^u = \{g \in G : r(g) = u\}$$

Alternatively, these are the preimages of units under the source and range maps, whence the name. In particular, they must both be closed subspaces of G. One of several reasons to tacitly demand that G is etalé is following.

Proposition 3.4.3. *The source - and range fibers associated to any unit in an etalé LCSC groupoid G are always discrete subspaces of G.*

Proof. Choose some morphisms g in G_u with u being any unit in G. Let U be an open neighbourhood around g such that s restricts to a homeomorphisms hereon. From injectivity of the range map, $U \cap G_u \cong r(U \cap G_u) = r(U) \cap r(G_u)$ follows. The second factor in the former intersection is a singleton and therefore $U \cap G_u = \{g\}$ for some morphism g. Upon G being second countable, G_u becomes a countable union of singletons, hence must be discrete. Discreteness of G^u may be derived analoguesly, upon replacing the source map with the range map throughout the argument. □

We now associate to any etalé LCSC groupoid G a C*-algebra. Consider the set $C_c(G)$ consisting of compactly supported functions $\xi \colon G \longrightarrow \mathbb{C}$. It clearly defines a complex vector space with pointwise operations. Endowing $C_c(G)$ with the maps

$$(\xi * \eta)(g) = \sum_{h \in G_{s(g)}} \xi(gh^{-1})\eta(h) \quad \text{and} \quad \xi^*(g) = \overline{\xi(g^{-1})},$$

defined for all g in G, forms an involutive algebraic structure on $C_c(G)$ according to the following reasoning. For two compactly supported functions $\xi, \eta \colon G \longrightarrow \mathbb{C}$, the convolution $(\xi * \eta)(g)$ evaluated at $g \in G$ is finite, since the sum above is indexed over $G_{s(g)}$ intersected with the supports of ξ and η, meaning an intersection of a discrete set with compact sets (hence finite). Thus $\xi * \eta$ belongs to $C_c(G)$. Associativity may be verified as follows. Let u be a morphism in G. Then

$$(\xi * \eta) * \mu(u) = \sum_{g \in G_{s(g)}} (\xi * \eta)(ug^{-1})\mu(g) = \sum_{g,h \in G_{s(g)}} \xi(u(hg)^{-1})\eta(h)\mu(g)$$

and

$$\xi * (\eta * \mu)(u) = \sum_{g \in G_{s(g)}} \xi(ug^{-1})(\eta * \mu)(g) = \sum_{g,h \in G_{s(g)}} \xi(ug^{-1})\eta(gh^{-1})\mu(h).$$

As such the substitution $\gamma = hg$ will bring the sought equality. For the sake of convenience, we note that for any pair $\xi, \eta \in C_c(G)$ and element $g \in G$ one has

$$(\xi^* * \eta)(g) = \sum_{h \in G_{s(g)}} \overline{\xi(hg^{-1})}\eta(h) = \sum_{t \in G_{r(g)}} \overline{\xi(t)}\eta(tg), \tag{3.5}$$

$$(\xi * \eta)^*(g) = \sum_{h \in G_{s(g)}} \overline{\xi(hg^{-1})\eta(h^{-1})} = (\xi^* * \eta^*)(g). \tag{3.6}$$

In order to craft a norm on $C_c(G)$, we produce a C*-algebra together with a faithful $*$-homomorphism mapping into it. For this purpose, we define a left action $C_0(G_0) \times C_c(G) \longrightarrow C_c(G)$ together with a $C_0(G_0)$-valued semi-inner product on $C_c(G)$ by declaring that

$$(f.\xi)(t) = \xi(t)f(s(t)) \quad \text{respectively,} \quad \langle \xi, \eta \rangle(u) = \sum_{t \in G_u} \overline{\xi(t)}\eta(t)$$

for all $t \in G$, $u \in G_0$ and $f \in C_0(G_0)$. Collecting all these observations yield the following example of a Hilbert C*-module over $C_0(G_0)$.

Definition. For every etalé LCSC groupoid G we define $L^2(G)$ to be the Hilbert space completion of $C_c(G)$ viewed as an involutive $C_0(G_0)$-module. In symbols,

$$L^2(G) := \overline{C_c(G)/\mathcal{L}_{\langle \cdot, \cdot \rangle}}^{\|\cdot\|}$$

where $\mathcal{L}_{\langle \cdot, \cdot \rangle} = \{\xi \in C_c(G) : \langle \xi, \xi \rangle = 0\}$ and the closure is taken via the induced norm.

Remark. The shape of $C_0(G_0)$-valued inner product in $L^2(G)$ is inspired from the ordinary $\ell^2(G)$ Hilbert space, whence the symbolic notation. Furthermore, one ought to compare the action with the latter example on page 53. In particular, the induced norm $\|\cdot\|_{L^2(G)}$ becomes the "2-norm" on G in the sense that

$$\langle \xi, \xi \rangle(u) = \sum_{t \in G_u} |\xi(g)|^2 \tag{3.7}$$

for every unit u in G and compactly supported functions $\xi \colon G \longrightarrow \mathbb{C}$, upon which

$$\|\xi\|^2_{L^2(G)} = \sup_{u \in G_0} \sum_{t \in G_u} |\xi(g)|^2, \quad x \in C_c(G).$$

Continuing our quest in discovering a C*-norm to $C_c(G)$, we shall investigate the following left-regular representation esque map of groupoids. We define $\lambda \colon C_c(G) \longrightarrow \mathrm{B}(L^2(G))$ by

$$\lambda(\xi)(\eta) = \xi * \eta.$$

in terms of elements $\xi, \eta \in C_c(G)$. Here $\mathrm{B}(L^2(G))$ reads the C*-algebra of adjointable operators that was addressed in the preceding section. We assert λ must be contractive map attaining values in $\mathrm{B}(L^2(G))$. For every pair of compactly supported function $\xi, \eta \colon G \longrightarrow \mathbb{C}$ and unit $u \in G$,

$$\langle \eta, \lambda(\xi^*)\zeta \rangle(u) = \sum_{w \in G_u} \overline{\eta(w)}(\xi^* * \zeta)(w) \overset{(3.5)}{=} \sum_{w \in G_v} \sum_{g \in G_{s(w)}} \overline{\eta(w)\xi(wg^{-1})}\zeta(w)$$

while

$$\langle \lambda(\xi)\eta, \zeta \rangle(u) = \sum_{w \in G_u} \overline{(\xi * \eta)(w)}\zeta(w) = \sum_{w \in G_u} \sum_{g \in G_{s(w)}} \overline{\xi(wg^{-1})\eta(w)}\zeta(w)$$

Substituting $w = u$ yields equality, so we infer that $\lambda(\xi)^* = \lambda(\xi^*)$, proving $\lambda(\xi)$ be an adjointable operator for each compact supported function $\xi \colon G \longrightarrow \mathbb{C}$, hence well-defined. Extending λ by linearity yields a $*$-representation of $C_c(G)$. On the other hand, $|(\xi * \eta)(t)| < \infty$ for $t \in G$, hence

$$\langle \lambda(\xi)\eta, \lambda(\xi)\eta \rangle(u) = \sum_{t \in G_u} |(\xi * \eta)(t)|^2 = \sum_{t \in G_u} \left| \sum_{g \in G_{s(t)}} \xi(tg^{-1})\eta(g) \right|^2 < \infty$$

Upon the latter being finite for every unit u inside G, we have $\|\lambda(\xi)\| < \infty$. Thus λ extends to a $*$-homomorphism, which must be faithful on the merits of $\lambda(\xi^* * \xi) = 0$ forcing $\langle \lambda(\xi)\eta, \lambda(\xi)\eta \rangle = 0$ for all η in $C_c(G)$ in conjunction with the above calculation entailing $\xi = 0$. The plethora of facts allows us build our C*-algebra.

Definition. Suppose G denotes a locally compact etalé groupoid. We define the *reduced groupoid C*-algebra* to be the norm closure of $\lambda(C_c(G)) \hookrightarrow \mathrm{B}(L^2(G))$.

To us groupoids serve as a class of C*-algebras wherein plenty of members belong to the UCT class. The underlying theory behind this derives from Tu's theorem, stating that the reduced groupoid C*-algebra associated to an etalé amenable LCSC groupoid belongs to the UCT class. To properly understand his assertion, we must introduce the notion of amenability for groupoids.

Definition. An etalé LCSC groupoid G is said to be amenable if there exists a net $(\xi_i)_{i \in J}$ consisting of compact supported nonnegative unctions $\xi_i \colon G \longrightarrow \mathbb{C}$ fulfilling

$$\sum_{t \in G_{r(g)}} \xi_i(t) \longrightarrow 1 \quad \text{together with} \quad \sum_{t \in G_{r(g)}} |\xi_i(g) - \xi_i(tg)| \longrightarrow 0$$

for all g belonging to G, uniformly on compact subsets in G.

Once more we must confine ourselves with merely stating various classification results associated to amenability modulo one variation using $L^2(G)$-language. However, it is worthwhile noting that amenable groupoids produce nuclear C*-algebras while the full and reduced version coincide, similar to its discrete group counterpart.

Proposition 3.4.4. *An etalé LCSC groupoid G is amenable if and only if there exists a net $(\eta_i)_{i\in J}$ of compactly supported functions $\eta_i\colon G \longrightarrow \mathbb{C}$ such that $\|\eta_i\| \leq 1$ and $(\eta_i^* * \eta_i)(g) \to 1$ for all g belonging to G, uniformly on compact subsets in G.*

Proof. We omit verifying each component in the proof, due to the tedious nature of the computations. Assume at first the G were amenable with respect to the net $(\xi_i)_{i\in J}$. Define accordingly

$$\mu_i(\cdot) = \max\big\{\langle \xi_i^{1/2}, \xi_i^{1/2}\rangle(\cdot), 1\big\} \quad \text{and} \quad \eta_i(\cdot) = \xi_i(\cdot)^{1/2}\mu_i(s(\cdot))^{-1/2}$$

for each index i in J. If $u \in G_0$ fulfills that $\langle \xi_i^{1/2}, \xi_i^{1/2}\rangle(u)$ exceeds the constant map $u \mapsto 1$, then $\langle \eta_i, \eta_i\rangle(u) = \sum_{t\in G_u} |\eta_i(t)|^2 \to 1$ according to (3.7) for every unit u in G, whereof we infer $\|\eta_i\| \leq 1$. The remaining part is verified through similar computations. Conversely, given a net $(\eta_i)_{i\in J}$ described in the statement, one merely defines $\xi_i(\cdot) = |\eta_i(\cdot)|^2$ for all indices i. Thus,

$$\begin{aligned}
\sum_{t\in G_{r(g)}} |\xi_i(t) - \xi_i(tg)| &\leq \sum_{t\in G_{r(g)}} \big|\,|\eta_i(t)| - |\eta_i(tg)|\,\big|^2 \\
&= \sum_{t\in G_{r(g)}} |\eta_i(t)|^2 + |\eta_i(tg)|^2 - 2|\eta_i(t)|\cdot|\eta_i(tg)| \\
&\overset{(3.5)}{=} \sum_{t\in G_{r(g)}} |\eta_i(t)|^2 + \sum_{t\in G_{r(g)}} |\eta_i(tg)|^2 - 2(\eta_i^* * \eta_i)(g) \\
&\to 1 + 1 - 2 = 0
\end{aligned}$$

for every morphism g in G. We omit calculating further and consider our work done. □

In conjunction with Tu's theorem, we shall apply a rather curious identification occurring for transformation groupoids. We collect the observation into a single swoop for the sake of reference together with a product property of the reduced groupoid C*, proofs being omitted. Afterwards we state Tu's theorem and the nuclearity-amenability correspondence.

Proposition 3.4.5. *Suppose G denotes a discrete group acting continuously on a locally compact topological space X via homeomorphisms. Let $\alpha\colon G \longrightarrow \mathrm{Aut}(X)$ be the action in play and write*

$$X \rtimes G = \big\{(x,t,y) \in X \times G \times X : x = \alpha_t(y)\big\}.$$

Under these premises, $X \rtimes G$ determines a groupoud having X as unit set, the formula $s(x,t,y) = y$ as source map, the formula $r(x,t,y) = x$ as range map and having, for decomposable elements, the rule $(x,s,y)(y,t,z) = (x,st,z)$ as composition. The resulting groupoid $X \rtimes G$ is called "the transformation groupoid" of the pairing (X,G) and it carries an isomorphism $C^*_\lambda(X \rtimes G) \cong C(X) \rtimes_{\alpha,r} G$

Theorem 3.4.6. *An etalé LCSC groupoid G is amenable if and only if $C^*_\lambda(G)$ is nuclear.*

Proof. See [2, theorem 5.6.18] for a rigorous proof. □

Theorem 3.4.7 (Tu)**.** *The reduced groupoid C*-algebra, hence the full, associated to any amenable etalé locally compact second countable groupoid belongs to $\mathcal{U}_N$.*

Proof. We refer to [14] for a rigorous proof. □

Chapter 4

Elementary Amenable Groups and Quasidiagonality

Rosenberg proved a remarkable theorem concerning the reduced group C*-algebra and quasidiagonality, revealing that a discrete countable group must be amenable whenever its associated reduced group C*-algebra is quasidiagonal. The validity of the converse, which Rosenberg himself conjectured to be true, remained unanswered prior to recent discoveries with an affirmative being provided by Tikuisis, White and Winter. However, a partial converse due to Ozawa, Rørdam and Sato was established in 2014. The aforementioned result specifically treats the case of elementary amenable groups and the proof is based on a, so-called, "bootstrap" argument using a C*-algebra arising from a dynamical system encoding quasidiagonality. The chapter seeks to pursue the proof of Sato, Ozawa and Rørdam. Unfortunately, certain classification results concerning $A\mathbb{T}$-algebras of real-rank zero are merely stated and regarded as divine intervention. To begin with, we thoroughly the investigate the dynamical system.

4.1 The Bernoulli Crossed Product

Let $\beta\colon G \curvearrowright S$ be a group action onto some set S written as $g \mapsto g.s$ for every $s \in X$ and $g \in G$. The groups in play will be discrete, so topological aspects are set aside even though one may prove various assertions in the continuous case as well. Suppose A denotes any unital C*-algebra and define the induced *non-commutative Bernoulli shift* action $\sigma^\beta : G \curvearrowright \bigotimes_S A$ by

$$\sigma_g^\beta\left(\bigotimes_{s\in S} a_s\right) = \bigotimes_{s\in S} a_{g^{-1}.s}$$

on elementary tensors, for all g in G. Extend hereafter σ^β $\mathbb{C}$-linearly to an action defined on the entirety of $\bigotimes_S A$. Intuitively speaking, σ^β encodes the "permutation" that the action $G \curvearrowright S$ creates onto the infinite tensor product $\bigotimes_S A$ by permuting the tensor indexes via the formula provided by β. Due to the involutive and algebraic structure on infinite tensor product C*-algebras occurring factorwise, any discrete group G hereby acts on $\bigotimes_G A$ by $*$-automorphisms $\sigma\colon G \longrightarrow \mathrm{Aut}\,(\bigotimes_G A)$ via the Bernoulli shift arising from the left-translation on G, hence determines a unital discrete C*-dynamical system $(\bigotimes_G A, \sigma, G)$. This forms the following dynamical unital C*-algebra.

Definition. For any discrete group G acting on itself by left-translation, the corresponding reduced crossed product $B(G) = (\bigotimes_G \mathbb{M}_{2^\infty}) \rtimes_{\sigma,r} G$ arising from the unital discrete C*-dynamical system $(\bigotimes_G \mathbb{M}_{2^\infty}, \sigma, G)$ is called *the Bernoulli shift crossed product* associated to G.

Notation. Throughout the entire chapter, the C*-algebra $\mathbb{M}_{2^\infty}$ will emerge in a plethora of disguises. To emphasize on its appearances, we abbreviate $\mathbb{M}(I) = \bigotimes_I \mathbb{M}_{2^\infty}$ for any countable indexing set I. We simplify whenever the indexing set consists of a single element, writing $\mathbb{M}(g) = \mathbb{M}(\{g\})$ instead for every element g in G. As such, we have $\mathbb{M}(I) \cong \mathbb{M}(J) \cong \mathbb{M}_{2^\infty}$ for any two not necessarily distinct countable sets I and J due to $\mathbb{M}_{2^\infty}$ being self-absorbing; see for instance proposition 1.4.9.

In the countable case, $B(G)$ inherits powerful invariants from $\mathbb{M}(G)$. In fact, our first objective will be to derive these properties, for we shall require it when applying classification theorems. The sought invariants of $B(G)$ inherited from $\mathbb{M}(G)$ are the existence of a unique tracial state and simplicity. The proof of simplicity relies heavily on a theorem due to Kishimoto. We state the theorem independently below, referring to [1] for the original proof. Recall that an action $\alpha\colon G \longrightarrow \mathrm{Aut}(A)$ on a unital C*-algebra is *inner* if and only if for every g in G there exists some unitary u_s in A fulfilling $\alpha_s = \mathrm{Ad}_{u_s}$. The obtained normal subgroup in $\mathrm{Aut}(A)$ consisting of inner $*$-automorphism is commonly denoted $\mathrm{Inn}(A)$ and elements the quotient $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ are called *outer*.

Theorem 4.1.1 (Kishimoto [1])**.** *Suppose (A, α, G) denotes a unital discrete C*-dynamical system, where A denotes a simple C*-algebra. Then $A \rtimes_{\alpha,r} G$ is simple if and only if α is outer.*

The proof establishing the monotracial property of $B(G)$ exploits a more general observation, which we independently prove because of its independent intriguing features. For those still slightly unacquainted with crossed products and conditional expectations, please read or skim section 1.7.

Proposition 4.1.2. *Suppose (A, α, G) denotes some discrete unital C*-dynamical system admitting a tracial state τ which is α-invariant, meaning $\tau \circ \alpha_g = \tau$ for all g in G. Under these premises, the C*-algebra $A \rtimes_{\alpha,r} G$ admits a tracial state given by the composition $\tau \circ E\colon A \rtimes_{\alpha,r} G \longrightarrow \mathbb{C}$, where E denotes the canonical conditional expectation on $A \rtimes_{\alpha,r} G$.*

Proof. Consider the canonical faithful conditional expectation $E\colon A \rtimes_{\alpha,r} G \longrightarrow A$ defined on the dense involutive subalgebra $C_c(G, A) \hookrightarrow A \rtimes_{\alpha,r} G$ by $E(\sum_{s\in G} a_s u_s) = a_e$ and extended via continuity onto the norm closure. The obtained composed positive linear functional $\tau \circ E$ on $A \rtimes_{\alpha,r} G$ becomes a tracial state hereon due to the following reasoning. Let $\{u_s\}_{s\in G}$ be the collection of unitaries implementing the action α. Suppose one has elements $a = \sum_{s\in G} a_s u_s$ and $b = \sum_{t\in G} b_t u_t$ in $C_c(G, A) \hookrightarrow A \rtimes_{\alpha,r} G$ and note that $t = s^{-1}$ whenever $st = e$. One hereof computes:

$$\begin{aligned}
(\tau \circ E)(ab) = (\tau \circ E)\Bigg(\sum_{s,t\in G} a_s \alpha_s(b_t) u_{st}\Bigg) &= \tau\Bigg(\sum_{s\in G} a_s u_s b_{s^{-1}} u_s^* u_e\Bigg) \\
&= \sum_{s\in G} \tau(\alpha_{s^{-1}}(a_s) b_{s^{-1}} u_e) \\
&= \sum_{s\in G} \tau(b_{s^{-1}} \alpha_{s^{-1}}(a_s) u_e) \\
&= \sum_{s\in G} \tau(b_s \alpha_s(a_{s^{-1}}) u_e) \\
&= (\tau \circ E)(ba).
\end{aligned}$$

The second equality stems from the definitions of E and α being implemented via the unitaries $\{u_s\}_{s\in G}$, the third equality from the α-invariance property of τ combined with $\alpha_{s^{-1}} = \mathrm{Ad}_{u_s^*}$, the fifth is based on the trace property of τ whereas the sixth follows from substituting s with s^{-1}. This verifies the trace property of $\tau \circ E$, hence $\tau \circ E$ becomes a trace acting on a dense subalgebra in $A \rtimes_{\alpha,r} G$, so normalizing yields a tracial state. Continuity of the maps involved entails that $\tau \circ E$ extends to a tracial state on $A \rtimes_{\alpha,r} G$, proving the claim. □

Proposition 4.1.3. *The Bernoulli shift crossed product associated to a countable discrete group is unital, separable, monotracial and simple.*

Proof. Suppose G is some discrete countable group. Let us tackle the easy parts first. The algebra $\mathbb{M}_{2^\infty}$ is a UHF-algebra having $\bigcup_{n=1}^\infty \mathbb{M}_{2^n}$ as dense subset. Due to the latter being a countable union of separable spaces, it becomes separable itself. If A is any unital separable C*-algebra admitting an action $G \curvearrowright A$, the involutive algebra $C_c(G, A)$ becomes separable. Indeed it consists of formal sums $\sum_{s\in G} a_s s$ with $a_s \in A_0$ being zero for all save finitely many $s \in G$, where A_0 denotes some countable dense subset of A. Upon the involutive algebra $C_c(G, A)$ being dense in $A \rtimes_{\alpha,r} G$, the latter contains a dense countable subset. Since $C_c(G, A)$ is generated by A and a collection of unitaries $\{u_s\}_{s\in G}$ implementing the action $G \curvearrowright A$ via inner $*$-automorphisms, wherein $1_A u_e$ acts as the identity, the reduced crossed product $A \rtimes_\alpha G$ becomes unital in addition. Thus $B(G)$ becomes unital and separable due to $\mathbb{M}_{2^\infty}$ being both unital and separable.

In order to prove simplicity of $B(G)$, we shall verify that the induced Bernoulli shift action arising from the left-translation on G, i.e., $\sigma\colon G \longrightarrow \mathrm{Aut}(\mathbb{M}(G))$ must be outer[1]. To achieve this, let some nontrivial element s in G be fixed a projection $(p_n)_{n\geq 1}$ inside the product $\ell^\infty(\mathbb{M}_2, \mathbb{N})$, meaning p_n defines a projection for every positive integer n. We may arrange the sequence $(p_n)_{n\geq 1}$ in a manner that it becomes central, that is, one has $\|[p_n, a]\| \to 0$ as $n \to \infty$ for every a in $\mathbb{M}_2$. For instance, a sequence of diagonal 2×2-matrices having the value 1 along the diagonal will work. The natural embedding $\mathbb{M}_2 \cong \bigotimes_{\{e\}} \mathbb{M}_2 \hookrightarrow \bigotimes_{\mathbb{N}} \mathbb{M}_2$ permits us to regard the projections p_n as projections inside $\mathbb{M}_{2^\infty}$. Letting q_n be the point-image of p_n under the embedding, one obtains a new central sequence $(q_n)_{n\geq 1}$ of projections in $\mathbb{M}_{2^\infty}$, as opposed to $\mathbb{M}_2$, because

$$\left\| q_n\left(\bigotimes_{s\in G} a_s\right) - \left(\bigotimes_{s\in G} a_s\right) q_n\right\| = \left\|(p_n a_s - a_s p_n) \otimes \bigotimes_{s\in G\setminus\{e\}} a_s\right\| = \|[p_n a_s]\| \cdot \left\|\bigotimes_{s\in G\setminus\{e\}} a_s\right\| \to 0$$

for all elementary tensors $\bigotimes_{s\in G} a_s$, hence everywhere by linearity. Adapting the same argument, wherein one exploits the natural embedding $\mathbb{M}_{2^\infty} \cong \bigotimes_e \mathbb{M}_{2^\infty} \hookrightarrow \mathbb{M}(G)$ instead, allows us to regard $(q_n)_{n\geq 1}$ as being a central sequence in $\mathbb{M}(G)$. The non-triviality assumption $s \neq e$ ensures that $\sigma_s(q_n) \neq q_n$ while a straightforward computation reveals that $\sigma_s(q_n)$ commutes with q_n. Due to any pair consisting of two distinct commuting projections p and q in a C*-algebra being of distance 1 apart[2], one has $||\sigma_s(q_n) - q_n|| = 1$. If σ were inner, say with respect to unitaries $\{u_t\}_{t\in G}$, then $\sigma_s = \mathrm{Ad}_{u_s}$ holds for all n in $\mathbb{N}$, whereupon we infer

$$1 = ||\sigma_s(q_n) - q_n|| = ||u_s q_n u_s^* - q_n|| = ||u_s q_n - q_n u_s|| \to 0$$

becomes valid. We hereby arrive at a contradiction when n becomes sufficiently large. This shows that σ must be outer and invoking Kishimoto's theorem guarantees simplicity of $B(G)$.

We proceed to proving that $B(G)$ must be monotracial, starting with existence by applying the previous proposition. In our specific scenario, the C*-algebra in play is the UHF-algebra $\mathbb{M}_{2^\infty}$ and therefore admits a unique tracial state τ. Let now E denote the conditional expectation on $B(G)$. Verifying that the functional $\tau \circ E$ on $B(G)$ determines a tracial state thus amounts to establishing invariance of τ under the Bernoulli shift. The sought invariance in fact follows immediately from a more general observations: If A denotes a unital monotracial C*-algebra admitting a group action $\alpha\colon G \curvearrowright A$ by $*$-automorphisms, then $\tau_s := \tau \circ \mathrm{Ad}_{u_s}$ with $\{u_s\}_{s\in G}$ being the canonical unitaries implementing the action α, defines a tracial state on A for each $s \in G$. Uniqueness of trace entails $\tau_s = \tau$ for every $s \in G$, which is precisely α-invariance of the trace τ.

[1] Kishimoto's theorem applies due to UHF-algebras being simply via proposition 1.4.3

[2] In case you have not seen this: the spectrum of the difference $p - q$ must necessarily be contained inside the set $\{-1, 0, 1\}$ according to the spectral mapping theorem, whereof one has $||p - q|| = r_\sigma(p - q) = 1$ with $r_\sigma(p - q)$ being the spectral radius of $p - q$.

To achieve uniqueness of τ, suppose $\varrho\colon B(G) \longrightarrow \mathbb{C}$ is another tracial state. If $\tau \circ E = \tau$ occurs for any tracial state τ on $B(G)$, one may infer that

$$\varrho(E(a)) = (\varrho_{|\mathbb{M}_{2^\infty}} \circ E)(a) = (\tau_{|\mathbb{M}_{2^\infty}} \circ E)(a) = \tau(a)$$

for all a in $\mathbb{M}(G)$ by uniqueness of trace on $\mathbb{M}(G) \cong \mathbb{M}_{2^\infty}$. Hence proving that $\tau \circ E = \tau$ for all tracial states on $B(G)$ suffices, so suppose τ denotes some tracial state on $B(G)$. To ease the notation, let M be the dense subalgebra $\bigcup_n (\bigotimes_{k=1}^n \mathbb{M}_{2^k})$ of the $\mathbb{M}_{2^\infty}$. Due to the involutive algebra generated by $\bigotimes_F M$ and $\{u_s\}_{s\in G}$, with $F \subseteq G$ finite, being dense in $B(G)$, we are only required to prove that $\tau(au_s) = 0$ for any $s \neq e$ in G together with a belonging to $\bigotimes_F M$ holds to prove that $\tau \circ E = \tau$. Certainly, having accomplished this permits us to deduce that

$$\tau\Big(\sum_{s\in G} a_s u_s\Big) = \tau(a_e u_e) = (\tau \circ E)\Big(\sum_{s\in G} a_s u_s\Big)$$

becomes valid for any element $\sum_{s\in G} a_s u_s$ on a dense subalgebra in $B(G)$, whence onto $B(G)$ by continuity of τ. As such, we seek to verify that $\tau(au_s) = 0$ for all $s \neq e$ in G.

For any positive integer n, one may find an isomorphic copy B of $\mathbb{M}_{2^n}$ inside $\mathbb{M}(e)$ commuting with a via the induced $*$-monomorphism $\mathbb{M}_{2^n} \hookrightarrow B \cong \bigotimes_e \mathbb{M}_{2^\infty} \hookrightarrow \mathbb{M}(G)$. Consider the collection $p_1, p_2, \ldots, p_{2^n}$ consisting of the rank one projections p_k in $\mathbb{M}_{2^n}$ attaining the value 1 on the k'th diagonal entry with zeroes elsewhere. Let hereafter $\tau_0\colon B \longrightarrow \mathbb{C}$ be the trace

$$\tau_0(\cdot) = \tau(\cdot\ \sigma_s(p_n))$$

for some fixed nontrivial $s \in G$ and $n \in \mathbb{N}$. Due to $\sigma_s(p_n)$ being a projection commuting with $\mathbb{M}_{2^n}$, the spectrum of $a\sigma_s(p_n)$ belongs to the positive reals so $a\sigma_s(p_n) \geq 0$. Therefore τ_0 must necessarily be a positive trace on B. Uniqueness of trace on $\mathbb{M}_{2^\infty}$ implies $\tau_0 = \lambda\tau$ for some λ in $\mathbb{R}$. Observing that $\tau(\sigma_s(p_n)) = \tau_0(1_B) = \lambda\tau(1_B)$ hereby permits one to deduce that $\lambda = \tau(\sigma_s(p_n))$. Formulated differently, one has $\tau(a\sigma_s(p_n)) = \tau(a)\tau(\sigma_s(p_n))$ for all $a \in B$. Now, the projections $p_1, p_2, \ldots, p_{2^n}$ evidently sum to the identity on $\mathbb{M}_{2^n}$ and since the unique tracial state τ_{2^n} on $\mathbb{M}_{2^n}$ applied to projections counts the amount of eigenvalues equal to 1 divided by 2^n, one has

$$\tau(p_n\sigma_s(p_n)) = \tau(p_n)\tau(\sigma_s(p_n)) = 2^{-n}2^{-n} = 2^{-2n}$$

wherein the second equality stems from the aforementioned multiplicative property of τ (σ_s permutes the entry e to the entry s by definition). Combining our observations yields

$$\begin{aligned}
|\tau(au_s)| = \Big|\tau\Big(\sum_{k=1}^{2^n} p_k^2 a u_s\Big)\Big| &\leq \sum_{k=1}^{2^n} |\tau(p_k a u_s p_k)| \\
&= \sum_{k=1}^{2^n} |\tau(u_s a p_k)| \\
&= \sum_{k=1}^{2^n} |\tau(u_s a p_k \sigma_s(p_k)\sigma_{s^{-1}}(p_k))| \\
&\leq \sum_{k=1}^{2^n} \|u_s a\| \cdot |\tau(p_k\sigma_s(p_k))| \\
&\leq \|a\| \cdot 2^{-n}.
\end{aligned}$$

The first equality is based on $\sum_{k=1}^{2^n} p_k = 1$, the third stemming from $\mathbb{M}_{2^n}$ commuting with the element a implying that $\tau(p_k a u_s p_k) = \tau(u_s p_k a) = \tau(u_s a p_k)$ and the fifth arising from σ being a $*$-automorphism entailing $\sigma_{s^{-1}}(p_k) = \sigma_s(p_k^*) = \sigma_s(p_k)$. The latter expressions tends to zero as n tends to infinite, whence $\tau(au_s) = 0$ follows, completing the proof in view of our initial remarks. $\square$

Definition. The class of countable discrete groups G whose associated reduced crossed product $\bigotimes_G \mathbb{M}_2 \rtimes_\sigma G$ determines a quasidiagonal C*-algebra is represented by $\mathscr{P}$.

Proposition 4.1.4. *The inclusions $\mathscr{P} \subseteq \mathscr{G}_Q \subseteq \mathscr{A}$ of classes are all valid, where $\mathscr{A}$ denotes the class of discrete amenable groups and $\mathscr{G}_Q$ the class of discrete groups whose associated reduced group C*-algebra is quasidiagonal. Furthermore, a countable discrete group G belongs to the class $\mathscr{P}$ if and only if the Bernoulli crossed product $B(G)$ defines a quasidiagonal C*-algebra.*

Proof. Only the first inclusion together with the equivalent formulation of the class $\mathscr{P}$ requires justification thanks to Rosenberg's theorem. To each discrete unital C* dynamical system (A, α, G) one has a $*$-monomorphism $C^*_\lambda(G) \hookrightarrow A \rtimes_\alpha G$ based on the following observation. Fix some faithful separable representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ of A and let $\pi_\alpha \times \lambda$ be associated integrated form of π extended to $A \rtimes_\alpha G$. Since the right-hand side of

$$(\pi_\alpha \times \lambda)(au_s)(\xi \otimes \delta_t) = \pi(\alpha_s(a))\xi \otimes \lambda_s\delta_t = (\pi(\alpha_s(a)) \otimes \lambda_s)(\xi \otimes \delta_t)$$

determines an element in $A \otimes C^*_\lambda(G)$ for all $s, t \in G$ and $a \in A$, we may infer that $A \rtimes_{\alpha,r} G$ contains an isomorphic copy of $C^*_\lambda(G)$ (the representation $1 \otimes \lambda$ is faithful being the tensor product map of faithful representations). Therefore, $C^*_\lambda(G)$ embeds into $\bigotimes_G \mathbb{M}_2 \rtimes_\alpha G$ and thus $\mathscr{P}$ becomes a subclass of $\mathscr{G}_Q$ due to quasidiagonality passing to subalgebras, proving the former inclusion.

In order to verify the remaining assertion, the equivalence, let G be a countable discrete group such that $B(G)$ is quasidiagonal. The canonical embedding $\iota\colon \mathbb{M}_2 \hookrightarrow \bigotimes_G \mathbb{M}_2 \cong \mathbb{M}_{2^\infty}$ induces a G-equivariant $*$-monomorphism $\varrho\colon \bigotimes_G \mathbb{M}_2 \hookrightarrow \bigotimes_G \mathbb{M}_{2^\infty}$ by making G-many copies of ι tensorially. Upon the functor $(\cdot) \rtimes_{\alpha,r} G$ assigning equivariant $*$-monomorphisms to embeddings, one obtains an embedding $\bigotimes_G \mathbb{M}_2 \rtimes_{\sigma,r} G \subseteq \mathbb{M}(G) \rtimes_{\sigma,r} G$ in the C*-algebraic framework. The latter $B(G)$, hence quasidiagonal by hypothesis, whereof the former inherits quasidiagonality.

Conversely, suppose G denotes a group in the class $\mathscr{P}$. Let $\delta\colon G \curvearrowright G \times G$ be the diagonal action associated to the left-translation, i.e., $\delta_s(g, h) = (s^{-1}g, s^{-1}h)$ for all $s, g, h \in G$. Letting V denote the subset of all diagonal elements $(s, s) \in G \times G$ for $s \in G$ ensures that the induced Bernoulli shift action σ^δ agrees with the product action $\sigma \times \sigma$ on V. As such one may deduce that

$$\bigotimes_{G^2} \mathbb{M}_2 \rtimes_{\sigma^\delta} G = \bigotimes_{G^2} \mathbb{M}_2 \rtimes_{(\sigma\times\sigma)_{|V}} G^2 \hookrightarrow \bigotimes_{G^2} \mathbb{M}_2 \rtimes_{(\sigma\times\sigma)} G^2$$

via the embeddings on the corresponding dense subalgebras, again the functorial properties of the reduced crossed product being exploited. Now, $\bigotimes_G(\bigotimes_G A)$ of any unital C* algebra is obviously $*$-isomorphic to the G^2-fold of A, that is, $\bigotimes_{G^2} A$. Hence selecting an appropriate bijection $G^2 \longrightarrow G^2$ permits one to identify the Bernoulli shift action $G \curvearrowright \bigotimes_{G^2} A$ induced by δ with the usual Bernoulli shift $\sigma\colon G \curvearrowright \bigotimes_G(\bigotimes_G A)$. Altogether, one has

$$\Big(\bigotimes_G \bigotimes_G \mathbb{M}_2\Big) \rtimes_\sigma G \cong \bigotimes_{G^2} \mathbb{M}_2 \rtimes_{\sigma^\delta} G \hookrightarrow \bigotimes_{G^2} \mathbb{M}_2 \rtimes_{(\sigma\times\sigma)} G^2 \quad \cong \Big(\bigotimes_G \mathbb{M}_2 \rtimes_\sigma G\Big) \otimes \Big(\bigotimes_G \mathbb{M}_2 \rtimes_\sigma G\Big).$$

The last isomorphism may be justified in the following manner. The assignment

$$\bigotimes_{(s,t)\in G^2} a_{(s,t)} \mapsto \bigotimes_{(s,e)} a_{(s,e)} \otimes \bigotimes_{(e,t)} a_{(e,t)}$$

extends to an isomorphism $\bigotimes_{G^2} \mathbb{M}_2 \cong (\bigotimes_G \mathbb{M}_2) \otimes (\bigotimes_G \mathbb{M}_2)$ of C*-algebras and the tensor-action $\sigma \otimes \sigma$ meaning $(\sigma \otimes \sigma)_{(s,t)} = \sigma_s \otimes \sigma_t$ for each pair $s, t \in G$, is readily seen to correspond to the product action $\sigma \times \sigma\colon G^2 \curvearrowright \bigotimes_{G^2} \mathbb{M}_2$ under the appropriate identifications. Regarding $\bigotimes_{G^2} \mathbb{M}_2 \rtimes_{\sigma\times\sigma} G^2$ in this fashion allows us to invoke proposition 1.6.2 to obtain the desired.

The C*-algebra on the right-hand side above is a spatial tensor product of quasidiagonal C*-algebras by hypothesis, hence quasidiagonal itself. The C^*-algebra on the left-hand side is isomorphic to $\mathbb{M}(G) \rtimes_\sigma G$ in the event of G having infinite order. In the finite case, $B(G)$ embeds into $\mathbb{M}_{2^\infty} \otimes \mathbb{M}_{|G|}$ according to the corollary below (stated in a more general setting). The latter algebra is quasidiagonal being the spatial tensor product of a matrix algebra and an UHF-algebra. Regardless of the outcome, $B(G)$ embeds into a quasidiagonal C*-algebra, hence it inherits quasidiagonality from the ambient algebras and voila. □

Lemma 4.1.5. *Suppose $H \subseteq G$ is a finite index inclusion of discrete groups with $N = |G : H|$ and let (A, α, G) be a unital C*-dynamical system. Then $A \rtimes_\alpha G \hookrightarrow (A \rtimes_\alpha H) \otimes \mathbb{M}_N$*

Proof. Let $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$ be a faithful representation. The map π induces a faithful representation of $C_c(G, A)$ on $\mathcal{H} \otimes \ell^2(G)$, namely the integrated form $\pi_\alpha \times \lambda$ with respect to the left regular representation λ of G. Similarly, the restrictions $\lambda_{|H}$ together with $\beta = \alpha_{|H}$ induce a faithful representation of $C_c(H, A)$ via the integrated form $\pi_\beta \times \lambda$. By hypothesis, one may choose finitely many suitable elements $s_1, s_2, \ldots, s_N$ in G partitioning G via the left cosets $s_i G$, so one obains

$$\ell^2(G) \cong \bigoplus_{i=1}^N \ell^2(s_i H) \quad \text{and} \quad \mathcal{H} \otimes \ell^2(G) \cong \bigoplus_{i=1}^N \mathcal{H} \otimes \ell^2(s_i H).$$

To ease the notation, let $\mathcal{K} = \mathcal{H} \otimes \ell^2(G)$ and $\mathcal{K}_i = \mathcal{H} \otimes \ell^2(s_i G)$. Since $\ell^2(s_i H)$ determines a subspace of $\ell^2(G)$, it admits an orthogonal projection $p_i \colon \mathcal{K} \longrightarrow \mathcal{K}_i$. Recall that $C_c(G, A)$ is the involutive algebra generated by A and the unitaries $u_g = 1_A g$ for every g in G. Therefore, one may define a family of unitaries v_i on $\mathcal{H} \otimes \ell^2(G)$ by $v_i = (\pi_\alpha \times \lambda)(u_{s_i})$ for all indices $1 \le i \le N$. Expanding the expressions reveals that these unitaries necessarily must be of the form $v_i = (\pi_\alpha \times \lambda)(u_{s_i}) = 1_{\mathcal{H}} \otimes \lambda_{s_i}$ for every $1 \le i \le N$.

As such v_i maps an elementary tensor $\xi \otimes \delta_t$, with t being an element of H into $\mathcal{K}$. We deduce the identity $v_i(\mathcal{K}_0) = \mathcal{K}_i$ follows when setting $\mathcal{K}_0 = \mathcal{H} \otimes \ell^2(H)$. Based on the preceding, one readily verifies that the $*$-homomorphism $\varrho \colon \mathrm{B}(\mathcal{K}) \longrightarrow \mathrm{B}(\mathcal{K}_0) \otimes \mathbb{M}_N$ defined via the assignment

$$a \overset{\varrho}{\longmapsto} \sum_{i,j=1}^N v_i^* p_i a p_j v_j \otimes e_{ij},$$

where e_{ij} denotes the canonical basis elements in $\mathbb{M}_N$, becomes an isomorphism. Moreover,

$$\begin{aligned}(v_i^* p_i (\pi_\alpha \times \lambda)(a u_s) p_j v_j)(\xi \otimes \delta_t) &= \begin{cases} u_i^* p_i (\pi_\alpha(a) \otimes \lambda_s)(\xi \otimes \delta_{s_j t}), & \text{if } s_j t \in s_j H \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \pi(\alpha_{(t s_j)^{-1}}(a)) \xi \otimes \delta_{s_i^{-1} s s_j t}, & \text{if } s_i^{-1} s s_j \in H \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} (\pi_\alpha \times \lambda)(\alpha_{s_j^{-1}}(a) u_{s_i^{-1} s s_j})(\xi \otimes \delta_t), & \text{if } s_i^{-1} s s_j \in H \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

for any choice of indices $1 \le i, j \le N$, $\xi \in \mathcal{H}$ and $s, t \in G$. The last expression belongs to the image of $A \rtimes_\alpha H$ under $\pi \times \lambda$, so ϱ maps $(\pi_\alpha \times \lambda)(A \rtimes_\alpha H)$ onto $(\pi_\alpha \times \lambda)(A \rtimes_\alpha H) \otimes \mathbb{M}_N$ which grants us the desired embedding when restricting ϱ accordingly. □

Corollary 4.1.6. *There exists a $*$-monomorphism of $B(H)$ into $B(G) \otimes \mathbb{M}_{|G:H|}$ whenever $H \subseteq G$ is a finite index inclusion of discrete groups.*

Proof. Let N denote the index of the subgroup H in G. Since $H \times G/H$ may be identified with G as sets, we have $\bigotimes_H \mathbb{M}(G/H) \cong \mathbb{M}(G)$ while $\mathbb{M}(G/H) \cong \mathbb{M}_{2^\infty}$. Hence invoking the lemma,

$$\mathbb{M}(G) \rtimes_\sigma G \hookrightarrow (\mathbb{M}(G) \rtimes_{\sigma'} H) \otimes \mathbb{M}_N \cong (\bigotimes_H \mathbb{M}(G/H) \rtimes_{\sigma'} H) \otimes \mathbb{M}_N \cong \mathbb{M}(H) \rtimes_{\sigma'} H) \otimes \mathbb{M}_N$$

The right-hand side is precisely $B(H) \otimes \mathbb{M}_N$ whereas the left-hand side is $B(G)$. □

4.2 Elementary Amenable Groups

Proposition 4.1.4 leans towards the salient feature of the Bernoulli shift crossed product algebra along with certain permanence properties; it shall be revealed that it is stable under basic group operations such as direct limits. In order to aptly capture the essence, we state various definitions and results required to describe a theorem due to Chou, which was improved upon significantly by Osin. We initially briefly introduce elementary amenable groups despite of the terminology appearing being described afterwards.

Definition. Define EG as the smallest subclass of discrete amenable groups containing the finite and abelian groups while being stable under taking direct limits, subgroups, quotients and group extensions in EG. The subclass of these restricted to countable groups is denoted EG_c.

The aforementioned mathematicians succeeded in weakening the minimality condition imposed on elementary amenable groups: The theorem of Osin and Chou asserts that the conditions of being closed under taking quotients and passing to subgroups are superfluous. For the proof, we shall adopt an approach inspired by both of these participants to the theory.

Definition. Let $\mathscr{A}$ denote any class of groups. Another class $\mathscr{B}$ of groups is said to be

- *closed under passing to subgroups* if every subgroup H of some G in $\mathscr{B}$ must belong to $\mathscr{B}$;
- *closed under taking quotients* if every quotient G/N of groups within $\mathscr{B}$ remains inside $\mathscr{B}$;
- *closed under taking direct limits* if every direct limit of groups within $\mathscr{B}$ remains inside $\mathscr{B}$.
- *closed under $\mathscr{A}$ extensions* if, for every short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1,$$

  wherein $N, G/N \in \mathscr{A}$ one has $G \in \mathscr{B}$.

The corresponding operations are commonly referred to as being *the elementary operations.*

Definition. Given any class $\mathscr{B}$, we define the *elementary class* $\mathrm{E}(\mathscr{B})$ associated to $\mathscr{B}$ to be the minimal class of groups containing the class $\mathscr{B}$ such that $\mathrm{E}(\mathscr{B})$ is closed under passing to subgroups, taking quotients, taking direct limits and $\mathscr{B}$ extensions. We call $\mathscr{B}$ the *base class* of $\mathrm{E}(\mathscr{B})$.

Remark. The notion of elementary classes of course applies to the elementary amenable ones. In this case, the class EG merely equals $\mathrm{E}(\mathscr{Z})$ with $\mathscr{Z}$ denoting the class consisting of all finite and abelian groups. Therefore, statements concerning the general properties of elementary classes pass to the elementary amenable case as well.

Terminology. Suppose $\mathscr{B}$ and $\mathscr{A}$ be any pair of classes. We refer to $\mathscr{A}$ as being *closed under $\mathscr{B}_0$ extensions* if for every short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

wherein N belongs to $\mathscr{A}$ and G/N lies in $\mathscr{B}$, the group G must belong to $\mathscr{B}$.

Remark. The principle behind $\mathscr{B}_0$ extensions is slightly mysterious at first glance. However, we shall use techniques concerning these now and towards the end of the chapter. Essentially, working with $\mathscr{B}_0$ extensions becomes easier to deal with should the underlying class $\mathscr{B}$ be relatively small; this is the desired reduction we seek from Osin's/Chou's theorem.

The proof of Chou and Osin exploits transfinite induction. Those unfamiliar with ordinals and the transfinite induction principle are encouraged to take a detour and read the corresponding part in the first section. Chou and Osin's theorem succeeds in reformulating the elementary classes inductively by successively performing elementary operations to the elements in the preceding class. We specify the corresponding procedure further.

The construction of elementary classes. Let $\mathscr{B}$ be any class of groups and write $\mathrm{E}_0(\mathscr{B}) := \mathscr{B}$ with 0 being the zero-ordinal. Assume hereafter that classes $\mathrm{E}_\beta(\mathscr{B})$ have been prescribed for every ordinal $\beta < \alpha$, where α denotes some nonzero ordinal. We recursively define

- if α is a successor ordinal, then $\mathrm{E}_\alpha(\mathscr{B})$ is defined to consists of all groups arising as either a direct limit of groups in $\mathrm{E}_{\alpha-1}(\mathscr{B})$ or as an $\mathrm{E}_{\alpha-1}(\mathscr{B})_0$ extension;
- $\mathrm{E}_\alpha(\mathscr{B}) = \bigcup_{\beta<\alpha} \mathrm{E}_\beta(\mathscr{B})$ whenever α is a limit ordinal.

Without further ado, we prove Osin's theorem.

Theorem 4.2.1 (Chou, Osin)**.** *Suppose $\mathscr{B}$ denotes any class of groups closed under passing to subgroups and taking quotients. Under these premises, the following hold.*

(i) For every ordinal α, $E_\alpha(\mathscr{B})$ is closed under passing to subgroups and taking quotients.

(ii) We have $E(\mathscr{B}) = \bigcup_\alpha E_\alpha(\mathscr{B})$ with the union being indexed over all ordinals α.

(iii) $E(\mathscr{B})$ is the minimal class containing $\mathscr{B}$ while simultaneously being closed under $\mathscr{B}_0$ extensions and taking direct limits.

Proof. (i): The proof revolves around an application of the transfinite induction principle, the case $\alpha = 0$ trivially being true by the hypothesis imposed upon $\mathscr{B}$. Therefore, we assume that the assertion is valid for every ordinal $\beta < \alpha$ for some fixed nonzero ordinal α. Before proceeding, we settle some setup. Let in the following H be a normal subgroup of some group G belonging to the class $\mathrm{E}_\alpha(\mathscr{B})$ and let $\varphi\colon G \longrightarrow G/H$ be the canonical quotient homomorphism. We must verify that $\mathrm{E}_\alpha(\mathscr{B})$ contains both H and $\varphi(G)$.

Step 1. Suppose at first α is some limit ordinal, meaning $\mathrm{E}_\alpha(\mathscr{B}) = \bigcup_{\beta<\alpha} \mathrm{E}_\beta(\mathscr{B})$, so that G must belong to $\mathrm{E}_\beta(\mathscr{B})$ for some ordinal $\beta < \alpha$. According to the induction hypothesis, $\mathrm{E}_\beta(\mathscr{B})$ is closed under restricting to subgroups and taking quotients, hence H together with $\varphi(G)$ must be members of $\mathrm{E}_\beta(\mathscr{B}) \subseteq \mathrm{E}_\alpha(\mathscr{B})$ as desired.

Step 2(a). The case wherein α is a successor ordinal requires more meticulous care. Suppose at first $G = \varinjlim G_\lambda$ denotes a direct limit of groups G_λ indexed over some set I, where each occurring group G_λ is a member of the preceding class $\mathrm{E}_{\alpha-1}(\mathscr{B})$. The induction hypothesis enables us to deduce that $H_\lambda = H \cap G_\lambda$ must belong $\mathrm{E}_{\alpha-1}(\mathscr{B})$ for every such λ. The same argument reveals that $\varphi(G_\lambda)$ belongs to $\mathrm{E}_{\alpha-1}(\mathscr{B})$ for all λ in I. It is apparent that the inclusions $G_\lambda \longrightarrow G$ restricted to $H \cap G_\lambda$ yield morphisms $\sigma_\lambda \colon H_\lambda \longrightarrow H$ such that the directed system $(H_\lambda, \sigma_\lambda)_{\lambda \in I}$ becomes a model for H. In symbols, we have $H = \varinjlim H_\lambda$ whereas a parallel observation yields $\varphi(G) = \varinjlim \varphi(G_\lambda)$. Thus, H and $\varphi(G)$ define direct limits of members in $\mathrm{E}_{\alpha-1}(\mathscr{B}) \subseteq \mathrm{E}_\alpha(\mathscr{B})$.

Step 2(b). The scenario in which G is an extension of groups in $\mathrm{E}_{\alpha-1}(\mathscr{B})$ remains to be tackled. To achieve the sought conclusion, let

$$1 \longrightarrow N \longrightarrow G \longrightarrow E \longrightarrow 1$$

be a short-exact sequence wherein $N \in \mathrm{E}_{\alpha-1}(\mathscr{B})$ and $E \in E_{\alpha-1}(\mathscr{B})$. Due to N being normal in G, the intersection $N \cap H$ becomes a normal subgroup of H belonging to the class $\mathrm{E}_{\alpha-1}(\mathscr{B})$ whose associated quotient $H/(N \cap H)$ must likewise be a member of $\mathrm{E}_{\alpha-1}(\mathscr{B})$ by the induction hypothesis. Exactness of the corresponding sequence

$$1 \longrightarrow N \cap H \longrightarrow H \longrightarrow H/(N \cap H) \longrightarrow 1$$

thus implies $H \in \mathrm{E}_\alpha(\mathscr{B})$. Concerning quotients, notice that $\varphi(N)$ must be the kernel of the quotient homomorphism $\varphi(G) \longrightarrow \varphi(E)$ meaning $\varphi(G)$ is an extension of $\varphi(E)$ by $\varphi(N)$. Since $\varphi(N), \varphi(E) \in \mathrm{E}_{\alpha-1}(\mathscr{B})$, we may infer that $\varphi(G)$ lies inside $\mathrm{E}_\alpha(\mathscr{B})$. This finalizes the proof of (i) when invoking the principle of transfinite induction.

(ii)+(iii): The property (iii) is an obvious consequence of (ii). The inclusion $\supseteq$ in (ii) is immediate. Indeed, any member G of either of the classes $\mathrm{E}_\alpha(\mathscr{B})$ will eventually belong to some elementary class $\mathrm{E}_\beta(\mathscr{B})$ for some successor ordinal β. Therefore, G arises as either an extension or direct limit of groups in the preceding elementary class $\mathrm{E}_{\beta-1}(\mathscr{B})$, which $\mathrm{E}(\mathscr{B})$ contains by definition. Ergo, to prove equality it suffices to show that $\bigcup_\alpha \mathrm{E}_\alpha(\mathscr{B})$ is closed under restricting to subgroups, taking quotients, direct limits and $\mathscr{B}$ extensions by minimality. Part (i) establishes the first two conditions, from which the proof amounts to deriving the claims below.

Claim 1. Given any collection of groups G_λ in $\bigcup_\alpha \mathrm{E}_\alpha(\mathscr{B})$ indexed over some directed set I, the directed limit $G = \varinjlim G_\lambda$ remains in the union $\bigcup_\alpha \mathrm{E}_\alpha(\mathscr{B})$.

Proof of claim 1. By hypothesis, for any index $\lambda \in I$ there exists some ordinal α_λ satisfying $G_\lambda \in \mathrm{E}_{\alpha_\lambda}(\mathscr{B})$. Set hereof $\alpha = \sum_{\lambda \in I} a_\lambda$. From the very construction we must have $G_\lambda \in \mathrm{E}_\alpha(\mathscr{B})$ for every index λ. Therefore G must be an inductive limit of groups in the elementary class $\mathrm{E}_\alpha(\mathscr{B})$, meaning G has to lie in the class $\mathrm{E}_{\alpha+1}(\mathscr{B}) \subseteq \bigcup_\beta \mathrm{E}_\beta(\mathscr{B})$, proving the the first claim.

Claim 2. Suppose N, G and E are groups for which the sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 1$$

becomes exact. If N belongs to $\mathrm{E}_\alpha(\mathcal{B})$ and H belong to $\mathrm{E}_\beta(\mathcal{B})$ for some ordinal α and β, then there exists another ordinal γ fulfilling $G \in \mathrm{E}_\gamma(\mathcal{B})$.

Proof of claim 2. We will argue by transfinite induction on the ordinal β, the case $\beta = 0$ being trivial. Let $\beta > 0$ be some fixed ordinal and suppose the claim has been verified for all lesser ordinals. If β defines a limit ordinal, the argument presented during part (i) in the limit ordinal case may be copied without hindrances. Now, suppose β defines a successor ordinal meaning the group H arises as either a $\mathscr{B}_0$ extension or a direct limit of groups within the class $\mathrm{E}_{\beta-1}(\mathscr{B})$. Consider initially the case where the sequence

$$1 \longrightarrow E \longrightarrow H \longrightarrow F \longrightarrow 1$$

is exact, having the containments $E \in \mathrm{E}_{\beta-1}(\mathscr{B})$ together with $F \in \mathscr{B}$ granted. The preimage $\varphi^{-1}(E)$ obviously defines a normal subgroup belonging to G such that φ restricts to en epimorphism $\varphi^{-1}(E) \longrightarrow E$, whose kernel M determines some subgroup of N. Due to M being a subgroup of some member in $\mathrm{E}_\alpha(\mathscr{B})$, we deduce that M lies therein. Exactness of the sequence

$$1 \longrightarrow M \longrightarrow \varphi^{-1}(E) \longrightarrow E \longrightarrow 1$$

hereby entails the containment $\varphi^{-1}(E) \in \mathrm{E}_\mu(\mathscr{B})$ for some ordinal μ according to the induction hypothesis. Consider hereafter the group homomorphism $\psi\colon G/\varphi^{-1}(E) \longrightarrow H/E$ defined by

$$\psi(g + \varphi^{-1}(E)) := \varphi(g) + \varphi(\varphi^{-1}(E)) = \varphi(g) + E.$$

The kernel of ψ coincides with the preimage $\varphi^{-1}(E)$, hence ψ must be an isomorphism. Through the containment $H/E \cong F \in \mathscr{B}$ we infer that $G/\varphi^{-1}(E)$ belongs to $\mathscr{B}$. As such, G becomes a $\mathscr{B}_0$ extension in $\mathrm{E}_\mu(\mathscr{B})$, hence G must belong to $\mathrm{E}_{\mu+1}(\mathscr{B})$.

We still need to handle the direct limit case, however, we omit including a plethora of details since the proof mimics the one in (i). Suppose H is a direct limit of groups H_λ in $\mathrm{E}_{\beta-1}$ with respect to some directed set I. Write $H_\lambda^0 = \varphi^{-1}(H_\lambda)$ for every index $\lambda \in I$ and consider the extensions

$$1 \longrightarrow N_\lambda \longrightarrow H_\lambda^0 \longrightarrow H_\lambda \longrightarrow 1$$

wherein N_λ is some subgroup of N, i.e., a member of $\mathrm{E}_\alpha(\mathscr{B})$ due to (i). The induction hypothesis yields the containment $H_\lambda^0 \in \mathrm{E}_{\lambda_\mu}(\mathscr{B})$ for some ordinal λ_μ and every $\lambda \in I$. Letting $\mu = \sum_{\lambda \in I} \lambda_\mu$ guarantees that $H_\lambda^0 \in \mathrm{E}_\mu(\mathscr{B})$ for each λ. Due to the inclusions (we assume injectivity without loss of generality) $H_\lambda \hookrightarrow H_{\lambda'}$ inducing inclusions $H_\lambda^0 \hookrightarrow H_{\lambda'}^0$, the identifications $\varinjlim H_\lambda^0 = \varphi^{-1}(H) = G$ reveals that G is the direct limit of groups in $\mathrm{E}_\mu(\mathscr{B})$, whereof $G \in \mathrm{E}_{\mu+1}(\mathscr{B})$ as desired. □

We supply the theorem of Osin with some added attention to the countable case, including an explicit construction. We emphasize on the fact that certain constructions occurring in the upcoming discussion actually form sets as opposed to general classes, however, proving these assertion have been omitted as they deter from the overall theme of the chapter.

The countable elementary amenable groups class. Here we construct the class EG_c and afterwards prove certain permanence property required towards the end of the chapter. Let $\mathscr{G}_c$ denote the set of all countable groups and let $\mathscr{D} = \{\mathscr{D}_\alpha : \alpha \in J\}$ be an exhausting list of subsets $\mathscr{D}_\alpha$ in $\mathscr{G}_c$ subject to the constraints below:

- the subset $\mathscr{D}_\alpha$ contains all countable abelian - and finite groups for all $\alpha \in J$;
- the subset $\mathscr{D}_\alpha$ is closed under restricting to subgroups, passing to quotients, taking direct limits and group extensions for every index $\alpha \in J$.

We define $\mathrm{EG}_c^0 = \bigcap_{\alpha \in J} \mathscr{D}_\alpha$. Note that the intersection is nonempty by the first condition.

Proposition 4.2.2. *The class EG_c exists in the sense that* $\mathrm{EG}_c = \mathrm{EG}_c^0$.

Proof. We must verify that EG_c^0 defines the minimal class consisting of countable groups, which is closed under performing either of the elementary operations. Fortunately, accomplishing this almost follows automatically. Indeed, suppose G is a group arising as either a direct limit, quotient or a subgroup of some member inside EG_c^0. Regardless of the disguise G chooses, it arises from elementary operations in each of the sets $\mathscr{D}_\alpha$, each of these being stable under performing the elementary operations so that G belongs in some $\mathscr{D}_\alpha \subseteq \mathrm{EG}_c^0$. Minimality is easy; if $\mathscr{A}$ denotes a class of countable groups closed under the elementary operations, then $\mathscr{A}$ would coincide with $\mathscr{D}_\alpha$ for some indix α, hence in $\mathrm{EG}_c^0 \subseteq \mathscr{A}$. Voila. □

Proposition 4.2.3. *The following are equivalent for a group G.*

(i) G is elementary amenable.

(ii) Every countable subgroup of G is elementary amenable, hence belong to EG_c.

(iii) There exists a directed system $(G_j, j \in J)$ consisting of countable elementary amenable subgroup in G whose direct limit equals G.

Proof. (i) $\Rightarrow$ (ii) stems from EG being closed under restricting to subgroups, whereas (iii) $\Rightarrow$ (i) follows from EG being closed under taking direct limits. It remains to be shown that the second condition implies the third. For this, let J be the directed set consisting of all finite subsets in G ordered by inclusion. Any finite subset $F \subseteq G$ generates a countable subgroup $G_F = \langle F \rangle$ such that an inclusion of finite subsets $F \subseteq F_0$ in G induce a group monomorphism $F \hookrightarrow F_0$. Write hereof G as the directed limit of the groups G_F having the iaforementioned embeddings as connecting morphisms. This turns G into a directed limit of countable subgroups, each being elementary amenable by hypothesis, proving the claim. □

4.3 A Partial Answer to Rosenberg's Conjecture

Osin's theorem provides us with a strategy to answer Rosenberg's conjecture for elementary amenable groups; it reduces the task, in this specific class, into verifying that $\mathscr{P}$ is closed under direct limits, subgroups and extensions by countable elementary amenable groups. Certainly, we may conclude that $\mathscr{P}$ must contain the class EG_c through minimality, thereby ensuring the inclusion $\mathrm{EG}_c \subseteq \mathscr{G}_Q$. In view of our previous results, providing an affirmative answer to this amounts to investigating functorial properties of the Bernoulli shift crossed product, so we are inclined to start here.

Proposition 4.3.1 (Ozawa, Rørdam, Sato)**.** *The functor $G \mapsto B(G)$ from the category of countable discrete groups into the category of C*-algebras satisfies the following properties.*

(i) If H is a subgroup of G, then $B(H)$ is a subalgebra of $B(G)$. In particular, the class $\mathscr{P}$ is closed under restrictions to subgroups.

(ii) Given a chain of subgroups $G_1 \subseteq G_2 \subseteq \dots$ one has $B\left(\bigcup_n G_n\right) = \overline{\bigcup_n B(G_n)}$. Consequently, the class $\mathscr{P}$ is closed under taking countable direct limits.

(iii) Suppose H acts on G by group automorphisms. If so, the action $H \curvearrowright G$ extends to an action τ of $*$-automorphisms $H \curvearrowright B(G)$ such that $B(G) \rtimes_\tau H \cong B(G \rtimes H)$.

Proof. (i) The Bernoulli shift crossed product $B(G)$ is the closure of the involutive algebra generated by $\mathbb{M}(G)$ and a collection of unitaries $\{u_s\}_{s\in G}$ under the left-regular representation. Thus, for any subgroup inclusion $H \subseteq G$ of discrete countable groups, the dense subalgebra of $B(H)$ must necessarily be an involutive subalgebra of $B(G)$, since the generating sets are identical, except for the generating collection $\{u_g\}_{g\in H}$ of unitaries which is a subcollection of the generating ones in $B(G)$. The remaining part is due to quasidiagonality passing to subalgebras.

(ii) Write $G = \bigcup_{n=1}^\infty G_n$ to simplify the notation. The algebra $\bigcup_n B(G_n)$ consists of a union of C*-algebras generated by $\mathbb{M}(G_n)$ and unitaries $\{u_s\}_{s\in G_n}$ implementing the action whereas $B(\bigcup_n G_n)$ is generated by $\mathbb{M}(G)$ and unitaries $\{u_s\}_{s\in G}$. Consider the following identity concerning the assignment $H \mapsto \mathbb{M}(H)$ from the category of discrete groups to the category of C*-algebras:

$$\mathbb{M}(G) = \bigotimes_{\bigcup_n G_n} \mathbb{M}_{2^\infty} \cong \bigcup_{n=1}^{\infty} \bigotimes_{G_n} \mathbb{M}_{2^\infty} = \bigcup_{n=1}^{\infty} \mathbb{M}(G_n).$$

The identification in the middle is not difficult to convince oneself to be true, granted one ought to expand the definitions to do so. The observation permits us to regard any element in $\mathbb{M}(G_n)$ as precisely one element in $\mathbb{M}(G)$. The obtained inclusion extends to a $*$-isomorphism on the union $\bigcup_n \mathbb{M}(G_n)$. Since the generating sets of unitaries on the norm-closure of $\bigcup_n B(G_n)$ and $B(G)$ coincide, we may deduce that they are identical.

In order to justify the second assertion in (ii), note that the previous part forces $B(\bigcup_n G_n)$ to be the norm closure $\bigcup_n B(G_n)$, where the latter may be regarded as an inductive limit of quasidiagonal C*-algebras in, which the connecting morphisms are inclusions, whenever each G_n lies in the $\mathscr{P}$ class. Due to inductive limits of quasidiagonal C*-algebras with monic connecting morphisms being quasidiagonal, $\mathscr{P}$ must be closed under taking countable inductive limits of groups.

(iii) Now, for the difficult aspect. By hypothesis, we have a homomorphism $\alpha\colon H \longrightarrow \operatorname{Aut}(G)$ of groups and the action becomes inner in $G \rtimes H$ when regarding the respective groups as subgroups of $G \rtimes H$, meaning $\alpha_g(\cdot) = g(\cdot)g^{-1}$ for all g belonging to $G \hookrightarrow G \rtimes H$[3]. The group G acts on the semidirect product $G \rtimes H$ via the homomorphism $\beta\colon G \longrightarrow \operatorname{Aut}(G \rtimes H)$ defined by

$$\beta_s(t, g) := (s^{-1}, 1_G) \cdot (t, g) = (s^{-1}\varphi_{1_G}(t), g1_G) = (s^{-1}t, g)$$

for all $s, t \in G$ and each $g \in H$. The action β thus acts on $G \rtimes H$ by translating elements in the copy of G while ignoring the copy of H. As such, the Bernoulli action σ induced from the left translation $G \curvearrowright G$ coincides with σ^{β} when regarding these as maps on $G \hookrightarrow G \rtimes H$. The preceding allows one to infer $B(G) \cong \mathbb{M}(G \rtimes H) \rtimes_{\sigma^{\beta}} G$ as their dense involutive subalgebras are isomorphic. For the sake of avoiding confusion we shall abbreviate the latter isomorphic version of $B(G)$ as $B_{\beta}(G)$. The identification $B(G) \cong B_{\beta}(G)$ reduces the task into producing an action $H \curvearrowright B_{\beta}(G)$. For this purpose, consider the action $\gamma\colon G \longrightarrow \operatorname{Aut}(G \rtimes H)$ given by

$$\gamma_g(s, h) = (\alpha_g(s), g) \cdot (1_G, h) = (\alpha_g(s), gh)$$

for every $g, h \in H$ and $s \in G$. By faithfully representing $\mathbb{M}(G \rtimes H)$ into some Hilbert space $\mathcal{H}$ and exploiting independence of choice of faithful representation for the reduced crossed product, we may identify $\mathbb{M}(G \rtimes G)$ with its image herein. We proceed towards extending the action $H \curvearrowright G$ to an action on $B_{\beta}(G)$. For each t in H, let $\pi_t\colon \mathbb{M}(G \rtimes H) \longrightarrow \mathbb{M}(G \rtimes H)$ be the $*$-endomomorphism given by $\pi_t(\cdot) = \sigma_t^{\gamma}(\cdot)$. If $\{u_s\}_{s\in G}$ denotes the collection of unitaries implementing the action α, then set $\nu_t(\cdot) = u_{\alpha_t(\cdot)}$ for each t in H to obtain a unitary representation $\nu\colon G \longrightarrow \mathcal{U}(\mathcal{H})$. Due to

$$\nu_g(t)(\pi_g(a))\nu_g^*(t) = u_{\alpha_g(t)}\sigma_g^{\gamma}(a)u_{\alpha_g(t)}^* = \sigma_{\alpha_g(t)}^{\gamma}\sigma_g^{\gamma}(a) = \sigma_{gtg^{-1}g}^{\gamma}(a) = \pi_g(\sigma_t^{\gamma}(a))$$

being valid for every $a \in \mathbb{M}(G \rtimes H)$, $g \in H$ and $t \in G$, the triple $(\pi_t, \nu_t, \mathcal{H})$ determines a covariant representation of $B_{\beta}(G)$ for every $t \in H$. In the calculations, the second equality stems from the action σ^{γ} being inner whereas the third is based on α_g corresponding to conjugation by g in $G \rtimes H$. Since the triple defines a covariants representation of $B_{\beta}(G)$, the map $\tau\colon H \longrightarrow \operatorname{Aut}(B_{\beta}(H))$ defined on a generic element au_s in $C_c(G \rtimes H, \mathbb{M}(G \rtimes H))$ via $\tau_t(au_s) = \pi_t(a)\nu_t(s)$ is an action. According to proposition 1.6.3,

$$B(G) \rtimes_{\tau} H \cong B_{\beta}(G) \rtimes_{\tau} H = (\mathbb{M}(G \rtimes H) \rtimes_{\sigma^{\beta}} G) \rtimes_{\tau} H \cong \mathbb{M}(G \rtimes H) \rtimes_{\sigma} (G \rtimes H) = B(G \rtimes H).$$

This finalizes the proof. □

Perhaps the reader may find the last property slightly unrelated to the section. The precise reasoning will be postponed for now, so we merely promise that it permits us to invoke powerful classification theorems concerning $A\mathbb{T}$-algebras of real rank zero. For the convenience of the reader, we remind them that a unital C*-algebra is referred to as being of *real rank zero* provided that any self-adjoint element may be norm-approximated by self-adjoint invertible elements within any tolerance, i.e., the

[3]see appendix $A.1$.

collection of self-adjoint invertible elements is dense in the collection of the self-adjoint ones. For a proof of Matui's theorem, see [**?**, theorem 2].

Definition. A C*-algebra A is an $A\mathbb{T}$-algebra if there exists a direct sequence $(F_n, \varphi_n)_{n\geq 1}$ consisting of finite dimensional C*-algebras F_n fulfilling $A \cong \varinjlim(C(\mathbb{T}) \otimes F_n, 1 \otimes \varphi_n)$.

Theorem 4.3.2 (Matui)**.** *Suppose A denotes a unital separable simple $A\mathbb{T}$-algebra of real rank zero. Under these premises, any C*-dynamical system $(A, \alpha, \mathbb{Z})$ with integer action admits a crossed product $A \rtimes_\alpha \mathbb{Z}$ which is AF-embeddable. In particular, it becomes quasidiagonal.*

How does Matui's theorem come to our aid? Assuming one may arrange the wealth of assumptions, then combining Matui's theorem with the classification theorem below provides us with a fruitful result. A proof may be recovered in the original article [7]. For the record, $\mathcal{Q}$ denotes the *universal UHF-algebra* meaning the UHF-algebra whose associated supernatural number is the one wherein each exponent is infinite. After presenting the theorem, we adapt it to our scenario.

Theorem 4.3.3 (Niu, Winter)**.** *Any unital separable simple nuclear monotracial quasidiagonal C*-algebra in the UCT class turns the spatial tensor product $A \otimes \mathcal{Q}$ into an $A\mathbb{T}$-algebra of real rank zero. In particular, A becomes AF-embeddable, hence quasidiagonal.*

Corollary 4.3.4. *The crossed product $A \rtimes_\alpha \mathbb{Z}$ associated to any C*-dynamical system $(A, \alpha, \mathbb{Z})$ consisting of a unital separable simple nuclear monotracial quasidiagonal C*-algebra A in the UCT class is AF-embeddable. In particular, $A \rtimes_\alpha \mathbb{Z}$ must be quasidiagonal.*

Proof. According to *theorem* 4.3.3, the spatial tensor product $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{Q}$ becomes AF-embeddable. However, the crossed product $A \rtimes_\alpha \mathbb{Z}$ lies therein due to $A \rtimes_\alpha \mathbb{Z} \hookrightarrow \mathcal{Q} \otimes (A \rtimes_\alpha \mathbb{Z}) \cong (A \otimes \mathcal{Q}) \rtimes_{\alpha\otimes \mathrm{id}} \mathbb{Z}$ with the identification being verified in the following manner. Suppose $\tau = \alpha \otimes \mathrm{id}$ is the natural induced action $\mathbb{Z} \curvearrowright A \otimes \mathcal{Q}$. Let $(a \otimes b)u_s$ be a generating element $C_c(\mathbb{Z}, A \otimes \mathcal{Q})$. Fix some faithful representation $\varrho = \pi \otimes \pi_{\mathcal{Q}}$ of $A \otimes \mathcal{Q}$ onto of Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$. Upon

$$\begin{aligned}[\varrho_\tau \times \lambda]((b \otimes a)u_s)(\eta \otimes \xi \otimes \delta_t) &= \pi_{\mathcal{U}}(b)\eta \otimes \pi(\alpha_{t^{-1}}(a))\xi \otimes \delta_{st} \\ &= [\pi_{\mathcal{Q}} \otimes (\pi \otimes \lambda)](b \otimes au_s)(\eta \otimes \xi \otimes \delta_t)\end{aligned}$$

being valid for every pair of vectors $\xi, \eta \in \mathcal{H}$, $b \in \mathcal{Q}$, $a \in A$ and $t \in \mathbb{Z}$, independence of the choice of faithful representation permits us to deduce that $\mathcal{U} \otimes (A \rtimes_\alpha \mathbb{Z}) \cong (A \otimes \mathcal{Q}) \rtimes_{\alpha\otimes \mathrm{id}} \mathbb{Z}$ by continuity and density of the maps involved. Since $A \otimes \mathcal{Q} \rtimes_\tau \mathbb{Z}$ is AF-embeddable on the merits of Matui's theorem, the subalgebra $A \rtimes_\alpha \mathbb{Z}$ becomes AF-embeddable, thereof quasidiagonal as quasidiagonality passes to subalgebras and AF-algebras are quasidiagonal. This completes the proof. □

The plethora of terminology occurring in the previous classification theorems may seem intimidating, especially the mysterious UCT-class condition. We settle this particular matter swiftly using the paramount work of Tu.

Proposition 4.3.5. *the Bernoulli-crossed product $B(G)$ associated to any discrete amenable group G is nuclear and belong to the UCT-class.*

Proof. It is a well-established fact that the reduced crossed product of a nuclear C*-algebra by an amenable group must be nuclear. Indeed, the result applies to our scenario since UHF-algebras are nuclear being direct limits of nuclear C*-algebras with monic connecting morphisms.

To verify the last assertion, we shall recognize $B(G)$ as the C*-algebra associated to an amenable etalé locally compact groupoid, thereby invoking Tu's theorem. Let $X = \prod_{G\times\mathbb{N}} \mathbb{Z}_2$ be the countable product of the compact space $\mathbb{Z}_2$ with itself. Then X must be compact due to Tychonoff's theorem and the discrete subgroup $H = \bigoplus_{G\times\mathbb{N}} \mathbb{Z}_2$ herein acts on X by left-multiplication entrywise. Suppose γ denotes this particular action $H \curvearrowright X$. The group G acts on both X and H by permuting the indicies occurring as elements in $G \times \mathbb{N}$, leaving the $\mathbb{N}$ unaffected while translating the G factor. We shall refer to these actions as α and β, respectively. Since G acts on H, we may form the semidirect product $H \rtimes G$ thereby. The group $H \rtimes G$ acts on X through the composition $\sigma : (h, g) \mapsto \gamma_h \circ \alpha_g$ meaning $H \rtimes G$ acts on X by letting G act on X and thereafter on H. Applying proposition 3.4.5 in conjunction with **??** repetitively yields

$$C(X) \rtimes_r H \cong C^*_\lambda(X \rtimes H) = C^*_\lambda\Big(\prod_{G\times\mathbb{N}} \mathbb{Z}_2 \rtimes \bigoplus_{G\times\mathbb{N}} \mathbb{Z}_2\Big) \cong \bigotimes_{G\times\mathbb{N}} C^*_\lambda(\mathbb{Z}_2 \rtimes \mathbb{Z}_2) \cong \bigotimes_{G\times\mathbb{N}} C(\mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong \mathbb{M}_{2^\infty}.$$

Exploiting that the action σ is compatible with the actions γ and α, proposition 3.4.5 grants

$$C(X) \rtimes_r (H \rtimes G) \cong (C(X) \rtimes_r H) \rtimes_r G \cong \mathbb{M}_{2^\infty} \rtimes G \cong B(G).$$

We arrive at the sought conclusion due to the left-hand side being the reduced C*-algebra associated to the etalé locally compact groupoid $X \rtimes (H \rtimes G)$, viewing the group as being discrete. □

Having set the stage, we arrive at the grand finale. The strategy will be to use a bootstrap argument on the class EG, the class of elementary amenable discrete groups, using Osin/Chou's theorem by verifying the closure properties therein for the class $\mathscr{P}$. We declare that a discrete countable amenable group H belongs to the $\mathscr{G}_{QQ}$ class if and only if whenever

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

is a short exact sequence such that N belongs to $\mathscr{P}$, then G belongs to $\mathscr{P}$ as well. The class $\mathscr{P}$ contains all members of $\mathscr{G}_{QQ}$ due to the trivial group being a member in $\mathscr{P}$ in conjunction with exactness of the sequence above for $N = \{1\}$ and $G = H$ forcing H to belong in $\mathscr{P}$.

Theorem 4.3.6 (Ozawa, Rørdam, Sato)**.** *The following are valid.*

(i) The class $\mathscr{P}$ is closed under performing the following operations: subgroups, direct limits and extensions by countable elementary amenable groups.

(ii) The class of countable amenable discrete groups EG_c determines a subclass of $\mathscr{P}$ and $C^*_\lambda(G)$ is AF-embeddable for any member G of $\mathscr{P}$. In particular, $C^*_\lambda(G)$ is quasidiagonal provided G is elementary amenable.

Proof. (i) Initially, we will prove that $\mathscr{G}_{QQ}$ is closed under countable direct limits and $\mathscr{Z}$ extensions in order to invoke Osin's theorem. Having established this, we may conclude that $\mathrm{EG}_c \subseteq \mathscr{G}_{QQ} \subseteq \mathscr{P}$ according to our previous remark. For this purpose, suppose

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\psi} H \longrightarrow 1 \tag{4.1}$$

denotes a short exact sequence of discrete groups. Assume that H decomposes into the countable direct limit $\bigcup_n H_n$ of discrete groups H_n in $\mathscr{G}_{QQ}$. We must verify that H belongs to the class $\mathscr{G}_{QQ}$, which amounts to establishing the containment $G \in \mathscr{P}$ class whenever one has $N = \ker\psi \in \mathscr{G}_{QQ}$. For this, define G_n to be the preimage of H_n under ψ for every positive integer n. Then G_n becomes a subgroup in G for which

$$1 \longrightarrow \ker\psi_{|G_n} \longrightarrow G_n \xrightarrow{\psi} H_n \longrightarrow 1$$

becomes exact. Since H_n belongs to $\mathscr{G}_{QQ}$ while $\ker \psi_{|G_n}$ belongs to $\mathscr{P}$ by hypothesis, we have $G_n \in \mathscr{P}$, whereupon $G = \bigcup_n G_n$ determines a countable direct limit of groups whose associated Bernoulli crossed product is quasidiagonal. However, the second functoriality property in *proposition* 4.3.1 ensures that $B(G)$ must be quasidiagonal, yielding $H \in \mathscr{P}$ on the merits of the final statement in *proposition* 4.1.4. Therefore, $\mathscr{G}_{QQ}$ must be closed under taking countable direct limits.

Proceeding towards verifying that $\mathscr{G}_{QQ}$ is closed under $\mathscr{Z}$ extensions, let H_0 be some normal finite index subgroup in H satisfying $H_0 \in \mathscr{G}_{QQ}$. The subgroup $G_0 = \psi^{-1}(H_0)$ in G has finite index once more and from $\ker \psi_{|G_0} \in \mathscr{P}$ combined with exactness of the corresponding sequence

$$1 \longrightarrow \ker \psi_{|G_0} \longrightarrow G_0 \xrightarrow{\psi} H_0 \longrightarrow 1 \tag{4.2}$$

it follows that G_0 must be a member of $\mathscr{P}$. As the groups in play are all discrete, *corollary 4.1.6* grants us an embedding $B(G) \hookrightarrow B(G_0) \otimes \mathbb{M}_N$ where N denotes the index of G_0 in G. The latter C*-algebra is quasidiagonal being the spatial tensor product of quasidiagonal C*-algebras. Hence $B(G)$ inherits quasidiagonality, proving that H lies in $\mathscr{G}_{QQ}$ whenever it contains a finite index subgroup therein. If H is some finite discrete group, then the trivial group clearly determines a finite index subgroup of H subject to the containment $\{1\} \in \mathscr{G}_{QQ}$. Thus $\mathscr{G}_{QQ}$ becomes closed under extensions by finite groups, leaving only the integer scenario to be tackled.

Suppose H denotes a discrete group having a normal subgroup H_0 in the $\mathscr{G}_{QQ}$ class fulfilling $H/H_0 \cong \mathbb{Z}$. We must show that H belongs to the $\mathscr{G}_{QQ}$ class, i.e., the sequence (4.1) with N belonging to $\mathscr{P}$ yields $G \in \mathscr{P}$. To this end, consider the short exact sequence

$$1 \longrightarrow H_0 \longrightarrow H \xrightarrow{q} H/H_0 \longrightarrow 1$$

Define G_0 to be the preimage of H_0 under ψ, which determines a subgroup inside G. Exactness of the preceding sequence implies exactness of the sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{q\psi} \mathbb{Z} \longrightarrow 1$$

Due to $\mathbb{Z}$ being a free group, the set-map $\omega\colon \{1\} \longrightarrow \psi^{-1}(\{1\})$ sending the generator 1 of $\mathbb{Z}$ into some fixed element a in the fiber $\psi^{-1}(\{1\})$ induces a homomorphism $\varphi : \mathbb{Z} \longrightarrow G$ agreeing with ω on the generator 1 via the universal property of free groups. Since these maps agree on the generator, the induced homomorphism φ satisfies $\psi\varphi = \mathrm{id}_{\mathbb{Z}}$, meaning the short exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{q\psi} \mathbb{Z} \longrightarrow 1$$

splits, whereupon G must be isomorphic to the semidirect produt $G_0 \rtimes \mathbb{Z}$. According to the third functoriality property of $B(G)$ in *proposition* 4.3.1, so $\mathbb{Z}$ admits an action on $B(G_0)$ fulfilling $B(G_0) \rtimes \mathbb{Z} \cong B(G_0 \rtimes \mathbb{Z}) \cong B(G)$. The former C*-algebra is quasidiagonal due to the containment $G_0 \in \mathscr{P}$ (an argument similar to the one exploiting exactness of (4.2) verifies this) in conjunction with $\mathbb{Z}$ being amenable permitting us to invoke *corollary* 4.3.4. This entails that $B(H)$ must be quasidiagonal as well, so $\mathscr{G}_{QQ}$ must be a class closed under performing $\mathscr{Z}$ extensions.

(ii) Concerning the second statement, the inclusion $\mathrm{EG}_c \subseteq \mathscr{G}_{QQ} \subseteq \mathscr{P}$ are immediate consequences of $\mathscr{P}$ being closed under the required group operations in EG_c and minimality of EG_c. The following statement stems $B(H)$ having an isomorphic copy of the reduced group C*-algebra implying that $C^*_\lambda(H)$ becomes AF-embeddable in the event of $B(H)$ being AF-embeddable. However, $B(H)$ is AF-embeddable whenever H is a member of $\mathscr{P}$ according to Matui's theorem, which evidently verifies the second statement.

For the final statment, note that $C^*_\lambda(H)$ is quasidiagonal if H denotes a member in EG_c on the merits of the two preceding observations. In the uncountable case, H may be written as a directed union of all its countable elementary amenable subgroups H_n according to proposition 4.2.3. However, any direct limit of quasidiagonal C*-algebras remains quasidiagonal, whence the desired stems from section 2.2, completing the proof. □

Chapter 5

Ingredients in the Theorem of White, Winter and Tikusis

An incredible amount of C*-algebraic theory is applied to confirm Rosenberg's conjecture. The proof was established by Tikuisis, White, and Winter, in accordance with the statement: Every faithful trace defined on a separable nuclear C*-algebra in the UCT-class is automatically quasidiagonal. This final chapter attempts to unravel a portion of the essential ingredients used in the proof. The proof thereof requires KK-theoretic properties, due to Dardalat and Eilers, in conjunction with the apparatus of order zero maps. The current objective will be to deduce the necessary tools of order zero maps and unveiling the intriguing features of quasidiagonal traces.

Quasidiagonal traces form a notion aptly build to restate Rosenberg's conjecture in terms of traces. Consider any discrete group G and its associated reduced group C*-algebra $C^*_\lambda(G)$. If G is countable, then $C^*_\lambda(G)$ becomes separable, amenability will grant us nuclearity and Tu's theorem supplies the UCT-criterion. Therefore we may invoke the theorem of White, Winter and Tikuisis to conclude that the canonical faithful trace on $C^*_\lambda(G)$ must be quasidiagonal. As such Rosenberg's conjecture follows provided that existing faithful quasidiagonal trace imply quasidiagonality.

5.1 A Lifting Theorem

Our initial objective will be to introduce the notion of quasidiagonal traces and supply the reader with a characterization that Tikuisis, White and Winter use a technical variation of. The characterization originates from the Chou/Effros lifting theorem, which we prove for the sake of completeness and since the author never saw the proof prior to writing this project, plus it encapsulates a potent lifting property for nuclear C*-algebras.

Definition. Suppose A and B are unital C*-algebras with B unital. Let I be a $*$-ideal in B and let $\varrho\colon B \longrightarrow B/I$ be the canonical quotient map. A contractive completely positive map $\varphi\colon A \longrightarrow B/I$ is said to be *liftable via* ψ if there exists a contractive completely positive map $\psi\colon A \longrightarrow B$ such that the diagram

$$\begin{array}{ccc} & & B \\ & \overset{\psi}{\nearrow} & \big\downarrow \varrho \\ A & \xrightarrow[\varphi]{} & B/I \end{array}$$

commutes, i.e., $\varrho\psi = \varphi$. We call ψ a c.c.p (resp. u.c.p) lift should it fulfill these constraints.

Lemma 5.1.1. *Every liftable contractive completely positive map between unital C*-algebras admits a unital completely positive lift.*

Proof. Suppose $\psi\colon A \longrightarrow B$ denotes the c.c.p lift of some c.c.p map $\varphi\colon A \longrightarrow B/I$ between C*-algebras. Define accordingly $\psi_u\colon A \longrightarrow B$ via the formula

$$\psi_u(\cdot) = \psi(\cdot) + (1_B - \psi(1_A))\omega(\cdot)$$

for any chosen state ω acting on B. Plainly, that ψ_u remains completely positive due to $\psi(1_A) \leq 1_B$ stemming from ψ being c.c.p and $\psi_u(1_A) = 1_B$, proving the claim. □

Lemma 5.1.2. *Suppose $\varphi\colon A \longrightarrow B/I$ is a contractive completely positive map admitting a completely positive lift $\psi\colon A \longrightarrow B$. If so, φ becomes liftable in the c.c.p sense.*

Proof. For the sake of avoiding unnecessary technical adjustments, we assume that both A and B are unital C*-algebras. Choose some approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ in I and define for each $\alpha \in \Lambda$ a linear map $\psi_\alpha\colon A \longrightarrow B$ by $a \mapsto (1_B - e_\alpha)\psi(a)(1_B - e_\alpha)$. The map ψ_α must be completely positive being the conjugation of one by bounded elements[1]. Moreover, one has $\varrho\psi_\alpha = \psi$ for every index $\alpha \in \lambda$ due to $\varrho(e_\alpha) = 0$, while $\lim_{\alpha\in\Lambda} \|\psi_\alpha\| = 1$. We deduce that $\lim_{\alpha\in\Lambda} \psi_\alpha$ is the sought contractive completely lift of φ, proving the claim. □

The lifting theorem is separated into two parts: Remove any topological hindrances, then exploit the correspondence of completely positive maps and positive elements in matrix algebras. The first part is achieved through Arveson's lemma, which we proceed to immediately.

Proposition 5.1.3 (Arveson's Lemma)**.** *Suppose I denotes a closed two-sided $*$-ideal in a unital C*-algebra B and let A be a separable C*-algebra. Under these premises, the collection of all liftable c.c.p maps $A \longrightarrow B/I$ is closed in the point-norm topology.*

Proof. Assume $\omega\colon A \longrightarrow B/I$ arises as the point-norm limit of c.c.p maps $\omega_n\colon A \longrightarrow B/I$, each admitting a c.c.p lift $\varphi_n\colon A \longrightarrow B$. Throughout the entire proof, $\varrho\colon B \longrightarrow B/I$ will be the canonical quotient map. By separability of A, we may assume that there exists some dense sequence $\{a_n\}_{n\geq 1} \subseteq A$. Passing to a suitable sequence if necessary, we may further arrange that

$$\|\varrho\omega_n(a_k) - \omega(a_k)\| < \frac{1}{2^n}, \quad k \leq n. \tag{5.1}$$

It suffices to verify the existence of c.c.p maps $\psi_n\colon A \longrightarrow B$ fulfilling the constraints

$$\|\varrho\psi_n(a_k) - \omega(a_k)\| < \frac{1}{2^n}, \quad k \leq n; \tag{5.2}$$

$$\|\psi_{n+1}(a_k) - \psi_n(a_k)\| < \frac{1}{2^{n-1}}, \quad k \leq n. \tag{5.3}$$

Indeed, (5.3) shows that the net $(\psi_n(a_k))_{n\geq 1}$ converges for every positive integer k, hence over all of A through density and continuity, to some c.c.p map $\psi\colon A \longrightarrow B$. On the other hand, (5.1) combined with (5.2) entails that $\|\varrho\psi - \omega\| \to 0$ on a dense subset, hence on the ambient space A via continuity of the maps involved. To produce such a sequence of c.c.p maps, we proceed by induction where the initial step is obviously achieved by setting $\psi_1 = \varphi_n$. Suppose the maps $\psi_1, \psi_2, \ldots, \psi_n$ subject to (5.1) - (5.2) have been constructed and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an approximate unit in I. Passing to the convex hull

[1] Perhaps the reader has only seen that conjugation of a $*$-homomorphism by a bounded operator defines a c.p. However, this version may be verified by applying Stinespring's dilation theorem to ψ after having representing the C*-algebras faithfully into $B(\mathcal{H})$.

of the approximate unit, we may extract a quasicentral approximate unit hereof, so we simply assume it to be quasicentral instead. An application of [2, proposition 1.2.2] yields

$$\lim_{\alpha\in\Lambda} \|(1-e_\alpha)^{1/2}\psi_n(a_k)(1-e_\alpha)^{1/2} + e_\alpha^{1/2}\psi_n(a_k)e_\alpha^{1/2} - \psi_n(a_k)\| = 0$$

for all $k \leq n$. Abbreviate next $b_k = \omega_{n+1}(a_k) - \psi_n(a_k)$ for every integer $k \leq n$. Moreover, for each b belonging to B we have the estimate

$$\begin{aligned}\lim_{\alpha\in\Lambda} \|(1-e_\alpha)^{1/2}b_k(1-e_\alpha)^{1/2}\| &= \|\varrho\varphi_{n+1}(a_k) \pm \omega(a_k) - \varrho\psi_n(a_k)\| \\ &\overset{(5.2),(5.3)}{\leq} \|\varrho\omega_{n+1}(a_k) - \omega(a_k)\| + \|\omega(a_k) - \varrho\psi_n(a_k)\| = \frac{3}{2^{n+1}}.\end{aligned}$$

The estimate on the second term stems from the induction hypothesis imposed on ψ_n; more precisely the induction hypothesis (1). Therefore, there must exist some α in Λ such that

$$\|(1-e_\alpha)^{1/2}\psi_n(a_k)(1-e_\alpha)^{1/2} + e_\alpha^{1/2}\psi_n(a_k)e_\alpha^{1/2} - \psi_n(a_k)\| < \frac{1}{2^{(n+1)}},$$

together with

$$\|(1-e_\alpha)^{1/2}b_k(1-e_\alpha)^{1/2}\| < \frac{3}{2^{(n+1)}}.$$

Upon e_α being trivial in the image of ϱ, the linear map

$$\psi_{n+1}(\cdot) = (1-e_\alpha)^{1/2}\omega_{n+1}(\cdot)(1-e_\alpha)^{1/2} + e_\alpha^{1/2}\psi_n(\cdot)e_\alpha^{1/2}$$

clearly determines a contractive completely positive map subject to $\varrho\psi_{n+1} = \varrho\omega_{n+1}$. Thus (5.2) becomes valid. For the condition (5.3), notice that for all $k \leq n$,

$$\begin{aligned}\|\psi_{n+1}(a_k) - \psi_n(a_k)\| &= \|\psi_{n+1}(a_k) \pm (1-e_\alpha)^{1/2}b_k(1-e_\alpha)^{1/2} - \psi_n(a_k)\| \\ &< \frac{1}{2^{(n+1)}} + \frac{3}{2^{(n+1)}} = \frac{1}{2^{n-1}}\end{aligned}$$

wherein the two preceding estimates have been exploited. By the principle of induction, the sought sequence $(\psi_n)_{n\geq 1}$ consisting of c.c.p maps satisfying (5.2)-(5.3) exists, completing the proof. □

Theorem 5.1.4 (Choi, Effros). *Nuclear c.c.p maps from a separable C*-algebra into a quotient are liftable. In particular, c.c.p maps from a nuclear separable C*-algebra are nuclearly liftable.*

Proof. Suppose A denotes a separable C*-algebra and let $\omega\colon A \longrightarrow B/I$ be a nuclear c.c.p map between C*-algebras. Nuclearity of φ amounts φ factoring through matrix algebras $\mathbb{M}_{k(n)}$ in the point-norm limit via a c.c.p maps $A \longrightarrow \mathbb{M}_{k(n)} \longrightarrow B/I$. Due to Arveson's lemma, verifying that every c.c.p map $\mathbb{M}_{k(n)} \longrightarrow B/I$ lifts to a c.c.p map $\mathbb{M}_{k(n)} \longrightarrow B$ suffices.

To acquire the lifts, let $\varphi\colon \mathbb{M}_n \longrightarrow B/I$ be any c.c.p map of C*-algebras. Remember the correspondence between completely positive maps $\mathbb{M}_n \longrightarrow A$ and positive elements in $\mathrm{M}_n(A)$ given by assigning $[\psi(e_{ij})]_{i,j=1}^n$ to each such completely positive map $\psi\colon \mathbb{M}_n \longrightarrow A$, where e_{ij} denotes the (i,j)'th unit matrix in $\mathbb{M}_n$ per usual. Let a be the corresponding positive element in $\mathrm{M}_n(B/I)$ stemming from φ. Then due to the n'th amplification $\varrho_n = \varrho \otimes \mathrm{id}_n\colon \mathrm{M}_n(B) \longrightarrow \mathrm{M}_n(B/I)$ of the quotient map being a $*$-epimorphism, one may infer that a lifts to some positive element b belonging to $\mathrm{M}_n(B)$. The unique completely positive map $\omega\colon \mathbb{M}_n \longrightarrow B$ corresponding to b therefore satisfies $\varrho\omega = \varphi$ under the appropriate identifications, proving the first assertion.

The second claim stems from the composition of any two c.c.p maps, with either one of them being nuclear, becoming nuclear. By definition, the identity map of a nuclear C*-algebra $A \longrightarrow A$ is nuclear so given a c.c.p map $\varphi\colon A \longrightarrow B/I$ with A nuclear, the composition $A \longrightarrow A \longrightarrow B/I$ coincides with φ and must be nuclear. Voila. □

Having taken a smaller detour into the category of nuclear C*-algebras we return to the quasidiagonal framework. Throughout the remainder of the section, ω will be a fixed free ultrafilter on the $\mathbb{N}$ and denote by $\mathcal{Q}_\omega$ the ultrapower of the universal UHF-algebra associated to ω. Furthermore,

$$\mathbb{M}_\infty := \ell^\infty(\mathbb{M}_n, \mathbb{N})/c_0(\mathbb{M}_n, \mathbb{N}).$$

The intriguing features of the mysterious device $\mathcal{Q}_w$ is its capability of witnessing quasdiagonality in the nuclear framework. The information concerning quasidiagonality encoded herein is what Tikuisis, White and Winter exploit. Thus, we are inclined to discover the properties emerging from $\mathcal{Q}_\omega$. In essence the characterization of quasidiagonality is described in terms of lifts, giving a more algebraic description of quasidiagonality.

Theorem 5.1.5. *Suppose A is a separable C*-algebra. Under this premise, A is quasidiagonal if and only if there exists a liftable $*$-monomorphism $A \hookrightarrow \mathbb{M}_\infty$, meaning the diagram*

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{M}_\infty \\ & \searrow & \uparrow \varrho \\ & & \ell^\infty(\mathbb{M}_n, \mathbb{N}) \end{array}$$

commutes with the diagonal map being some contractive completely positive map[2].

Proof. Suppose initially A is quasidiagonal. By hypothesis, A admits a sequence $(\varphi_k)_{k\geq 1}$ of asymptotically isometric - and multiplicative c.c.p maps $\varphi_k \colon A \longrightarrow \mathbb{M}_{n(k)}$. Let π be the induced c.c.p map given by the composition $A \longrightarrow \ell^\infty(\mathbb{M}_{n(k)}, \mathbb{N}) \twoheadrightarrow \mathbb{M}_\infty$ with φ being the mapping $a \mapsto (\varphi_k(a))_{k\geq 1}$. The c.c.p map φ evidently lifts π, hence only the $*$-homomorphism and isometric property of π remains to be justified. However, for every $a \in A$ one has

$$\|\pi(a)\| = \|\varrho\varphi_k(a)\| = \limsup_{k\to\infty} \|\varphi_k(a)\| = \|a\|,$$

verifying the isometric property, the third equality stemming from the asymptotic isometric property. Multiplicativity stems from $(\varphi_k)_{k\geq 1}$ being asymptotically multiplicative in a similar manner so that π becomes a $*$-homomorphisms; completely positive maps are automatically involutive.

Conversely, suppose one has a commutative diagram as described in the theorem with ψ being the c.c.p lift and $A \hookrightarrow \mathbb{M}_\infty$ being denoted by π for simplicity. Let σ_n be the n'th projection map $\ell^\infty(\mathbb{M}_n, \mathbb{N}) \longrightarrow \mathbb{M}_n$. Using this notation, one obtains a diagram

$$\begin{array}{cccccc} A & \xrightarrow{\ \pi\ } & \mathbb{M}_\infty & & \\ & \searrow^{\psi} & \uparrow \varrho & & \\ & & \ell^\infty(\mathbb{M}_n, \mathbb{N}) & \xrightarrow[\sigma_n]{} & \mathbb{M}_n \end{array}$$

in which the triangle commutes. Our prime candidate sequence $(\psi_n)_{n\geq 1}$ consisting of asymptotically multiplicative - and isometric c.c.p maps will be the compositions $\psi_n = \sigma_n\psi$. Each ψ_n is clearly c.c.p being the composition of such morphisms. It remains to be verified that the sequence satisfies the sought asymptotic properties. To accomplish this, note that commutativity of the assumed diagram entails

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = \|\varrho\psi(ab) - \varrho\psi(a)\varrho\psi(b)\| = \|\pi(ab) - \pi(a)\pi(b)\| = 0$$

Furthermore,

$$\lim_{n\to\infty} \|\psi_n(a)\| = \limsup_{n\to\infty} \|\sigma_n\psi(a)\| = \|\pi(a)\| = \|a\|$$

for every $a \in A$, since π is isometric. This completes the proof. □

[2] Liftability may be omitted whenever A is nuclear and separable according to the Choi/Effros theorem.

We proceed to deriving the version used by Tikuisis, White and Winter.

Theorem 5.1.6. *Let A be a unital separable nuclear C*-algebra. If so, A is quasidiagonal if and only if there exists a unital $*$-monomorphism $A \hookrightarrow \mathcal{Q}_\omega$.*

Proof. We commence the proof with the "only if" part. Due to $\mathcal{Q}$ containing an isomorphic copy of $\mathbb{M}_n$ for each $n \in \mathbb{N}$, the induced map $\ell^\infty(\mathbb{M}_n) \hookrightarrow \ell^\infty(\mathcal{Q})$ becomes a $*$-monomorphism. Under the assumption that quasidiagonality of A is witnessed via c.c.p maps $\varphi_k\colon A \longrightarrow \mathbb{M}_{n(k)}$, one selects

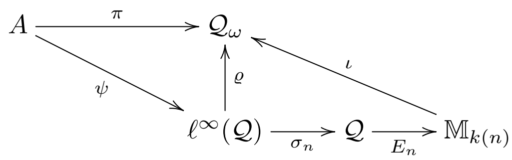
$$\pi\colon A \xrightarrow{\ \varphi\ } \ell^\infty(\mathbb{M}_{n(k)}) \longrightarrow \ell^\infty(\mathcal{Q}) \xrightarrow{\ \varrho\ } \mathcal{Q}_\omega$$

Here φ denotes the infinite inflation map associated to the sequence $(\varphi_k)_{k\geq 1}$. Let us prove that π is isometric. Suppose $a \in A$ and let σ_k be the k'th projection map. One hereof deduces that

$$\|\pi(a)\| = \|\varrho\varphi(a)\| = \lim_{k\to\omega} \|\varphi_k(a)\| = \|a\|.$$

Again the asymptotic isometric property comes into play. The verification of multiplicativity runs completely parallel whereas the involution part stems from complete positivity.

The converse is also inspired by the previous proof, however, we must overcome some obstacles. Suppose $\pi\colon A \longrightarrow \mathcal{Q}_\omega$ denotes a unital $*$-monomorphism. The nuclearity hypothesis permits us to invoke Choi-Effros' theorem to produce a c.c.p lift $\varphi\colon A \longrightarrow \ell^\infty(\mathcal{Q})$ of π. To construct the sequence of c.c.p maps witnessing quasdiagonality, consider for each $n \in \mathbb{N}$ the diagram



Here σ_n and E_n are the canonical coordinate projection - and conditional expectation (we refer to the one occurring in proposition 1.4.10; $\mathcal{Q}$ has each configuration of these, which is somewhat the point here). Define accordingly c.c.p maps by $\psi_n = \sigma_n \circ \psi$ for each positive integer n. An argument completely resembling the one in the preceding proof yields corresponding sequence $(\psi_n)_{n\geq 1}$ to be asymptotically multiplicative - and isometric. Define c.c.p maps φ_n by $\varphi_n = E_n \circ \psi_n$. The induced sequence $(\varphi_n)_{n\geq 1}$ is readily checked to inherit the c.c.p property while maintaining the asymptotic features of ψ_n according to the norm-observation in *proposition* 1.5.10. For instance,

$$\lim_{n\to\infty} \|\varphi_n(a)\| = \lim_{n\to\omega} \|E_n\sigma_n\psi(a)\| = \|\varrho(\psi(a))\| = \|a\|$$

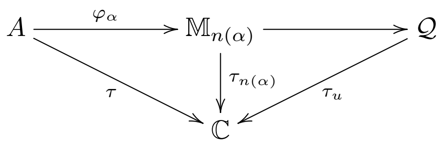
for every $a \in A$, and so on. We consider our work finished. □

5.2 Quasidiagonal Traces

The significance that $\mathcal{Q}_\omega$ carries is staggering. More adequately, $\mathcal{Q}$ the algebra to study should one seek to understand the converse of Rosenberg's conjecture. This section continues the investigation of $\mathcal{Q}_\omega$ by adding traces into the mix, so lets start there.

Definition. Suppose A denotes a unital C*-algebra. A tracial state τ on A is called *quasidiagonal* if there exists a net consisting of asymptotically multiplicative u.c.p maps $\varphi_\alpha\colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ fulfilling $\tau_{n(\alpha)} \circ \varphi_\alpha \to \tau$ weak*-wise. We demand the net to be a sequence whenever A is separable.

Remark. The unique tracial state on $\mathcal{Q}$ restricts to the ordinary tracial state τ_n on $\mathbb{M}_n$, hence any quasidiagonal trace τ on a unital C*-algebra A witnessed by u.c.p maps φ_α admits a diagram



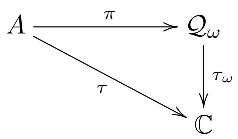
wherein the left-hand triangle commutes in the weak*-limit and the right-hand commutes directly.

Our current purpose will be to establish that the existence of a faithful quasidiagonal trace implies quasidiagonality in the unital separable case. The first vital observation is that $\mathcal{Q}_\omega$ admits a tracial state: for every $a = \varrho_\omega(a_1, a_2, \ldots)$ in $\mathcal{Q}_\omega$, the sequence $(\tau_{\mathcal{Q}}(a_n))_{n\geq 1}$ is bounded in $\mathbb{C}$, hence in some precompact set. Consequently, the sequence converges along ω by theorem 1.2.2, so we may set

$$\tau_\omega(a) = \lim_{n\to\omega} \tau_u(a_n)$$

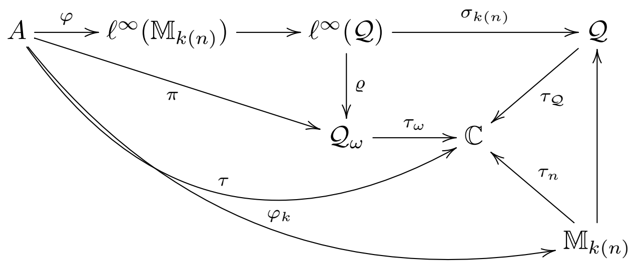
It is routine to verify τ_ω defines a tracial state on $\mathcal{Q}_\omega$. To some extend, the next proposition is the reason why $\mathcal{Q}_\omega$ intrigues us. In fact, having established it the remainder becomes plain sailing.

Proposition 5.2.1. *Let A be a unital separable C*-algebra admitting a tracial state τ. Under these premises, if τ is quasidiagonal, then there exists a unital $*$-homomorphism $\pi\colon A \longrightarrow \mathcal{Q}_\omega$ recovering τ in the sense that $\tau = \tau_\omega \circ \pi$ or pictorially; the diagram*



commutes. Conversely, with A nuclear, quasidiagonality is assured provided π exists

Proof. Suppose τ is a quasidiagonal trace on A witnessed via u.c.p maps $\varphi_k\colon A \longrightarrow \mathbb{M}_{n(k)}$ and let φ be the infinite inflation of these, $a \mapsto (\varphi_k(a))_{k\geq 1}$, which remains u.c.p. Define another u.c.p map $\pi\colon A \longrightarrow \mathcal{Q}_\omega$ by $\pi = \varrho \circ \varphi$, where ϱ denotes the ordinary quotient map $\ell^\infty(\mathcal{Q}) \twoheadrightarrow \mathcal{Q}_\omega$. Maintaining the notation established till now, our situation may be captured in the following diagram [3]:



The arrows without a symbol attached are inclusions and the lower triangle commutes on the merits of the preceding remark. In other words, π is defined as the map making the upper-left triangle commute. The c.c.p map π becomes multiplicative; if $a, b \in A$ then $\|\pi(ab)\| = \lim_{n\to\omega} \|\varphi(ab)\|$, for $(\varphi_k)_{k\geq 1}$ is asymptotically multiplicative. Thus π becomes a $*$-homomorphisms and from

$$(\tau_\omega \circ \pi)(a) = \tau_\omega[\varrho\varphi(a)] = \lim_{n\to\omega} \tau_{\mathcal{Q}}[\sigma_n\varphi(a)] = \lim_{n\to\omega} \tau_{k(n)}\varphi_n(a) = \tau(a)$$

[3]Please appreciate the diagram, it took a while to tex.

we infer that τ is quasidiagonal with respect to the sequence $(\varphi_n)_{n\geq 1}$ and that π recovers τ. The first equality is based on the definition of π, the second on the definition of τ_ω, the third on $\sigma_n(\varphi(a)) = \varphi_n(a) \in \mathbb{M}_{k(n)}$ in conjunction with commutativity of the right-hand triangle and the fourth from the lowest triangle being commutative when passing to the limit.

Conversely, if such a unital $*$-homomorphism π exists with A nuclear, then the Choi-Effros' theorem applies to yield a u.c.p lift $\varphi\colon A \longrightarrow \ell^\infty(\mathcal{Q})$ of π. Composing with the projections together with conditional expectations $\ell^\infty(\mathcal{Q}) \longrightarrow \mathcal{Q} \longrightarrow \mathbb{M}_n$ will grant us an asymptotically multiplicative sequence of u.c.p maps $\varphi_n\colon A \longrightarrow \mathbb{M}_n$. Hence we need only verify that the sequences witnesses quasidiagonality. However, this is apparent since τ_u coincides with τ_n on the image of the φ_n and we omit the details. This finalizes the proof. □

Theorem 5.2.2. *Every unital nuclear separable C*-algebra admitting a faithful quasidiagonal tracial state must be quasidiagonal. In particular, the Rosenberg conjecture has an affirmative answer provided that the canonical faithful trace on the reduced group C*-algebra of an amenable countable discrete group is quasidiagonal.*

Proof. The first statement almost trivially follows from *proposition* 5.2.2. Let τ be any faithful tracial state on A and apply the proposition to produce a unital $*$-homomorphism $\pi\colon A \longrightarrow \mathcal{Q}_\omega$ satisfying $\tau = \tau_\omega \circ \pi$. Due to τ being faithful, π must be faithful as well, implying that π determines a $*$-monomorphism. Quasidiagonality of A therefore stems from the second lifting criterion. The second assertion was previously mentioned: the reduced group C*-algebra associated to a discrete countable group defines a separable unital C*-algebra admitting a faithful tracial state, so the first part immediately entails the second. Voila. □

Corollary 5.2.3. *Simple separable unital C*-algebra with a quasidiagonal trace are quasidiagonal.*

Proof. Suppose A denotes a separable unital simple C*-algebra admitting a quasidiagonal trace τ. The $*$-ideal $\mathcal{L}_\tau = \{a \in A : \tau(a^*a) = 0\} \subseteq A$ must be zero or all of A, the latter being false as $\tau \neq 0$. The sought thus conclusion stems from *theorem* 5.2.2. □

5.3 Order Zero Maps

Order zero maps are the apparatus to be discussed in this final section of the project. In essence, the applied property of order zero maps is their alternate characterization as $*$-homomorphism when tensoring with $C_0(0,1]$. We here present a thorough survey concerning order zero maps including a Stinespring dilation theorem esque version, a functional calculus and the $*$-homomorphism correspondence of these. However, we rely on the paramount work in [16], which are merely stated. We begin the survey with a discussion concerning orthogonality.

Definition. Suppose A denotes a C*-algebra. Two elements $a, b \in A$ are said to be *orthogonal*, symbolically represented by $a \perp b$, whenever $ab = ba = a^*b = ab^* = 0$.

The reader should convince themselves that $a \perp b$ inside some C*-algebra, say A, if and only if one has the relations $a^*a \perp b^*b$, $a^*a \perp bb^*$, $aa^* \perp b^*b$ and $aa^* \perp bb^*$. Order zero maps are precisely those completely positive maps preserving orthogonality.

Definition. A completely positive map $\varphi\colon A \longrightarrow B$ between C*-algebra is *of order zero* provided that $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ for all $a, b \in A$.

Lemma 5.3.1. *Suppose A and B denotes C*-algebras.*

(i) A completely positive map $\varphi\colon A \longrightarrow B$ is of order zero if and only if $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ for any two positive elements $a, b \in A$.

(ii) Let $\pi\colon A \longrightarrow B$ be a $*$-homomorphisms and fix some element $b \geq 0$ in the commutant of $\pi(A)$. Then the map $a \mapsto b\pi(a)$ defines an order zero map.

Proof. (i) The only if part trivially holds. For the converse, let a and b be two elements in A. The hypothesis imposed on φ forces $\varphi(a^*a) \perp \varphi(b^*b)$, $\varphi(a^*a) \perp \varphi(bb^*)$, $\varphi(aa^*) \perp \varphi(bb^*)$ together with $\varphi(aa^*) \perp \varphi(b^*b)$. Due to φ being involutive, one may infer that $0 \leq \varphi(c^*)\varphi(c) \leq \varphi(c^*c)$ by positivity for every element c inside A, whereupon orthogonality preservation grants

$$\varphi(a^*)\varphi(a)\varphi(b^*)\varphi(b) \leq \varphi(a^*a)\varphi(b^*b) = 0,\ \varphi(a^*)\varphi(a)\varphi(b)\varphi(b^*) \leq 0,\ \ldots \text{ etc.}$$

One may therefore conclude that $\varphi(a^*)\varphi(a) \perp \varphi(b^*)\varphi(b)$. In a completely similar fashion, one obtains the remaining relations required to deduce $\varphi(a) \perp \varphi(b)$.

(ii) Due to π being a $*$-homomorphism and $b \geq 0$, the assignment clearly becomes completely positive. Concerning orthgonality, let $a \perp a_0$ inside A. Since b lies in the commutant of $\pi(A)$, we have $b\pi(a)b\pi(a_0) = b\pi(aa_0)b = 0$ and likewise for the remaining products, proving the assertion. □

We shall witness that every order zero map arises in this disguise. We will require the aid of Jordan homomorphism or, more adequately formulated, the aforementioned result due to Wolf. The upcoming statements are proven in [16], and have been omitted here for the sake of brevity.

Definition. A linear involutive map $\pi\colon A \longrightarrow B$ of C*-algebras is called a *Jordan homomorphism* ifπ preserves squares, meaning one has $\pi(a^2) = \pi(a)^2$ whenever $0 \leq a \leq 1$.

To understand Wolf's theorem, we emphasize on the semi-obvious notion of orthogonaliy preserving maps. A map $\varphi\colon A \longrightarrow B$ between C*-algebras is said to be *orthogonal preserving* provided that any pair of self-adjoint orthogonal elements are mapped into self-adjoint orthogonal elements.

Theorem 5.3.2 (Wolff). *Suppose A and B are unital C*-algebras, $\varphi\colon A \longrightarrow B$ is some orthogonality preserving linear map and write $e_\varphi = \varphi(1_A)$. Under these premises, one has:*

(i) e_φ belongs to the center of the C-algebra generated by $\varphi(A)$.*

(ii) If φ is involutive and unital, then φ becomes a Jordan homomorphism.

In order to prove the classification statement of order zero maps, we isolate an intermediate step at first. Wolff's theorem depends highly on the units and the preservation of these, so we ought to tackle nonunital obstructions (to fully build a new subcategory in C* with morphisms having advantages over c.c.p maps, we need to regardless). Throughout the entire remainder of the section, given any linear involutive map $\varphi\colon A \longrightarrow B$ of C*-algebras we shall denote by B_φ the C*-algebra generated by the image of φ and abbreviate $\mathscr{M} \cong B_\varphi^{**}$ viewed a von Neumann algebra.

Proposition 5.3.3. *Let $\varphi\colon A \longrightarrow B$ be an order zero c.c.p map with B lacking a unit. If so, φ extends to a c.c.p order zero map $\varphi_+\colon A^+ \longrightarrow \mathscr{M}$.*

Proof. Suppose B_φ is represented non-degenerately on a Hilbert space $\mathcal{H}$, identifying B_φ with its image therein. Choose an approxmate unit $(e_\alpha)_{\alpha\in\Lambda}$ for A and let e be the strong-operator limit of $\varphi(e_\alpha)$, whose existence stems from φ being involutive and order preserving implying that the net

$(\varphi(e_\alpha))_{\alpha\in\Lambda}$ determines an increasing net consisting of self-adjoint bounded operators. We define $\varphi_+ : A^+ \longrightarrow \mathscr{M}$ via the formula

$$\varphi_+(a + \lambda 1_{A^+}) = \varphi(a) + \lambda e$$

for all a belonging to A and every complex number λ. We assert that φ_+ becomes a c.c.p order zero map attaining values in $\mathscr{M}$, starting with the c.c.p property. Let $(\sigma, v, \mathcal{K})$ be the Stinespring dilation associated to the c.c.p map φ meaning $v^*v \leq 1_{\mathcal{H}}$ and $\varphi(\cdot) = v^*\sigma(\cdot)v$, where $v : \mathcal{H} \longrightarrow \mathcal{K}$ is some bounded operator and $\sigma : A \longrightarrow \mathrm{B}(\mathcal{K})$ is some *non-degenerate* $*$-representation of A. Due to multiplication by a bounded operator being strong-operator continuous, one may infer that

$$\begin{aligned} e = \text{sot-}\lim_{\alpha\in\Lambda} \varphi(e_\alpha) &= \text{sot-}\lim_{\alpha\in\Lambda} v^*\sigma(e_\alpha)v \\ &= v^*(\text{sot-}\lim_{\alpha\in\Lambda} \sigma(e_\alpha))v \\ &= v^* 1_{\mathcal{H}} v \\ &= v^*v. \end{aligned}$$

During the third inequality, we exploited the fact that σ is non-degenerate. The $*$-homomorphism extension $\sigma_+ : A^+ \longrightarrow \mathrm{B}(\mathcal{H})$ defined by $a + \lambda 1_{A^+} \mapsto \sigma(a) + \lambda 1_{\mathcal{H}}$ satisfies

$$\begin{aligned} v^*\sigma_+(a + \lambda 1_{A^+})v &= v^*\sigma(a)v + \lambda v^*v \\ &= \varphi(a) + \lambda e \\ &= \varphi_+(a + \lambda 1_{A^+}) \end{aligned}$$

due to the previous computation, revealing that φ_+ is c.c.p being the conjugation of a representation via bounded operators. We proceed to proving that φ_+ must be of order zero. For this purpose, suppose $x = a + \lambda 1_{A^+}$ and $y = b + \mu 1_{A^+}$ are two positive orthogonal elements in the unitalization of A. Since $xy = 0$ forces that either λ or μ must be zero, we may assume without loss of generality that $\mu = 0$. This in turn implies $b \geq 0$ as $y \geq 0$ and one easily deduces from the products $0 = x^*y = xy^*$ that a has to be self-adjoint and $\lambda \geq 0$. Consider now the rewriting

$$x = a + \lambda 1_{A^+} = (a + \lambda 1_{A^+})^{1/2}(1_{A^+} - e_\alpha)(a + \lambda 1_{A^+})^{1/2} + (a + \lambda 1_{A^+})^{1/2} e_\alpha (a + \lambda 1_{A^+})^{1/2}.$$

Let z_0 denote the first term and z the second term on the right-hand side. The element $z \leq x$ lies in A. Hence we have $z \perp b$ from which we deduce that $\varphi_+(b)\varphi_+(z) = 0$. Exploiting continuity of φ_+ in conjunction with $1_{A^+} - e_\alpha$ being central in A^+, one obtains

$$\begin{aligned} 0 \leq \text{sot-}\lim_{\alpha\in\Lambda} \varphi_+(z_0) &= \text{sot-}\lim_{\alpha\in\Lambda} \varphi_+((1_{A^+} - e_\alpha)x) \\ &= \text{sot-}\lim_{\alpha\in\Lambda} \varphi_+(\lambda(1_{A^+} - e_\alpha)) \\ &= \lambda \cdot \text{sot-}\lim_{\alpha\in\Lambda} [\varphi_+(1_{A^+}) - \varphi^+(e_\alpha)] = 0 \end{aligned}$$

Altogether, we have $\varphi_+(b)\varphi_+(x) = \varphi_+(b)\varphi_+(z_0) \to 0$ in the strong-operator sense, so we obtain $\varphi_+(b)\varphi_+(x) = 0$ and thereby $\varphi_+(y)\varphi_+(x) = 0$. The remaining few orthogonality properties are verified in parallel manners. Thus φ_+ determines an order zero c.c.p map extending φ. □

As promised, the Stinespring dilation theorem for order zero maps.

Theorem 5.3.4 (Winter, Zacharias)**.** *Let $\varphi : A \longrightarrow B$ be a c.p order zero map between C^*-algebras. Under this premise, there exists a positive element e_φ in the C^*-algebra $\mathcal{M}(B_\varphi) \cap B'_\varphi$ of norm $\|e\| = \|\varphi\|$ together with a $*$-homomorphism $\pi_\varphi : A \longrightarrow \mathcal{M}(B_\varphi) \cap \{e_\varphi\}'$ fulfilling $\pi_\varphi(\cdot)e = \varphi(\cdot)$. Moreover, in the scenario where A admits a unit one may choose $e_\varphi = \varphi(1_A)$.*

Proof. The strategy revolves around building a c.c.p Jordan homomorphism π_φ, then derive that π_φ must be a bone-fide $*$-homomorphism. Establishing this requires us to assume that A admits a unit, however, the merits of the preceding proposition is the tool to extend any c.c.p order zero map to the unitalization, in which the unital case applies. Therefore, suppose A admits a unit 1_A and write $e_\varphi = \varphi(1_A)$. Passing to the universal representation if needed, we may assume that B_φ determines a nondegenerate C*-subalgebra in $\mathrm{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. According to part (i) in Wolff's theorem, the element $\varphi(1_A) \geq 0$ belongs to the center of B_φ, from which the set $N = e_\varphi B_\varphi e_\varphi$ coincides with the subspace $\{\varphi(a)e_\varphi^2 : a \in A\} \subseteq B_\varphi$. As e_φ is positive, it admits a positive square root in B_φ, hence

$$\varphi(b) = \varphi(b)e_\varphi^{1/2}e_\varphi^{1/2} \in N$$

becomes valid for every b belonging to A. This entails that the norm closure of N equals A or, formulated differently, e_φ is strictly positive. Regarded as an operator acting on $\mathcal{K}$, the orthogonal projection η_φ onto the range of e_φ must be the identity due to strict positivity and $B_\varphi \subseteq \mathrm{B}(\mathcal{K})$ non-degenerately. On the other hand, the strong operator limit

$$\eta_\varphi = \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1}e_\varphi$$

coincides with $1_{\mathcal{H}} = 1_{\mathscr{M}}$. The limit exists because the element $e_\varphi + n^{-1}1_{\mathcal{H}}$ decreases as n tends to infinity, whereupon its inverse must be increasing; some routine functional calculus applied to e_φ guarantees this. Define accordingly a linear map $\pi_\varphi \colon A \longrightarrow \mathscr{M}$ by the expression

$$\pi_\varphi(\cdot) = \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1}e_\varphi \cdot \varphi(\cdot).$$

Due to η_φ being monotone increasing and self-adjoint, based on φ being c.p, the strong-operator limit exists. We assert that π_φ must be completely positive. However, it is apparent the strong operator limit in the product $\mathrm{B}(\mathcal{H}\otimes\mathcal{K}) \cong \mathrm{B}(\mathcal{H})\otimes\mathrm{B}(\mathcal{K})$ is defined in terms of the seminorms $S\otimes T \mapsto \|T\xi\|\cdot\|S\eta\|$ indexed over all elementary tensors $\xi\otimes\eta$ in $\mathcal{H}\otimes\mathcal{K}$, so strong-operator convergence in $M_n(A)$ occurs entrywise. In particular, strong-operator limits of c.c.p maps are c.c.p. As such π_φ defines a completely positive map and we claim it must be of order zero. Indeed, if $a \perp b$ in A then due to e_φ commuting with B_φ we may deduce that

$$\begin{aligned}\pi_\varphi(a)\pi_\varphi(b) &= \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1}\varphi(a)\ \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1}\varphi(b)\\ &= \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1}\varphi(a)\varphi(b)\ \text{sot-}\lim_{\alpha\in\Lambda}(e_\varphi + n^{-1}1_{\mathcal{H}})^{-1} = 0\end{aligned}$$

with the latter equality stemming from φ being of order zero. Moreover, $\pi_\varphi(1_A) = 1_{\mathscr{M}}$ so Wolf's theorem part (ii) applies to ensure that π_φ must be Jordan. In fact, π_φ becomes an actual $*$-homomorphism according to the following reasoning. Let $(\sigma, V, \mathcal{H})$ be the Stinespring dilation with respect to π_φ. Due to π_φ being unital, we may replace the operator V by the unit in $1_{\mathscr{M}}$ to obtain $\pi_\varphi = \mathrm{Ad}_{1_{\mathscr{M}}} \circ \varrho$. Using this in conjunction with π_φ preserving squares in A, we obtain

$$\begin{aligned}\|1_{\mathscr{M}} - \varrho(a) - 1_{\mathscr{M}}\varrho(a)1_{\mathscr{M}}\|^2 &= \|1_{\mathscr{M}}\varrho(a)(1_{\mathcal{H}} - 1_{\mathscr{M}})\varrho(a)1_{\mathscr{M}}\|\\ &= \|\pi_\varphi(a^2) - \pi_\varphi(a)^2\| = 0\end{aligned}$$

whenever $0 \leq a \leq 1$. Thus $\pi_\varphi(ab) = 1_{\mathscr{M}}\varrho(a)1_{\mathscr{M}}\varrho(b)1_{\mathscr{M}} = \pi_\varphi(a)\pi_\varphi(b)$ for all positive elements $a, b \in A$. By linearity and the fact that element in C*-algebras are linear combinations of at most four positive elements, π_φ becomes a $*$-homomorphism. It remains to be shown that π_φ attains values in $\mathcal{M}(B_\varphi) \cap \{e_\varphi\}'$. To this end, consider the calculation $\varphi(a) - \pi_\varphi(a)e_\varphi = \varphi(a) - \varphi(a)\eta_\varphi = 0$ stemming from $\eta_\varphi = 1_{\mathcal{H}}$. Thus, $\varphi(a) = \pi_\varphi(a)e_\varphi = e_\varphi\pi_\varphi(a)$ holds for all a belonging to A, so in particular the image of π_φ commutes with e_φ. The preceding provides us with the containment

$$\pi_\varphi(a)\varphi(b) = \pi_\varphi(a)\pi_\varphi(b)e_\varphi = \pi_\varphi(ab)e_\varphi = \varphi(ab) \in B_\varphi$$

and similarly $\varphi(b)\pi_\varphi(a) = \varphi(ba) \in B_\varphi$ for all $a, b \in A$. Remembering that the multiplier algebra of B_φ is isomorphic to $\mathcal{M}(B_\varphi) = \{a \in \mathrm{B}(\mathcal{H}) : ab \in B_\varphi,\ ba \in B_\varphi,\ b \in B_\varphi\}$[4]. The two containments prove that $\pi_\varphi(A) \subseteq \mathcal{M}(B_\varphi)$. This tackles the unital case A.

In the absence of a unit in A, apply *proposition* 5.3.4 to extend φ to a completely positive order zero map $\varphi_+ \colon A^+ \longrightarrow \mathscr{M}$. The unital case produces a $*$-homomorphism π_{φ_+} together with a positive element $(e_\varphi)_+$ fulfilling the identity $\varphi_+(\cdot) = \pi_{\varphi_+}(\cdot)(e_\varphi)_+$. For every positive $b \in B^+$,

$$e_{\varphi_+}\varphi(b) = e_{\varphi_+}\varphi_+(b) = e_{\varphi_+}\pi_{\varphi_+}(b^{1/2})^2 e_{\varphi_+} = \varphi_+(b^{1/2})^2 \in B_\varphi.$$

Again, linearity yields the general sought property and a similar computations yields $e_{\varphi_+}\varphi(b) \in B_\varphi$ whereupon its image lies in the multiplier algebra. This completes the proof.

□

Notation. Suppose $\varphi \colon A \longrightarrow B$ denotes an order zero c.c.p map. The corresponding image C*-algebra B_φ, positive element e_φ and $*$-homomorphism $\pi_\varphi \colon A \longrightarrow \mathcal{M}(B_\varphi) \cap \{e_\varphi\}'$ recovering φ will be represented by these symbols and we shall collect these into triples, written as $(\pi_\varphi, e_\varphi, B_\varphi)$. The notation is inspired from the Stinespring dilation triple or GNS triples due to the resemblance.

In spite of the characterization describing the failure of being a $*$-homomorphism of an order zero map, one seeks, whenever possible, to have an analogue of order zero maps defined completely in terms of full-fledged $*$-homomorphism having some configurations attached to them. Certainly, this was provided for quasihomomorphism using free products of C*-algebras. The sought correspondence is precisely the properties exploited by Tikuisis, White and Winter to derive the so-called patching lemma. We abbreviate $(0, 1]$ by I_0 and recall that $C_0(I_0)$ is canonically generated by $\mathrm{id}_{(0,1]}$.

Corollary 5.3.5. *For every pair of C*-algebras A and B, there exists a one-to-one correspondence of sets between the collection of contractive order zero maps $\varphi \colon A \longrightarrow B$ and $*$-homomorphisms $C_0(I_0) \otimes A \longrightarrow B$. The correspondence is explicitly expressed via the set maps*

$$\varphi \colon A \longrightarrow B \ \text{ c.c.p order zero} \quad \overset{\Phi}{\longmapsto} \quad \varrho_\varphi \colon C_0(I_0) \otimes A \longrightarrow B; \quad \varrho_\varphi(\mathrm{id} \otimes a) = \varphi(a)$$

$$\varrho \colon C_0(I_0) \otimes A \longrightarrow B \ \ *\text{-homomorphism} \quad \overset{\Psi}{\longmapsto} \quad \varphi_\varrho \colon A \longrightarrow; \quad \varphi_\varrho(a) = \varrho(\mathrm{id} \otimes a)$$

meaning Φ and Ψ are mutual inverses of one another.

Proof. Suppose $\varphi \colon A \longrightarrow B$ denotes a c.c.p order zero map and let $(\pi_\varphi, e_\varphi, B_\varphi)$ be the corresponding triple. Define accordingly a $*$-homomorphism $\sigma \colon C_0(I_0) \longrightarrow \mathcal{M}(B_\varphi)$ by assigning $\mathrm{id} \mapsto e_\varphi$. Upon e_φ commuting with the image of π_φ, the induced tensor-product $*$-homomorphism $\varrho_\varphi = \sigma \otimes \pi_\varphi \colon C_0(I_0) \otimes A \longrightarrow \mathcal{M}(B_\varphi)$ defined by right-multiplication of π_φ, i.e.,

$$\varrho_\varphi(\mathrm{id} \otimes a) = e_\varphi \pi_\varphi(a), \quad a \in A,$$

becomes a $*$-homomorphism attaining values in B_φ since $e_\varphi\pi_\varphi(\cdot) = \varphi(\cdot) \in B\varphi \subseteq B$.

Conversely, given a $*$-homomorphism $\varrho \colon C_0(I_0) \otimes A \longrightarrow B$, one sets $\varphi_\varrho(\cdot) := \varrho \circ (\mathrm{id} \otimes \cdot)$ and checks that the sought conditions are fulfilled: Let $a \perp b$ occur within A, then observe that $\varphi_\varrho(a)\varphi_\varrho(b) = \varrho(\mathrm{id} \otimes ab) = 0$ whereupon orthogonality preservation follows. Due to φ_ϱ being the composition of c.c.p maps[5], it must be c.c.p itself hence an order zero c.c.p map. Therefore, the assignments Φ and Ψ are well-defined whereas verifying that these are mutual inverse of one another is easily deduced from the relation $e_\varphi\pi_\varphi = \varphi$. For instance, one has for every a in A that

$$\varphi_{\varrho_\varphi}(a) = \varrho_\varphi(\mathrm{id} \otimes a) = e_\varphi\pi_\varphi(a) = \varphi(a)$$

The remaining one is proven similarly, finalizing the proof. □

[4] see the proof of existence in the appendix $A.2$ if memory fails.

[5] the spatial tensor product of c.c.p maps remains c.c.p, see for instance [2, theorem 3.5.3].

Remark. As a courtesy towards having a keen eye on categorical points of views; the corollary above may be described in a concise manner. Suppose $\mathcal{CO}(A,B)$ denote the collection of c.c.p order zero maps - and and $\mathcal{O}(A,B)$ be the collection of all order zero maps from $A \longrightarrow B$. Using this notation, the corollary reads $\mathcal{CO}(A,B) \cong \mathrm{Hom}(C_0(I_0) \otimes A, B)$ in the category of sets.

For the sake of exhibiting the flexibility of order zero maps, we finalize the project by deriving a functional calculus unique to order zero maps and produce traces via order zero maps.

Proposition 5.3.6. *Suppose $\varphi \colon A \longrightarrow B$ is an order zero c.c.p map having $(\pi_\varphi, e_\varphi, B_\varphi)$ as associated triple. Under these premises, the assignment $\Delta_\varphi \colon C_0(I_0)_+ \longrightarrow \mathcal{CO}(A,B)$ given by*

$$\Delta_\varphi(f)(a) = f(e_\varphi)\pi_\varphi(a)$$

satisfies the following properties:

- *$\Delta_\varphi(f)$ attains values in B_φ;*
- *$[f(e_\varphi), \pi_\varphi(A)] = 0$ for every element f in $C_0(I_0)_+$;*
- *If f is some contractive element in $C_0(I_0)$, then $\Delta_\varphi(f)$ becomes contractive.*

Proof. Let us maintain the notation occurring in the statement. Due to π_φ attaining values in the commutant of $\{e_\varphi\}'$, we infer that $[f(e_\varphi)\pi_\varphi(a)] = 0$ for all a belonging to A due to the continuous functional calculus preserving commutativity. Let e_{ij} be the (i,j)'th unit matrix in M_n. Exploiting this yields that for every $a = [a_{ij}]$ in $\mathbb{M}_n(A)$ one has

$$\begin{aligned}(\Delta_\varphi(f) \otimes 1_n)(a^*a) &= \sum_{i,j=1}^{n} \Delta_\varphi(f)(a_{ij}^* a_{ij}) \otimes e_{ij} \\ &= \sum_{i,j=1}^{n} f(e_\varphi)\pi_\varphi(a_{ij})^*\pi_\varphi(a_{ij}) \otimes e_{ij} \\ &= \sum_{i,j=1}^{n} \pi_\varphi(a_{ij})^* f(e_\varphi)\pi_\varphi(a_{ij}) \otimes e_{ij} \geq 0\end{aligned}$$

because $f(e_\varphi) \geq 0$ as $e_\varphi \geq 0$. Therefore the map $\Delta_\varphi(f)$ must be completely positive, proving well-definedness in conjunction with the second property. The fact that $\Delta_\varphi(f)$ attains values in B_φ stems from $e\pi_\varphi(a)$ lying within B_φ for any element $a \in A$. Lastly, if $f \colon I_0 \longrightarrow [0,\infty)$, then

$$\|\Delta_\varphi(f)\| \leq \|f(e_\varphi)\| \cdot \|\pi_\varphi\| \leq 1$$

proving the assertion. □

Corollary 5.3.7. *Let $\varphi \colon A \longrightarrow B$ be a c.c.p order zero map and suppose B admits a trace. If so, A admits a trace via the composition $\tau \circ \varphi$.*

Proof. We solely consider the tracial property. Let $f \colon I_0 \longrightarrow [0,\infty)$ be the function $t \mapsto t^{1/2}$. Denote by $(\pi_\varphi, e_\varphi, B_\varphi)$ be the triple associated to φ. We will abbreviate $\Delta_\varphi(f) = \varphi^{1/2}$ for simplicity. Using the calculus of order zero maps, we obtain

$$\varphi^{1/2}(b)\varphi^{1/2}(a) = f(e_\varphi)^2\pi_\varphi(a)\pi_\varphi(b) = e_\varphi\pi_\varphi(ab) = \varphi(ab)$$

for all $a, b \in A$, due to e_φ commuting with the image of π_φ and the continuous functional calculus applied to e_φl. According to the above, for any such pair of elements $a, b \in A$ we have

$$\tau\varphi(ab) = \tau(\varphi^{1/2}(a)\varphi^{1/2}(b)) = \tau(\varphi^{1/2}(b)\varphi^{1/2}(a)) = \tau\varphi(ba)$$

for all $a, b \in A$, completing the proof. □

Appendix A

Group Extensions and Multiplier Algebras of C*-Algebras

A.1 Group Extensions and Semi-Direct Products

This minute appendix seeks to settle some terminology concerning extensions of groups, which appear during the chapters concerning elementary amenable groups and the universal coefficient theorem. We initiate the survey with basic notions, thereafter considering the special case of semi-direct products whose appearance are constant in the project. We refer to [12] for proofs.

Definition. Suppose G denotes any group. We call G an *extension* of H by N if

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

is short-exact. Warning: in the literature, some call G and extension of N by H instead.

One commonly wishes to count the amount of distinct extensions via a group, called the Ext-group. We define the abelian group $\mathrm{Ext}(G, H)$ associated to G and H is defined in the upcoming fashion. We declare that two group extensions

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varphi} H \longrightarrow 0$$

$$0 \longrightarrow N \xrightarrow{j} G' \xrightarrow{\psi} H \longrightarrow 0$$

in the category of abelian groups are *equivalent* provided the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

is guaranteed. We define the sum of any such pair of extensions to be the extension

$$0 \longrightarrow N \longrightarrow \Gamma \longrightarrow H \longrightarrow 0$$

where Γ is the quotient of the additive group $\{(g, g') \in G \oplus G' : \varphi(g) = \psi(g')\}$ by the additive subgroup wherein we identify $(\iota(n) + g, g') \sim (g, j(n) + g')$ for all $n \in N$. We define $\mathrm{Ext}(G, H)$ to be the additive group consisting of all equivalence classes having the above addition of extensions.

A reoccurring construction of groups in this project is the *semidirect product* of groups. For the sake of having some standards, we review the structure. Suppose H were a group acting by automorphism on an additional group via $\varphi\colon H \longrightarrow \mathrm{Aut}(G)$. The semidirect product of G by H, symbolically written as $G \rtimes H$, is the group whose underlying set is the cartesian product $G \rtimes H$ with the composition, neutral element and inversions defined in terms of the formulas

$$(g,h)(s,t) = (g\varphi_h(s), ht), \quad (g,h)^{-1} = (\varphi_h^{-1}(g^{-1}), h^{-1}) \quad \text{and} \quad 1 = (1_G, 1_H).$$

for all $g, s \in G$ and $t, h \in H$. The groups G and H become subgroups in $G \rtimes H$, where G becomes normal, via the embeddings $g \mapsto (g, 1_H)$ and $h \mapsto (1_G, h)$, respectively. On the merits of these embeddings, one typically write a product of elements in $(g,h) \in G \times H$ in the semidirect product as gh to shorten the notation. Under these identifications, the action φ transforms into conjugation,

$$\varphi_h(\cdot) = h^{-1}(\cdot)h$$

for every $h \in H$. Indeed note that

$$hgh^{-1} = (1_G, h^{-1})(g, 1_H)(1_G, h) = (\varphi_h(g), h^{-1})(1_G, h) = (\varphi_h(g), 1_H).$$

which translates into the above under the appropriate identifications. An alternative and highly fruitful characterization of a semidirect product may be phrased in the following manner. A proof may be recovered in practically any book mentioning semidirect products.

Proposition A.1.1. *A group G is isomorphic to the semidirect product $N \rtimes H$ of two additional groups N and G if and only if there exists a split short exact sequence*

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 1$$

meaning there exists a group homomorphism $\psi\colon H \longrightarrow G$ such that $\psi\varphi = id_H$.

A.2 The Multiplier Algebra

During the study of KK-theory and order zero maps, multiplier algebras tend to make an appearance. The author had prior to this project only heard of multiplier algebras, hence this appendix was included. The existence of multiplier algebras may be derived in several ways and herein we adopt one using representation theory. First, some preliminary notions.

Definition. An ideal I in a C*-algebra A is *essential* provided that its *orthogonal complement*

$$I^\perp = \{a \in A : aI = Ia = 0\}$$

is trivial, meaning it only contains the 0. This is equivalent to I intersecting any ideal in A.

Definition. A *multiplier algebra* of some C*-algebra A is a maximal unital C*-algebra $\mathcal{M}(A)$ containing A as an essential ideal and fulfilling the following universal property: For any additional C*-algebra B containing A as an essential ideal, there is a unique $*$-monomorphism $\Psi\colon B \longrightarrow \mathcal{M}(A)$ restricting to the identity on A.

Evidently, one desires two properties of multiplier algebras: We wish to establish uniqueness up to isomorphism and existence. The first part is almost immediate from the universal property whereas existence is more tricky. To prove existence, we derive a minor extension result from representation theory of C*-algebras. Since the write was unable to find a reference, here is a slightly concise proof.

Lemma A.2.1. *Suppose A is some C*-algebras containing an ideal I. Then any non-degenerate representation of I extends uniquely to a representation on A.*

Proof. Suppose $\pi\colon I \longrightarrow \mathrm{B}(\mathcal{H})$ is a non-degenerate representation. Choose an approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ of I and define accordingly $\pi_0\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ by setting

$$\pi_0(a)(\pi(b)\xi) = \pi(ab)\xi$$

for all $b \in I$ and $\xi \in \mathcal{H}$. This is well-defined due to $\pi(B)\mathcal{H}$ being dense in $\mathcal{H}$ via non-degeneracy in conjunction with I being an ideal. Furthermore, the net $(\pi_0(a)\pi(e_\alpha)\xi)_{\alpha\in\Lambda}$ must be Cauchy for all fixed elements a in A on the merits of

$$\begin{aligned}\|\pi_0(a)(\pi(b)\xi) - \pi_0(a)(\pi(b')\xi)\| &= \|\pi(ab - ab')\xi\| \\ &= \lim_{\alpha\in\Lambda} \|\pi(ae_\alpha)\pi(b-b')\xi\| \\ &\le \|a\| \cdot \|\pi(b-b')\xi\|\end{aligned}$$

being true for every pair $b, b' \in I$. Completeness thus guarantees the existence of a limit point, whereupon the map $\varrho\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ defined by

$$\varrho(a)\xi = \lim_{\alpha\in\Lambda} \pi_0(a)\pi(e_\alpha)\xi = \lim_{\alpha\in\Lambda} \pi(ae_\alpha)$$

becomes well-defined. One readily verifies that ϱ is a representation extending π, so only uniqueness remains to be justified. However, if $\sigma\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ were another representation extending π, then one obtains the identity $\sigma(a)(\pi(b)\xi) = \sigma(ab)\xi = \pi(ab)\xi$ for all $a \in A$, $b \in I$ and $\xi \in \mathcal{H}$. By density and continuity of the maps involved, the representation σ must equal ϱ, completing the proof. ☐

Theorem A.2.2. *Every C*-algebra A admits a multiplier algebra $\mathcal{M}(A)$ uniquely determined up to isomorphism. Furthermore, in the event of A being unital one has $A = \mathcal{M}(A)$.*

Proof. Let $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ be any faithful non-degenerate representation of a C*-algebra A. Define $\mathcal{M}(A)$ to be the idealizer[1] of A under π, i.e.,

$$\mathcal{M}(A) = \{b \in \mathrm{B}(\mathcal{H}) : b\pi(a), \pi(a)b \in A \text{ for all } a \in A\}.$$

Identifying A with its isomorphic image in $\mathrm{B}(\mathcal{H})$ under π permits one to deduce that A must be an ideal inside $\mathcal{M}(A)$ by construction. In fact, A is essential herein, for if an element b in $\mathrm{B}(\mathcal{H})$ satisfies $\pi(a)b = b\pi(a) = 0$ for every a in A, then one has $b\pi(a)\xi = 0$ for every vector ξ inside $\mathcal{H}$. By non-degeneracy of π, the set $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$ so that $b\mathcal{H} = \{0\}$ whereupon $b = 0$ follows immediately. It is apparent that $\mathcal{M}(A)$ contains the unit of $\mathrm{B}(\mathcal{H})$.

To verify maximality together with the universal property, we invoke the previous lemma. Suppose B is another C*-algebra containing A as an essential ideal. According to the lemma, the representation π extends uniquely to a representation $\varrho\colon B \longrightarrow \mathrm{B}(\mathcal{H})$ via the formula

$$\varrho(b) = \lim_{\alpha\in\Lambda} \pi(be_\alpha)$$

for some fixed choice of approximate unit $\{e_\alpha\}_{\alpha\in\Lambda}$ in A. In general, one may always extract a quasicentral approximate unit from the convex hull of the net $\{e_\alpha\}_{\alpha\in\Lambda}$ and we shall therefore assume that our chosen approximate unit is quasicentral. Remembering this until later, observe that any b in the kernel of ϱ must satisfy $\lim_{\alpha\in\Lambda} be_\alpha = 0$ due to π being faithful. By multiplying the latter expression with any $a \in A$ yields $ba = 0$ so the kernel of ϱ must belong to the orthogonal complement

[1] that is, the largest C*-algebra containing A as a $*$-ideal.

of A. Since $A^\perp = \{0\}$, the representation ϱ must be faithful. Now, due to the approximate unit being quasicentral, one has

$$\varrho(b)\pi(a) = \lim_{\alpha\in\Lambda}(\pi(be_\alpha))\pi(a) = \lim_{\alpha\in\Lambda}\pi(b)\pi(e_\alpha a) = \pi(ba) \in \pi(A)$$

for all $a \in A$ and $b \in B$. Hence we have a unique inclusion $B \cong \varrho(B) \subseteq \mathcal{M}(A)$, which clearly restricts to the identity on B, and therefore $\mathcal{M}(A)$ determines a multiplier algebra of A. This establishes the existence part, whereas uniqueness may be verified as follows.

Let M be another multiplier algebra of A. According to the universal property, there are unique $*$-homomorphisms $\varphi\colon M \longrightarrow \mathcal{M}(A)$ and $\psi\colon M \longrightarrow \mathcal{M}(A)$ restricting to the identity on A, so $\psi\varphi\colon \mathcal{M}(A) \longrightarrow \mathcal{M}(A)$ and $\varphi\psi\colon M \longrightarrow M$ must be $*$-homomorphisms restricting to the identity on A. However, the universal property applied once more demands that any $*$-homomorphism $M \longrightarrow M$ restricting to the identity on A must be the identity on M by uniqueness. Since the same argument applies to $\mathcal{M}(A)$, the maps φ and ψ must be mutual inverses, whereof $M \cong \mathcal{M}(A)$.

Concerning the remaining statement, assume A has a unit and regard A as being an essential ideal in its multiplier algebra $\mathcal{M}(A)$. Then $(1_A - 1_{\mathcal{M}(A)})A = \{0\}$ must be valid and thus $1_{\mathcal{M}(A)} = 1_A \in A$ due to A being an essential ideal, which entails that $A = \mathcal{M}(A)$. Voila. □

A.3 Strictly Positive Elements

This little minor section describes σ-unital C*-algebras, which occur occasionally in the project although perhaps without the use of this particular terminology. Moreover, the equivalence of being σ-unital and existence of a strictly positive is being exploited.

Definition. A positive element a in a C*-algebra A is called *strictly positive*, written $a > 0$, whenever aAa determines a norm-dense subalgebra in A.

Note that for A unital, a positive element a therein is strictly positive if and only if a is invertible. Indeed, invertibility yields $aAa = A$ event without taking the closure and conversely $a > 0$ entails that there exists some $b \in A$ fulfilling $\|aba - 1_A\| < 1$ from which aba becomes invertible in A, hence a must be invertible.

Proposition A.3.1. *For a positive element a in some C*-algebra A, the following are equivalent.*

(i) A is σ-unital.

(ii) a is strictly positive.

(iii) $\omega(a) > 0$ for every state ω on A.

Proof. The implication (i) ⇒ (ii) is obvious. For (i) ⇒ (iii), let $\{e_n\}_{n\geq 1}$ be an approximate unit in A and define accordingly $a = \sum_{n=1}^\infty 2^{-n}e_n$. Suppose ω denotes a state subject to $\omega(a) = 0$. Then $\omega(e_n) = 0$ for all $n \in \mathbb{N}$, so for every element $b \geq 0$ in A one has

$$\omega(b) = \lim_{n\to\infty}\omega(e_n^{1/2}e_n^{1/2}b) \leq \omega(e_n)\omega(b^*e_nb) = 0.$$

Since every element in A is the sum of at most four positive elements, linearity of ω yields $\omega = 0$. Therefore $\omega(e_n) > 0$ for at least one positive integer n, hence $\omega(a) > 0$. In order to prove (iii) ⇒ (i), define at first $e_n = a(n^{-1}1_{A^+} + a)^{-1}$ in the unitalization A^+. Since $a > 0$ regarded as an element in A^+, it must be invertible therein, meaning $\sigma(a)$ cannot contain zero. Thus some functional calculus yields $e_n \to 0$ proving that $\{e_n\}_{n\geq 1}$ determines a countable approximate unit of A. (i) ⇒ (iii): Due to each ω being non-trivial, a positive nonzero element b such that $\omega(aba) > 0$ exists. However, $0 < \omega(aba) \leq \|a\|^2 \cdot \|b\|$ thus forces $a \neq 0$ as desired. □

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