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On the structure of

Simple, nuclear C^*-algebras

The Toms-Winter Conjecture

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Abstract

The background for this thesis is to be found in a conjecture posited by Andrew Toms and Wilhelm Winter, which states that for a unital, simple, separable, nuclear, stably finite and non-elementary C^*-algebra A, the regularity properties $\mathcal{Z}$-stability, finite decomposition rank and strict comparison of positive elements are all equivalent. We are particularly concerned with a recent result of Hiroki Matui and Yasuhiko Sato which states that if a C^*-algebra is unital, simple, separable, nuclear, quasidiagonal, has a unique tracial state and strict comparison of positive elements, then it has finite decomposition rank. Together with results obtained by other authors, this shows that the Toms-Winter conjecture has an affirmative answer under the additional assumption that A is quasidiagonal and has a unique tracial state. The proof relies on a study of central sequence algebras, both tracial and metric, as well as deep results from the theory of von Neumann algebras. We also compare the notions of decomposition rank and nuclear dimension and, in particular, we show that they differ on Kirchberg algebras.

Resumé

Baggrunden for dette speciale skal findes i en formodning fremsat af Andrew Toms og Wilhelm Winter, der siger at for en unital, simpel, separabel, nukleær, stabilt endelig og ikke-elementær C^*-algebra A er regularitetsegenskaberne $\mathcal{Z}$-stabilitet, endelig dekompositionsrang og streng sammenligning af positive elementer ækvivalente. Vi fokuserer på et nyligt resultat af Hiroki Matui of Yasuhiko Sato, der siger at hvis en C^*-algebra er unital, simpel, separabel, nukleær, kvasidiagonal, har en entydig sportilstand og streng sammenligning af positive elementer, så har den endelig dekompositionsrang. Sammen med resultater opnået af andre forfattere, viser dette at Toms-Winter formodningen er sand, under antagelse af at A desuden er kvasidiagonal og har en entydig sportilstand. Beviset hviler på et studie af algebraer bestående af følger der er centrale med hensyn til den naturlig metrik, hhv. med hensyn til metrikken induceret af sportilstandene, samt dybe resulter fra teorien om von Neumann algebraer. Desuden sammenligner vi begreberne dekompositionsrang og nukleær dimension, og vi viser at de er uenige på Kirchberg algebraer.

Contents

Introduction

The background for the material covered in this thesis, as suggested by the title, is to be found in a conjecture posited by Wilhelm Winter and Andrew Toms. It is known as the Toms-Winter conjecture, for obvious reasons, and has been verified in wide generality, although a complete solution has not yet been given.

Conjecture 1 (Toms-Winter)**.** If A is a unital, separable, simple, nuclear, non-elementary and stably finite C^*-algebra, then the following are equivalent:

(i) A absorbs the Jiang-Su algebra, i.e., $A \otimes \mathcal{Z} \cong A$,

(ii) A has finite decomposition rank,

(iii) A has strict comparison of positive elements.

Each of the above, presumed equivalent, conditions has a distinct flavor; the first condition is of an algebraic nature, the second is topological, and the third is K-theoretic in spirit. The Jiang-Su algebra $\mathcal{Z}$ was first introduced by Xinhui Jiang and Hongbing Su in [11]. This algebra has proven fundamental for the classification program instigated by George Elliott, since it seems likely that the largest class of C^*-algebras for which the Elliott Conjecture is true, consists of those algebras that absorb $\mathcal{Z}$ tensorially, i.e., satisfies condition (i) in the above conjecture. However, Jiang-Su stability remains very difficult to detect in the structure of a C^*-algebra, and as of yet, it is still unknown whether a satisfying intrinsic characterization of Jiang-Su stability, for instance an analogue of [24, Theorem 7.2.6], is possible. An affirmative answer to the Toms-Winter conjecture would yield such a characterization, at least in the finite case. It should be mentioned that Wilhelm Winter and Joachim Zacharias has posited a revised conjecture, which states the following;

Conjecture 2 (Toms-Winter, revised)**.** If A is a unital, separable, simple, nuclear and non-elementary C^*-algebra, then the following are equivalent:

(i) A absorbs the Jiang-Su algebra,

(ii) A has finite nuclear dimension,

(iii) A has strict comparison of positive elements.

Turning our attention back to the original Toms-Winter conjecture, let us quickly explain what is known. Wilhelm Winter proved in [31] that (ii)⇒(i) and Mikael Rørdam proved in [25] that (i)⇒(iii). The implication (iii)⇒(i) was originally proven by Hiroki Matui and Yasuhiko Sato in [18] under the additional assumption that A only has finitely many extremal tracial states. Later this result was extended independently by Eberhard Kirchberg and Mikael Rørdam in [15], Yasuhiko Sato in [28] and Andrew Toms, Stuart White and Wilhelm Winter in [30] to the case where the extreme boundary of the trace simplex is closed and of finite topological dimension. Finally, Matui-Sato proved the implication (iii)⇒(ii) in [19] under the additional assumptions that A is quasidiagonal and has a unique tracial state. Summing up these results, along with the fact that any C^*-algebra with finite decomposition rank is strongly quasidiagonal, we obtain the following theorem.

Theorem 0.0.1. *Let A be a simple, separable, unital, nuclear and infinite-dimensional C^*-algebra with a unique tracial state. Then the following are equivalent:*

(i) A absorbs the Jiang-Su algebra and is quasidiagonal,

(ii) A has finite decomposition rank,

(iii) A has strict comparison and is quasidiagonal

The primary focus of this thesis is the implication (iii)⇒(ii), although we will also touch upon the implication (iii)⇒(i).

Let us explain the contents of this thesis in greater detail. The first chapter, helpfully titled Preliminaries, contains just that: the preliminary study. However, far from everything in this chapter is trivial or basic, quite the contrary in fact. The point is rather that we collect all the theory that we will consider to be background for the rest of the exposition, and as such, with a few exceptions, this chapter contains virtually no proofs. References are provided so that the reader may track down proofs of all the statements if she should wish to do so.

The second chapter is devoted to a study of central sequence algebras. We start with an introduction to ultrapowers before moving on to central sequence algebras, in particular we aim to introduce property (SI) and verify that a certain class of C^*-algebras have this property. Property (SI) is particularly useful when we wish to "lift" properties of the tracial central sequence algebra to corresponding properties of the metric central sequence algebra, a topic which is explored in the third chapter. This chapter is primarily based on [15].

The third chapter contains the main theorems of this thesis. More precisely we prove that, under certain restrictions, we may deduce Jiang-Su absorption and finite decomposition rank from strict comparison of positive elements. We also compare the notions of finite decomposition rank and nuclear dimension and, in particular, we show that they differ on any Kirchberg algebra. The main sources for the exposition in this chapter are to found in [18], [15] and [33].

A number of very interesting topics will not be covered in the manner they deserve. This is due to certain external factors, chief among which are the authors knowledge before starting this thesis, time constraints and the need for a certain narrative focus. Thus, von Neumann algebra theory will be taken for granted when deemed necessary, along with important and deep structure theorems for the Jiang-Su algebra and Kirchberg algebras. It is hoped that the reader will not find these omissions troublesome.

It may not be wise to read the thesis in a linear fashion. Instead, the reader should only use the first chapter to get acquainted with the concepts that are new to her, and otherwise refer back as needed. It could also be useful to browse through the proofs in the third chapter before venturing into the jungle of the second chapter, in order to properly understand the motivation for the study that is carried out there.

Notation And Terminology

We reserve the letters A, B, C and D for C^*-algebras and $\mathscr{M}$ and $\mathscr{N}$ for von Neumann algebras. If A is a unital C^*-algebra, then we let 1_A denote the unit of A. Sometimes we write $\mathbf{1}$ for the unit if the algebra is implied by the context, but we will usually reserve $\mathbf{1}$ for the identity map on a Hilbert space. We let $(A)_r$ denote the closed unit ball of A with radius $r > 0$.

We only consider C^*-subalgebras of C^*-algebras, and therefore when writing 'A is a subalgebra of B' we mean that A is a *-subalgebra which is also closed in the norm topology. Similarly the term 'ideal' is only applied to closed two-sided ideals. If we are considering other types of ideals (e.g. left, right or algebraic ideals), this will be explicitly pointed out. Furthermore, when we say that 'A is a unital subalgebra of B', we mean that A is a subalgebra which contains the unit of B. If A, B are both unital while the unit may differ, this will be indicated by saying 'A is a subalgebra of B with a unit'. An approximate unit for a C^*-algebra will always be an increasing net of positive contractions. If A is a subalgebra of B and $(e_\lambda)_\lambda \subseteq A$ is an approximate unit for A such that $\lim_\lambda \|e_\lambda b - be_\lambda\| = 0$ for all $b \in B$ then we say that $(e_\lambda)_\lambda$ is a quasicentral approximate unit for A in B.

Given a indexed set $(A_\lambda)_{\lambda\in\Lambda}$ of C^*-algebras we let $\prod_{\lambda\in\Lambda} A_\lambda$ denote the set of bounded indexed sets $(x_\lambda)_{\lambda\in\Lambda}$ such that $x_\lambda \in A_\lambda$. If Λ is a directed set, then we let $\bigoplus_{\lambda\in\Lambda} A_\lambda$ denote the set of nets $(x_\lambda)_{\lambda\in\Lambda}$ such that $x_\lambda \in A_\lambda$ and $\lim_\lambda \|x_\lambda\| = 0$. This notation will also be applied to von Neumann algebras, i.e., given a set of von Neumann algebras $(\mathscr{M}_\lambda)_{\lambda\in\Lambda}$, where $\mathscr{M}_\lambda \subseteq B(\mathcal{H}_\lambda)$ we let $\prod_{\lambda\in\Lambda} \mathscr{M}_\lambda \subseteq B(\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda)$ denote the von Neumann algebra consisting of bounded indexed sets $(T_\lambda)_{\lambda\in\Lambda}$, acting on $\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$ in the obvious way, i.e., $(T_\lambda)_{\lambda\in\Lambda}((\xi_\lambda)_{\lambda\in\Lambda}) = (T_\lambda(\xi_\lambda))_{\lambda\in\Lambda}$ (although the notation $\bigoplus_\lambda \mathscr{M}_\lambda$ is more common for von Neumann algebras). Given an inductive system of C^*-algebras $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$ we let $\varinjlim(A_n, \varphi_n)$ denote the inductive limit.

We reserve the lettes $\mathcal{H}$ and $\mathcal{K}$ for Hilbert spaces and we let $B(\mathcal{H})$ denote the bounded operators on the Hilbert space $\mathcal{H}$. The $n \times n$ matrices with complex entries are denoted M_n and if A is C^*-algebra then $M_n(A)$ denotes the $n \times n$ matrices with entries in A. Given a Hilbert space $\mathcal{H}$ we let $\mathbb{K}(\mathcal{H})$ denote the ideal of compact operators on that Hilbert space and $\mathbf{1}_\mathcal{H}$ denote the unit. Finally $\mathbb{K}$ denotes the compact operators on a separable infinite-dimensional Hilbert space, when the particular Hilbert space is of no importance.

If $p, q \in A$ are projections and p is Murray-von-Neumann equivalent with q, i.e., there exists a partial isometry $v \in A$ such that $v^*v = p$ an $vv^* = q$, then we write $p \simeq q$ (the symbol $\sim$ is reserved for a different purpose). Similarly, if p is subequivalent with q, i.e., there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* \leq q$, then we write $p \preceq q$.

Given a C^*-algebra, then we say that a linear functional $\tau : A \to \mathbb{C}$ is a trace if τ is positive and $\tau(xy) = \tau(yx)$ for all $x, y \in A$. We let $T(A)$ denote the set of tracial states on a C^*-algebra A, i.e., the set of states that are also traces, and $\partial_e T(A)$ denote the set of extreme points (within $T(A)$), i.e., the set of extremal tracial states. We let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Finally, if $\varphi : A \to B$ is a completely positive map between C^*-algebras, then we say that φ is a cp. map. Similarly, if φ is contractive, completely positive then we say that φ is ccp. and if φ is unital completely positive then we say that φ is ucp.

CHAPTER 1

Preliminaries

1.1 Von Neumann Algebras: General overview

The coming sections serve as an overview of the parts of the theory of von Neumann algebras that we will be needing. Most of the statements made will not be proven, as this would lead us too far astray from the main thread of the story we are trying to tell. However, the reader should not be cheated from some sort of introduction to the subjects, as it is a quite beautiful theory. Therefore, we have arrived at the compromise you see before you, and it is hoped that readers new to the subject will find the exposition useful.

Given a Hilbert space $\mathcal{H}$, there are several interesting topologies on $B(\mathcal{H})$ one might consider, besides the norm topology (also known as the uniform topology in the literature). We list a selection of them below, although it should be noted that the list is not exhaustive.

- ▷ **The weak operator topology**, abbreviated WO, is generated by the seminorms $x \mapsto |\langle x\xi, \eta\rangle|$, for $\xi, \eta \in \mathcal{H}$.

- ▷ **The strong operator topology**, abbreviated SO, is generated by the seminorms $x \mapsto \|x\xi\|$, for $\xi \in \mathcal{H}$.

- ▷ **The ultraweak operator topology**, abbreviated UW, is generated by the seminorms $x \mapsto \sum_{n=1}^{\infty} |\langle x\xi_n, \eta_n\rangle|$ for sequences in $\mathcal{H}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty$.

- ▷ **The ultrastrong operator topology**, abbreviated US, is generated by the seminorms $x \mapsto \left(\sum_{n=1}^{\infty} \|x\xi_n\|^2\right)^{1/2}$, for sequences in $\mathcal{H}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

The reason for introducing the abbreviations above, is that we wish to employ some notation for dealing with limits of nets in each of the topologies listed above. That is, when writing UW-$\lim_\alpha T_\alpha = T$, we mean that the net $(T_\alpha)_{\alpha\in A}$ converges to T in the ultraweak topology. Similar notation is applied when dealing with the other topologies.

Some quick comments on the topologies seems to be in order. The weak operator and ultraweak topologies agree on bounded sets, and similarly the strong and ultrastrong topologies agree on bounded sets. The closure of convex sets in the weak and strong (resp. ultraweak and ultrastrong topology) operator topologies coincide and the closure of *-subalgebras of $B(\mathcal{H})$ in each of the topologies coincide. Thus, a * subalgebra $\mathscr{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra if and only if it is closed in one of these topologies. We shall insist that a von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$ contains the identity $\mathbf{1}_{\mathcal{H}}$.

We shall be particularly interested in the ultraweakly continuous functionals on a von Neumann algebra, for reasons that will be made clear shortly. However, first we examine their properties. For a proof, see [29, Chapter 1, Theorem 2.6]

Proposition 1.1.1. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $f : \mathscr{M} \to \mathbb{C}$ a linear functional. Then the following conditions are equivalent;*

(i) f is ultrastrongly continuous.

(ii) f is ultraweakly continuous.

(iii) For every increasing, bounded net of self-adjoint operators $(T_\alpha)_{\alpha\in A}$ we have that $f(SO\text{-}\lim_\alpha T_\alpha) = \lim_\alpha f(T_\alpha)$.

(iv) There exist sequences $(\xi_n)_n, (\eta_n)_n \subseteq \mathcal{H}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty, \quad \text{and,} \quad f(x) = \sum_{n=1}^{\infty} \langle x\xi_n, \eta_n \rangle,$$

for all $x \in \mathscr{M}$.

We have a similar statement for weak operator continuous functionals (barring part (iii) above), see for instance [34, Theorem 16.1]. It is also clear from this proposition (more precisely, the equivalence (ii)⇔(iv)), that a net $(x_\alpha)_{\alpha\in A} \subseteq \mathscr{M}$ in a von Neumann algebra $\mathscr{M}$ converges ultraweakly to $x \in \mathscr{M}$ if and only if $f(x_\alpha) \to f(x)$ for all ultraweakly continuous linear functionals $f \in \mathscr{M}^*$.

Definition 1.1.2. Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $f : \mathscr{M} \to \mathbb{C}$ a linear functional on $\mathscr{M}$. Then f is said to be **normal** if it satisfies one of the equivalent conditions in Proposition 1.1.1.

We have a similar definition for positive linear maps between general von Neumann algebras.

Definition 1.1.3. Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras and $\varphi : \mathscr{M} \to \mathscr{N}$ a bounded, positive linear map. Then φ is said to be **normal** if it is ultraweak-to-ultraweak continuous.

One may wonder why we insist that the map should be positive. The reason is that conditions (ii) and (iii) in Proposition 1.1.1 need not be equivalent for general bounded linear maps between von Neumann algebras, and usually one reserves the term *normal* for maps satisfying condition (iii). However, we will not have any use for the term in the general situation, in fact it will almost only be applied to *-homomorphisms and functionals and in this setting the above definition is appropriate.

The story of these topologies is far from finished at this point, but any information we may require, will be explained along the way. For the moment we contend ourselves to proving the following proposition.

Proposition 1.1.4. *Let $\mathscr{M}$ be a von Neumann algebra and $\mathcal{I} \subseteq \mathscr{M}$ an ultraweakly closed ideal. Then there exists a central projection $p \in \mathscr{M}$ such that $\mathcal{I} = p\mathscr{M}$.*

In particular, if $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras and $\varphi : \mathscr{M} \to \mathscr{N}$ is a surjective, normal *-homomorphism, then $\mathscr{M} \cong \ker\varphi \oplus \mathscr{N}$.

Proof. Let $\mathcal{I} \subseteq \mathscr{M}$ be an ultraweakly closed ideal of $\mathscr{M}$. In particular, $\mathcal{I}$ is norm closed and thus we may choose a quasicentral approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ for $\mathcal{I}$. Then, as $\mathcal{I}$ is ultraweakly closed, $(e_\lambda)_\lambda$ converges in strong operator topology to a projection $p \in \mathcal{I}$ which acts as the identity on $\mathcal{I}$ and is central in $\mathscr{M}$. As $\mathcal{I}$ is an ideal, we therefore obtain that $\mathcal{I} = p\mathscr{M}$.

The second statement follows easily from the first. Namely, since φ is normal we see that $\ker\varphi$ is an ultraweakly closed ideal in $\mathscr{M}$ and we may therefore choose a central projection $p \in \mathscr{M}$ such that $p\mathscr{M} = \ker\varphi$, whence we see that $p\mathscr{M} \oplus (\mathbf{1} - p)\mathscr{M} = \mathscr{M}$. Obviously φ restricts to an isomorphism $\varphi : (1 - p)\mathscr{M} \to \mathscr{N}$. □

It turns out that any von Neumann algebra may be identified with the dual space of a certain Banach space, as witnessed by Theorem 1.1.6. First we introduce the following notation.

Definition 1.1.5. Let $\mathscr{M}$ be a von Neumann algebra. The **predual of** $\mathscr{M}$ is the subset $\mathscr{M}_* \subseteq \mathscr{M}^*$ consisting of the ultraweakly continuous linear functionals on $\mathscr{M}$.

As the intelligent reader will already have guessed, the set $\mathscr{M}_*$ is in fact a predual in the sense that $\mathscr{M}_*$ is a Banach space and $(\mathscr{M}_*)^*$ is isometrically isomorphic to $\mathscr{M}$ in a canonical way. This is indeed the contents of the next theorem. See [29, Chapter 1, Theorem 2.6] for a proof.

Theorem 1.1.6. *Let $\mathscr{M}$ be a von Neumann algebra. Then $\mathscr{M}_*$ is a closed linear subspace of $\mathscr{M}^*$ and if $\Phi : \mathscr{M} \to (\mathscr{M}_*)^*$ denotes the point-evaluation map, i.e., Φ is given by $\Phi(x)(f) = f(x)$, then Φ is a surjective linear isometry.*

It is, in fact, also true that the pre-dual of a von Neumann algebra is essentially unique, which justifies the terminology *the* predual. Note that it is an easy consequence of this theorem, if this was not already deduced by the reader from Proposition 1.1.1, that the ultraweakly continous linear functionals on a von Neumann algebra separates points.

Remark 1.1.7. In fact, the map Φ in Theorem 1.1.6 is an ultraweak-to-weak* homeomorphism. This follows easily from Proposition 1.1.1 and the remarks below. As a consequence, we may deduce from Alaouglu's Theorem that the closed unit ball of a von Neumann algebra is ultraweakly compact. It therefore follows that if $\varphi : \mathscr{M} \to B(\mathcal{H})$ is a normal representation, then $\varphi(\mathscr{M})$ is also a von Neumann algebra.

For reference, we remind the reader of the type decomposition of von Neumann algebras.

Definition 1.1.8. ?? Let $\mathscr{M}$ be a von Neumann algebra and $p \in \mathscr{M}$ a projection.

- ▷ We say that p is **abelian** if $p\mathscr{M}p$ is abelian, and we say that p is **finite** if whenever $v \in \mathscr{M}$ is a partial isometry such that $v^*v = p$ and $vv^* \leq p$ then $vv^* = p$. We say that p is infinite if it is not finite.

  We say that $\mathscr{M}$ is **finite** if the identity is a finite projection, and we say that $\mathscr{M}$ is **properly infinite** if it does not contain any central, non-zero finite projections.

- ▷ We say that $\mathscr{M}$ is; of **type** I if every non-zero projection majorizes a non-zero abelian projection; of **type** II if it does not contain any abelian projections and any non-zero projection majorizes a non-zero finite projection; of **type** III if it does not contain any finite projections.

It should be clear that the cases above are mutually exclusive.

Theorem 1.1.9. *For any von Neumann algebra $\mathscr{M}$, we have a unique decomposition $\mathscr{M} = \mathscr{M}_{\mathrm{I}} \oplus \mathscr{M}_{\mathrm{II}} \oplus \mathscr{M}_{\mathrm{III}}$, where $\mathscr{M}_{\mathrm{I}}, \mathscr{M}_{\mathrm{II}}$ and $\mathscr{M}_{\mathrm{III}}$ is either zero or of type* I*, type* II *or type* III*, respectively.*

Furthermore, any von Neumann algebra $\mathscr{M}$ can be uniquely decomposed as a direct sum $\mathscr{M} = \mathscr{M}_f \oplus \mathscr{M}_\infty$, where $\mathscr{M}_f$ is either finite or zero, and $\mathscr{M}_\infty$ is either properly infinite or zero.

As a consequence of the above theorem, along with the fact that any type III von Neumann algebra is automatically properly infinite, we see that any von Neumann algebra may be uniquely decomposed as a sum of at most 5 von Neumann algebras, where each summand is of type I_f, I_∞, II_f, II_∞ or type III. Although it is not immediately clear from the definition, a von Neumann algebra factor must be exactly one of these mentioned types, as witnessed by the following proposition (we remind the reader that $\mathscr{M}$ is a factor if $\mathscr{M} \cap \mathscr{M}' = \mathbf{1}\mathbb{C}$). If $\mathscr{M}$ is a von Neumann algebra and $p \in \mathscr{M}$ is a projection, then we let $c_p \in \mathscr{M}$ denote the central support of p, i.e., the least central projection which majorizes p.

Proposition 1.1.10. *Let $\mathscr{M}$ be a von Neumann algebra. Then the following holds:*

(i) The von Neumann algebra $\mathscr{M}$ is of type I *if and only if for for each non-zero projection $p \in \mathscr{M}$ there exists a non-zero abelian projection $q \in \mathscr{M}$ such that $c_p c_q \neq 0$.*

(ii) The von Neumann algebra $\mathscr{M}$ is of type II *if and only if $\mathscr{M}$ does not contain any non-zero abelian projections and for each non-zero projection $p \in \mathscr{M}$ there exists a non-zero finite projection $q \in \mathscr{M}$ such that $c_p c_q \neq 0$.*

In particular, if $\mathscr{M}$ is a factor then it must be of either type I*, type* II *or type* III.

Proof. During this proof, we need the fact that if $p, q \in \mathscr{M}$ are projections such that $q \leq p$ or $q \simeq p$ and p is abelian (resp. finite), then q is also abelian (resp. finite). This is left as an exercise for the reader. Furthermore, we need the fact that if $p, q \in \mathscr{M}$ are projections then $c_p c_q \neq 0$ if and only if there exist non-zero projections $q_1 \leq q$ and $p_1 \leq p$ such that $q_1 \simeq p_1$ (see [34, Proposition 24.7]).

(i): Let $p \in \mathscr{M}$ be an arbitrary non-zero projection. Assuming that $\mathscr{M}$, there exists a non-zero abelian projection $q \leq p$. Hence, $c_p c_q \neq 0$, thus proving one implication. On the other hand, if there exists a non-zero abelian projection $q \in \mathscr{M}$ such that $c_p c_q \neq 0$, then we find non-zero projections $q_1 \leq q$ and $p_1 \leq p$ such that $q_1 \simeq p_1$. Since q is abelian, $q_1 \leq q$ and $q_1 \simeq p_1$ we find that p_1 is an abelian projection. Since p was arbitrary, we have hereby proven the other implication.

(ii): This is proved in a completely similar manner to (ii), and the proof is therefore omitted.

With this in place, proving the last statement becomes easy. Indeed, assuming $\mathscr{M}$ is a factor, determining the type of $\mathscr{M}$ reduces to checking for the existence of a single non-zero abelian or finite projection.

In greater detail; assume that $\mathscr{M}$ is a factor and let $p \in \mathscr{M}$ be an arbitrary non-zero projection. Assume furthermore that there exists a non-zero abelian projection $q \in \mathscr{M}$. Since $q \neq 0$ and c_q is central, we find that $c_q = 1_{\mathscr{M}}$ and therefore $c_p c_q \neq 0$. Hence, it follows from (i) that $\mathscr{M}$ is type I. On the other hand, assume that $\mathscr{M}$ contains no non-zero abelian projections, while $q \in \mathscr{M}$ is a non-zero finite projection. Again, since $q \neq 0$ and c_q is central we see that $c_q = 1_{\mathscr{M}}$ and therefore $c_q c_p \neq 0$. Thus, part (ii) implies that $\mathscr{M}$ is type II. If $\mathscr{M}$ contains no non-zero finite projections, then $\mathscr{M}$ is of type III, thus completing the proof. □

The von Neumann algebras encountered in the present thesis will all be finite, and they will almost always be factors. It is well-known that if $\mathscr{M}$ is finite type I factor, then there exists $n \in \mathbb{N}$ such that $\mathscr{M} \cong M_n$, in which case we say that $\mathscr{M}$ is a type I_n factor. If $\mathscr{M}$ is a finite von Neumann algebra of type II, we say that $\mathscr{M}$ is a II_1 von Neumann algebra (it follows directly from Definition **??** that a factor is either finite or properly infinite).

It is a deep, but well-known result, that there exists a unique normal, faithful tracial state on any finite factor. In fact we have the following theorem:

Theorem 1.1.11. *Any finite von Neumann algebra has a separating family of normal, tracial states. Furthermore, any finite factor has a unique normal, faithful tracial state.*

The standard way of proving this theorem is via the existence of a center-valued trace on finite von Neumann algebras [12, Theorem 8.2.8]. We only really need the fact that any finite factor has a unique tracial state, but the full theorem is stated for completeness. The subject of the trace simplex will be examined further later on in the thesis, but for now we move on to other pastures. In particular, we need a few results concerning the hyperfinite II_1-factor.

Definition 1.1.12. Let $\mathscr{M}$ be a von Neumann II_1 factor. Then $\mathscr{M}$ is said to be **approximately finite** if there exists an increasing sequence $(\mathscr{N}_k)_{k\in\mathbb{N}}$ of von Neumann subalgebras of $\mathscr{M}$ such that $\bigcup_{k\in\mathbb{N}} \mathscr{N}_k$ is weak-operator dense in $\mathscr{M}$ and $\mathscr{N}_k$ is a type I_{n_k} factor, for some increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$.

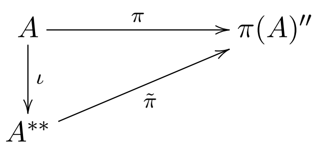
It is a classical theorem of Murray and von Neumann that all approximately finite II_1-factors are isomorphic (see [21]). This unique II_1-factor is referred to as **the hyperfinite** II_1-factor and is denoted $\mathscr{R}$. When proving uniqueness of $\mathscr{R}$ Murray and von Neumann also proved the following.

Proposition 1.1.13. *For every increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that n_k divides n_{k+1} there exists an increasing sequence $(\mathscr{N}_k)_{k\in\mathbb{N}}$ of von Neumann subalgebras of $\mathscr{R}$ such that $\bigcup_{k\in\mathbb{N}} \mathscr{N}_k$ is weak-operator dense in $\mathscr{R}$ and $\mathscr{N}_k$ is a type I_{n_k}-factor for each $k \in \mathbb{N}$.*

1.2 The Universal Enveloping von Neumann Algebra

Here we introduce the important concept of the universal enveloping von Neumann algebra of a C^*-algebra. That is, we wish to outline how, given a C^*-algebra A we might give the double dual (in the Banach space sense) A^{**} the structure of a von Neumann algebra. While this might not seem terribly impressive or remarkable at first, it does give us a quite powerful tool. This will hopefully be evident, upon completed reading of this thesis. But enough with the chit-chat, let us state some results. Proofs of almost all the statements in this section may be found in [29, Chapter 3: Section 2].

Proposition 1.2.1. *Suppose that A is C^*-algebra and $\pi : A \to B(\mathcal{H})$ be a representation of A on the Hilbert space $\mathcal{H}$. Then π extends to a weak*-to-ultraweak continuous map $\tilde{\pi} : A^{**} \to \pi(A)''$. In other words, we have the following commutative diagram:*



where ι denotes the natural inclusion and $\tilde{\pi}$ is weak*-to-ultraweak continuous. Moreover, if π is non-degenerate, then $\tilde{\pi}$ is surjective.

Before we go further, we remind the reader of the universal representation of a C^*-algebra: Let A be a C^*-algebra and $S(A)$ denote the set of states on A. For each $\phi \in S(A)$ we let $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ denote the GNS-triple corresponding to ϕ. Then we obtain a representation $\pi_u := \bigoplus_{\phi \in S(A)} \pi_\phi : A \to B(\bigoplus_{\phi \in S(A)} \mathcal{H}_\phi)$, and we say that π_u is **the universal representation** of A. Furthermore we let $\mathcal{H}_u := \bigoplus_{\phi \in S(A)} \mathcal{H}_\phi$. It is well-known, and not difficult to prove, that this representation is faithful, since the states on a C^*-algebra separates points, and non-degenerate. Furthermore, the previous proposition, along with the following theorem, shows that π_u satisfies a certain universal property (perhaps not all that surprising given the name).

Theorem 1.2.2. *Let A be a C^*-algebra. Then $\pi_u : A \to B(\mathcal{H}_u)$ extends to a surjective, isometry $\tilde{\pi}_u : A^{**} \to \pi_u(A)''$, which is also a weak*-to-ultraweak homeomorphism.*

It follows from this theorem that we have a canonical way of endowing A^{**} with a von Neumann algebra structure in such a way that the weak* topology becomes the ultraweak topology. The two previous results combine to yield the following result.

Corollary 1.2.3. *Let A be a C^*-algebra. If $\pi : A \to B(\mathcal{H})$ is a representation of A, then there exists a unique, normal *-homomorphism $\tilde{\pi} : \pi_u(A)'' \to \pi(A)''$ such that the following diagram commutes;*

$$\begin{array}{ccc} A & \xrightarrow{\pi_u} & \pi_u(A)'' \\ & \searrow^{\pi} & \downarrow{\tilde{\pi}} \\ & & \pi(A)'' \end{array}$$

Moreover, if π is non-degenerate, then $\tilde{\pi}$ is surjective.

It is not difficult to prove that a representation satisfying the above universal property is essentially unique, meaning that if $\rho : A \to B(\mathcal{K})$ is another such non-degenerate representation, then there exists a normal isomorphism $\varphi : \rho(A)'' \to \pi_u(A)''$ such that $\pi_u = \varphi \circ \rho$. This observation, along with Theorem 1.2.2 justifies that we may call A^{**} *the* universal enveloping von Neumann algebra, and identify it with $\pi_u(A)''$ at will. We conclude this section with some consequences of the already stated results. The only non-trivial part of the next proposition is part (vii) and (viii).

Proposition 1.2.4. *Let A and B be C^*-algebras, $\varphi : A \to B$ a linear map and let φ^{**} denote the double adjoint of φ. Then we have the following:*

(i) The map $\varphi^{**} : A^{**} \to B^{**}$ is ultraweak-to-ultraweak continuous.

(ii) The maps φ and φ^{**} have the same norm $\|\varphi\| = \|\varphi^{**}\|$.

(iii) If φ is a *-homomorphism, so is φ^{**}.

(iv) If φ is hermitian, so is φ^{**}.

(v) If φ is positive, so is φ^{**}.

(vi) If φ is unital, so is φ^{**}.

(vii) If φ is completely bounded, so is φ^{**} and $\|\varphi\|_{cb} = \|\varphi^{**}\|_{cb}$.

(viii) If φ is completely positive, so is φ^{**}.

Proof. (i)-(vi) are trivial. Consult [3, Proposition B.6] for an idea of how to prove (vii) and (viii). □

The polar decomposition of elements in a von Neumann algebra should be familiar to the reader, but the result is stated for reference.

Proposition 1.2.5. *Let A be a C^*-algebra and $a \in A$ be any element. Then there exists a partial isometry $v \in A^{**}$ such that $a = v|a|$ and $\ker v = \ker a$. Moreover, if $f \in C_0((0, \infty))$, then $vf(|a|) \in A$.*

Proof. The first statement is just the usual theorem on polar decomposition in von Neumann algebras, put into the context of C^*-algebras and the double dual (see for instance [34, Theorem 18.9]). Note that we implicitly identified A^{**} with $\pi_u(A)''$, so that the statement $\ker v = \ker a$ makes sense. The second statement follows from the first along with an easy application of the Stone-Weierstrass theorem. □

Remark 1.2.6. A few trivial, but useful remarks on the polar decomposition are in order, since they will be applied without mention in the rest of this thesis. First of all, in the setup of the above proposition, since v is a partial isometry with $\ker v = \ker a$ we find that v^*v is the range projection for $|a|$, whence $v^*v|a| = |a|$ and $v^*a = |a|$. Obviously, we have that $a^* = |a|v^*$, and therefore we see that

$$(v|a|v^*)^2 = v|a|^2v^* = (v|a|)(|a|v^*) = aa^* = |a^*|^2.$$

As the positive squareroot of a positive element is unique, we find that $v|a|v^* = |a^*|$.

1.3 Injective von Neumann Algebras

The theory of injective von Neumann algebras is vast, and we do not attempt to do it justice in this section. We contend ourselves to proving a few easy consequences of some very deep and beautiful results, since anything else would simply require too much effort. The definition may as well be stated for general C^*-algebras, and some of the results remain true in this setting. However, as we shall only need results concerning injective von Neumann algebras, we remain in this setting throughout.

Remark 1.3.1. Usually, we insist that a von Neumann algebra $\mathscr{M} \subseteq B(\mathcal{H})$ should contain the identity $1_{\mathcal{H}}$. However, when dealing with corners of von Neumann algebras, we will bypass this convention. To be precise, if $p \in \mathscr{M} \subseteq B(\mathcal{H})$ is a projection, then we will still regard $p\mathscr{M}p \subseteq B(\mathcal{H})$ as a von Neumann algebra acting on $\mathcal{H}$, although $1_{p\mathscr{M}p} = p \neq 1_{\mathcal{H}}$. This is merely a technicality since, upon replacing $\mathcal{H}$ with $\mathcal{K} := p(\mathcal{H})$ we see that $1_{\mathcal{K}} \in p\mathscr{M}p \subseteq B(\mathcal{K}) \cong pB(\mathcal{H})p$. In particular, if $\prod_{\alpha\in A} \mathscr{M}_\alpha \subseteq B(\mathcal{H})$ then we will regard each $\mathscr{M}_\alpha$ as a von Neumann algebra acting on $\mathcal{H}$.

Definition 1.3.2. Let $\mathscr{M}$ be a von Neumann algebra. Then $\mathscr{M}$ is said to be **injective** if, given any pair of von Neumann algebras $\mathscr{N}, \mathscr{N}_0$ such that $\mathscr{N}_0 \subseteq \mathscr{N}$ is a subalgebra, then any cp. map $\varphi_0 : \mathscr{N}_0 \to \mathscr{M}$ extends to a cp. map $\tilde{\varphi} : \mathscr{N} \to \mathscr{M}$.

It is an easy consequence of Arveson's extension theorem (see for instance [1, Theorem II.6.9.12]) that $B(\mathcal{H})$ is an injective von Neumann algebra for any Hilbert space $\mathcal{H}$. Recall also, that if A, B are C^*-algebras such that B is a subalgebra of A, then $E : A \to B$ is said to be a **conditional expectation** if $E(b) = b$ for all $b \in B$.

Proposition 1.3.3. *Let $\mathscr{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Then $\mathscr{M}$ is injective if and only if there exists a ucp. projection $E : B(\mathcal{H}) \to \mathscr{M}$.*

Proof. Assume that $\mathscr{M}$ is injective. Then $\mathrm{id}_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ is a completely positive map and therefore, by definition, extends to a cp. map $E : B(\mathcal{H}) \to \mathscr{M}$. As $1_{\mathcal{H}} \in \mathscr{M}$ it follows that E is unital.

Now, let $E : B(\mathcal{H}) \to \mathscr{M}$ be a ucp. projection and let $\mathscr{N}_0 \subseteq \mathscr{N}$ be von Neumann algebras. If $\varphi_0 : \mathscr{N}_0 \to \mathscr{M}$ is a cp. map then, since $B(\mathcal{H})$ is injective, it follows that φ_0 extends to a cp. map $\Phi : \mathscr{N} \to B(\mathcal{H})$. Therefore $\varphi := E \circ \Phi : \mathscr{N} \to \mathscr{M}$ is a cp. extension of φ_0. □

Note that it follows from Tomiyama's Theorem (see for instance [3, Theorem 1.5.9]), that it is sufficient to check the existence of a unital, contractive projection $E : B(\mathcal{H}) \to \mathscr{M}$. We will not dwell too long on the permanence properties of injective von Neumann algebras, but a few of them will be useful. In all honesty, the next proposition is fairly trivial, but it does collect some nice observations in a neat package, and once proven it is at least dealt with for good. To wit:

Proposition 1.3.4. *Let $\mathscr{M}, \mathscr{N}$ and $(\mathscr{M}_\alpha)_{\alpha \in A}$ be von Neumann algebras. Then the following holds:*

(i) For any projection $p \in \mathscr{M}$ the von Neumann algebra $p\mathscr{M}p$ is injective if $\mathscr{M}$ is injective.

(ii) The von Neumann algebra $\prod_{\alpha \in A} \mathscr{M}_\alpha$ is injective if and only if $\mathscr{M}_\alpha$ is injective for each α.

(iii) If $\varphi : \mathscr{M} \to \mathscr{N}$ is a normal, surjective *-homomorphism, then $\mathscr{M}$ is injective if and only if both $\ker \varphi$ and $\mathscr{N}$ are injective.

(iv) If $\pi : \mathscr{M} \to B(\mathcal{H})$ is a normal representation of $\mathscr{M}$ and $\mathscr{M}$ is injective, then $\pi(\mathscr{M})$ is also injective.

Proof. (i): Let $\Phi : \mathscr{M} \to p\mathscr{M}p$, be given by $\Phi(x) = pxp$ and let $E : B(\mathcal{H}) \to \mathscr{M}$ be a ucp. projection. Then $\Phi \circ E : B(\mathcal{H}) \to p\mathscr{M}p$ is a ucp. projection.

(ii): The 'only if' part of the statement is a consequence of (i). Therefore, assume that each $\mathscr{M}_\alpha$ is injective, and let $E_\alpha : B(\mathcal{H}) \to \mathscr{M}_\alpha$ be a ucp. projection. Then we may define $E : B(\mathcal{H}) \to \prod_{\alpha \in A} \mathscr{M}_\alpha$ by $E(x) = (E_\alpha(x))_{\alpha \in A}$. This is clearly a unital, contractive projection and, as noted before the statement of this proposition, the desired result therefore follows.

(iii) is a consequence of (ii) and Proposition 1.1.4, while (iv) is a special case of (iii). □

So far we have kept things light and easy, but next up are some very deep and very beautiful results, both due to Alain Connes.

Theorem 1.3.5. *If A is a nuclear C^*-algebra, then A^{**} is an injective von Neumann algebra.*

Proof. See [3, Section 9.3] and references therein. □

This should already be awe inspiring, but the next result is of an equally jaw dropping nature.

Theorem 1.3.6. *Let $\mathscr{M}$ be a II_1-factor acting on a separable Hilbert space. Then $\mathscr{M}$ is injective if and only if it is isomorphic to the hyperfinite II_1-factor.*

Proof. See [7, Theorem 7.1]. □

As an easy consequence of these wonderful theorems, we have the following result:

Corollary 1.3.7. *Let A be nuclear C^*-algebra. If $\pi : A \to B(\mathcal{H})$ is any non-degenerate representation of A on a separable Hilbert space, then $\pi(A)''$ is an injective von Neumann algebra. Moreover, if $\pi(A)''$ is a II_1-factor, then $\pi(A)''$ is isomorphic to the hyperfinite II_1-factor.*

Proof. It follows from Corollary 1.2.3 that there exists a surjective, normal *-homomorphism $\varphi : \pi_u(A)'' \to \pi(A)''$. Since A is nuclear it follows from Theorem 1.3.5 along with Theorem 1.2.2 that $A^{**} \cong \pi_u(A)''$ is injective, whence Proposition 1.3.4 part (iii) implies that $\pi(A)''$ is injective. The second statement follows from the first along with Theorem 1.3.6. □

1.4 Basic C^*-algebra Theory

The sections to come contain all the basic theory on C^*-algebras we will need in the thesis. Actually, not all the theory is that basic, but the reader should view this is a summary of what is seen to be background material for the main chapters. We do take the time to prove some results, but everything that requires more than a minimum amount of work is avoided. In other words, proofs are included precisely when they cost so little effort as to, in any reasonable sense, not cost any effort at all. However, there is a certain escalation in the depth of material we cover, and any non-trivial statement will be carefully pointed out to the reader and references for proofs are supplied. The present section contains the parts of the theory that will usually go unmentioned when applied.

Hereditary Subalgebras

A small collection of facts concerning hereditary subalgebras be quite useful at various points, although they are not major players in this particular thesis.

Definition 1.4.1. Let B be a C^*-algebra and $A \subseteq B$ a subalgebra. Then A is said to be a **hereditary subalgebra** of B if it holds that $a \in A$ whenever $0 \leq a \leq b$ for some $b \in A_+$.

There is a useful correspondence between hereditary subalgebras and left ideals. More precisely, let $A \subseteq B$ be a hereditary subalgebra of B and $L(A) := \{a \in B \mid a^*a \in A\}$. Furthermore, let $LI(B)$ denote the set of left ideals in B and similarly $H(B)$ denote the set of hereditary subalgebras (this notation should be forgotten by the reader as soon as the next statement has been read). Then the maps $\Lambda : LI(B) \to H(B)$ and $\Gamma : H(B) \to LI(B)$ given by

$$\Lambda(J) = J \cap J^*, \quad \Gamma(A) = L(A),$$

are inverse to each other. From this it is not difficult to deduce the following properties of hereditary subalgebras.

Proposition 1.4.2. *Let B be a C^*-algebra and $A \subseteq B$ a subalgebra.*

(i) A is hereditary if and only if $aba' \in A$ for all $b \in B$ and $a, a' \in A$. In particular, any ideal in B is hereditary.

(ii) If A is hereditary and $I \subseteq A$ is an ideal, then $I = A \cap J$ where $J \subseteq B$ is the ideal generated by the subalgebra $I \subseteq B$.

(iii) For any $x \in B$, the subalgebra $\overline{x^*Bx} = \overline{(x^*x)B(x^*x)} \subseteq B$ is the smallest hereditary subalgebra containing x^*x.

(iv) If A is separable and hereditary, then there exists $b \in A_+$ such that $\overline{bBb} = A$.

Proof. See [20, Section 3.2]. □

Note that it follows from (ii) in the above proposition that if $A \subseteq B$ is a hereditary subalgebra in a simple C^*-algebra, then A is itself simple. Next, we examine the polar decomposition in context with hereditary subalgebras.

Proposition 1.4.3. *Let A be a C^*-algebra, $x \in A$ be any element and let $x = v|x|$ be the polar decomposition of x, where $v \in A^{**}$. Then for every $y \in \overline{x^*Ax}$ the elements vy, yv^* and vyv^* belong to A.*

Proof. Clearly we have that $v(x^*x) = x|x| \in A$ and $(x^*x)v^* = |x|x^*$, whence we obtain that vy, yv^* and vyv^* belongs to A for any $y \in (x^*x)A(x^*x)$. Therefore the same holds true for any $y \in \overline{(x^*x)A(x^*x)}$. □

Nuclearity and Tensor Products

The subjects of nuclearity and tensor products are assumed to be known by the reader. In particular, the definitions of minimal and maximal tensor product of C^*-algebras are assumed to be known. Thus the results and definitions here are purely for reference (and for the readers convenience). We let $A \odot B$ denote the algebraic tensor product of the C^*-algebras A and B.

The only general facts we will need about tensor products are summarized in the following proposition.

Proposition 1.4.4. *Let A, B, C and D be C^*-algebras.*

(i) If $\varphi : A \odot B \to C$ is a *-homomorphism, then it extends to a unique *-homomorphism $A \otimes_{\max} B \to C$. In particular, if $\varphi_A \to C$ and $\varphi_B : B \to C$ are *-homomorphisms with commuting ranges, then there exists a unique *-homomorphism $\varphi_A \times \varphi_B : A \otimes_{\max} B \to C$ such that $\varphi_A \times \varphi_B(a \otimes b) = \varphi_A(a)\varphi_B(b)$ for all $a \in A$ and $b \in B$.

(ii) If $\varphi : A \to C$ and $\psi : B \to D$ are *-homomorphisms, then there exists a unique *-homomorphism $\varphi \otimes \psi : A \otimes_{\min} B \to C \otimes_{\min} D$ such that $\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$ for all $a \in A$ and $b \in B$.

(iii) For every $n \in \mathbb{N}$, the map $M_n \odot A \to M_n(A)$ given by $\sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \mapsto [a_{ij}]_{i,j=1}^{n}$ is a *-isomorphism.

Proof. See [3, Theorem 3.3.7] or [20, Theorem 6.5.7] for a proof of (i), [20, Theorem 6.5.1] for a proof of (ii). Part (iii) is trivial. □

Having dealt with tensor products, we move on swiftly to nuclear maps and two equivalent formulations of nuclearity.

Definition 1.4.5. Let A and B be C^*-algebras. A ccp. map $\theta : A \to B$ is said to be **nuclear** if there exist nets $\varphi_\alpha : A \to M_{n(\alpha)}$ and $\psi_\alpha : M_{n(\alpha)} \to B$ of ccp. maps, for some indexed set of natural numbers $(n(\alpha))_\alpha$, such that

$$\lim_{\alpha} \|\psi_\alpha \circ \varphi_\alpha(a) - \theta(a)\| = 0, \tag{1.1}$$

for all $a \in A$.

Remark 1.4.6. If A and B are both unital, and θ is a nuclear ucp. map, then we may choose the approximating net in the above definition to consist of ucp. maps as well (see [3, Proposition 2.2.6]).

Furthermore, note that we only define the concept of nuclearity for ccp. maps instead of general cp. maps. However it is clear that any cp. map satisfying the above will automatically be contractive.

The following theorem is well-known, albeit highly non-trivial. The reader may consult [5, Theorem 3.1] or [3, Theorem 3.8.7] for a proof.

Theorem 1.4.7. *Let A be a separable C^*-algebra. Then the following conditions are equivalent.*

(i) For any C^*-algebra B, the canonical quotient map

$$A \otimes_{\max} B \to A \otimes_{\min} B$$

is also injective.

(ii) The identity map $id_A : A \to A$ is nuclear.

Definition 1.4.8 (Nuclear C^*-algebras)**.** A C^*-algebra A is said to be **nuclear** if it satisfies one of the equivalent conditions in Theorem 1.4.7.

As most, if not all, C^*-algebras in this thesis will be nuclear, we will freely use the universal property of both the maximal and minimal tensor product (part (i) and (ii) of Proposition 1.4.4 respectively), without specifying which tensor product we are currently considering. Similarly the above theorem will go unmentioned when applied. Before moving on, we provide the reader with a construction scheme, of sorts, for obtaining nuclear C^*-algebras. The deepest statement by far is part (v).

Proposition 1.4.9 (Permanence Properties)**.** *The following hold:*

(i) Every finite dimensional C^*-algebra is nuclear.

(ii) Every commutative C^*-algebra is nuclear.

(iii) If A and B are both nuclear, then $A \otimes B$ is also nuclear.

(iv) If $A \subseteq B$ is a hereditary subalgebra of a nuclear C^*-algebra B, then A is also nuclear. Furthermore, if $A \subseteq B$ is a full, hereditary subalgebra of B, B is separable and A is nuclear, then B is also nuclear.

(v) If A is nuclear and $I \subseteq A$ an ideal, then A/I is nuclear. Furthermore, if $I \subseteq A$ is an ideal such that both I and A/I are nuclear, then A is also nuclear.

(vi) Any inductive limit of nuclear C^*-algebras is nuclear.

(vii) If A is nuclear and G is a discrete, amenable group acting on A, then $A \rtimes G$ is also nuclear.

Proof. (i) and (iii) are easy to prove, and (vi) is an easy consequence of (v). The reader may consult [3, Prop 2.4.2] for a proof of (ii), [3, Corollary 9.4.4] for a proof of the first statement in (v) and [20, Theorem 6.5.3] for a proof of the second statement. A (sketch) of the proof of (vii) may be found in [3, Theorem 4.2.6]. A proof of the first statement in (iv) is also quite easy, once it has been noted that for the ccp. maps $\Phi_\lambda : B \to A$, given by $\Phi(b) = e_\lambda b e_\lambda$ where $(e_\lambda)_\lambda \subseteq A$ is an approximate unit, we have $\lim_\lambda \|\Phi_\lambda(b) - b\| = 0$, for all $b \in A$. A proof of the second statement requires Brown's Theorem (see [2, Theorem 2.8]) □

Lifting Results

The title is an umbrella for a range of quite different lifting results. Some of them are almost trivial, but useful, while others are certainly deeper. As before, we only prove the trivial results.

Proposition 1.4.10. *Let A and B be C^*-algebras and $\varphi : A \to B$ be a surjective *-homomorphism. Then φ maps the closed unit ball of A onto the closed unit ball of B.*

Proof. We may extend φ to a (surjective) *-homomorphism $\Phi : A^{**} \to B^{**}$. Let $b \in B$ be a contraction and $a \in A$ be any lift of A. Then we may find a partial isometry $v \in A^{**}$ such that $a = v|a|$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$f(t) = \begin{cases} t & \text{if } t \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\tilde{a} = vf(|a|)$ and note that since $f(0) = 0$ we have that $\tilde{a} \in A$. Furthermore $\|f(|a|)\| \leq 1$, whence $\|\tilde{a}\| \leq 1$, and

$$\begin{aligned} \Phi(\tilde{a}) &= \Phi(v)\Phi\big(f(|a|)\big) \\ &= \Phi(v)f\big(|\Phi(a)|\big) \\ &= \Phi(v)\Phi(|a|) \\ &= \Phi(a) = b. \end{aligned}$$

□

The above propositions will be used throughout this thesis without mention, so the reader should hereby consider herself warned.

Proposition 1.4.11. *Let A and B be C^*-algebras and $\varphi : A \to B$ a surjective *-homomorphism. If $e, f \in B$ are positive elements such that $ef = 0$, then there exist positive elements $a, b \in A$ such that $\|a\|, \|b\| \leq \max\{\|e\|, \|f\|\}$, $\varphi(a) = e$, $\varphi(b) = f$ and $ab = 0$.*

In particular, if both A and B are unital and $e, f \in B$ are positive contractions such that $ef = f$, then there exist positive contractions $a, b \in A$ such that $\varphi(a) = e$, $\varphi(b) = f$ and $ab = b$.

Proof. Let x be a self-adjoint lift of $e - f$, such that $\|x\| \leq \|e - f\| \leq \max\{\|e\|, \|f\|\}$. Let $a := x_+$ and $b : x_-$. Since $ab = 0$ we find that $\varphi(a)\varphi(b) = 0$ and therefore

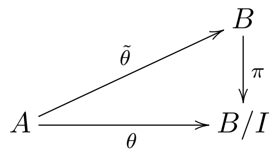
$$\varphi(a) = \varphi(x_+) = \varphi(x)_+ = (e - f)_+ = e.$$

In the same way we obtain that $\varphi(b) = f$, and by construction $\|a\|, \|b\| \leq \|x\| \leq \max\{\|e\|, \|f\|\}$.

The second statement follows from the first. Indeed, we see that $1-e$ and f are orthogonal positive contractions, hence we may choose positive contractions $x, b \in A$ such that $\varphi(x) = 1-e$ and $\varphi(b) = f$ and $xb = 0$. Letting $a := 1_A - x$ we therefore obtain the desired result. □

With these trivialities out of the way it is time to get down to business.

Theorem 1.4.12 (Choi–Effros)**.** *Let A and B be C^*-algebras, with A separable, and $I \subseteq B$ an ideal. Then any nuclear map $\theta : A \to B/I$ lifts to a ccp. map $\tilde{\theta} : A \to B$, i.e., there exists a ccp. map $\tilde{\theta} : A \to B$ such that the following diagram commutes;*



where $\pi : B \to B/I$ denotes the quotient map.

Proof. See [4, Theorem 3.10]. □

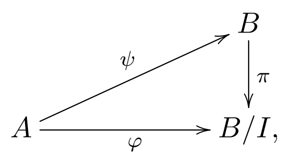
Obviously, this result in conjunction with Theorem 1.4.7 provides us with the following corollary.

Corollary 1.4.13. *Let A and B be C^*-algebras, with A separable and nuclear, and let $I \subseteq B$ be an ideal. Then any ccp. map $\theta : A \to B/I$ lifts to a ccp. map $\tilde{\theta} : A \to B$.*

Remark 1.4.14. If θ is ucp. (obviously A and B must also be unital), then the lift $\tilde{\theta}$ may be chosen to be ucp. as well. One way of achieving this is to choose a state κ on B and replace $\tilde{\theta}$ with $\tilde{\theta} + \big(1_B - \tilde{\theta}(1_A)\big)\kappa$.

The next, and final, lifting result will be somewhat more hidden in the background than the previous statements. However, it is still a wonderful result, and thus it is included in the treatment. Before stating it we need a definition.

Definition 1.4.15 (Projectivity)**.** A C^*-algebra A is said to be **projective** if for any pair (B, I), where B is a C^*-algebra and $I \subseteq B$ is an ideal, and any *-homomorphism $\varphi : A \to B/I$, there exists a *-homomorphism $\psi : A \to B$ which lifts φ. In other words, there exists a *-homomorphism ψ such that the following diagram commutes;



where $\pi : B \to B/I$ denotes the quotient map.

Projective C^*-algebras are extremely rare, but some examples are known, as witnessed by the following theorem. For a C^*-algebra algebra A we refer to $C_0\big((0,1]\big) \otimes A$ as the **cone over A**.

Theorem 1.4.16. *The cone over any finite-dimensional C^*-algebra is projective.*

Proof. This may deduced from [17, Theorem 4.9] in conjunction with [17, Theorem 5.1]. □

1.5 Quasidiagonal C^*-algebras

The subject of quasidiagonal C^*-algebras plays a fundamental role for the Toms-Winter conjecture. In fact, at this point in time it appears the quasidiagonality represents the dividing line between C^*-algebras with finite decomposition rank and those with finite nuclear dimension (compare 3.2.10 with [**?**, Theorem B]). It is therefore somewhat remarkable, that we only really need the definition of quasidiagonality to prove our results. However, given the importance of quasidiagonality in the context of this thesis, it is felt by the author that simply stating the definition is unsatisfactory. Besides, it is a fascinating subject, and the reader who is unfamiliar with it, will certainly not be a lesser person for knowing the basics of the theory. Hence we devote a (small) amount of our time to an introduction. All the definitions and results are stated for separable C^*-algebras and Hilbert spaces, but the majority of them have natural generalizations to the non-separable world. We do not need the full theory however, so we make the blanket assumption that everything in this section is separable.

Voiculescu's Theorem

The term "Voiculescu's Theorem" is reminiscent of the term Hahn-Banach in the sense that it really refers to a collection of related results and corollaries. We will not attempt to list all of them, instead we simply collect the forms that will be needed. It is a useful tool when proving some equivalences between different formulations of quasidiagonality, but this does not exhaust the applications, as we shall see later on. The reader should know that the terms "Weyl-von Neumann's Theorem" or "Voiculescu's Weyl-von Neumann Theorem" are frequently used in the litterature instead of "Voiculescu's Theorem".

Definition 1.5.1. A representation $\pi : A \to B(\mathcal{H})$ of a C^*-algebra A, is said to be **essential** if $\pi(A) \cap \mathbb{K}(\mathcal{H}) = \{0\}$.

Another definition is in order.

Definition 1.5.2. Let A be a separable C^*-algebra. Two maps $\pi : A \to B(\mathcal{H})$, $\sigma : A \to B(\mathcal{K})$ are said to be **approximately unitarily equivalent** if there exists a sequence of unitaries $U_n : \mathcal{H} \to \mathcal{K}$ such that

$$\lim_{n\to\infty} \|\sigma(a) - U_n\pi(a)U_n^*\| = 0,$$

for all $a \in A$. If it also happens that $\sigma(a) - U_n\pi(a)U_n^*$ is a compact operator for each $n \in \mathbb{N}$ and $a \in A$, then π and σ are said to be **approximately unitarily equivalent relative to the compacts.**

The reader may consult [9, Corollary II.5.8] for a proof of the next theorem. Recall that the **rank** of an operator $T \in B(\mathcal{H})$, written $\text{rank}(T)$ is defined to be the dimension of the image, if this dimension is finite, and $\text{rank}(T) = \infty$ otherwise

Theorem 1.5.3. *Let $\sigma : A \to B(\mathcal{H})$ and $\rho : A \to B(\mathcal{K})$ be non-degenerate representations on separable Hilbert spaces, of a separable C^*-algebra A. Then σ and ρ are approximately unitarily relative to the compacts if and only if* $\text{rank}(\sigma(a)) = \text{rank}(\rho(a))$ *for all $a \in A$.*

Corollary 1.5.4. *Let $\pi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$ be faithful, essential representations on separable Hilbert spaces of a separable C^*-algebra A. If A is unital and both π_1, π_2 are unital, then they are approximately unitarily equivalent relative to the compacts. If A is non-unital, then π_1 and π_2 are always approximately unitarily equivalent relative to the compacts.*

Before pressing on, we introduce some terminology and notation. Let $\pi : B(\mathcal{H}) \to B(\mathcal{H})/\mathbb{K}(\mathcal{H})$ denote the quotient map, A be a unital C^*-algebra and $\varphi : A \to B(\mathcal{H})$ be a ucp. map. Then we say that φ is a **representation modulo the compacts** if $\pi \circ \varphi$ is a *-homomorphism. If $\pi \circ \varphi$ is faithful, then we say that φ is a **faithful representation modulo the compacts**. In this situation, we define constants $\eta_\varphi(a)$ by

$$\eta_\varphi(a) := 2\max\{\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2}\},$$

for every $a \in A$.

The next theorem is a technical variation of Voiculescu's Theorem and will be useful in the next part. A proof may be found in [3, Theo 1.7.6].

Theorem 1.5.5. *Let A be a unital, separable C^*-algebra and $\varphi : A \to B(\mathcal{H})$ be a faithful representation modulo the compacts on a separable Hilbert space $\mathcal{H}$. If $\sigma : A \to B(\mathcal{K})$ is any faithful, unital, essential representation on a separable Hilbert space $\mathcal{K}$, then there exist unitaries $U_n : \mathcal{H} \to \mathcal{K}$ such that*

$$\limsup_{n\to\infty} \|\sigma(a) - U_n\varphi(a)U_n^*\| \leq \eta_\varphi(a),$$

for every $a \in A$.

Various Formulations of Quasidiagonality

Definition 1.5.6. A separable C^*-algebra A is said to be **quasidiagonal** if there exists a sequence of ccp. maps $\rho_n : A \to M_{k(n)}$, for some sequence of integers $(k(n))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|\rho_n(a)\rho_n(b) - \rho_n(ab)\| = 0, \quad \lim_{n\to\infty} \|\rho_n(a)\| = \|a\|, \tag{1.2}$$

for every $a, b \in A$.

The origins for the term quasidiagonality is not immediately clear from this definition, but all good things come to those that wait, and by the end of this section, an explanation should be superfluous. Before moving on we note the following useful lemma (the reader may consult [3, Lemma 7.1.4] for a proof).

Lemma 1.5.7. *Let A be a unital and quasidiagonal C^*-algebra. Then there exists a sequence of natural numbers $(k(n))_n \subseteq \mathbb{N}$ and a sequence of ucp. maps $\varphi_n : A \to M_{k(n)}$ that are both asymptotically isometric and asymptotically multiplicative, i.e., satisfies the equations* (1.2).

Definition 1.5.8. Let $\mathcal{H}$ be a separable Hilbert space and $X \subseteq B(\mathcal{H})$ be some subset. Then X is said to be a **quasidiagonal set of operators** if there exists an increasing sequence of finite rank projections $(P_n)_{n\in\mathbb{N}}$ such that SO-$\lim_{n\to\infty} P_n = 1_{\mathcal{H}}$ and $\lim_{n\to\infty} \|P_n x - xP_n\| = 0$ for every $x \in X$.

If A is a C^*-algebra and $\pi : A \to B(\mathcal{H})$ is a *-homomorphism, then π is said to be a **quasidiagonal representation** if $\pi(A) \subseteq B(\mathcal{H})$ is a quasidiagonal set of operators.

We only sketch the proof of the following theorem. However, the reader should not experience any difficulties in filling out the blanks.

Theorem 1.5.9. *For a separable, unital C^*-algebra A the following are equivalent.*

(i) A is quasidiagonal.

(ii) A admits a unital, faithful, quasidiagonal representation (on a separable Hilbert space).

(iii) Every faithful, essential and unital representation of A (on a separable Hilbert space) is quasidiagonal.

Proof. We prove (i)⇒(iii)⇒(ii)⇒(i).

Assume that A is quasidiagonal and let $\varphi_n : A \to M_{k(n)}$ be a sequence of asymptotically isometric and asymptotically multiplicative ucp. maps (see Lemma 1.5.7). Furthermore let $\pi : A \to B(\mathcal{H})$ be any unital, faithful, essential representation. Let $F \subseteq A$, $\chi \subseteq \mathcal{H}$ be finite subsets and $\varepsilon > 0$ be arbitrary. Note that we may assume that

$$\eta_{\varphi_n}(a) \leq \varepsilon/2,$$

for all $a \in F$ and all $n \in \mathbb{N}$.

Let

$$\Phi : A \to \prod_{n\in\mathbb{N}} M_{k(n)} \subseteq B\Big(\bigoplus_{n\in\mathbb{N}} \mathbb{C}^{k(n)}\Big)$$

be given by $\Phi(a) = \bigoplus_{\mathbb{N}} \varphi_n(a)$ and $Q_n \in B\big(\bigoplus_{\mathbb{N}} \mathbb{C}^{k(n)}\big)$ be the projection on to the n'th coordinate of $\bigoplus_{\mathbb{N}} \mathbb{C}^{k(n)}$. Then we see that $\varphi_n(a) = Q_n\Phi(a)Q_n$. Letting $P_n = \sum_{i=1}^n Q_n$, we find that $(P_n)_n$ is an increasing sequence of finite rank projections converging strongly to the identity and therefore $(P_n)_n$ is an approximate unit for $\mathbb{K}\big(\bigoplus_{\mathbb{N}} \mathbb{C}^{k(n)}\big)$. As a consequence we find that an operator $T \in \prod_{n\in\mathbb{N}} M_{k(n)}$ is compact if and only if $\|Q_nTQ_n\| = \|(P_n - P_{n-1})T(P_n - P_{n-1})\| \to 0$ and $n \to \infty$. Therefore, Φ is a faithful representation modulo the compacts and the desired result then follows from Theorem 1.5.5.

Assume now that (iii) holds and let $\pi : A \to B(\mathcal{H})$ be any unital, faithful representation. Then $\pi^\infty := \bigoplus_{\mathbb{N}} \pi : A \to B(\bigoplus_{\mathbb{N}} \mathcal{H})$ is unital, faithful and essential and therefore, by assumption, quasidiagonal.

Let $\pi : A \to B(\mathcal{H})$ be a faithful quasidiagonal representation and let $(P_n)_n \subseteq B(\mathcal{H})$ be an increasing sequence of finite rank projections converging to $1_{\mathcal{H}}$ in the strong operator topology such that $\|[\pi(a), P_n]\| \to 0$ for every $a \in A$. Then the sequence of ccp. maps $\varphi_n : A \to P_nB(\mathcal{H})P_n \cong M_{k(n)}$, where $k(n) := \text{Rank}P_n$, given by $\varphi_n(a) := P_n\pi(a)P_n$ has the desired properties. ☐

With this in place, we establish some basic permanence properties.

Proposition 1.5.10. *Let A be a unital, separable and quasidiagonal C^*-algebra. Then*

(i) Every subalgebra of A is also quasidiagonal.

(ii) If B is another unital, separable and quasidiagonal C^*-algebra, then $A \otimes_{\min} B$ is also unital and quasidiagonal.

(iii) If B is a separable C^*-algebra and $(A_n)_n$ is an increasing sequence of quasidiagonal subalgebras of B, each with a unit, such that $\overline{\bigcup_{n\in\mathbb{N}} A_n} = B$, then B is also quasidiagonal.

Proof. (i): Obviously the restriction of a quasidiagonal representation is also a quasidiagonal representation.

(ii): Let $\pi : A \to B(\mathcal{H})$ and $\rho : B \to B(\mathcal{K})$ be unital, faithful and quasidiagonal representations. It is not difficult to see that $\pi \otimes \rho : A \otimes_{\min} B \to B(\mathcal{H} \otimes \mathcal{K})$ is a unital, faithful quasidiagonal representation.

(iii): Let $\{a_1, \dots, a_n\} \subseteq B$ and $\varepsilon > 0$ be arbitrary. For any $\delta > 0$ we may choose some $k \in \mathbb{N}$ and elements $b_1, \dots, b_n \in A_k$ such that

$$\|a_j - b_j\| < \delta$$

for all $1 \le j \le n$. Furthermore, we can find a ccp. map $\psi : A_k \to M_n$ such that

$$\|\psi(b_i b_j) - \psi(b_i)\psi(b_j)\| < \delta, \quad \|\psi(b_i)\| > \|b_i\| - \delta$$

for all $1 \le i, j \le n$. By Arveson's Extension Theorem, we may extend ψ to a ccp. map $\varphi : B \to M_n$. Thus an appropriate choice of δ yields that φ is multiplicative and isometric up to ε on the elements $a_1, \dots, a_n$. □

One property of quasidiagonal C^*-algebras we shall be particularly interested in is that of being stably finite.

Definition 1.5.11. A unital C^*-algebra A is said to be stably finite if $\mathbf{1}_n \otimes 1_A \in M_n \otimes A$ is a finite projection for every $n \in \mathbb{N}$, where $\mathbf{1}_n \in M_n$ denotes the unit. Equivalently, every isometry in $M_n \otimes A$ is unitary.

Proposition 1.5.12. *Every unital, separable and quasidiagonal C^*-algebra is stably finite.*

Proof. Note that since M_n is quasidiagonal for each $n \in \mathbb{N}$ (this is trivial), it follows that $M_n \otimes A$ is also quasidiagonal for each $n \in \mathbb{N}$, and it is therefore sufficient to show that A is finite.

Let $\varphi_n : A \to M_{k(n)}$ be an asymptotically multiplicative and asymptotically isometric sequence of ucp. maps and $s \in A$ be an isometry. Since

$$\|\varphi_n(s^*)\varphi_n(s) - 1_{k(n)}\| \to 0,$$

we may as well assume that $\|\varphi_n(s)^*\varphi_n(s) - 1_{k(n)}\| < 1$ for all $n \in \mathbb{N}$. This implies that $\varphi_n(s)^*\varphi_n(s)$ is invertible, and since $\varphi_n(s)^*\varphi_n(s)$ is a matrix for every $n \in \mathbb{N}$, this implies that $\varphi_n(s)$ is invertible for all $n \in \mathbb{N}$, whence

$$\begin{aligned}
&\lim_{n\to\infty} \|\varphi_n(s)\varphi_n(s)^* - 1_{k(n)}\| \\
&= \lim_{n\to\infty} \|\varphi_n(s)(1_{k(n)} - (\varphi_n(s))^{-1}(\varphi_n(s)^*)^{-1})\varphi_n(s)^*\| \\
&\le \lim_{n\to\infty} \|\varphi_n(s)\|^2 \|1_{k(n)} - (\varphi_n(s)^*\varphi_n(s))^{-1}\| = 0
\end{aligned}$$

□

We mention one more property of quasidiagonal C^*-algebras, before we move on.

Proposition 1.5.13. *Every unital, separable and quasidiagonal C^*-algebra has a tracial state.*

Proof. See [3, Proposition 7.1.16]. □

1.6 Order Zero Maps

The common reference of decomposition rank and nuclear dimension, to be defined in the coming sections, is that they involve the notion of completely positive maps of order zero. This is a particularly well-behaved class of cp. maps, as should be evident once this section has been read.

Definition 1.6.1. Let A and B be C^*-algebras and $\varphi : A \to B$ a completely positive map. Then φ is said to be an **order zero map** if $\varphi(a)\varphi(b) = 0$, whenever $a, b \in A$ are positive elements such that $ab = 0$.

We have the following wonderful characterization of order zero maps (note the similarity with Stinespring's Dilation Theorem).

Theorem 1.6.2. *Let A and B be C^*-algebras and $\varphi : A \to B$ be a cp. order zero map. Set $C := C^*\big(\varphi(A)\big) \subseteq B$.*

Then there exists a positive element $h \in \mathcal{M}(C) \cap C'$, where $\mathcal{M}(C)$ denotes the multiplier algebra of C, such that $\|h\| = \|\varphi\|$ and a *-homomorphism

$$\pi_\varphi : A \to \mathcal{M}(C) \cap \{h\}'$$

such that

$$\pi_\varphi(a)h = \varphi(a), \quad \text{for } a \in A.$$

If A is unital, then $h = \varphi(1_A)$ and π_φ is unital.

Proof. See [32, Theorem 3.3]. □

Obviously any *-homomorphism and positive element as above give rise to an order zero cp. map, so the above description gives a complete characterization of such maps. As a straightforward consequence of Theorem 1.6.2, we obtain the following result.

Corollary 1.6.3. *Let A, B be C^*-algebras and $\varphi : A \to B$ be an order zero ccp. map. Then there exists a unique *-homomorphism $\rho_\varphi : C_0\big((0,1]\big) \otimes A \to B$ such that $\rho_\varphi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a)$.*

Conversely, any *-homomorphism $\rho : C_0\big((0,1]\big) \otimes A \to B$ induces an order zero ccp. map $\varphi_\rho : A \to B$ via $\varphi_\rho(a) = \rho(\mathrm{id}_{(0,1]} \otimes a)$.

These mutual assignments yield bijections between the space of ccp. order zero maps $A \to B$ and the space of *-homomorphisms $C_0\big((0,1]\big) \otimes A \to B$.

Remark 1.6.4. It is easy to see that if A, B are C^*-algebras, $\varphi : A \to B$ is an order zero ccp. map and $\tau : B \to \mathbb{C}$ is a trace, then the composition $\tau \circ \varphi : A \to \mathbb{C}$ is a also a trace. Indeed, letting $\rho_\varphi : C_0((0,1]) \otimes A \to B$ denote the induced *-homomorphism and $x, y \in A$ be arbitrary elements, we find that

$$\begin{aligned}\tau(\varphi(xy)) &= \tau\big(\rho_\varphi(\mathrm{id}_{(0,1]} \otimes x)\rho_\varphi(\mathrm{id}_{(0,1]} \otimes y)\big)\\ &= \tau\big(\rho_\varphi(\mathrm{id}_{(0,1]} \otimes y)\rho_\varphi(\mathrm{id}_{(0,1]} \otimes x)\big)\\ &= \tau(\varphi(yx)).\end{aligned}$$

Since $\tau \circ \varphi$ is obviously positive we obtain the desired result.

In connection with both notions of non-commutative covering dimension, to be discussed in the following sections, order zero maps from finite-dimensional C^*-algebras are of particular interest, whence we take the time to prove the next proposition.

Proposition 1.6.5. *Let A be an inductive limit C^*-algebra, i.e., suppose $(A, \{\mu_n\}_{n=1}^{\infty}) = \varinjlim(A_n, \varphi_n)$ and $\psi : F \to A$ be an order zero ccp. map, where F is a finite-dimensional C^*-algebra. Then, for every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ and an order zero ccp. map $\tilde{\psi} : F \to \mu_k(A_k)$ such that $\|\psi - \tilde{\psi}\| < \varepsilon$.*

Proof. This statement essentially boils down to the fact that the cone over a finite-dimensional C^*-algebra is projective (in fact this is not necessary - semi-projectivity would suffice).

To prove the statement, we use a particular model of the direct limit. To that end, let $\prod_{n=1}^{\infty} A_n$ and $\bigoplus_{n=1}^{\infty} A_n$ be given as usual, and let

$$\pi : \prod_{n=1}^{\infty} A_n \to \prod_{n=1}^{\infty} A_n / \bigoplus_{n=1}^{\infty} A_n$$

denote the quotient map. Then we let $\mu_k : A_k \to \prod_{n=1}^{\infty} A_n / \sum_{n=1}^{\infty} A_n$ be given by

$$\mu_n(a) := \pi\big((\varphi_{m,n}(a))_{m=1}^{\infty}\big),$$

where $\varphi_{m,n} : A_n \to A_m$ is defined in the standard way. It is not difficult to see that $\overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$ is a model for the direct limit $\varinjlim(A_n, \varphi_n)$.

Let $\psi : F \to A$ be an order zero ccp. map and $\rho : C_0\big((0,1]\big) \otimes F \to A$ denote the induced *-homomorphism from Corollary 1.6.3. It follows from Theorem 1.4.16 that we can represent ρ by a sequence of *-homomorphisms, i.e., there exists a sequence of *-homomorphisms $\rho_n : C_0\big((0,1]\big) \otimes F \to A_n$ such that

$$\rho(x) = \pi\big((\rho_n(x))_{n=1}^{\infty}\big)$$

for all $x \in C_0\big((0,1]\big) \otimes F$. Thus, we obtain a sequence of *-homomorphisms $\sigma_n : C_0\big((0,1]\big) \otimes F \to \mu_n(A_n)$ by setting $\sigma_n := \mu_n \circ \rho_n$. We aim to prove that there exists some $n_0 \in \mathbb{N}$ such that the order zero ccp. map $\tilde{\psi}$ induced by σ_{n_0} (again via Corollary 1.6.3) satisfies the conclusion of the proposition.

Let ψ_n be the order zero ccp. map $F \to \mu_n(A_n)$ induced by σ_n, and $\delta > 0$, $b \in F$ be arbitrary. Furthermore, let $h \in C_0\big((0,1]\big)$ denote the identity, i.e., $h := \mathrm{id}_{(0,1]}$. Choose $N \in \mathbb{N}$ and $a \in A_N$ such that

$$\|\rho(h \otimes b) - \mu_N(a)\| = \limsup_{k\to\infty} \|\rho_k(h \otimes b) - \varphi_{k,N}(a)\| < \delta. \tag{1.3}$$

Then we see that

$$\begin{aligned}\|\psi(b) - \psi_n(b)\| &= \|\rho(h \otimes b) - \sigma_n(h \otimes b)\| \\ &\leq \delta + \limsup_{k\to\infty} \|\varphi_{k,N}(a) - \varphi_{k,n} \circ \rho_n(h \otimes b)\|.\end{aligned}$$

Comparing the last limit with (1.3), we see that, upon ensuring that n is large enough, we have

$$\|\psi(b) - \psi_n(b)\| < \delta + \limsup_{k\to\infty} \left\| \varphi_{k,n}\big(\varphi_{n,N}(a) - \rho_n(h \otimes b)\big) \right\| \leq 2\delta.$$

As a straightforward consequence we see that for any $\varepsilon > 0$ and any finite set $\{b_1, \dots, b_m\} \in F$ we may choose $k \in \mathbb{N}$ such that

$$\|\psi(b_i) - \psi_k(b_i)\| < \varepsilon/2$$

for all $1 \leq i \leq m$. Letting $\tilde{\psi} := \psi_k$ and applying a compactness argument to the unit ball of F will thus finish the proof. □

Before moving on to non-commutative covering dimension we note the following proposition.

Proposition 1.6.6. *Let A, B, C and D be nuclear C^*-algebras and $\varphi : A \to C$, $\psi : B \to D$ be order zero ccp. maps. Then the induced ccp. map*

$$\varphi \otimes \psi : A \otimes B \to C \otimes D$$

has order zero.

Proof. See [32, Corollary 4.3]. □

Nuclearity is not necessary in the above proposition. The statement remains true if this assumption is removed and $\otimes$ is replaced by $\otimes_{\min}$ or $\otimes_{\max}$.

1.7 Decomposition Rank

In this section and the next we seek to introduce the reader to the notions of decomposition rank and nuclear dimension. These concepts are two examples of what is known in the litterature as non-commutative covering dimension, and the motivation for this term is given in Proposition 1.8.8. The definitions are very similar, but lead to quite different notions, as we shall see later in the thesis. For the moment we contend ourselves with stating the definitions and proving some easy permanence properties. The deeper statements are left unproven, since we really only have to work with the definitions to obtain our results. However, the reader should not be without some sort of introduction, hence we do a small amount of work.

Definition 1.7.1 (Decomposition Rank)**.** Let A be separable C^*-algebra. Then A is said to have **decomposition rank at most d** $\in \mathbb{N}_0$, written $\mathrm{dr}A \leq d$, if there exists a sequence (F_n, φ_n, ψ_n), where each F_n is a finite dimensional C^*-algebra and $\varphi_n : A \to F_n$ and $\psi_n : F_n \to A$ are cp. maps satisfying:

(i) $\lim_{n\to\infty} \|\psi_n \circ \varphi_n(a) - a\| = 0$, for every $a \in A$.

(ii) $\|\varphi_n\|, \|\psi_n\| \leq 1$.

(iii) For each $n \in \mathbb{N}$, F_n decomposes into $d+1$ ideals $F_n = \bigoplus_{i=0}^{d} F_{i,n}$ such that $\psi_{i,n} := \psi_n|_{F_{i,n}}$ is an order zero map for each $i = 0, 1, \ldots, d$.

If no such d exists we write $\mathrm{dr}(A) = \infty$. Otherwise we say that **A has decomposition rank d** if $d \in \mathbb{N}_0$ is the least integer such that $\mathrm{dr}A \leq d$.

It is easy to see that any separable C^*-algebra with finite decomposition rank is necessarily nuclear. Furthermore, the above definition has a natural extension to the non-separable case, but this is sufficient for our purposes.

Before examining permanence properties, we introduce some terminology: let $\psi : F \to B$ be a cp. map between C^*-algebras, where F is finite dimensional. Then ψ is said to be n**-decomposable** if there exists a decomposition $F = \bigoplus_{i=0}^{n} F_i$ such that $\psi|_{F_i}$ is order zero. If A is a C^*-algebra with finite decomposition rank and $(F_n, \varphi_n, \psi_n)_{n\in\mathbb{N}}$ is a cp. approximating system for the identity satisfying the conditions in Definition 1.7.1, then we refer to $(F_n, \varphi_n, \psi_n)_{n\in\mathbb{N}}$ as an n**-decomposable ccp. approximating system for** A. Finally, if $\chi \subseteq A$ is a finite set, $\varepsilon > 0$ is given and (F, φ, ψ) is a triple consisting of a finite dimensional C^*-algebra F, a ccp. map $\varphi : A \to F$ and a n-decomposable map ccp. map $\psi : F \to A$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for all $a \in \chi$, then we say we say that (F, φ, ψ) is a n**-decomposable ccp. approximation for** χ **within** ε.

Remark 1.7.2. Clearly, a separable C^*-algebra A has decomposition rank at most n if and only if for each finite subset $\chi \subseteq A$ and each $\varepsilon > 0$ we can find a n-decomposable ccp. approximation for χ within ε.

Furthermore, note that if $\psi : F \to A$ is a n-decomposable ccp. map, then ψ is also m-decomposable for any $m \geq n$.

Proposition 1.7.3. *Let A, B be nuclear, separable C^*-algebras. Then the following hold:*

(i) If $J \subseteq A$ is an ideal, then $\mathrm{dr}(A/J) \leq \mathrm{dr}A$.

(ii) $\mathrm{dr}(A \oplus B) = \max\{\mathrm{dr}A, \mathrm{dr}B\}$.

(iii) If $A = \varinjlim A_n$ *then* $\mathrm{dr}A \leq \liminf_{n\to\infty} \mathrm{dr}A_n$.

(iv) $\mathrm{dr}(A \otimes B) \leq (\mathrm{dr}A + 1)(\mathrm{dr}B + 1) - 1$

Proof. Note that the above statements are only interesting when the indicated decomposition ranks are finite, and thus we only deal with this case.

(i): Let $\pi : A \to A/J$ denote the quotient map. Since A/J is nuclear, we can find a ccp. lift $\sigma : A/J \to A$ of π. Let $\{a_1, \dots, a_k\} \subseteq A/J$ and $\varepsilon > 0$ be given, and suppose that (F, φ, ψ) is a n-decomposable ccp. approximation for $\{\sigma(a_1), \dots, \sigma(a_k)\}$ within ε. As π is *-homomorphism we see that $\pi \circ \psi$ is still a n-decomposable ccp. map, whence $(F, \varphi \circ \sigma, \pi \circ \psi)$ is a n-decomposable ccp. approximation for $\{a_1, \dots, a_k\}$. This proves that if $\mathrm{dr}A \leq n$ then $\mathrm{dr}(A/J) \leq n$ from which the conclusion obviously follows.

(ii): Let $\{(a_1, b_1), \dots, (a_k, b_k)\} \subseteq A \oplus B$ and $\varepsilon > 0$ be given. Suppose that (F_A, φ_A, ψ_A) is a n-decomposable ccp. approximation for $\{a_1, \dots, a_k\}$ within ε and (F_B, φ_B, ψ_B) is a m-decomposable ccp. approximation for $\{b_1, \dots, b_k\}$ within ε. Let $N := \max\{n, m\}$. Then $\psi_A \oplus \psi_B : F_A \oplus F_B \to A \oplus B$ is a N-decomposable ccp. map and therefore $(F_A \oplus F_B, \varphi_A \oplus \varphi_B, \psi_A \oplus \psi_B)$ is a N-decomposable ccp. approximation for $\{(a_1, b_1), \dots (a_k, b_k)\}$ within ε. This shows that $\mathrm{dr}(A \oplus B) \leq \max\{\mathrm{dr}(A), \mathrm{dr}(B)\}$, while the other inequality follows from (i).

(iii): Let $\rho_i : A_i \to A$ be the induced maps. Then $A = \overline{\bigcup_{i \in \mathbb{N}} \rho_i(A_i)}$. Let $\{x_1, \dots, x_k\} \subseteq A$ and $\varepsilon > 0$ be given. Choose $j \in \mathbb{N}$ and $a_1, \dots, a_k \in \rho_j(A_j)$ such that $\|x_i - a_i\| < \varepsilon/2$ for each $1 \leq i \leq k$ and $\mathrm{dr}A_j \leq \liminf_{n \to \infty} \mathrm{dr}A_n$ (this is possible since the decomposition rank is integer-valued).

By (i), we may choose n-decomposable ccp. approximation $(F, \tilde{\varphi}, \psi)$ for $\{a_1, \dots, a_k\}$ within $\varepsilon/2$ (in $\rho_j(A_j)$), where $n \leq \mathrm{dr}(A_j)$. By Arveson's Extension Theorem we may extend $\tilde{\varphi}$ to a ccp. map $\varphi : A \to F$, whence we see that (F, φ, ψ) a n-decomposable ccp. approximation for $\{x_1, \dots, x_k\}$ within ε.

(iv): The proof is very similar to the previous ones. Simply note that it is sufficient to find suitable ccp. approximations for elementary tensors and that for finite-dimensional C^*-algebras (not exclusively in this situation) $F_{A,1}, \dots, F_{A,n}; F_{B,1}, \dots, F_{B,m}$ we have that

$$\Big(\bigoplus_{i=1}^{n} F_{A,i}\Big) \otimes \Big(\bigoplus_{j=1}^{m} F_{B,j}\Big) \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (F_{A,i} \otimes F_{B,j}).$$

□

One prominent permance property is missing in the above proposition, as it is somewhat more difficult to prove:

Proposition 1.7.4. *If $A \subseteq B$ is a hereditary subalgebra a separable C^*-algebra B, then $\mathrm{dr}A \leq \mathrm{dr}B$.*

Proof. See [16, Proposition 3.8]. □

Before moving on to nuclear dimension, we state the following remarkable theorem. The reader may consult [16, Theorem 5.3] for a proof.

Theorem 1.7.5. *Let A be a separable C^*-algebra with $\mathrm{dr}A < \infty$. Then every representation of A is quasidiagonal.*

The conclusion of the above theorem may be restated by saying that A is *strongly* quasidiagonal. As a particular consequence of the above theorem and Proposition 1.5.12 we find that if A is a C^*-algebra with finite decomposition rank, then A is stably finite. This statement is certainly not true for C^*-algebras of finite nuclear dimension, as we shall see later on.

1.8 Nuclear Dimension

Definition 1.8.1 (Nuclear Dimension)**.** Let A be a separable C^*-algebra. Then A is said to have **nuclear dimension at most d** $\in \mathbb{N}_0$, written $\dim_{\mathrm{nuc}} A \leq d$, if there exists a sequence (F_n, φ_n, ψ_n), where F_n is a finite dimensional C^*-algebra and $\varphi_n : A \to F_n$, $\psi_n : F_n \to A$ are cp. maps satisfying:

(i) $\lim\limits_{n\to\infty} \|\psi_n \circ \varphi_n(a) - a\| = 0$ for every $a \in A$.

(ii) $\|\varphi_n\| \leq 1$.

(iii) For each $n \in \mathbb{N}$, F_n decomposes into $d+1$ ideals $F_n = \bigoplus_{i=0}^{d} F_{i,n}$ such that $\psi_{i,n} := \psi_n|_{F_{i,n}}$ is a ccp. order zero map for $i = 0, 1, \ldots, d$.

We say that the nuclear dimension of A is equal to d if d is the smallest integer such that $\dim_{\mathrm{nuc}} A \leq d$.

As is the case for decomposition rank, if A is a C^*-algebra such that $\dim_{\mathrm{nuc}} A < \infty$, then A is nuclear, although this is not an obvious consequence of our definition of nuclearity (see [33, Remark 2.2]). On the face of it one could easily mistake the definition of decomposition rank for the definition of nuclear dimension. Indeed, the only difference is that we cannot ensure that each ψ_n is a contraction. However, this seemingly minor difference gives rise to rather drastic changes in the corresponding notions. The permanence properties do remain more or less the same. The proof of the following proposition is also almost the same as that of Proposition 1.7.3.

Proposition 1.8.2. *Let A, B be separable, nuclear C^*-algebras. Then the following holds:*

(i) If $J \subseteq A$ is an ideal, then $\dim_{\mathrm{nuc}}(A/J) \leq \dim_{\mathrm{nuc}} A$.

(ii) $\dim_{\mathrm{nuc}}(A \oplus B) = \max\{\dim_{\mathrm{nuc}}(A), \dim_{\mathrm{nuc}}(B)\}$.

(iii) $\dim_{\mathrm{nuc}}(A \otimes B) \leq (\dim_{\mathrm{nuc}}(A) + 1)(\dim_{\mathrm{nuc}}(B) + 1) - 1$.

(iv) If $A = \varinjlim A_n$ then $\dim_{\mathrm{nuc}} A \leq \liminf\limits_{n\to\infty} \dim_{\mathrm{nuc}}(A_i)$.

A significant permance property of nuclear dimension, which it does not share with decomposition rank (see [16, Section 6]), is given in the following proposition.

Proposition 1.8.3. *Given a short exact sequence of C^*-algebras*

$$0 \to J \to A \to B \to 0,$$

we have

$$\max\{\dim_{\text{nuc}} J, \dim_{\text{nuc}} B\} \leq \dim_{\text{nuc}} A \leq \dim_{\text{nuc}} J + \dim_{\text{nuc}} B + 1$$

Proof. See [33, Proposition 2.9]. □

In addition we have the following result, which will be somewhat more important for us than the above proposition.

Proposition 1.8.4. *Let A be a separable, nuclear C^*-algebra and $B \subseteq A$ a full hereditary subalgebra. Then $\dim_{\text{nuc}}(B) = \dim_{\text{nuc}}(A)$.*

Proof. See [33, Corollary 2.8]. □

Remark 1.8.5. As a special case of the above proposition, we see that if A is simple, and $p \in A$ is a non-trivial projection, then $pAp \subseteq A$ is a full hereditary subalgebra of A with unit p, and therefore $\dim_{\text{nuc}}(pAp) = \dim_{\text{nuc}}(A)$ for any non-zero projection $p \in A$.

The case $\dim_{\text{nuc}} A = 0$ is interesting as it characterizes the AF-algebras:

Proposition 1.8.6. *Let A be a separable C^*-algebra. Then $\dim_{\text{nuc}} A = 0$ if and only if $\text{dr} A = 0$. Furthermore, this is case if and only if A is an AF-algebra.*

Proof. See [16, Example 4.1] along with [33, Remark 2.2] □

We have yet to explain the term non-commutative covering dimension, but this will remedied. First, we need a definition of covering dimension, to be found in [16, Definition 1.1]. Recall that given an open covering $\mathscr{O}$ of a topological space, then an open covering $\mathscr{O}'$ is said to **refine** $\mathscr{O}$ if, for each set $O' \in \mathscr{O}'$ there is a set $O \in \mathscr{O}$ such that $O' \subseteq O$.

Definition 1.8.7 (Covering Dimension)**.** Let X be a topological space.

(i) The order of a collection of subsets $(U_\lambda)_{\lambda \in \Lambda}$ does not exceed $n \in \mathbb{N}_0$ if for each $n+2$ distinct indices $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$ we have $\bigcap_{i=0}^{n+1} U_{\lambda_i} = \emptyset$.

(ii) The **covering dimension of X** does not exceed n, written $\dim X \leq n$, if any open covering of X has a refinement of order not exceeding n which covers X. We say that $\dim X = n$ if n is the least integer such that $\dim X \leq n$.

We have no particular use for this definition, excepting the next proposition, which should explain the term non-commutative covering dimension.

Proposition 1.8.8. *Let X be a second-countable, locally compact Hausdorf space. Then*

$$\dim_{\text{nuc}} C_0(X) = \text{dr } C_0(X) = \dim X$$

1.9 Comparison Theory

In this part we seek to introduce comparison theory of positive elements of C^*-algebras. The theory is quite diverse and we only cover a small part of it. We will focus on introducing the reader to C^*-algebras with strict comparison of positive elements (to be defined) and investigate the subject to such an extend that readers will no longer find the term mysterious. For the most part of this thesis, the basic definition of strict comparison will suffice, but we also take the time to relate it to the Cuntz semi-group.

This section is by far the most technical of this chapter, as it is deemed necessary for a proper understanding of the subject. However, the major result of the section (Theorem 1.9.12) will not be proven, as it lies somewhat outside the scope of the present exposition. Throughout this section, we make the blanket assumption that every C^*-algebra is unital.

We start by defining a pre-ordering on the positive cone of a C^*-algebra. This is not the usual definition, but we shall see later on (Proposition 1.9.7) that no harm is done.

Definition 1.9.1 (Cuntz Comparison)**.** Let A be a C^*-algebra and $a, b \in A_+$ be given. Then we write $a \precsim b$ if for all $\varepsilon > 0$ there exists $r \in A$ such that $(a-\varepsilon)_+ \leq rbr^*$. Furthermore, we write $a \sim b$ if $a \precsim b$ and $b \precsim a$.

It is immediately clear that this relation is reflexive. To see that $\precsim$ is transitive assume that $a, b, c \in A_+$ are given such that $a \precsim b$ and $b \precsim c$ and let $\varepsilon > 0$ be arbitrary. Let $0 < \varepsilon_1 < \varepsilon$ and choose $s_1 \in A$ such that $(a - \varepsilon_1/2)_+ \leq s_1 b s_1^*$. By choosing $\delta > 0$ sufficiently small we may ensure that $a - \varepsilon_1 \leq s_1(b-\delta)_+ s_1^*$, and an application of functional calculus yields an element $s_2 \in C^*(1, a)$ such that $(a-\varepsilon)_+ = s_2(a-\varepsilon_1)s_2^*$. By letting $s := s_1 s_2$ we therefore obtain that $(a-\varepsilon)_+ \leq s(b-\delta)_+ s^*$. By assumption we can find $t \in A$ such that $(b-\delta)_+ \leq tct^*$, whence we finally obtain that $(a-\varepsilon)_+ \leq rcr^*$ when $r := st$.

We now aim for Proposition 1.9.7, but first we need a few results, which are quite useful and interesting in their own right.

Lemma 1.9.2. *Let A be a C^*-algebra and $a, b \in A_+$. If $\|a - b\| < \varepsilon$, then there exists $r \in A$ such that $(a-\varepsilon)_+ \leq rbr^*$. In particular $(a-\varepsilon)_+ \precsim b$.*

Proof. Let $\delta := \|a-b\| < \varepsilon$. Then $a-\delta 1_A \leq b$, whence a standard functional calculus argument yields that

$$(\varepsilon-\delta)(a-\varepsilon)_+ \leq (a-\varepsilon)_+^{1/2}(a-\delta 1_A)(a-\varepsilon)_+^{1/2} \leq (a-\varepsilon)_+^{1/2} b (a-\varepsilon)_+^{1/2}.$$

Thus, upon letting $r = (\varepsilon-\delta)^{-1/2}(a-\varepsilon)_+^{1/2}$ we obtain the first statement. The second statement follows from the first, along with the observation that for any $\varepsilon_1 > 0$ we have $\big((a-\varepsilon)_+ - \varepsilon_1\big)_+ = \big(a-(\varepsilon+\varepsilon_1)\big)_+ \leq (a-\varepsilon)_+$. ☐

Furthermore, the pre-ordering $\precsim$ extends the canonical ordering $\leq$ on A_+. This is witnessed by Lemma 1.9.6, along with Proposition 1.9.7. However, we need a few technical results before we get that far. Recall the following theorem (the easy and well-known proof is omitted).

Proposition 1.9.3 (Dini's Theorem)**.** *Let X be a compact topological space and $f_n : X \to \mathbb{R}$ be a monotonically increasing sequence of continuous functions converging pointwise to a continuous function $f : X \to \mathbb{R}$. Then the sequence $(f_n)_n$ converges uniformly to f.*

The next two lemmas are due to Gert K. Pedersen and may be found in [22] (Lemma 1.4.4 and Proposition 1.4.5 respectively).

Lemma 1.9.4. *Let A be a C^*-algebra and $x, y \in A$. If $a \in A_+$ and $\alpha, \beta \in \mathbb{R}_+$ are given such that $\alpha + \beta > 1$, $x^*x \leq a^\alpha$ and $yy^* \leq a^\beta$, then the sequence $u_n = x(1/n + a)^{-1/2} y$ converges in norm to some element $u \in A$ with $\|u\| \leq \left\| a^{(\alpha+\beta-1)/2} \right\|$.*

Proof. Let $d_{nm} := (1/n + a)^{-1/2} - (1/m + a)^{-1/2}$. Then we see that

$$\begin{aligned} \|u_n - u_m\|^2 &= \|x d_{nm} y\|^2 \\ &= \|y^* d_{nm} x^* x d_{nm} y\| \\ &\leq \left\| a^{\alpha/2} d_{nm} y \right\|^2 \\ &\leq \left\| a^{\alpha/2} d_{nm} a^{\beta/2} \right\|^2 \\ &= \left\| d_{nm} a^{(\alpha+\beta)/2} \right\|^2. \end{aligned}$$

Since $\alpha + \beta > 1$ we see that the sequence of functions $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f_n(t) = (1/n + t)^{-1/2} t^{(\alpha+\beta)/2}$ monotonically increasing and converges pointwise to $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(t) = t^{(\alpha+\beta-1)/2}$. It therefore follows from Dini's theorem, that the f_n's converge uniformly to f, and therefore, by the above calculations, that $(u_n)_n$ is a Cauchy sequence. The same line of reasoning shows that for each $n \in \mathbb{N}$ we have

$$\|u_n\| \leq \left\| a^{(\alpha+\beta-1)/2} \right\|,$$

thus completing the proof. ☐

Lemma 1.9.5. *Let A be a C^*-algebra and $x, a \in A$ be given such that $a \geq 0$ and $x^*x \leq a$. If $0 < \alpha < 1/2$, then there exists $u \in A$ such that $x = ua^\alpha$ and $\|u\| \leq \left\| a^{1/2-\alpha} \right\|$.*

Proof. Let $(u_n)_n \subseteq A$ be the sequence where $u_n := x(1/n + a)^{-1/2}a^{1/2-\alpha}$. It follows from Lemma 1.9.4 that this is a convergent sequence, so let $u \in A$ denote the limit. We see that

$$\|u\| \leq \left\| a^{(1+1-2\alpha-1)/2} \right\| = \left\| a^{1/2-\alpha} \right\|.$$

Furthermore, we find that

$$\|x - u_n a^\alpha\|^2 \leq \left\| a^{1/2}\big(1 - (1/n + a)^{-1/2}a^{1/2}\big) \right\|$$

and upon applying Dini's Theorem we see that this converges to 0. Therefore the desired result has been proven. □

Lemma 1.9.6. *Let $a, b \in A_+$ be given such that $a \leq b$.*

(i) There exists $r \in A$ such that $a = rb^{1/2}r^*$.

(ii) There exists a sequence $(r_n)_n \subseteq A$ such that $\lim_{n\to\infty} \|a - r_n b r_n^*\| = 0$.

Proof. (i): It follows from Lemma 1.9.5 that we may find $r \in A$ such that $a^{1/2} = rb^{1/4}$.

(ii): Let $g_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $g_\delta(t) = \min\{\delta^{-1}, t^{-1}\}$ for each $\delta > 0$. Let $r_\delta = a^{1/2}g_\delta(b)^{1/2}$. We aim to show that $\|a - r_\delta b r_\delta^*\| \to 0$ as $\delta \to 0$. To this end, let $s_\delta = a^{1/2}\big(1_A - g_\delta(b)b\big)^{1/2}$. Note that $s_\delta s_\delta^* = a - r_\delta b r_\delta^*$, while

$$\begin{aligned} s_\delta^* s_\delta &= \big(1_A - g_\delta(b)b\big)^{1/2} a \big(1_A - g_\delta(b)b\big)^{1/2} \\ &\leq b - g_\delta(b)b^2. \end{aligned}$$

A standard functional calculus argument shows that $\|s_\delta^* s_\delta\| \to 0$ as $\delta \to 0$, which completes the proof. □

We are now ready to give some equivalent formulations of $\precsim$, including the standard definition.

Proposition 1.9.7. *Let A be a C^*-algebra and $a, b \in A_+$. Then the following are equivalent.*

(i) $a \precsim b$.

(ii) $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.

(iii) There exists a sequence $(r_n)_n \subseteq A$ such that $\lim_{n\to\infty} \|a - r_n b r_n^*\| = 0$.

(iv) For all $\varepsilon > 0$ there exist $\delta > 0$ and $r \in A$ such that $(a - \varepsilon)_+ = r(b - \delta)_+ r^*$.

Proof. (i)⇒(ii): Since $\big(a-(\varepsilon+\delta)\big)_+ \leq (a-\varepsilon)_+$ for all $\delta,\varepsilon>0$, this follows directly from the definition of $\precsim$.

(ii)⇒(iii): Let $\delta>0$ be given and find some $s\in A$ such that $(a-\delta)_+ \leq sbs^*$. It follows from Lemma 1.9.6 that we may find $t\in A$ such that

$$\|(a-\delta)_+ - tsbs^*t^*\| < \delta.$$

Since $\|a-(a-\delta)_+\| \leq \delta$, we see that $\|a-rbr^*\|<2\delta$, when $r=ts$.

(iii)⇒(iv): Let $\varepsilon>0$ be given and choose $s_1\in A$ such that

$$\|a-s_1bs_1^*\| < \varepsilon.$$

As $\|b-(b-\delta)_+\|\to 0$ as $\delta\to 0$, we can choose $\delta>0$ such that

$$\|a-s_1(b-2\delta)_+s_1^*\| < \varepsilon.$$

For each $\gamma>0$ let $f_\gamma:\mathbb{R}_+\to\mathbb{R}_+$ be given by

$$f_\gamma(t)=\begin{cases}0 & \text{when } t\leq\gamma\\ \gamma^{-1}(t-\gamma) & \text{when } \gamma\leq t\leq 2\gamma\\ 1 & \text{when } t\geq 2\gamma\end{cases}.$$

By the functional calculus we may find $s_2\in A$ such that $s_1(b-2\delta)_+s_1^* = s_2f_{2\delta}(b)s_2^*$. Together with Lemma 1.9.2 this implies that we may find $s_3\in A$ such that

$$(a-\varepsilon)_+ \leq s_3f_{2\delta}(b)s_3^*.$$

Let $z=s_3f_{2\delta}(b)^{1/2}$ and $z=v|z|$ be the polar decomposition of z in A^{**}. Note that $zf_\delta(b)=z$, and therefore $|z|f_\delta(b)=v^*zf_\delta(b)=|z|$ and it follows that $v|z|^{1/2}f_\delta(b)=v|z|^{1/2}$. Letting $s_4:=v|z|^{1/2}\in A$ we therefore find that

$$(s_4f_\delta(b)s_4^*)^2=(s_4s_4^*)^2=(v|z|v^*)^2=zz^*=s_3f_{2\delta}(b)s_3^*\geq(a-\varepsilon)_+.$$

Finally we may apply Lemma 1.9.6 part (i) along with functional calculus to obtain $r\in A$ such that

$$(a-\varepsilon)_+ = r(b-\delta)_+r^*.$$

(iv)⇒(i): This is obvious, since $(b-\delta)_+\leq b$ for all $\delta>0$. □

Lemma 1.9.8. *Let A be a C^*-algebra. Let $a,e,f\in A$ be positive contractions such that $fe=e$ and suppose there exists $\delta>0$ such that $f\precsim(a-\delta)_+$. Then there exists an element $r\in A$ such that $\|r\|\leq\delta^{-1/2}$ and $rar^*=e$.*

Proof. We may assume that $<\delta<1$, since otherwise the statement is trivial. Let $g:[0,1]\to\mathbb{R}_+$ be given by

$$g(t)=\begin{cases}0 & \text{when } t\leq\delta\\ (1-\delta)^{-1}(t-\delta), & \text{when } \delta\leq t\leq 1.\end{cases}$$

It follows from Proposition 1.9.7, and an application of functional calculus, that we may find $s \in A$ such that

$$s(a-\delta)_+ s^* = g(f).$$

Since $g(1) = 1$ it follows that $g(f)e = e$, and obviously $\|(a-\delta)_+^{1/2} s^*\| \le 1$. Now, let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$h(x) = \begin{cases} \delta^{-2}x & \text{when } 0 \le x \le \delta \\ x^{-1} & \text{when } x \ge \delta, \end{cases}$$

and let $t := sh(a)^{1/2}(a-\delta)_+^{1/2}$. Then we see that

$$tat^* = s(h(a)a(a-\delta)_+)s^* = s(a-\delta)_+ s^* = g(f),$$

and $\|t\| \le \|h\|_\infty^{1/2} \le \delta^{-1/2}$. Upon setting $r = e^{1/2}t$, we therefore obtain the desired result. □

Now, the reader might find the situation of the above proposition rather specific, and she would be right. Indeed, we will not need the result for some time, but it does come into play at a crucial point. We now have all the results on $\precsim$ we need, and therefore proceed to consider strict comparison of positive elements and the Cuntz semi-group.

Strict Comparison

We are now ready to introduce the notion of strict comparison of positive elements. We will briefly outline the connection to the Cuntz semi-group, since it lurks in the background of this theory. However, the Cuntz semi-group will not be a recurring feature in this thesis, so the introduction will be kept short and sweet.

Let A be a C^*-algebra. Given $a \in M_n \otimes A$ and $b \in M_m \otimes A$, we let $a \oplus b \in M_{n+m} \otimes A$ denote the orthogonal sum, i.e.,

$$a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

For each $k \in \mathbb{N}$ we obtain an embedding $M_k \otimes A \to M_{k+1} \otimes A$ by $a \mapsto a \oplus 0$. We let $M_\infty(A)$ denote the *-algebra $\bigcup_{k\in\mathbb{N}}(M_k \otimes A)$, and $M_\infty(A)_+ := \bigcup_{k\in\mathbb{N}}(M_k \otimes A)_+$. We can extend the pre-order $\precsim$ on A to a pre-order on $M_\infty(A)_+$ as follows:

Definition 1.9.9. Let A be a C^*-algebra. For positive elements $a \in M_n \otimes A$ and $b \in M_m \otimes A$, we write $a \precsim b$ if there exists a sequence $(r_n)_{n\in\mathbb{N}} \subseteq M_{n,m}(A)$ such that $\|a - r_n b r_n^*\| \to 0$ as $n \to \infty$. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$.

Of course, the above definition is equivalent to the following: For two elements $a, b \in M_\infty(A)_+$ we say that $a \precsim b$ if there exists a sequence $(r_n)_n \subseteq M_\infty(A)$ such that $\|a - r_n^* b r_n\| \to 0$ as $n \to \infty$ ($M_\infty(A)$ carries a natural norm, since embeddings of C^*-algebras are isometric).

Let $\mathrm{Tr}_k : M_k \to \mathbb{C}$ denote the unnormalized trace on M_k. Given a tracial state τ on A we can define a function $d_\tau : M_\infty(A)_+ \to [0, \infty)$ by setting

$$d_\tau(a) = \lim_{n\to\infty} (\mathrm{Tr}_k \otimes \tau)(a^{1/n}),$$

when $a \in (M_k \otimes A)_+$. Note the importance of using the *unnormalized* trace Tr_k on M_k. We usually omit the reference to Tr_k and simply consider τ to be an unnormalized trace on $M_k \otimes A$ for every $k \in \mathbb{N}$. Furthermore, it is not difficult to see that the indicated limit exists for all elements $a \in M_\infty(A)_+$; Assume first that a is a contraction in $(M_k \otimes A)_+$. Then $\big(\tau(a^{1/n})\big)_n$ is an increasing sequence bounded above by k and therefore convergent. For a general $a \in (M_k \otimes A)_+$ we see that

$$\lim_{n\to\infty} \tau(a^{1/n}) = \lim_{n\to\infty} \tau\left(\|a\|^{1/n}\left(\frac{a}{\|a\|}\right)^{1/n}\right) = \lim_{n\to\infty} \tau\left(\left(\frac{a}{\|a\|}\right)^{1/n}\right),$$

and therefore everything works out in this case as well.

Definition 1.9.10 (Strict Comparison)**.** Let A be a unital, simple and exact C^*-algebra. Then A is said to have **strict comparison for positive elements** if $a \precsim b$ whenever $a, b \in M_\infty(A)_+$ satisfies $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$.

Remark 1.9.11. It may puzzle the reader that we only define strict comparison for C^*-algebras that are unital and exact. There is, of course, a definition which is valid for general C^*-algebras, but the most natural way to state this definition would involve the notion of lower semi-continuous dimension functions on A, a concept we do not intend to discuss. However, a well-known, albeit as of yet unpublished, result of Uffe Haagerup ([10]) states that all 2-quasitraces on a unital and exact C^*-algebra are genuine traces and as a result, the general definition and the definition above coincide on unital, exact C^*-algebras.

The reasons for requiring simplicity are somewhat more complicated. Essentially a unital and exact C^*-algebra may satisfy the above definition without having good comparison properties. However, if A is unital, simple and exact, then A satisfies the above definition if and only if $W(A)$ is almost unperforated (see [25] for a proof of this along with a definition of almost unperforation). As a consequence, the author is reasonably certain that, under the indicated restrictions, no one will be offended by the given definition.

We will not require a great deal of results on C^*-algebras with strict comparison, the one exception being the following theorem due to Mikael Rørdam (see [25, Theorem 4.5] for a proof). The readers who are unfamiliar with the Jiang-Su algebra can find a brief introduction in Section 3.1 (or alternatively consult the original article [11]).

Theorem 1.9.12. *Let A be a simple, exact and unital C^*-algebra and $\mathcal{Z}$ denote the Jiang-Su algebra. Then $A \otimes \mathcal{Z}$ has strict comparison of positive elements.*

We remind the reader of the following definition.

Definition 1.9.13 (Strict Comparison for Projections)**.** Let A be a unital, exact C^*-algebra with $T(A) \neq \emptyset$. We say that A has **strict comparison for projections** if, for any pair projections $p, q \in M_k \otimes A$ satisfying $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, there exists a partial isometry $v \in M_k \otimes A$ such that $v^*v = p$ and $vv^* \leq q$, i.e., $p \preceq q$.

Note that strict comparison of positive elements imply strict comparison for projections (of course the C^*-algebra has to be unital, simple, exact and posses at least one tracial state for this to make sense). Indeed, for a projection p we see that $p^{1/n} = p$ whence $d_\tau(p) = \tau(p)$. The statement then follows from [23, Proposition 2.1].

As a closing statement, let us (very!) briefly put the above exposition in the context of the Cuntz semi-group. Let $W(A) := (M_\infty(A)_+)/\sim$ denote the set of equivalence classes with respect to $\sim$, and $\langle a \rangle \in W(A)$ denote the equivalence class containing a. We may define a composition on $W(A)$ by $\langle a \rangle + \langle b \rangle := \langle a \oplus b \rangle$. The reader may check that this is well-defined. In fact, $W(A)$ becomes an ordered, abelian semi-group when equipped with the ordering defined by

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b.$$

The functions d_τ, defined earlier, satisfy the conditions

- ▷ If $a \precsim b$ then $d_\tau(a) \leq d_\tau(b)$.
- ▷ If a, b are orthogonal, then $d_\tau(a + b) = d_\tau(a) + d_\tau(b)$.

and therefore induce well-defined functions on $W(A)$. It is clear that we may reformulate the definition of strict comparison of positive elements as follows; A has strict comparison for positive elements if $\langle a \rangle \leq \langle b \rangle$ whenever $d_\tau(\langle a \rangle) < d_\tau(\langle b \rangle)$ for all $\tau \in T(A)$ (again this is only the correct definition when A is unital, simple and exact). There are a number of things that needs to be proven here, but we do not intend to delve into the subtleties of the theory. Let us simply remark that strict comparison of positive elements is really a property pertaining to the Cuntz semi-group, and therefore of a K-theoretical nature. This aspect will not be explored in the present thesis.

CHAPTER 2

Ultrapowers

2.1 The Metric Ultrapower

We take the basics concerning metric ultrapowers for granted, and therefore this section serves primarily to fix notation and remind the reader of the most important general results.

Notation: Throughout this thesis we let ω denote a free ultrafilter on $\mathbb{N}$, and for a C^*-algebra A we let

$$c_\omega(A) := \{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}.$$

It easy to see that $c_\omega(A)$ is an ideal in $\ell^\infty(A)$, and we let $A_\omega := \ell^\infty(A)/c_\omega(A)$. We refer to A_ω as **the metric ultrapower of** A, or, more commonly, simply the **ultrapower** of A, with respect to the filter ω. We let $\pi_\omega : \ell^\infty(A) \to A_\omega$ denote the quotient map and $\iota_\omega : A \to A_\omega$ denote the inclusion of A as the constant sequences. We usually omit any reference to ι_ω and identify A with the image under ι_ω.

The ultrapower of a C^*-algebra is thoroughly uninteresting when ω is not free, and an indication of the advantages of assuming that ω is an ultrafilter is given by the next proposition. For any filter ω on $\mathbb{N}$ and any $(a_n)_n \in \ell^\infty(\mathbb{R})$ we define:

$$\limsup_\omega a_n := \inf_{X\in\omega} \sup_{n\in X} a_n.$$

Proposition 2.1.1. *Let ω be a free filter on $\mathbb{N}$. Then $\big\| \pi_\omega\big((a_n)_n\big) \big\| = \limsup_\omega \|a_n\|$. If ω is also an ultrafilter, then $\big\| \pi_\omega\big((a_n)_n\big) \big\| = \lim_{n\to\omega} \|a_n\|$.*

Actually, the above theorem is usually just a convenience, and the usefullness of assuming ω to be an ultrafilter is not exhausted at this point. It should however indicate why we do it. There is a nice relation between A and the ultrapower A_ω as demonstrated by the following proposition.

Proposition 2.1.2. *Let A be a C^*-algebra. Then the following hold:*

(i) Every projection in A_ω lifts to a projection in $\ell^\infty(A)$.

(ii) If A is unital, then every isometry in A_ω lifts to an isometry in $\ell^\infty(A)$.

(iii) If A is unital, then every unitary in A_ω lifts to a unitary in $\ell^\infty(A)$.

This summarizes the primary properties of ultrapowers that we will use without proof. For a general introduction to ultrapowers the reader should consult [24] (possibly [6] if the reader does not object to reading another project by the author) or [13] for a more thorough investigation.

The next lemma, although technical in nature, summarizes the essentials of ultrapowers in a wonderful way. It will be used all through this thesis, in almost any proof involving ultrapowers. To give the reader an intuitive idea of the content, she should keep the application to ultrapowers in mind and think of the lemma in the following setting: for the moment, let us consider a fixed subset $X \subseteq A$ of some C^*-algebra A (i.e., $X = X_n$ for all $n \in \mathbb{N}$ in the situation of the lemma), and imagine that each of the functions $f^{(k)}$ determine a one-variable relation R_k on A, i.e., $x \in A$ satisfies R_k if and only if $f^{(k)}(x) = 0$. Then the lemma states that, if we, for each $m \in \mathbb{N}$, can find a sequence $(x_n)_n \in \prod_{i=1}^\infty X$ that almost satisfies the relations $R_1, R_2, \dots, R_m$, at least in the limit, then there exists a sequence $(y_n)_n \in \prod_{i=1}^\infty X$ which satisfies all the relations $R_1, R_2, \dots,$ again at least in the limit. Hopefully, the applications to come will make this idea more clear to the reader.

Lemma 2.1.3 (The ε-test)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be any sequence of sets and suppose that for each $k \in \mathbb{N}$ we are given a sequence $\big(f_n^{(k)}\big)_{n\in\mathbb{N}}$ of functions $f_n^{(k)} : X_n \to [0,\infty)$. For each $k \in \mathbb{N}$ define a new function $f_\omega^{(k)} : \prod_{i=1}^\infty X_n \to [0,\infty]$ by*

$$f_\omega^{(k)}(s_1, s_2, ...) = \lim_{n\to\omega} f_n^{(k)}(s_n), \quad (s_n)_{n\in\mathbb{N}} \in \prod_{n=1}^\infty X_n.$$

Suppose that for each $m \in \mathbb{N}$ and each $\varepsilon > 0$ there exists $s = (s_1, s_2, \dots) \in \prod_{n=1}^\infty X_n$ such that $f_\omega^{(k)}(s) < \varepsilon$ for $k = 1, 2, \dots, m$. Then there is $t = (t_1, t_2, \dots) \in \prod_{n=1}^\infty X_n$ with $f_\omega^{(k)}(t) = 0$ for all $k \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ define a decreasing sequence $(X_{n,m})_{m\geq 0}$ of subsets of X_n by: $X_{n,0} := X_n$ and

$$X_{n,m} := \{s \in X_n \mid \max\{f_n^{(1)}(s), \dots, f_n^{(m)}(s)\} < 1/m\},$$

for $m \geq 1$. We let $m(n) := \max\{m \leq n \mid X_{n,m} \neq \emptyset\}$ and for each $k \in \mathbb{N}$ let $Y_k := \{n \in \mathbb{N} \mid k \leq m(n)\}$.

Fix some $k \geq 1$. By assumption there exists $s = (s_n)_n \in \prod_{n=1}^{\infty} X_n$ such that $f_\omega^{(j)}(s) < 1/k$ for all $1 \leq j \leq k$, hence

$$Z_k = \{n \in \mathbb{N} \mid \max\{f_n^{(1)}(s_n), \dots, f_n^{(k)}(s_n)\} < 1/k\}$$

is a member of ω for each $k \in \mathbb{N}$. It follows that $X_{n,k} \neq \emptyset$ and hence $\min\{n,k\} \leq m(n) \leq n$ for all $n \in Z_k$. This in turn implies that $Z_k \setminus \{1, \dots, k-1\} \subseteq Y_k$, whence $Y_k \in \omega$ (it was assumed that ω is a free filter and therefore $\{k, k+1, \dots\} \in \omega$). This almost finishes the proof since

$$\lim_{n\to\infty} \frac{1}{m(n)} = \liminf_{n\to\infty} \frac{1}{m(n)} \leq \inf_k \sup_{n \in Y_k} \frac{1}{m(n)} \leq \inf_k \frac{1}{k} = 0.$$

Now we are really done since we can by definition find $t_n \in X_{n,m(n)}$. Setting $t = (t_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} X_n$ we obtain

$$f_\omega^{(k)}(t) = \lim_{n\to\omega} f_n^{(k)}(t_n) \leq \lim_{n\to\omega} \frac{1}{m(n)} = 0.$$

□

In the applications to come the sets X_n will rarely depend on n, and rather be some fixed subset of a C^*-algebra placing an upper bound on the norm of the elements (e.g. the set of unitaries, or the set of positive contractions). The functions $f_n^{(k)}$ will be referred to as the test functions.

In an effort to guide the reader towards the appreciation of the above lemma that it deserves, we take the time to prove Proposition 2.1.5. The proof serves as a nice application of the ε-test for a number of reasons. It is short and easy and the basic form of the proof is very similar to later applications of the above lemma. More importantly, it demonstrates a principle which is fundamental for the philosophy behind ultrapowers of C^*-algebras. Namely, if A satisfies an approximation property, then A_ω satisfies the exact version of the same property, and if A_ω satisfies an approximation version of some property, then it automatically satisfies the exact version of the same property. We start off with the following definition.

Definition 2.1.4. Let A, B be C^*-algebras, with A separable and B unital, and $\varphi, \psi : A \to B$ be *-homomorphisms. Then we say that φ and ψ are **unitarily equivalent**, and write $\varphi \sim_u \psi$, if there exists a unitary $v \in B$ such that $\mathrm{Ad}_v \circ \varphi = \psi$. Similarly, given a topology $\mathcal{T}$ on B we say that φ and ψ are **approximately unitarily equivalent in** $\mathcal{T}$, if there exists a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subseteq B$ such that $\mathrm{Ad}_{u_n} \circ \varphi$ converges to ψ in the point-$\mathcal{T}$ topology. If the topology $\mathcal{T}$ is the norm topology, then we simply say that φ and ψ are approximately unitarily equivalent.

Usually the topology $\mathcal{T}$ above will be given by a norm, or possibly a family of semi-norms.

Proposition 2.1.5. *Let A and B be C^*-algebras with A separable and B unital, ω a free ultrafilter on $\mathbb{N}$, and $\varphi, \psi : A \to B_\omega$ a pair of *-homomorphisms. If φ and ψ are approximately unitarily equivalent, then they are unitarily equivalent.*

Proof. Fix a dense sequence $(a_m)_{m\in\mathbb{N}} \subseteq A$ and for each $n \in \mathbb{N}$ let X_n denote the set of unitaries in B. Furthermore, let the test-functions $f_n^{(k)} : X_n \to [0,\infty]$ be given by

$$f_n^{(k)}(u) = \|u\varphi_n(a_k)u^* - \psi_n(a_k)\|,$$

where $(\varphi_n)_n$ and $(\psi_n)_n$ are representative bounded sequences of maps for φ and ψ respectively, i.e., $\varphi(a) = \pi_\omega\big((\varphi_n(a))_n\big)$ and $\psi(a) = \pi_\omega\big((\psi_n(a))_n\big)$ (no particular properties of these maps are required). By assumption we may, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, find a unitary $\tilde{u} \in B_\omega$ such that

$$\|\tilde{u}\varphi(a_k)\tilde{u}^* - \psi(a_k)\| < \varepsilon$$

for all $1 \leq k \leq m$. It follows from Proposition 2.1.1 that there exists a sequence of unitaries $(\tilde{u}_n)_n \subseteq B$ such that $\pi_\omega\big((\tilde{u}_n)_n\big) = \tilde{u}$. Hence, we find that

$$\begin{aligned} f_\omega^{(k)}(\tilde{u}_1, \tilde{u}_2, \dots) &= \lim_{n\to\omega} \|\tilde{u}_n\varphi_n(a_k)\tilde{u}_n^* - \psi_n(a_k)\| \\ &= \|\tilde{u}\varphi(a_k)\tilde{u}^* - \psi(a_k)\| < \varepsilon, \end{aligned}$$

for all $1 \leq k \leq m$. The ε-test therefore implies the existence of a sequence of unitaries $(u_n)_n \subseteq B$ such that

$$f_\omega^{(k)}(u_1, u_2, \dots) = \lim_{n\to\omega} \|u_n\varphi_n(a_k)u_n^* - \psi_n(a_k)\| = 0,$$

for all $k \in \mathbb{N}$. Letting $u := \pi_\omega\big((u_1, u_2, \dots)\big)$ we see that $\mathrm{Ad}_u \circ \varphi = \psi$, since $(a_m)_m \subseteq A$ is dense. □

As with most applications of the ε-test, one could prove the above statement directly, using only basic properties of free ultrafilters. However, the ε-test formalizes these types of arguments in an extremely versatile form, which is very close to optimal. Hopefully, upon completed reading of this chapter, the reader will share (at least some of) the authors enthusiasm for this lemma.

2.2 Traces and The Trace Kernel Ideal

We start by defining certain semi-norms on C^*-algebras. It may not be immediately clear that it is indeed semi-norms we define, but that will be proven shortly.

Definition 2.2.1 (Trace-seminorms)**.** Let A be C^*-algebra with $T(A) \neq \emptyset$. For any non-empty subset $S \subseteq T(A)$ and $p = 1, 2$, let $\|\cdot\|_{p,S}$ be given by

$$\|a\|_{p,S} := \sup_{\tau\in S} \tau\big((a^*a)^{p/2}\big)^{1/p} = \big[\sup_{\tau\in S} \tau\big((a^*a)^{p/2}\big)\big]^{1/p}.$$

If $S = \{\tau\}$, then we write $\|\cdot\|_{p,\tau}$ instead of $\|\cdot\|_{p,\{\tau\}}$. Furthermore, let

$$\|a\|_p := \|a\|_{p,T(A)} = \sup_{\tau\in T(A)} \tau\big((a^*a)^{p/2}\big)^{1/p}.$$

If $\mathcal{S} = (S_1, S_2, \dots)$ is a sequence of non-empty subsets of $T(A)$, and $\mathcal{T} = (T(A), T(A), \dots)$, then we define seminorms on A_ω by

$$\|a\|_{p,\mathcal{S}} = \lim_{n\to\omega} \|a_n\|_{p,S_n}, \quad \|a\|_{p,\omega} := \|a\|_{p,\mathcal{T}},$$

where $a = \pi_\omega\big((a_1, a_2, \dots)\big)$.

Remark 2.2.2. It is not difficult to see that $\|\cdot\|_{2,\tau}$ is a semi-norm, for any C^*-algebra and any $\tau \in T(A)$. Namely, let $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ be the associated GNS-triple. Then $\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau\rangle$. We therefore see that for any $a \in A$ we have that $\|a\|_{2,\tau} = \langle \pi_\tau(a^*a)\xi_\tau, \xi_\tau\rangle^{1/2} = \|\pi_\tau(a)\xi_\tau\|$, which clearly shows that $\|\cdot\|_{2,\tau}$ is a semi-norm. Furthermore, if τ is faithful, then it is a genuine norm. We have to work a little harder to show that $\|\cdot\|_{1,\tau}$ defines a semi-norm, but it is certainly doable. Note, that once we have proven this, all the maps defined above do give rise to semi-norms.

Additionally, we see that for any $a \in A$ we have $\|a\|_{p,S} \leq \|a\|_{p,T}$, whenever $S \subseteq T \subseteq T(A)$. Moreover, if $S \subseteq T$ and T is contained in the weak*-closed convex hull of S, then $\|a\|_{p,S} = \|a\|_{p,T}$ (this is a straightforward argument). In particular, $\|a\|_{p,\partial_e T(A)} = \|a\|_p$, an observation that will be of some importance later on.

Let us show that $\|\cdot\|_{1,\tau}$ defines a semi-norm for any C^*-algebra A and any $\tau \in T(A)$. First up is a collection of useful facts concerning tracial states on von Neumann algebras.

Lemma 2.2.3. *Let $\mathscr{M}$ be a von Neumann algebra and τ be a tracial state on $\mathscr{M}$. Then the following holds for all $x, y \in \mathscr{M}$:*

(i) $\tau(|y||x|) \leq \|y\|\tau(|x|)$;

(ii) $|\tau(yx)| \leq \tau(|yx|) \leq \|y\|\tau(|x|)$;

(iii) $\tau(|x|) = \sup\{|\tau(zx)| \mid z \in \mathscr{M}, \|z\| \leq 1\}$.

Proof. (i): This follows easily from the fact that τ is positive:

$$\tau(|y||x|) = \tau(|x|^{1/2}|y||x|^{1/2}) \leq \tau(|x|^{1/2}\|y\||x|^{1/2}) = \|y\|\tau(|x|).$$

(ii): Since $\mathscr{M}$ is a von Neumann algebra we may find partial isometries $u, v \in \mathscr{M}$ such that $x = u|x|$ and $y = v|y|$. Recalling that $u|x|u^* = |x^*|$ and $v|y|v^* = |y^*|$, we find that Cauchy-Schwartz along with part (i) implies;

$$\begin{aligned}
|\tau(yx)|^2 &= |\tau(v|y|u|x|)|^2 \\
&= \left|\tau\big((|y|^{1/2}v^*|x|^{1/2})^*(|y|^{1/2}u|x|^{1/2})\big)\right|^2 \\
&\leq \tau\big(|x|^{1/2}v|y|v^*|x|^{1/2}\big)\tau\big(|x|^{1/2}u^*|y|u|x|^{1/2}\big) \\
&= \tau\big(|y^*||x|\big)\tau\big(|y||x^*|\big) \\
&\leq \|y\|\|y^*\|\tau\big(|x|\big)\tau\big(|x^*|\big) \\
&= \|y\|^2\tau(|x|)^2.
\end{aligned}$$

Thus, we have obtained the inequality

$$|\tau(yx)| \leq \|y\|\tau(|x|), \tag{2.1}$$

and from this the desired result will follow. Indeed, the first inequality is a trivial consequence;

$$|\tau(yx)| = |\tau(1_{\mathscr{M}}yx)| \leq \|1_{\mathscr{M}}\|\tau(|yx|) = \tau(|yx|).$$

For the second inequality let $yx = w|yx|$ be the polar decomposition of yx. Then we have that $w^*yx = |yx|$, whence it follows from (2.1) that:

$$\tau(|yx|) = |\tau(w^*yx)| \leq \|w^*y\|\tau(|x|) \leq \|y\|\tau(|x|).$$

(iii): Evidently, it follows from (ii) that

$$\tau(|x|) \geq \sup\{|\tau(zx)| \mid z \in \mathscr{M}\|z\| \leq 1\}.$$

On the other hand, if $x = u|x|$ is the polar decomposition of x, then $u^*x = |x|$, and since u is a partial isometry we have that $\|u\| \leq 1$, thereby completing the proof. □

Corollary 2.2.4. *Let A be a unital C^*-algebra and τ be a tracial state on A. Then $\|\cdot\|_{1,\tau}$ defines a semi-norm on A, and a norm if τ is faithful.*

Proof. This follows from Lemma 2.2.3, once we have done a small amount of work. By taking adjoints twice (in the Banach space sense) of τ, we obtain a weak*-continuous linear functional $\tilde{\tau}$ on A^{**} such that $\|\tilde{\tau}\| = \|\tau\|$. Thus $\tilde{\tau}$ is ultraweakly continuous, when A^{**} is endowed with the canonical von

Neumann algebra structure. Since multiplication is ultraweakly continuous in each variable separately, it is easy to see that $\tilde{\tau}$ is a normal tracial state on the von Neumann algebra A^{**}. From part (iii) of Lemma 2.2.3 it therefore follows that $\|\cdot\|_{1,\tilde{\tau}}$ defines a semi-norm on A^{**}. Obviously, the restriction of $\|\cdot\|_{1,\tilde{\tau}}$ to A, identified with a subalgebra of A^{**}, is still a semi-norm and this restriction is precisely $\|\cdot\|_{1,\tau}$.

If τ is faithful and $\|x\|_{1,\tau} = \tau(|x|) = 0$, then $|x| = 0$ which clearly implies that $x = 0$, thus proving that $\|\cdot\|_{1,\tau}$ is a genuine norm in this case. □

Remark 2.2.5. It is not particularly difficult to see that

$$\|a\|_{1,\tau} \leq \|a\|_{2,\tau} \leq \|a\|_{1,\tau}^{1/2},$$

for any contraction $a \in A$. Indeed, let $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ be the GNS-triple associated with τ and let $a \in A$ be given. Then, by the Cauchy-Schwartz inequality, we have that

$$\begin{aligned} \|a\|_{1,\tau}^2 &= \langle \pi_\tau(a^*a)^{1/2}\xi_\tau, \xi_\tau\rangle^2 \\ &\leq \langle \pi_\tau(a^*a)\xi_\tau, \xi_\tau\rangle \cdot \|\xi_\tau\|^2 \\ &= \tau(a^*a) = \|a\|_{2,\tau}^2, \end{aligned}$$

which proves the first inequality. Next, let $a \in A$ be a contraction. Then it follows from Lemma 2.2.3 part (i) that

$$\|a\|_{2,\tau} = \tau\big(|a|^2\big)^{1/2} \leq \|a\|^{1/2}\tau(|a|)^{1/2} \leq \|a\|_{1,\tau}^{1/2},$$

which proves the second inequality.

Before we define the tracial ultrapower of a C^*-algebra, a few remarks are in order. Note that, for any sequence $\mathcal{S} = (S_1, S_2, \dots)$ of non-empty subsets of $T(A)$ and $a, b \in A_\omega$ we have that $\|a\|_{2,\mathcal{S}} = \|a^*\|_{2,\mathcal{S}}$, $\|ab\|_{2,\mathcal{S}} \leq \|a\|\|b\|_{2,\mathcal{S}}$ and $\|a\|_{2,\mathcal{S}} \leq \|a\|$, and thus it is easy to see that the claimed ideals in the next definition are just that – ideals. Usually J_A is referred to as the trace kernel ideal, hence the title of this section.

Definition 2.2.6. Let A be a unital C^*-algebra such that $T(A) \neq \emptyset$ and define ideals J_A, J_τ of A_ω by

$$\begin{aligned} J_A &:= \{a \in A_\omega \mid \|a\|_{2,\omega} = 0\}; \\ J_\tau &:= \{\pi_\omega((a_n)_n) \in A_\omega \mid \lim_{n\to\omega} \|a_n\|_{2,\tau} = 0\}. \end{aligned}$$

Then we let $A^\omega = A_\omega/J_A$, and refer to A^ω as **the tracial ultrapower of** A. Moreover, we let $A_\tau^\omega = A_\omega/J_\tau$ and refer to A_τ^ω as **the tracial ultrapower of** A **with respect to** τ.

When $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ we will sometimes write $[(x_n)_n]$ to denote the equivalence class of $(x_n)_n$ in the tracial ultrapower, when it is clear which particular ultrapower we are considering. It is easy to deduce from Remark 2.2.5 that we could have replaced $\|\cdot\|_{2,\omega}$ with $\|\cdot\|_{1,\omega}$ and $\|\cdot\|_{2,\tau}$ with $\|\cdot\|_{1,\tau}$ in the above definition, without changing the ideals. The reader should hereby consider herself warned that we use this observation without comment in the rest of the thesis.

The ultrapower we are really interested in is the "uniform" tracial ultrapower A^ω, as this takes the full trace simplex into consideration. Excepting the case $T(A) = \{\tau\}$, there is really no canonical choice for τ when considering A^ω_τ. However, it is useful to consider the latter tracial ultrapower before investigating the former, since, in certain cases, it does provide useful insight. Therefore we shall expend some effort towards understanding A^ω_τ.

2.3 Ultrapowers Of Factors

In the following, we make the standing assumption that A is a unital C^*-algebra and that τ is a tracial state on A. In order to treat A^ω_τ, we first determine when $\pi_\tau(A)''$ is a factor. First we need a basic, but still very useful, observation. Given a Hilbert space $\mathcal{H}$ and a *-subalgebra $\mathfrak{X} \subseteq B(\mathcal{H})$ we say that $\xi \in \mathcal{H}$ is a **tracial** vector for $\mathfrak{X}$, if the map $\mathfrak{X} \to \mathbb{C}$, given by $x \mapsto \langle x\xi, \xi\rangle$ is a trace. In other words, $\xi \in \mathcal{H}$ is a trace vector for $\mathfrak{X}$ if

$$\langle xy\xi, \xi\rangle = \langle yx\xi, \xi\rangle,$$

for all $x, y \in \mathfrak{X}$. Furthermore, we remind the reader that $\xi \in \mathcal{H}$ is said to be **cyclic** for $\mathfrak{X}$ if $\mathfrak{X}\xi \subseteq \mathcal{H}$ is dense, and $\xi \in \mathcal{H}$ is said to be **separating** for $\mathfrak{X}$ if $x\xi = 0$ implies that $x = 0$ whenever $x \in \mathfrak{X}$.

Proposition 2.3.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{X} \subseteq B(\mathcal{H})$ be a *-subalgebra. If $\xi \in \mathcal{H}$ is vector that is both tracial and cyclic for $\mathfrak{X}$, then ξ is also separating.*

Proof. Let $\mathfrak{X} \subseteq B(\mathcal{H})$ be a *-subalgebra and $\xi \in \mathcal{H}$ be a vector which is both tracial and cyclic for $\mathfrak{X}$. Let $x \in \mathfrak{X}$ be given such that $x\xi = 0$. Then for any pair of elements $y, z \in \mathfrak{X}$ we see that

$$\langle xy\xi, z\xi\rangle = \langle z^*xy\xi, \xi\rangle = \langle yz^*x\xi, \xi\rangle = 0.$$

Since ξ is also cyclic for $\mathfrak{X}$ we therefore find that

$$\langle x\eta, \eta'\rangle = 0,$$

for any pair of vectors $\eta, \eta' \in \mathcal{H}$, which clearly implies that $x = 0$. □

In the course of proving Proposition 2.3.3, we need the following theorem. The reader may consult [12, Theorem 7.3.6] for a proof.

Theorem 2.3.2 (Sakai-Radon-Nikodym). *Let κ_0, κ be positive, linear functionals on a von Neumann algebra $\mathscr{M}$, such that $\kappa_0 \leq \kappa$. Then there exists a positive element $H \in (\mathscr{M})_1$ such that $\kappa_0(T) = \kappa(HTH)$ for all $T \in \mathscr{M}$.*

Proposition 2.3.3. *Let $\mathscr{N}$ denote the weak operator closure of $\pi_\tau(A) \subseteq B(\mathcal{H}_\tau)$, where $\tau \in T(A)$ and $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ is the associated GNS-triple. Then there is a order-preserving one-to-one correspondence*

$$\Lambda : \{x \in Z(\mathscr{N}) \mid 0 \leq x \leq 1_{\mathscr{N}}\} \to \{\sigma \in A^* \mid \sigma \text{ is tracial and } 0 \leq \sigma \leq \tau\},$$

given by $x \mapsto \tau_x$, where $\tau_x \in A^*$ is given by

$$\tau_x(a) = \langle \pi_\tau(a) x \xi_\tau, \xi_\tau \rangle.$$

Proof. First of all, we see that τ gives rise to a normal, faithful tracial state $\tau_{\mathscr{N}}$ on $\mathscr{N}$. Indeed, for $a \in A$ we have that

$$\tau(a) = \langle \pi_\tau(a) \xi_\tau, \xi_\tau \rangle.$$

Thus we may let $\tau_{\mathscr{N}}$ be given by

$$\tau(z) = \langle z \xi_\tau, \xi_\tau \rangle, \quad z \in \mathscr{N}.$$

Since τ is a tracial state on A, $\pi_\tau(A)$ is weakly dense in $\mathscr{N}$ and multiplication is weakly continuous in each variable we see that $\tau_{\mathscr{N}}$ defines a normal, tracial state on $\mathscr{N}$. As $\xi_\tau \in \mathcal{H}_\tau$ is a vector which is both tracial and cyclic for $\mathscr{N}$ (since ξ_τ is cyclic for $\pi_\tau(A)$ it is also cyclic for $\mathscr{N}$), it follows from Proposition 2.3.1 that ξ_τ is separating for $\mathscr{N}$, i.e., that $\tau_{\mathscr{N}}$ is a normal, faithful tracial state on $\mathscr{N}$ (the reader should note that tracial state on $\pi_\tau(A)$ induced by τ is indeed faithful, although τ itself may not be faithful on A).

It is therefore clear, that Λ is a well-defined map, and also easy to check that it is injective. Indeed, assume that $\tau_x = \tau_y$. Since $\pi_\tau(A)$ is weakly dense in $\mathscr{N}$ and multiplication is weakly continuous in each variable, it follows that τ_x and τ_y also agree on $\mathscr{N}$. Furthermore, we see that $\tau_x(a) = \tau_{\mathscr{N}}(\pi_\tau(a))$. We therefore have that

$$\begin{aligned}\tau_{\mathscr{N}}((x-y)^*(x-y)) &= \tau_{\mathscr{N}}(x^2 - xy - yx - y^2) \\ &= \tau_x(x) - \tau_y(x) - \tau_x(y) + \tau_y(y) = 0,\end{aligned}$$

and since $\tau_{\mathscr{N}}$ is faithful, this proves that $x = y$.

Assume that σ is a positive, tracial functional on A such that $\sigma \leq \tau$. First, we show that there exists a vector $\eta \in \mathcal{H}_\tau$ such that

$$\sigma(a) = \langle \pi_\tau(a) \xi_\tau, \eta \rangle, \quad a \in A$$

Once this has been proven, we find that $\tilde{\sigma} : \mathscr{N} \to \mathbb{C}$ given by

$$\tilde{\sigma}(z) = \langle z \xi_\tau, \eta \rangle$$

is a positive, tracial functional $\tilde{\sigma}$ on $\mathscr{N}$ such that $\tilde{\sigma} \leq \tau_{\mathscr{N}}$ and $\tilde{\sigma}\big(\pi_\tau(a)\big) = \sigma(a)$.

Consider the map $\varphi_0 : \pi_\tau(A)\xi_\tau =: \mathcal{H}_0 \to \mathbb{C}$ given by $\varphi_0(\pi_\tau(x)\xi_\tau) = \sigma(x)$. Since ξ_τ is separating for $\pi_\tau(A)$ it follows that φ_0 is well-defined. It is obviously linear, and it is also bounded since, for any $a \in A$ we have

$$|\varphi(\pi_\tau(a)\xi_\tau)| = |\sigma(a)| \leq \sigma(a^*a)^{1/2}\sigma(1_A)^{1/2} \leq \tau(a^*a)^{1/2} = \|\pi_\tau(a)\xi_\tau\|.$$

Hence, φ_0 extends to a linear functional φ on $\mathcal{H}_\tau$ and we may find $\eta \in \mathcal{H}_\tau$ such that

$$\sigma(a) = \varphi(\pi_\tau(a)\xi_\tau) = \langle \pi_\tau(a)\xi_\tau, \eta\rangle,$$

as desired. Now we can apply Theorem 2.3.2 to obtain a positive element $x \in (\mathscr{N})_1$ such that for all $z \in \mathscr{N}$

$$\tilde{\sigma}(z) = \tau_{\mathscr{N}}(x^{1/2}zx^{1/2}) = \tau_{\mathscr{N}}(zx).$$

Clearly this implies that

$$\sigma(a) = \tilde{\sigma}(\pi_\tau(a))\tau_{\mathscr{N}}\big(\pi_\tau(a)x\big) = \langle \pi_\tau(a)x\xi, \xi\rangle,$$

for all $a \in A$, and all that remains is to prove that x is central in $\mathscr{N}$. However, this follows easily from the fact that $\tau_{\mathscr{N}}$ is faithful. Indeed, since $\tilde{\sigma}$ is a tracial functional, we see that for all $z \in \mathscr{N}$ we have

$$\begin{aligned}\tau_{\mathscr{N}}((zx - xz)^*(zx - xz)) &= \tau_{\mathscr{N}}(xz^*zx - xz^*xz - z^*xzx + z^*xxz) \\ &= \tau_{\mathscr{N}}(xz^*zx - xzz^*x - xz^*zx + xzz^*x) = 0\end{aligned}$$

which finishes the proof. □

Corollary 2.3.4. *Let $\mathscr{N}$ be given as before. Then $\mathscr{N}$ is a factor if and only if τ is an extremal tracial state.*

Proof. In view of Proposition 2.3.3 it is sufficient to prove that τ is an extremal tracial state if and only if it holds that for all positive tracial functionals $\sigma \leq \tau$ on A there exists $0 \leq \alpha \leq 1$ such that $\sigma = \alpha\tau$.

Assume that τ is an extremal tracial state and let σ be a positive tracial funtioncal such that $\sigma \leq \tau$. We may further assume that $\sigma \neq 0$ and $\tau - \sigma \neq 0$ (since otherwise the conclusion is trivial). Letting $\alpha = \sigma(1_A) \neq 0$ and $\beta = \tau(1_A) - \sigma(1_A) \neq 0$ we see that $\alpha + \beta = 1$ and

$$\tau = \alpha(\alpha^{-1}\sigma) + \beta(\beta^{-1}(\tau - \sigma)),$$

whence it follows that $\sigma = \alpha\tau$. On the other hand, if $\tau = (1-t)\tau_1 + t\tau_2$ for some $\tau_1, \tau_2 \in T(A)$ then $(1-t)\tau_1, t\tau_2 \leq \tau$ which proves the other implication. □

Now, the next series of results aims towards Theorem 2.3.10. The results should be well-known to readers who are well-versed in the theory of von Neumann algebras, and these may as well skip the proofs. However, they are included in the interest of completeness.

Proposition 2.3.5. *If $\mathscr{M}$ is a von Neumann algebra with a faithful, normal tracial state τ, and τ_x denotes the linear functional on $\mathscr{M}$ given by $\tau_x(y) = \tau(xy)$, then the map*

$$x \mapsto \tau_x, \quad (\mathscr{M}, \|\cdot\|_{1,\tau}) \to (\mathscr{M}^*, \|\cdot\|),$$

is a linear isometry with image contained in $\mathscr{M}_*$. Moreover, the image is dense in $\mathscr{M}_*$.

Proof. The described map is obviously linear, and it follows directly from Lemma 2.2.3 that it is also an isometry. It remains to show that the image is a dense subset of $\mathscr{M}_*$.

First we prove that $\tau_x \in \mathscr{M}_*$ for each $x \in \mathscr{M}$. To this end, let $(y_\alpha)_{\alpha\in A} \subseteq \mathscr{M}$ be a net converging to $y \in \mathscr{M}$ in the ultraweak topology. Since multiplication is ultraweakly continuous in each variable and τ is normal, we see that

$$\lim_\alpha \tau_x(y_\alpha) = \tau(\text{UW-}\lim_\alpha xy_\alpha) = \tau(xy) = \tau_x(y),$$

and hence τ_x is ultraweakly continuous.

Let $M \subseteq \mathscr{M}_*$ denote the closure of the image, and suppose towards a contradiction that $M \neq \mathscr{M}_*$. Choose some $\varphi \in \mathscr{M}_* \backslash M$. It follows from the Hahn-Banach Separation Theorem that we may choose an element $y \in (\mathscr{M}_*)^* \cong \mathscr{M}$ (see Theorem 1.1.6) such that $\varphi(y) \neq 0$ and $\psi(y) = 0$ for all $\psi \in M$. Let $y = v|y|$ be the polar decomposition of y in $\mathscr{M}$. Then, since $\tau_{v^*} \in M$, we have that

$$\tau_{v^*}(y) = \tau(v^*y) = \tau(|y|) = 0.$$

Since τ is faithful, we have that $|y| = 0$ and therefore that $y = 0$. But since $\varphi(y) \neq 0$ this is a contradiction, thus completing the proof. □

Proposition 2.3.6. *Let $\mathscr{M}$ be a von Neumann algebra with a faithful, normal tracial state τ. Then the strong operator topology and the topology induced by $\|\cdot\|_{2,\tau}$ agree on bounded sets.*

Proof. Note that it is sufficient to prove that these topologies agree on $(\mathscr{M})_1$, since the general case follows by scaling.

Let $(T)_{\lambda\in\Lambda}$ be a net in $(\mathscr{M})_1$ converging to T in strong operator topology. By the polarization identity it follows that $(T_\lambda - T)^*(T_\lambda - T)$ converges to 0 in the weak operator topology. Since the weak operator topology and the ultraweak topology agree on bounded sets (See [12, Remark 7.4.4]) and

$\|(T_\lambda - T)^*(T_\lambda - T)\| \leq 4$ for all λ it follows that $(T_\lambda - T)^*(T_\lambda - T)$ converges to 0 in the ultraweak topology. Since τ is normal we therefore deduce that $\|T_\lambda - T\|_{2,\tau}$ also converges to 0.

On the other hand, let $(T_\lambda)_{\lambda\in\Lambda}$ be a net converging to T in $\|\cdot\|_{2,\tau}$. Let $y \in \mathscr{M}$ and consider the linear functional τ_y on $\mathscr{M}$ as given in the statement of Proposition 2.3.5. Then τ_y is ultraweakly continuous and if y is self-adjoint we see that

$$\begin{aligned}\tau_y\big((T_\lambda - T)^*(T_\lambda - T)\big) &= \tau\big((T_\lambda - T)^*(T_\lambda - T)y\big)\\ &\leq \|y\|\|T_\lambda - T\|_{2,\tau}^2,\end{aligned}$$

whence $\tau_y\big((T_\lambda - T)^*(T_\lambda - T)\big)$ converges to 0. As the self-adjoint elements span $\mathscr{M}$ it follows that the same is true for any $y \in \mathscr{M}$. Furthermore, the set $\{\tau_y \mid y \in \mathscr{M}\} \subseteq \mathscr{M}_*$ is norm-dense and therefore $(T_\lambda - T)^*(T_\lambda - T)$ converges to 0 in the ultraweak topology. Almost the same arguments as before then yields that T_λ converges to T in the strong operator topology. □

Proposition 2.3.7. *Suppose that $\mathcal{H}$ is a Hilbert space and $A \subseteq B(\mathcal{H})$ is a unital C^*-subalgebra equipped with a normal, faithful tracial state τ. If $(A)_1$ is complete in $\|\cdot\|_{2,\tau}$ then A is a von Neumann algebra.*

Proof. Let $\mathscr{M}$ denote the ultraweak closure of A and extend τ to a normal, faithful trace on $\mathscr{M}$. Let $(T_\lambda)_{\lambda\in\Lambda}$ be a net in $(A)_1$ converging to $T \in \mathscr{M}$ in the strong operator topology. Then it follows from Proposition 2.3.6 that $\|T_\lambda - T\|_{2,\tau}$ converges to 0. Therefore $(T_\lambda)_\lambda$ is Cauchy in $\|\cdot\|_{2,\tau}$ which, by assumption, implies that the net converges to some operator in $(A)_1$ and of course this operator must be T. It follows that the closed unit ball of A is closed in the strong operator topology and therefore A is a von Neumann algebra. □

Lemma 2.3.8. *A von Neumann algebra $\mathscr{M}$ with a faithful tracial state τ is a factor if and only if for each non-zero projection $p \in \mathscr{M}$ with $\tau(p) \leq \frac{1}{2}$, we have $p \preceq 1 - p$.*

Proof. Assume that $\mathscr{M}$ is a factor. Then either $p \preceq 1 - p$ or $1 - p \preceq p$. But since $\tau(p) \leq \frac{1}{2}$ we have that $\tau(1-p) \geq \tau(p)$ and therefore, if $\tau(p) < \frac{1}{2}$ the latter cannot be the case. If $\tau(p) = \frac{1}{2}$, it follows that $\tau(p) = \tau(1-p)$ and therefore $p \simeq 1 - p$.

Assume that $\mathscr{M}$ is not a factor and let p be a non-trivial central projection in $\mathscr{M}$. We may assume that $\tau(p) \leq \frac{1}{2}$ (otherwise replace p with $1_{\mathscr{M}} - p$). Let q be a projection such that $q \simeq p$ and choose a partial isometry such that $v^*v = p$ and $vv^* = q$. As p is central we see that $q = vv^* = vp^2v^* = pvv^*p \leq p$, which shows that $q \not\leq 1 - p$, since p is non-trivial. Thus it is not possible that $p \preceq 1 - p$, which finishes the proof. □

Proposition 2.3.9. *Suppose that $\mathscr{M}$ is a von Neumann algebra, with at least one tracial state. Then each projection in $\mathscr{M}^\omega$ lifts to a projection in $\ell^\infty(\mathscr{M})$.*

Proof. Suppose that $p \in \mathscr{M}^\omega$ is a projection and let $(x_n)_n \in \ell^\infty(\mathscr{M})$ be a positive lift. Let $q_n = 1_{[\frac{1}{2},\infty)}(x_n)$ and $q = [(q_n)_n]$. Since $\frac{1}{2}q_n \leq x_n$ and $\|(1-q_n)x_n\| \leq \frac{1}{2}$ for each $n \in \mathbb{N}$, we see that

$$2^{-1}q \leq p, \quad \|(1-q)p\| \leq \frac{1}{2}.$$

Furthermore, the function $t \mapsto t^{1/2}$ is operator monotone, and therefore $2^{-1/2}q \leq p$. Iterating this argument we see that $2^{-1/2^n}q \leq p$ for all $n \in \mathbb{N}$ whence $q \leq p$. Therefore $p-q$ is a projection and since $\|p-q\| = \|(1-q)p\| \leq \frac{1}{2}$ we conclude that $q = p$. □

Theorem 2.3.10. *Suppose that A is unital C^*-algebra equipped with a tracial state τ. Then the tracial ultrapower A^ω_τ is a finite von Neumann algebra.*

Proof. It is clear that τ induces a faithful tracial state τ_ω on A^ω_τ by setting $\tau_\omega\big([(a_n)_n]\big) := \lim_{n\to\omega}\tau(a_n)$, which is well-defined since ω is an ultrafilter. Upon identifying A^ω_τ with $\pi_{\tau_\omega}(A^\omega_\tau) \subseteq B(\mathcal{H}_{\tau_\omega})$, where $(\pi_{\tau_\omega}, \mathcal{H}_{\tau_\omega}, \xi_{\tau_\omega})$ is the GNS-triple associated with τ_ω, we may furthermore assume that $A^\omega_\tau \subseteq B(\mathcal{H}_{\tau_\omega})$ is a unital subalgebra and that τ_ω is normal. Therefore, by Proposition 2.3.7, it suffices to prove that the closed unit ball of A^ω_τ is complete in $\|\cdot\|_{2,\tau_\omega}$. Once this has been proven, the fact that A^ω_τ is a finite von Neumann algebra is implied by the existence of a faithful tracial state.

Let $(x_n)_n \subseteq (A^\omega_\tau)_1$ be a Cauchy sequence with respect to $\|\cdot\|_{2,\tau_\omega}$. It suffices to prove that $(x_n)_n$ has a convergent subsequence, and we may therefore assume that $\|x_n - x_{n+1}\|_{2,\tau_\omega} < 2^{-n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we may choose lifts $\big(x_n^{(k)}\big)_k \in \ell^\infty(A)$ such that $\sup_k \|x_n^{(k)}\| \leq 1$. By assumption, each of the sets

$$F_n := \{l \in \mathbb{N} \mid \|x_k^{(l)} - x_{k+1}^{(l)}\|_{2,\tau} < 2^{-k}, k = 1,\dots,n\}$$

belongs to ω. Furthermore, upon replacing F_n with $F_n \cap \{n, n+1, \dots\}$ (these sets still belong to ω, since ω is a free filter), we may assume that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. Setting $F_0 = \mathbb{N}$ we see that

$$\mathbb{N} = \bigsqcup_{n=0}^{\infty} F_n \backslash F_{n+1}.$$

Set $x^{(l)} = x_n^{(l)}$, when $l \in F_n \backslash F_{n+1}$ and $x = [(x^{(l)})_l] \in A^\omega_\tau$. We aim to show that x is precisely the limit of $(x_n)_n$, by proving that $\|x_n - x\|_{2,\tau_\omega} < 2^{1-n}$. Note that

$$\|x_n - x\|_{2,\tau_\omega} = \lim_{l\to\omega} \|x_n^{(l)} - x^{(l)}\|_{2,\tau}.$$

Let $l \in F_n$ be given. It follows that there exists $k \geq n$ such that $l \in F_k \backslash F_{k+1}$. If $k = n$, then $x^{(l)} = x_n^{(l)}$, whence $\|x_n^{(l)} - x^{(l)}\|_{2,\tau} = 0$. If $k > n$ we see that

$$\begin{aligned}
\|x_n^{(l)} - x_n\|_\tau &= \|x_n^{(l)} - x_k^{(l)}\|_{2,\tau} \\
&= \left\| \sum_{m=n}^{k-1} x_m^{(l)} - x_{m+1}^{(l)} \right\|_{2,\tau} \\
&\leq \sum_{m=n}^{k-1} \|x_m^{(l)} - x_{m+1}^{(l)}\|_{2,\tau} \\
&< \sum_{m=n}^{k-1} 2^{-m} < 2^{1-n}.
\end{aligned}$$

Thus we see that $\|x_n^{(l)} - x^{(l)}\|_{2,\tau} < 2^{1-n}$ for all $l \in F_n$, which completes the proof. □

It is easy to see that, if τ is a faithful tracial state, we can use $\mathscr{N} := \pi_\tau(A)''$ instead of A when constructing A_τ^ω, i.e., it follows from the above that $\mathscr{N}_\tau^\omega$ is also a von Neumann algebra, and from Kaplanski's density theorem (see for instance [34, Theorem 19.5]) that

$$A_\tau^\omega \cong \mathscr{N}_\tau^\omega.$$

Indeed, consider the following commutative diagram:

$$\begin{array}{ccc}
\ell^\infty(A) & \xrightarrow{\tilde{\pi}_\tau} & \ell^\infty(\mathscr{N}) \\
\downarrow & & \downarrow \Phi \\
A_\tau^\omega & \longrightarrow & \mathscr{N}_\tau^\omega,
\end{array}$$

where $\tilde{\pi}_\tau$ is the map induced by $\pi_\tau : A \to \pi_\tau(A)''$. Let $x \in \mathscr{N}_\tau^\omega$ be positive and let $(x_n)_n \in \ell^\infty(\mathscr{N})$ be a positive lift. Since $\|x_n\|_{2,\tau} = \|x_n \xi_\tau\|$, it follows from Kaplanski's density theorem that for each $n \in \mathbb{N}$ we may choose $a_n \in A$ such that $\|a_n\| = \|\pi_\tau(a_n)\| \leq \|x_n\|$ and $\|\pi_\tau(a_n) - x_n\|_{2,\tau} < \frac{1}{n}$. Therefore $\Phi\big((\pi_\tau(a_n))_n\big) = \Phi\big((x_n)_n\big)$, which shows that $\Phi \circ \tilde{\pi}_\tau$ is surjective. Let $I := \ker(\Phi \circ \tilde{\pi}_\tau)$. By construction, we see that $J_\tau = \pi_\omega(I)$, which shows that $A_\tau^\omega \cong \mathscr{N}_\tau^\omega$.

Theorem 2.3.11. *Suppose that $\mathscr{M}$ is a von Neumann algebra equipped with a faithful, normal tracial state τ. If $\mathscr{M}$ is a factor, then $\mathscr{M}^\omega$ is a finite factor. Moreover, in this case we can determine the type of $\mathscr{M}^\omega$ as follows:*

(i) The tracial ultrapower $\mathscr{M}^\omega$ is a II_1-factor if and only if $\mathscr{M}$ is a II_1-factor.

(ii) The tracial ultrapower $\mathscr{M}^\omega$ is a I_n-factor if and only if $\mathscr{M}$ is a I_n-factor.

Proof. Note that it follows from the assumptions that $\mathscr{M}$ is a finite factor, and thus comes equipped with a unique tracial state. Thus $\mathscr{M}^\omega = \mathscr{M}^\omega_\tau$ and we therefore suppress τ in notation. We use Lemma 2.3.8 to prove that $\mathscr{M}^\omega$ is a factor.

Let $p \in \mathscr{M}^\omega$ be a non-zero projection and $(p_n)_n \in \ell^\infty(\mathscr{M})$ be a lift consisting of projections (see Proposition 2.3.9). We may assume, possibly after substituting $1-p$ for p and $1-p_n$ for p_n, that $F := \{n \in \mathbb{N} \mid \tau(p_n) \leq \frac{1}{2}\}$ belongs to ω. Thus, by Lemma 2.3.8 we find that $p_n \preceq 1_{\mathscr{M}} - p_n$ for each $n \in F$. That is, for each $n \in F$ we may find partial isometries $v_n \in \mathscr{M}$ such that

$$v_n^* v_n = p_n, \quad v_n v_n^* \leq 1_{\mathscr{M}} - p_n.$$

For each $n \in \mathbb{N} \backslash F$ let $v_n = 0$, and set $v = [(v_n)_n]$. Then it is easy to see that $v \in \mathscr{M}^\omega$ is a partial isometry such that $v^*v = p$, $vv^* \leq 1 - p$, thus proving that $\mathscr{M}^\omega$ is a factor. Since τ induces a faithful tracial state on $\mathscr{M}^\omega$, we find that $\mathscr{M}^\omega$ must also be finite.

As $\mathscr{M}$ is a finite factor, it will either be a II_1 factor or a I_k factor for some $k \in \mathbb{N}$. Assume that the former is the case. Then for any $n \in \mathbb{N}$ we may find mutually orthogonal, non-zero projections $p_1, \dots, p_n \in \mathscr{M}$. Letting $\iota : \mathscr{M} \to \mathscr{M}^\omega$ denote the canonical embedding, we obtain that $\iota(p_1), \dots, \iota(p_n) \in \mathscr{M}^\omega$ are mutually orthogonal, non-zero projections, and therefore $\mathscr{M}^\omega$ cannot be a I_k factor for $k < n$. Since this is valid for any $n \in \mathbb{N}$ we see that $\mathscr{M}^\omega$ is a II_1-factor.

Assume that $\mathscr{M}$ is a I_k factor for some $k \in \mathbb{N}$ and let a projection $p \in \mathscr{M}^\omega$ be given. Let $(p_n) \in \ell^\infty(\mathscr{M})$ be a lift consisting of projections. As $\mathscr{M}$ is a I_k-factor we see that $\tau(p_n) \geq \frac{1}{k}$ or $\tau(p_n) = 0$ for each $n \in \mathbb{N}$ and therefore $\tau_\omega(p) \geq \frac{1}{k}$ or $\tau_\omega(p) = 0$ and it follows that $\mathscr{M}^\omega$ cannot be a II_1 factor or I_n factor for any $n > k$, as both types of von Neumann algebras contain projections with trace value strictly between $\frac{1}{k}$ and 0. On the other hand, with virtually the same argument as in the II_1 case, $\mathscr{M}^\omega$ cannot be a I_n factor for any $n < k$ which shows that $\mathscr{M}^\omega$ is a I_k factor.

In fact this proof is hereby complete. Indeed, as $\mathscr{M}^\omega$ is a finite factor, it must either be a type I_k-factor for some $k \in \mathbb{N}$, or a type II_1-factor, and these are mutually exclusive possibilities. Hence, the reverse implications have automatically been established. □

Corollary 2.3.12. *Let A be a unital and infinite-dimensional C^*-algebra and τ an extremal tracial state which is also faithful. Then A^ω_τ is a type II_1 von Neumann algebra factor.*

Proof. Since A is infinite-dimensional and τ is faithful, it follows from Corollary 2.3.4 that $\mathscr{N} := \pi_\tau(A)'' \subseteq B(\mathcal{H}_\tau)$ is a II_1-factor. Thus, it follows from

Theorems 2.3.10 and 2.3.11 along with the comments made before the statement of the latter theorem, that $A_\tau^\omega \cong \mathscr{N}^\omega$ is a II_1-factor. □

2.4 The Uniform Tracial Ultrapower

We now seek to determine the uniform tracial ultrapower of a C^*-algebra in the case where $\partial_e T(A)$ is a finite set. First however, we need some basic facts concerning the ideal lattice in a C^*-algebra.

Proposition 2.4.1. *The set $\mathcal{I}(A) := \{I \mid I \text{ is an ideal in } A\}$ is a distributive lattice with operations $I \wedge J := I \cap J$ and $I \vee J := I + J$.*

Proof. It is easy to see that the indicated operations are well-defined and determine a lattice structure on $\mathcal{I}(A)$. Indeed, it is obvious that $I \cap J$ is an ideal and that $I + J$ is an algebraic ideal, for any pair of ideals $I, J \subseteq A$. Letting $\pi : A \to A/I$ denote the quotient, we see that $I + J = \pi^{-1}(\pi(J))$, whence $I + J$ also defines an ideal, for any pair of ideals.

We only show that $(I_1 \wedge I_2) \vee I_3 = (I_1 \vee I_3) \wedge (I_2 \vee I_3)$ as the other case is proven in a very similar way. Given an element $a \in (I_1 \wedge I_2) \vee I_3$, we may write $a = x + y$ where $x \in I_1 \cap I_2$ and $y \in I_3$. Therefore

$$x + y = \frac{x+y}{2} + \frac{x+y}{2} \in (I_1 \vee I_3) \wedge (I_2 \vee I_3)$$

thus proving one inclusion. On the other hand if $0 \leq x \in (I_1 \vee I_3) \wedge (I_2 \vee I_3)$, then we may write $x = y_1 + z_1 = y_2 + z_2$, where $y_1 \in I_1$, $y_2 \in I_2$ and $z_1, z_2 \in I_3$. Therefore

$$x^2 = y_1 y_2 + (y_1 z_2 + z_1 y_2 + z_1 z_2) \in (I_1 \wedge I_2) \vee I_3,$$

which implies that $x \in (I_1 \wedge I_2) \vee I_3$, thereby proving the other inclusion. □

Corollary 2.4.2. *If $I_1, \ldots, I_n \in \mathcal{I}(A)$ are different maximal ideals and $I = \bigcap_{k=1}^n I_k$ then*

$$A/I \cong \bigoplus_{k=1}^n A/I_k.$$

Proof. Let $\varphi : A \to \bigoplus_{k=1}^n A/I_k$ be given by

$$\varphi(a) = (a + I_1, a + I_2, \ldots, a + I_n).$$

Clearly $I = \ker \varphi$ and we therefore obtain an embedding $\tilde{\varphi} : A/I \to \bigoplus_{k=1}^n A/I_k$. Once we have shown that φ is surjective, the proof will be finished.

Let $(a_1 + I_1, \ldots, a_n + I_n) \in \bigoplus_{k=1}^n A/I_k$ be given. Note that since each I_k is a maximal ideal, it follows from Proposition 2.4.1 that

$$(I_1 \cap \cdots \cap I_{j-1}) + I_j = A,$$

for each $2 \leq j \leq n$. Therefore, fix $2 \leq j \leq n$ assume that we can find $\tilde{a} \in A$ such that

$$(a_1 + I_1, \ldots, a_n + I_n) = (\tilde{a} + I_1, \ldots, \tilde{a} + I_{j-1}, a_j + I_j, \ldots, a_n + I_n)$$

Then we may write

$$\tilde{a} - a_j = x + y, \quad x \in (I_1 \cap \cdots \cap I_{j-1}),\ y \in I_j.$$

Setting $a := \tilde{a} - x = a_j + y$ we find that

$$\begin{aligned} &(a + I_1, \ldots, a + I_{j-1}, a + I_j, a_{j+1} + I_{j+1}, \ldots, a_n + I_n) \\ =&(\tilde{a} + I_1, \ldots, \tilde{a} + I_{j-1}, a_j + I_j, a_{j+1} + I_{j+1}, \ldots, a_n + I_n). \end{aligned}$$

It clearly follows that φ is surjective. □

Now we are ready to determine the uniform tracial ultrapower of A in certain cases. The first result is a trivial observation, which is nonetheless very useful.

Proposition 2.4.3. *Let A be a unital C^*-algebra and assume that $0 < |\partial_e T(A)| < \infty$. Then for any $a = \pi_\omega\big((a_n)_n\big) \in A_\omega$ we have that*

$$\|a\|_{2,\omega} = \max_{\tau \in \partial_e T(A)} \lim_{n \to \omega} \|a_n\|_{2,\tau}.$$

In particular, we have that

$$J_A = \bigcap_{\tau \in \partial_e T(A)} J_\tau.$$

Proof. To ease notation, let $S = \partial_e T(A) = \{\tau_1, \tau_2, \ldots, \tau_n\}$. As $\|a\|_{2,\tau_i} \leq \max_{\tau \in S} \|a\|_{2,\tau}$ for each $1 \leq i \leq n$ and every $a \in A$, we easily deduce (see Remark 2.2.2) that for any $a = \pi_\omega\big((a_n)_n\big)$ we have

$$\|a\|_{2,\omega} = \lim_{n \to \omega} \|a_n\|_{2,S} \geq \max_{\tau \in S} \lim_{n \to \omega} \|a_n\|_{2,\tau},$$

proving one inequality. To see the other, let $a = \pi_\omega\big((a_n)_n\big)$ and note that for each $1 \leq i \leq n$ and $\varepsilon > 0$ we may find $X_i \in \omega$ such that

$$\|a_k\|_{2,\tau_i} < \max_{\tau \in S} \lim_{n \to \omega} \|a_n\|_{1,\tau} + \varepsilon,$$

for all $k \in X_i$. Hence, for all $k \in \bigcap_{i=1}^n X_i \in \omega$ we have that

$$\max_{\tau \in S} \|a_k\|_{2,\tau} < \max_{\tau \in S} \lim_{n \to \omega} \|a_n\|_{2,\tau} + \varepsilon,$$

which proves the second inequality.

The second statement in the proposition follows trivially from the first. □

Theorem 2.4.4. *Let A be a unital C^*-algebra, with $\partial_e T(A) = \{\tau_1, \tau_2, \dots, \tau_n\}$. Then the uniform tracial ultrapower A^ω is isomorphic to a finite direct sum of finite factors. More precisely:*

$$A^\omega \cong \bigoplus_{i=1}^{n} A^\omega_{\tau_i}$$

Proof. To ease notation, we let $J_i := J_{\tau_i}$ for each $1 \leq i \leq n$. Note that it follows from Corollary 2.3.12 that for each $1 \leq i \leq n$ we have that $A^\omega_{\tau_i}$ is a finite factor with the unique tracial state $\tau^{(i)}_\omega\big([(a_n)_n]\big) = \lim_{n\to\omega} \tau_i(a_n)$. This immediately yields that if $J_i = J_j$ then $\tau_i = \tau_j$. Indeed, if $J_i = J_j$ then both $\tau^{(i)}_\omega$ and $\tau^{(j)}_\omega$ define faithful tracial states on $A^\omega_{\tau_i}$, whence it follows that $\tau^{(i)}_\omega = \tau^{(j)}_\omega$. In particular we see that for all $a \in A$ we have

$$\tau_i(a) = \tau^{(i)}_\omega\big(\iota(a)\big) = \tau^{(j)}_\omega\big(\iota(a)\big) = \tau_j(a),$$

where $\iota : A \to A^\omega_{\tau_i}$ denotes the canonical inclusion. Therefore $\tau_i = \tau_j$.

Next we prove that every finite factor is a simple C^*-algebra. This will imply that each J_i is a maximal ideal in A_ω and then Corollary 2.4.2 will yield the desired result. Note that since any type I_k-factor is a full matrix algebra, the result is trivial in this case. Hence, we concentrate on the II_1-factor case.

Let $\mathscr{M}$ be any II_1-factor, $I \neq 0$ be a norm closed ideal in $\mathscr{M}$ and let τ denote the unique tracial state on $\mathscr{M}$. Since $I \neq 0$ we may find an element $a \in I$ of norm 1. Let $p = 1_{[\frac{1}{2},1]}(a^*a)$ and note that $a^*a \geq \frac{1}{2}p$. As ideals are hereditary, this implies that $p \in I$, and, since $p \neq 0$, that $\tau(p) > 0$. If $q \in \mathscr{M}$ is any projection such that $\tau(q) \leq \tau(p)$, then q is subequivalent with p, i.e., there exists a partial isometry $v \in \mathscr{M}$ such that $v^*v = q$ and $vv^* \leq p$. It follows that $pv = v$, whence $v \in I$ and therefore that $q \in I$. Since $\mathscr{M}$ is a II_1-factor we may for each $\varepsilon > 0$ find projections $p_1, \dots, p_m \in \mathscr{M}$ such that $\tau(p_k) \leq \varepsilon$ for all $1 \leq k \leq m$ and $p_1 + \cdots + p_m = 1_{\mathscr{M}}$ which finally implies that $1_{\mathscr{M}} \in I$ and therefore that $I = \mathscr{M}$. □

2.5 The Tracial Central Sequence Algebra

In the remaining sections we aim to give an introduction, of sorts, to central sequence algebras of C^*-algebras and von Neumann algebras and to introduce property (SI). This property, while technical in nature, has proven fundamental in showing that strict comparison of positive elements implies both Jiang-Su stability and finite decomposition rank, under certain assumptions naturally. Loosely speaking, property (SI) is a tool for lifting properties of the tracial central sequence algebra to related properties of the metric central sequence algebra. Therefore, we start by establishing the properties of the tracial central sequence algebra that we are interested in.

For the rest of this chapter, we assume the following setup: Let A and B be unital, separable C^*-algebras such that B is nuclear, $T(A) \neq \emptyset$ and there exists a unital embedding

$$\pi : B \to A_\omega.$$

We will add further conditions to A and B along the way to obtain the desired results, but always with the above setup in mind. We let $F\big(A, \pi(B)\big) := A_\omega \cap \pi(B)'$ and $F(A) := F\big(A, \iota_\omega(A)\big)$, to ease notation and obtain a cleaner aesthetic. We refer to $F\big(A, \pi(B)\big)$ as the metric central sequence algebra of A relative to $\pi(B)$ and $F(A)$ as the metric central sequence algebra of A (or more commonly, simply *the* central sequene algebra).

Recall that A^ω is, by definition, a quotient of A_ω (see Definition 2.2.6), and we let $\Phi : A_\omega \to A^\omega$ denote the quotient map. Note that since B is assumed nuclear, we may lift π to a ucp. map $\tilde{\pi} : B \to \ell^\infty(A)$ such that $\pi_\omega \circ \tilde{\pi} = \pi$. We may therefore find a sequence of ucp. maps $\pi_n : B \to A$ such that $\pi_\omega\big((\pi_n(b))_n\big) = \pi(b)$ for every $b \in B$.

Property (SI) will only be applied, at least in this thesis, along with the following theorem.

Theorem 2.5.1. *Let A and B be unital, separable C^*-algebras such that B is nuclear and $T(A) \neq \emptyset$, and suppose there exists a unital embedding $\pi : B \to A_\omega$. Then the restriction of the quotient map $\Phi : A_\omega \cap \pi(B)' \to A^\omega \cap \Phi\big(\pi(B)\big)'$ is surjective.*

Proof. Let $\tilde{\Phi} : \ell^\infty(A) \to A^\omega$ denote the composition $\Phi \circ \pi_\omega$ and fix an element $a = (a_1, a_2, \dots) \in \ell^\infty(A)$ such that $\tilde{\Phi}(a) \in A^\omega \cap \Phi\big(\pi(B)\big)'$. It is sufficient to prove that there exists $c \in \ell^\infty(A)$ such that $\pi_\omega(c) \in A_\omega \cap \pi(B)'$ and $\tilde{\Phi}(c) = \tilde{\Phi}(a)$.

Let $D := C^*(\tilde{\pi}(B), a) \subseteq \ell^\infty(A)$ and $J := D \cap \ker \tilde{\Phi}$. Note that $x = (x_1, x_2, \dots) \in \ell^\infty(A)$ belongs to $\ker \tilde{\Phi}$ if and only if $\lim_{n\to\omega} \|x_n\|_2 = 0$ and that for any $b \in B$ we have that $a\tilde{\pi}(b) - \tilde{\pi}(b)a \in J$. Let $(e^\lambda)_{\lambda\in\Lambda}$ be a quasicentral approximate unit for J in D. Then for all $b \in B$ we have that

$$\begin{aligned} 0 &= \big\| \tilde{\Phi}\big(a\tilde{\pi}(b) - \tilde{\pi}(b)a\big) \big\| && (2.2)\\ &= \lim_\lambda \big\| (1-e^\lambda)\big(a\tilde{\pi}(b) - \tilde{\pi}(b)a\big)(1-e^\lambda) \big\| && (2.3)\\ &= \lim_\lambda \|(1-e^\lambda)a(1-e^\lambda)\tilde{\pi}(b) - \tilde{\pi}(b)(1-e^\lambda)a(1-e^\lambda)\| && (2.4) \end{aligned}$$

We aim to use the above along with the ε-test to reach the desired conclusion. To this end, let $(b_k)_k$ be a dense sequence in B and X_n be the set of positive contractions in A. Now, let:

$$\begin{aligned} f_n^{(1)}(x) &= \|x\|_2;\\ f_n^{(k+1)}(x) &= \|(1-x)a_n(1-x)\pi_n(b_k) - \pi_n(b_k)(1-x)a_n(1-x)\|, \ k \geq 1. \end{aligned}$$

Since $(e^\lambda)_{\lambda\in\Lambda} \subseteq J$, we see that $f_\omega^{(1)}(e^\lambda) = \lim_{n\to\omega} \|e_n^\lambda\|_2 = 0$ for all $\lambda \in \Lambda$, where $e^\lambda = (e_1^\lambda, e_2^\lambda, \dots)$. Furthermore, it follows from (2.4) that an appropriate choice of $\lambda \in \Lambda$ corresponding to $m \in \mathbb{N}$ and $\varepsilon > 0$ leads to fulfillment of the conditions in the ε-test. That is, by choosing λ large enough we may ensure that

$$f_\omega^{(k)}(e_1^\lambda, e_2^\lambda, \dots) < \varepsilon$$

for all $2 \leq k \leq m$ and any $\varepsilon > 0$. Hence we may choose a positive contraction $e \in D$ such that $f_\omega^{(k)}(e) = 0$ for all $k \in \mathbb{N}$. In particular $e \in J$, since $f_\omega^{(1)}(e) = 0$.

Let $c = (1-e)a(1-e)$, and note that, since $e \in J$ we have that $\tilde\Phi(c) = \tilde\Phi(a)$. Furthermore, since $f_\omega^{(k+1)}(e) = 0$ for all $k \geq 1$, we see that

$$\pi_\omega\big(c\tilde\pi(b_k) - \tilde\pi(b_k)c\big) = 0$$

for all $k \geq 1$, which implies that $\pi_\omega(c) \in A_\omega \cap \pi(B)'$. □

We refer to $A^\omega \cap \Phi\big(\pi(B)\big)'$ as the tracial central sequence algebra of A relative to $\pi(B)$, and the above theorem states that we may regard $A^\omega \cap \Phi\big(\pi(B)\big)'$ as a quotient of $A_\omega \cap \pi(B)'$ in a natural way. Note, that if $B = A$ and $\pi : A \to A_\omega$ is the canonical inclusion, then $A^\omega \cap \Phi\big(\pi(A)\big)' = A^\omega \cap A'$, where A is identified with a subalgebra A^ω in the obvious way. In case $\partial_e T(A)$ is a finite set and A is simple (thus assuring that all tracial states are faithful), we may go a step further and describe $A^\omega \cap A'$ quite explicitely, as witnessed by the next proposition.

Proposition 2.5.2. *Let A be a unital, simple, separable and nuclear C^*-algebra with $\partial_e T(A) = \{\tau_1, \dots, \tau_n\}$ and let $\mathscr{N}_i$ denote the strong operator closure of $\pi_{\tau_i}(A) \subseteq B(\mathcal{H}_{\tau_i})$. Then the natural map*

$$\varphi : A^\omega \cap A' \to \bigoplus_{i=1}^n (\mathscr{N}_i^\omega \cap \mathscr{N}_i')$$

is an isomorphism. In other words

$$F(A)/(F(A) \cap J_A) \cong \bigoplus_{i=1}^n (\mathscr{N}_i^\omega \cap \mathscr{N}_i')$$

Proof. It follows from Theorem 2.5.1 that $F(A)/(F(A) \cap J_A) \cong A^\omega \cap A'$, since $\Phi \circ \iota_\omega : A \to A^\omega$ is just the canonical inclusion of A in A^ω. Thus the second statement above follows from the first.

We saw in the last section (see the comments before Theorem 2.3.11) that $\mathscr{N}_i^\omega \cong A_{\tau_i}^\omega$ via the map induced by $\pi_{\tau_i} : A \to \mathscr{N}_i$. It therefore follows that this map restricts to an isomorphism

$$A_{\tau_i}^\omega \cap A' \cong \mathscr{N}_i^\omega \cap \pi_{\tau_i}(A)' = \mathscr{N}_i^\omega \cap \mathscr{N}_i',$$

where the last equality follows from the fact that $\pi_{\tau_i}(A) \subseteq \mathscr{N}_i$ is dense in $\|\cdot\|_2$. Furthermore, also from the last section (Theorem 2.4.4), we have that the map $\varphi : A^\omega \to \bigoplus_{i=1}^n A^\omega_{\tau_i}$, given by

$$\varphi\big(\Phi(a)\big) = \big(\Phi_1(a), \dots, \Phi_n(a)\big),$$

is an isomorphism, where $\Phi_i : A_\omega \to A^\omega_{\tau_i}$ denotes the quotient for each $1 \le i \le n$. Thus, φ restricts to an isomorphism

$$A^\omega \cap A' \cong \Big(\bigoplus_{i=1}^n A^\omega_{\tau_i}\Big) \cap \varphi(A)' = \bigoplus_{i=1}^n (A^\omega_{\tau_i} \cap A').$$

These two observations in combination yield the desired isomophism. $\square$

The properties of $A^\omega \cap \Phi\big(\pi(B)\big)'$ we shall be interested in are stated in the next two propositions.

Proposition 2.5.3. *The central sequence algebra of the hyperfinite* II_1-*factor* $\mathscr{R}$ *contains a unital copy of* M_2.

In particular, if A *is a unital, simple, separable, nuclear and infinite-dimensional* C^**-algebra with* $0 < |\partial_e T(A)| < \infty$*, then* M_2 *embeds unitally in* $A^\omega \cap A'$.

Proof. Let $(\mathscr{N}_k)_{k\in\mathbb{N}}$ be an increasing sequence of unital von Neumann subalgebras of $\mathscr{R}$ such that $\mathscr{N}_k$ is a type I_{2^k}-factor for each $k \in \mathbb{N}$ and $\bigcup_{k\in\mathbb{N}} \mathscr{N}_k \subseteq \mathscr{R}$ is strong operator dense. It follows from Kaplansky's Density Theorem and Proposition 2.3.6 that $\bigcup_{k\in\mathbb{N}} \mathscr{N}_k \subseteq \mathscr{R}$ is dense in the $\|\cdot\|_2$ topology.

Since $\mathscr{N}_k \otimes M_2 \cong \mathscr{N}_{k+1}$, we may find a unital copy of M_2 in $\mathscr{N}_{k+1}$ which commutes with $\mathscr{N}_k$. For each $k \in \mathbb{N}$, let $\big(e^{(k)}_{ij}\big) \subseteq \mathscr{N}_{k+1}$ denote a set of matrix units for this copy of M_2. It follows that the elements

$$e_{ij} := \Big[\big(e^{(k)}_{ij}\big)_k\Big] \in \mathscr{R}^\omega$$

defines a set of matrix units for M_2, which gives rise to a unital embedding $M_2 \to \mathscr{R}^\omega$. It remains to see that each matrix unit is central.

By construction we have that $\lim_{n\to\omega} \big\| \big[x, e^{(n)}_{ij}\big] \big\|_2 = 0$ for each $x \in \bigcup_{k\in\mathbb{N}} \mathscr{N}_k$ (in fact $[x, e^{(n)}_{ij}] = 0$ for large enough n). Let $y \in \mathscr{R}$ be arbitrary and choose $x \in \bigcup_{k\in\mathbb{N}} \mathscr{N}_k$ such that $\|x - y\|_2 < \varepsilon/2$. Then we see that

$$\begin{aligned}\lim_{n\to\omega} \big\| \big[y, e^{(n)}_{ij}\big] \big\|_2 &= \lim_{n\to\omega} \big\| y e^{(n)}_{ij} - e^{(n)}_{ij} y \big\|_2 \\ &\le 2\|(x-y)\|_2 + \lim_{n\to\omega} \big\| x e^{(n)}_{ij} - e^{(n)}_{ij} x \big\|_2 < \varepsilon.\end{aligned}$$

Since this is valid for any $\varepsilon > 0$, we have hereby completed the proof of the first statement.

The second statement is a consequence of the first. To see this let $\partial_e T(A) = \{\tau_1, \ldots, \tau_n\}$. Then, in the notation of Proposition 2.5.2, $\mathscr{N}_i$ is a II_1-factor on a separable Hilbert space for each $1 \leq i \leq n$, and since A is nuclear, each II_1-factor is isomorphic to $\mathscr{R}$ (see Corollary 2.3.4 and Corollary 1.3.7 respectively). The desired conclusion therefore follows from Proposition 2.5.2 along with the first statement. □

Proposition 2.5.4. *Suppose that A is a unital, simple C^*-algebra with $0 < |\partial_e T(A)| < \infty$ and let $p, q \in A^\omega \cap \Phi(\pi(B))'$ be projections such that $\tau(p) \leq \tau(q)$ for every $\tau \in T\big(A^\omega \cap \Phi(\pi(B))'\big)$. Then $p \preceq q$, i.e., there exists a partial isometry $v \in A^\omega \cap \Phi\big(\pi(B)\big)'$ such that $v^*v = p$ and $vv^* \leq q$. In other words, $A^\omega \cap \Phi\big(\pi(B)\big)'$ has strict comparison for projections.*

Proof. It follows immediately from Theorem 2.4.4 that $A^\omega \cap \Phi\big(\pi(B)\big)'$ is a von Neumann algebra, which admits a faithful tracial state, namely the restriction of $\sigma := \frac{1}{n}\sum_{i=1}^n \tau_i$. Let p, q be projections as required in the statement. It follows from standard von Neumann algebra theory that there exists a central projection $z \in A^\omega \cap \Phi\big(\pi(B)\big)'$ such that $zp \preceq zq$ and $(\mathbf{1} - z)q \preceq (\mathbf{1} - z)p$. We want to show that $(\mathbf{1} - z)p \simeq (\mathbf{1} - z)q$, since this will basically complete the proof.

If $z = \mathbf{1}$ there is nothing to prove. Hence, we assume that $z \neq \mathbf{1}$. Then, since σ is faithful, we find that $\sigma(\mathbf{1} - z) > 0$ and therefore that the map $\tau : A^\omega \cap \Phi(\pi(B))' \to \mathbb{C}$ given by

$$\tau(x) = \frac{\sigma\big((\mathbf{1} - z)x\big)}{\sigma(\mathbf{1} - z)},$$

is a tracial state. By assumption, this implies that $\tau(p) \leq \tau(q)$ and therefore that $\sigma\big((\mathbf{1} - z)p\big) \leq \sigma\big((\mathbf{1} - z)q\big)$. This observation, in conjunction with the fact that $(\mathbf{1} - z)q \preceq (\mathbf{1} - z)p$, implies that $(\mathbf{1} - z)q \simeq (\mathbf{1} - z)p$.

Now the proof is almost done. Indeed, letting $q_1 \in A^\omega \cap \Phi(\pi(B))'$ be a projection such that $zp \simeq q_1$ and $q_1 \leq zq$, we find that

$$p = zp + (\mathbf{1} - z)p \simeq q_1 + (\mathbf{1} - z)q \leq zq + (\mathbf{1} - z)q = q,$$

and the proof is hereby completed. □

Remark 2.5.5. The above proof will work for any finite von Neumann algebra, with only minor alterations (see Theorem 1.1.11), and is thus not specific to the stated case. However, the above is sufficient for the purposes of this thesis, which is why is stated as is.

These propositions mark the end of our investigation of the tracial central sequence algebra. For the remainder of the chapter we shall prove that certain C^*-algebras possess property (SI), and in chapter 3 we shall collect the strands to obtain the desired properties of $F(A, \pi(B))$.

2.6 Property (SI)

We resume the setup from the previous section, i.e., we let A, B be unital, separable C^*-algebras such that B is nuclear and $T(A) \neq \emptyset$. Furthermore, we assume the existence of a unital embedding

$$\pi : B \to A_\omega.$$

These assumptions will remain fixed throughout this section. As the title suggests we are primarily interested in property (SI) and the corresponding relative notion, but it turns out that it is fruitful to go via excision of cp. maps.

Definition 2.6.1 (Excision in small central sequences)**.** Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. A completely positive map $\varphi : B \to B$ can be **excised in small central sequences in** A if, for all positive contractions $e, f \in F\big(A, \pi(B)\big)$ with $e \in J_A$ and $\sup_n \|1 - f^n\|_{2,\omega} < 1$ there exists $s \in A_\omega$ with $fs = s$ and $s^*\pi(b)s = \pi\big(\varphi(b)\big)e$.

If $B = A$ and $\varphi : A \to A$ satisfies the above, then we simply say that φ can be **excised in small central sequences**.

Note that since π is unital, we have that $\|s\|^2 \leq \|\varphi\|$. In particular, if φ is a ccp. map, then s is automatically a contraction.

Definition 2.6.2 (Property (SI))**.** A unital C^*-algebra A with $T(A) \neq \emptyset$ is said to have **property (SI) relative to** $\pi(B)$ if, for all positive contractions $e, f \in F\big(A, \pi(B)\big)$ with $e \in J_A$ and $\sup_n \|1 - f^n\|_{2,\omega} < 1$ there exists $s \in F\big(A, \pi(B)\big)$ such that $fs = s$ and $s^*s = e$.

If $B = A$ and satisfies the above, then we simply say that A has **property (SI)**.

It follows from Remark 2.2.2 that we could equally well use $\|\cdot\|_{1,\omega}$ in place of $\|\cdot\|_{2,\omega}$ in both definitions above, an observation that will be of some importance later, but will go unmentioned when applied.

The above definitions are reformulations, due to Kirchberg and Rørdam (see [15]) of the concepts introduced by Matui and Sato in [18] (see [15, Lemma 5.2, Lemma 5.4] for proofs that the different formulations are equivalent).

Before investigating excision of cp. maps we note the following easy proposition, which should indicate that such an investigation will not be futile.

Proposition 2.6.3. *Let A, B be C^*-algebras with A unital and $T(A) \neq \emptyset$. Then A has property (SI) relative to $\pi(B)$ if and only if the identity map $id_B : B \to B$ can be excised in small central sequences in A.*

Proof. Fix positive contractions $e, f \in F\big(A, \pi(B)\big)$ such that $e \in J_A$ and $\sup_k \|1 - f^k\|_{2,\omega} < 1$. First, assume that the cp. map $\mathrm{id}_B : B \to B$ can be excised in small central sequences in A, i.e., there exists $s \in A_\omega$ such that $fs = s$ and $s^*\pi(b)s = \pi(b)e$ for all $b \in B$. Since π is unital, it follows that $s^*s = e$. Furthermore, we see that

$$\begin{aligned}\|[s, \pi(b)]\|^2 &= \|(s^*\pi(b)^* - \pi(b)^*s^*)(s\pi(b) - \pi(b)s)\| \\ &= \|\pi(b^*b)e - \pi(b^*b)e - \pi(b^*b)e + \pi(b^*b)e\| = 0,\end{aligned}$$

for all $b \in B$. On the other hand, if A has property (SI) relative to $\pi(B)$, we may find $s \in F\big(A, \pi(B)\big)$ such that $fs = s$ and $s^*s = e$. Then, for any $b \in B$, we see that

$$s^*\pi(b)s = \pi(b)s^*s = \pi(b)e,$$

which finishes the proof. □

The aim of this section is to prove Theorem 2.6.12, a weaker version of which may be found in [19]. However, rather than going through the proof in the indicated article, we follow the approach of Kirchberg and Rørdam (see [15]), although we generalise a few statements slightly to fit in the relative picture painted above. The changes are so slight as to make virtually no difference, and will therefore go umentioned.

Lemma 2.6.4. *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Then for every separable subalgebra $C \subseteq A_\omega$ and every element $c \in C \cap J_A$ there exists a positive contraction $e \in C' \cap J_A$ such that $ec = c$*

Proof. It suffices to prove that there exists e as indicated above, for every positive contraction $c \in C \cap J_A$, since these span $C \cap J_A$.

Let $\big(c_1^{(k)}, c_2^{(k)}, \dots\big) \in \ell^\infty(A)$ be sequences of contractions such that $\big(c^{(k)}\big)_k$, where $c^{(k)} := \pi_\omega\big(c_1^{(k)}, c_2^{(k)}, \dots\big)$, is dense in $(C)_1$, and let $(c_1, c_2, \dots) \in \ell^\infty(A)$ be a representative sequence of $c \in C \cap J_A$ consisting of positive contractions. Furthermore choose a quasi-central approximate unit $\big(e^{(k)}\big)_{k\in\mathbb{N}}$ for $C \cap J_A$ in C and for each $k \in \mathbb{N}$ let $\big(e_1^{(k)}, e_2^{(k)}, \dots\big) \in \ell^\infty(A)$ be a representative sequence consisting of positive contractions.

We aim to apply the ε-test in order to obtain the desired element $e \in J_A \cap C'$. To this end, let X_n be the set of positive contractions in A and the test functions $f_n^{(k)} : X_n \to [0, \infty)$ be given by

$$\begin{aligned}f_n^{(1)}(x) &= \|(1 - x)c_n\|; \\ f_n^{(2)}(x) &= \|x\|_2; \\ f_n^{(k+2)}(x) &= \big\|\, c_n^{(k)}x - xc_n^{(k)} \,\big\|, \quad k \geq 1.\end{aligned}$$

Then we see that

$$\begin{aligned} f_\omega^{(1)}((e_1^{(l)}, e_2^{(l)}, \dots)) &= \|(1-e^{(l)})c\| \to 0; \\ f_\omega^{(2)}((e_1^{(l)}, e_2^{(l)}, \dots)) &= \|e^{(l)}\|_{2,\omega} = 0; \\ f_\omega^{(k+2)}((e_1^{(l)}, e_2^{(l)}, \dots)) &= \|c^{(k)}e^{(l)} - e^{(l)}c^{(k)}\| \to 0, \quad \text{as } l \to \infty. \end{aligned}$$

It therefore follows that we may find a sequence $(e_1, e_2, \dots) \in \ell^\infty(A)$ such that $f_\omega^{(k)}(e_1, e_2 \dots) = 0$ for all $k \in \mathbb{N}$. Obviously, $e := \pi_\omega((e_1, e_2, \dots))$ satisfies $e \in J_A$ and $ec = c$. Furthermore, since $f^{(k)}(e_1, e_2, \dots) = 0$ for all $k \geq 3$ it follows that e commutes with every $c^{(k)}$ and therefore that $e \in J_A \cap C'$. □

Lemma 2.6.5. *Let $e, f \in F\big(A, \pi(B)\big)$ be positive contractions with $e \in J_A$ and $\sup_n \|1 - f^n\|_{1,\omega} < 1$.*

(i) There are positive contractions $e_0, f_0 \in F\big(A, \pi(B)\big)$ with $e_0 \in J_A$ such that

$$e_0 e = e, \quad f_0 f = f_0, \quad \|1 - f_0\|_{1,\omega} = \sup_n \|1 - f_0^n\|_{1,\omega} = \sup_n \|1 - f^n\|_{1,\omega}.$$

(ii) If, in addition, A is simple, exact and has strict comparison and B is simple, then for every non-zero positive element $b \in B$ and for every $\varepsilon > 0$ there exists $t \in A_\omega$ such that

$$t^* \pi(b) t = e, \quad ft = t, \quad \|t\| \leq \|b\|^{1/2} + \varepsilon.$$

Proof. (i): The existence of $e_0 \in F\big(A, \pi(B)\big) \cap J_A$ with $e_0 e = e$ follows directly from Lemma 2.6.4, since B, and therefore also $C^*\big(e, \pi(B)\big)$, is a separable C^*-algebra (note that we do not need to assume that $e \in F\big(A, \pi(B)\big)$ for this).

Once again we aim to apply the ε-test to obtain f_0. To this end, let $(f_1, f_2, \dots) \in \ell^\infty(A)$ be a lift of f consisting of positive contractions, set $\rho := \sup_k \|1 - f^k\|_{1,\omega}$ and let $(b_k)_{k \in \mathbb{N}}$ be a dense sequence in B. Let X_n be the set of positive contractions in A and define the test functions $g_n^{(k)} : X_n \to [0, \infty)$ by

$$\begin{aligned} g_n^{(1)}(x) &= \|x(1-f_n)\|; \\ g_n^{(2k)}(x) &= \max\{\|1 - x^k\| - \rho, 0\}, \quad k \geq 1; \\ g_n^{(2k+1)}(x) &= \|x\pi_n(b_k) - \pi_n(b_k)x\|, \quad k \geq 1. \end{aligned}$$

Fix $l \in \mathbb{N}$ and let $s_n = f_n^l \in X_n$. Then

$$\begin{aligned} g_\omega^{(1)}(s_1, s_2, \dots) &= \|f^l(1-f)\| \to 0, \quad \text{as } l \to \infty; \\ g_\omega^{(2k)}(s_1, s_2, \dots) &= \max\{\|1 - f^{l \cdot k}\|_{1,\omega} - \rho, 0\} = 0; \\ g_\omega^{(2k+1)}(s_1, s_2, \dots) &= \|f^l \pi(b_k) - \pi(b_k) f^l\| = 0. \end{aligned}$$

Hence, we find a sequence $(f_{0,1}, f_{0,2}, \dots) \in \ell^\infty(A)$ of positive contractions such that $g_\omega^{(k)}(f_{0,1}, f_{0,2}, \dots) = 0$ for all $k \in \mathbb{N}$. Letting $f_0 = \pi_\omega(f_{0,1}, f_{0,2}, \dots) \in A_\omega$, we see that f_0 is a positive contraction satisfying $f_0 f = f_0$, $\|1 - f_0^l\|_{1,\omega} \leq \sup_n \|1 - f^n\|_{1,\omega}$ for all $l \in \mathbb{N}$ and $f_0 \in F\big(A, \pi(B)\big)$ (since f_0 commutes with all the elements $\pi(b_k)\big)$. Hence, we conclude that $f^k f_0 = f_0$ and therefore that

$$f_0^2 = f^k f_0^2 f^k \leq f^{2k}.$$

Since the function $t \mapsto t^{1/2}$ is operator monotone we find that $0 \leq f_0 \leq f^k \leq 1_{A_\omega}$ for all $k \in \mathbb{N}$, whence;

$$\sup_k \|1 - f_0^k\|_{1,\omega} \leq \sup_k \|1 - f^k\|_{1,\omega} \leq \|1 - f_0\|_{1,\omega} \leq \sup_k \|1 - f_0^k\|_{1,\omega},$$

which finishes the proof.

(ii): Let $b \in B$ and $\varepsilon > 0$ be given. We may assume that $\|b\| = 1$, since a successful proof in this case will imply the general case. Using the continuous functional calculus, we may therefore find positive elements in $g, h \in C^*(1, b)$ such that $\|g\| = 1$, $\|h\| = 1 + \varepsilon$ and $bhg = g$.

Let e_0, f_0 be given as in (i) and $(e_{0,n})_n, (f_{0,n})_n \in \ell^\infty(A)$ be positive, contrative lifts of e_0 and f_0 respectively. Note that it follows from Proposition 1.4.11, that we may assume $e_{0,n} e_n = e_n$ and $f_{0,n} f_n = f_{0,n}$ for all $n \in \mathbb{N}$. Put

$$\eta := \lim_{n \to \omega} \min_{\tau \in T(A)} \tau(f_{0,n}) = 1 - \|1 - f_0\|_{1,\omega} > 0.$$

Since $g \in B$ is non-zero and B is simple and unital, we may find finitely many elements $r_1, \dots, r_k \in B$ such that $\sum_{i=1}^k r_i^* g r_i = 1_B$. We let $\alpha = \big(\sum_{i=1}^k \|r_i\|^2\big)^{-1} > 0$. Note that since $f_0 \in F\big(A, \pi(B)\big)$ and π is unital, we have that

$$\begin{aligned}
&\lim_{n \to \omega} \min_{\tau \in T(A)} \tau(f_{0,n}) \\
=&\lim_{n \to \omega} \min_{\tau \in T(A)} \Big(\sum_{i=1}^k \tau\big(f_{0,n}^{1/2} \pi_n(r_i^*) \pi_n(g) \pi_n(r_i) f_{0,n}^{1/2}\big)\Big) \\
=&\lim_{n \to \omega} \min_{\tau \in T(A)} \Big(\sum_{i=1}^k \tau\big(f_{0,n}^{1/2} \pi_n(g)^{1/2} \pi_n(r_i r_i^*) \pi_n(g)^{1/2} f_{0,n}^{1/2}\big)\Big) \\
\leq&\alpha^{-1} \lim_{n \to \omega} \min_{\tau \in T(A)} \tau\big(f_{0,n}^{1/2} \pi_n(g) f_{0,n}^{1/2}\big).
\end{aligned}$$

Set $\delta := (\alpha\eta)/2 > 0$ and $b_n := \big(f_{0,n}^{1/2} \pi_n(g) f_{0,n}^{1/2} - \delta\big)_+$. Since b_n is a positive contraction for each $n \in \mathbb{N}$, it follows that $d_\tau(b_n) \geq \tau(b_n)$ for all $\tau \in T(A)$

and we therefore obtain;

$$\begin{aligned}\lim_{n\to\omega}\inf_{\tau\in T(A)} d_\tau(b_n) &\geq \lim_{n\to\omega}\inf_{\tau\in T(A)} \tau(b_n)\\ &\geq \lim_{n\to\omega}\inf_{\tau\in T(A)} \tau\big(f_{0,n}^{1/2}\pi_n(g)f_{0,n}^{1/2}\big) - \delta\\ &\geq \alpha \lim_{n\to\omega}\inf_{\tau\in T(A)} \tau(f_{0,n}) - \delta\\ &= \alpha\eta - \delta > 0.\end{aligned}$$

We furthermore claim that $\lim_{n\to\omega}\sup_{\tau\in T(A)} d_\tau(e_{0,n}) = 0$. To see this, apply (i) once again to e_0 to obtain a positive contraction $e' \in F\big(A, \pi(B)\big) \cap J_A$ such that $e'e_0 = e_0$. Let $(e'_1, e'_2, \dots) \in \ell^\infty(A)$ be a positive contractive lift of e' such that $e'_n e_{0,n} = e_{0,n}$ for each $n \in \mathbb{N}$, and note that this implies that $e'_n\gamma(e_{0,n}) = \gamma(e_{0,n})$ for all $\gamma \in C_0\big((0,1]\big)$. Therefore, it follows that $d_\tau(e_{0,n}) \leq \tau(e'_n)$ for all $n \in \mathbb{N}$ and $\tau \in T(A)$ and $\lim_{n\to\omega}\sup_{\tau\in T(A)} \tau(e'_n) = 0$, yielding the desired conclusion.

Now we are ready to apply strict comparison to finish the proof. It follows from the above, that set X of all $n \in \mathbb{N}$ such that

$$\sup_{\tau\in T(A)} d_\tau(e_{0,n}) < \inf_{\tau\in T(A)} d_\tau(b_n),$$

belongs to ω. Since A has strict comparison, this implies that $e_{0,n} \precsim b_n = \big(f_{0,n}^{1/2}\pi_n(g)f_{0,n}^{1/2} - \delta\big)_+$. Since $e_{0,n}e_n = e_n$, we may therefore find $v_n \in A$ such that $v_n^* f_{0,n}^{1/2}\pi_n(g)f_{0,n}^{1/2}v_n = e_n$ and $\|v_n\| \leq \delta^{-1/2}$ for all $n \in X$ (see Lemma 1.9.8). Put $t_n = \pi_n(h^{1/2})\pi_n(g^{1/2})f_{0,n}^{1/2}v_n$ when $n \in X$ and $t_n = 0$ otherwise, and put $t \in A_\omega := \pi_\omega((t_1, t_2, \dots))$. Observe that $\|\pi_n(g)^{1/2}f_{0,n}^{1/2}v_n\|^2 = \|e_n\| \leq 1$, from which it follows that

$$\|t_n\|^2 \leq \|\pi_n(h)\|\|e_n\| \leq 1 + \varepsilon,$$

and hence $\|t\| \leq 1 + \varepsilon$. Furthermore, since the sequence $(\pi_n)_n$ is asymptotically multiplicative, we may find a set $X_1 \in \omega$ such that for all $n \in X_1$ we have that

$$\|t_n^*\pi_n(b)t_n - e_n\| \leq \|v_n^* f_{0,n}^{1/2}\pi_n(bhg)f_{0,n}^{1/2}v_n - e_n\| + \varepsilon = \varepsilon,$$

whence it follows that $t^*\pi(b)t = e$. Finally, we see that for $n \in X$ we have that

$$\begin{aligned}\|(1-f_n)t_n\| &= \big\|\big(\pi_n(h^{1/2})\pi_n(g^{1/2}) - f_n\pi_n(h^{1/2})\pi_n(g^{1/2})\big)f_{0,n}^{1/2}v_n\big\|\\ &= \big\|\big(\pi_n(h^{1/2})\pi_n(g^{1/2})f_n - f_n\pi_n(h^{1/2})\pi_n(g^{1/2})\big)f_{0,n}^{1/2}v_n\big\|\\ &\leq \big\|\big[f_n, \pi_n(g^{1/2})\pi_n(h^{1/2})\big]\big\|\,\|f_{0,n}^{1/2}v_n\|.\end{aligned}$$

Since $\|f_{0,n}^{1/2}v_n\| \leq \delta^{-1/2}$ for all $n \in X$ and $f_0 \in F\big(A, \pi(B)\big)$, we find that $\|(1-f)t\| = 0$ and this finishes the proof. □

Definition 2.6.6. Let $C \subseteq D$ be C^*-algebras. A completely positive map $\varphi : C \to D$ is said to be **one-step elementary** if there exists a pure state κ on D and elements $c_1, \dots, c_n; d_1, \dots, d_n \in D$ such that

$$\varphi(x) = \sum_{i,j=1}^{n} \kappa(d_j^* x d_i) c_j^* c_i.$$

During the proof of the next proposition we will need the fact that for any pure state κ on a C^*-algebra we have the equality $\ker \kappa = L + L^*$ where L is the left kernel of κ, i.e., $L = \{x \in A \mid \kappa(x^*x) = 0\}$ (see [22, Prop 3.13.6] for a proof of this fact).

Proposition 2.6.7 (Akemann-Anderson-Pedersen)**.** *Let D be a unital C^*-algebra and κ be a pure state on D. For each finite subset $F \subseteq D$ and $\varepsilon > 0$ there is an element $d \in D_+$ of norm 1 such that $\|dxd - \kappa(x)d^2\| < \varepsilon$ for all $x \in F$.*

Proof. Let $N := L \cap L^*$, where L is the left kernel of κ. Then N is a (hereditary) subalgebra of D and hence we may choose an approximate unit $(e_\lambda)_{\lambda \in \Lambda}$ for N. Set $d_\lambda := 1 - e_\lambda$ and note that $\kappa(d_\lambda) = 1$ for each $\lambda \in \Lambda$. The proof will be done once we have succesfully proven that

$$d_\lambda(x - \kappa(x) \cdot 1_D)d_\lambda = d_\lambda x d_\lambda - \kappa(x) d_\lambda^2 \to 0.$$

for all $x \in D$. To this end note that for each $x \in D$, we have that $x - \kappa(x) \cdot 1 \in \ker \kappa = L + L^*$, and hence we need only prove that $y d_\lambda \to 0$ for $y \in L$. But if $y \in L$ then $y^* y \in N$ and hence

$$\|y d_\lambda\|^2 \le \|y^* y d_\lambda\| = \|y^* y(1 - e_\lambda)\| \to 0.$$

□

Lemma 2.6.8. *If A is a unital, separable, simple and exact C^*-algebra with strict comparison, then every one-step elementary map $\varphi : B \to B$ can be excised in small central sequences in A.*

Proof. Since B is separable, it suffices to prove that for each pair of positive contractions $e, f \in F\big(A, \pi(B)\big)$ such that $e \in J_A$ and $\sup_n \|1 - f^n\|_{1,\omega} < 1$ and each finite subset $F \subseteq B$, there exists $s \in A_\omega$ such that $fs = s$ and $\|s^* \pi(b) s - \pi(\varphi(b)) e\| < \varepsilon$ for all $b \in F$. This is really an application of the ε-test and since we automatically have that $\|s\|^2 \le \|\varphi\| + \varepsilon$, there is a uniform bound on the norm of s, independent of the set F (of course we can assume that $1_B \in F$ with no loss of generality).

Hence, let $2^{1/2} - 1 > \varepsilon > 0$ be given and $\varphi : B \to B$ be a non-zero one-step elementary map. Let κ be a pure state on B and $c_1, \dots, c_n; d_1, \dots, d_n \in B$ be non-zero elements such that

$$\varphi(b) = \sum_{i,j=1}^{n} \kappa(d_j^* b d_i) c_j^* c_i.$$

It follows from Proposition 2.6.7 that we may choose a positive element $h \in B$ such that $\|h\| = 1$ and

$$\|h^{1/2}(x - \kappa(x) \cdot 1_B)h^{1/2}\| < \frac{1}{2}\Big(\sum_{i=1}^{n} \|c_i\|\Big)^{-2}\varepsilon,$$

for all $x \in \{d_j^* b d_i \mid 1 \leq i, j \leq n, b \in F\}$. Then, by the previous lemma, we may choose $t \in A_\omega$ such that $t^*\pi(h)t = e$, $ft = t$ and $\|t\|^2 \leq 2$. Put

$$s = \sum_{i=1}^{n} \pi(d_i h^{1/2}) t \pi(c_i).$$

Then we see that for all $b \in F$ we have;

$$\begin{aligned}\big\| s^*\pi(b)s - \pi\big(\varphi(b)\big)e \big\| &\leq \sum_{i,j=1}^{n} \big\| \pi(c_j)^* t^* \pi\big(h^{1/2}(d_j^* b d_i - \kappa(d_j^* b d_i))h^{1/2}\big) t \pi(c_i) \big\| \\ &< \Big(\sum_{i,j=1}^{n} \|c_j\|\|c_i\|\Big)\Big(\sum_{i=1}^{n} \|c_i\|\Big)^{-2}\varepsilon = \varepsilon.\end{aligned}$$

□

Lemma 2.6.9. *The family of all cp. maps $\varphi : B \to B$ that can be excised in small central sequences in A is closed under point-norm limits.*

Proof. Let $\varphi_n : B \to B$ be a sequence of cp. maps, which converges pointwise to a cp. map $\varphi : B \to B$, and such that each φ_n can be excised in small central sequences in A.

Let $e, f \in F\big(A, \pi(B)\big)$ be positive contractions such that $e \in J_A$ and $\sup_k \|1 - f^k\|_{2,\omega} < 1$. For each $n \in \mathbb{N}$ we find an element $s_n \in A_\omega$ such that $f s_n = s_n$ and $s_n^* \pi(b) s_n = \pi(\varphi_n(b))e$ for all $b \in B$. Since B is unital and $\varphi_n(1_B) \to \varphi(1_B)$ it follows that $\|\varphi_n\| \to \|\varphi\|$. In particular it follows that we have an uniform bound $\|s_n\| \leq r$ for some $r \in \mathbb{R}_+$. Lift e and f to sequences of positive contractions $(e_1, e_2, \dots)$ and $(f_1, f_2, \dots)$, respectively, in $\ell^\infty(A)$. Let $(b_n)_{n\in\mathbb{N}}$ be a dense sequence in B and let $X_n = (A)_r$.

Once again we aim to use the ε-test to finish to proof. To this end let the test functions $g_n^{(k)} : X_n \to [0, \infty)$ be given by

$$\begin{aligned} g_n^{(1)}(x) &= \|(1 - f_n)x\| \\ g_n^{(k+1)}(x) &= \|x^* \pi_n(b_k) x - \pi_n(\varphi(b_k)) e_n\|. \end{aligned}$$

For each $m \in \mathbb{N}$, let $s'_m = (s_{m,1}, s_{m,2}, \dots) \in \ell^\infty(A)$ be a lift of s_m such that $s_{m,n} \in X_n$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} f_\omega^{(1)}(s'_m) &= \|(1 - f)s_m\| = 0 \\ f_\omega^{(k+1)}(s'_m) &= \big\| s_m^* \pi(b_k) s_m - \pi\big(\varphi(b_k)\big)e \big\| = \big\| \pi\big(\varphi_m(b_k) - \varphi(b_k)\big)e \big\| \end{aligned}$$

and since $\varphi_m \to \varphi$ pointwise, the conditions of the ε-test is fullfilled, thus completing the proof. □

The reader may well already have guessed what the next step is, namely to prove that any nuclear map $B \to B$ can be approximated in point-norm by one-step elementary maps, which is certainly true, under some mild assumptions. During the proof, we will need Kadison's Transitivity Theorem, which is stated for the readers convience, but not proven (the reader may find a proof in [20, Theorem 5.2.2]).

Theorem 2.6.10 (Kadison's Transitivity Theorem). *Let D be a non-zero C^*-algebra and $\pi : D \to B(\mathcal{H})$ be an irreducible representation. If $\xi_1, \dots, \xi_n$ and $\eta_1, \dots, \eta_n$ are elements of $\mathcal{H}$ such that $\xi_1, \dots, \xi_n$ are linearly independent, then there exists $d \in D$ such that $\pi(d)\xi_i = \eta_i$ for each $1 \leq i \leq n$.*

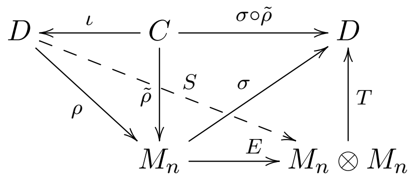
Proposition 2.6.11. *Let $C \subseteq D$ be separable C^*-algebras, and assume that there is a pure state κ on D such that the associated GNS representation $\pi_\kappa : D \to B(\mathcal{H})$ satisfies $\pi_\kappa^{-1}(\mathbb{K}) = \{0\}$. Let $\varphi : C \to D$ be a ccp. map. If φ is nuclear, then it is the point-norm limit of a sequence of cp. one-step elementary maps $\varphi_n : C \to D$.*

Proof. Suppose that $\varphi : C \to D$ is nuclear. Then φ can be approximated in the point-norm topology by maps of the form $\sigma \circ \tilde{\rho}$, where $\tilde{\rho} : C \to M_n$ and $\sigma : M_n \to D$ are cp. maps. Therefore the proposition need only be proved for maps of this form.

First note that it follows from Arveson's Extension Theorem that $\tilde{\rho}$ extends to a cp. map $\rho : D \to M_n$. Furthermore we may decompose $\sigma : M_n \to D$ as $\sigma = T \circ E$, where $E \colon M_n \to M_{n^2}$ is given by $E(x) = x \otimes \mathbf{1}_n \in M_n \otimes \mathbf{1}_n \subseteq M_{n^2}$ ($\mathbf{1}_n \in M_n$ denotes the identity), and $T : M_{n^2} \to D$ is of the form $T(y) = c^*yc$ for a suitable column matrix in $c \in M_{n^2,1}(D)$. Indeed, letting $(e_{ij})_{i,j}$ be the standard matrix units for M_n, we see that $\sum_{i,j=1}^n e_{ij} \otimes e_{ij}$ is positive, since $\frac{1}{n} \sum_{i,j=1}^n e_{ij} \otimes e_{ij}$ is a projection. As σ is a cp. map, we therefore find that

$$p = (\mathrm{id}_n \otimes \sigma)\Big(\sum_{i,j=1}^n e_{ij} \otimes e_{ij} \Big) \in M_n \otimes D$$

is positive and hence has a positive square root. Write $p^{1/2} = \sum_{i,j=1}^n e_{ij} \otimes c_{ij}$ and observe that $\sigma(e_{ij}) = \sum_{k=1}^n c_{ki}^* c_{kj}$ for each i, j. Thus, $c = (c_1, \dots, c_n^2)$ can be obtained from a suitable rearrangement of the matrix $(c_{ij})_{i,j}$. We now see that $\sigma \circ \rho = T \circ S$, where $S = E \circ \rho$. Thus we have the following commutative diagram:



(2.5)

where $\iota : C \to D$ denotes the inclusion.

Let κ be a pure state on D such that $\pi_\kappa^{-1}(\mathbb{K}) = \{0\}$, which exists by assumption, and let $(\pi_\kappa, \mathcal{H}_\kappa, \xi_\kappa)$ be the associated GNS-triple. Note that $\mathcal{H}_\kappa$ is a separable Hilbert space and π_κ is faithful by assumption. Furthermore, we let $\mathcal{H}$ be a Hilbert space, $\pi_S : D \to B(\mathcal{H})$ and $V : \mathbb{C}^{n^2} \to \mathcal{H}$ be given such that $S(d) = V^*\pi_S(d)V$ for all $d \in D$ (this is a consequence of Stinespring's Theorem, see [1, Theorem II.6.9.7] for a proof). Letting $\{\zeta_1, \ldots, \zeta_{n^2}\}$ be the standard orthonomal basis for $\mathbb{C}^{n^2}$ and $\eta_i = V\zeta_i$ we see that

$$\langle S(d)\zeta_j, \zeta_i\rangle = \langle \pi_S(d)\eta_j, \eta_i\rangle$$

for each $d \in D$. We may assume that $\pi_S^{-1}(\mathbb{K}) = \{0\}$ (if this is not the case at the outset, we can replace π_S with $\pi_S \oplus \pi_\kappa$). Furthermore, since both D and $\mathbb{C}^{n^2}$ are separable spaces, we may assume that $\mathcal{H}$ is separable as well (this follows from the construction of π_S and $\mathcal{H}$). Therefore Voiculescu's Theorem yields that π_S and π_κ are approximately unitarily equivalent (see Corollary 1.5.4). It follows that for every finite subset $F \subseteq D$ and $\varepsilon > 0$ we may find vectors $\xi_1, \ldots, \xi_{n^2}$ such that

$$|\langle S(x)\zeta_j, \zeta_i\rangle - \langle \pi_\kappa(x)\xi_j, \xi_i\rangle| < \frac{\varepsilon}{n^2}$$

for all $x \in F$. By Kadison's Transitivity Theorem (see Theorem 2.6.10) we find elements $d_1, \ldots, d_{n^2} \in B$ such that $\pi_\kappa(d_i)\xi_\kappa = \xi_i$, which finally yields that

$$|\langle S(x)\zeta_j, \zeta_i\rangle - \kappa(d_j^* x d_i)| = |\langle \pi_\kappa(d_j^* x d_i)\xi_\kappa, \xi_\kappa\rangle - \langle S(x)\zeta_j, \zeta_i\rangle| < \frac{\varepsilon}{n^2},$$

for all $x \in F$. Letting $(f_{ij})_{i,j}$ denote the standard matrix units in M_{n^2} we see that

$$\left\| S(x) - \sum_{i,j=1}^{n^2} f_{ij} \otimes \kappa(d_j^* x d_i) \right\| < \varepsilon$$

for all $x \in F$. Since $F \subseteq D$ and $\varepsilon > 0$ was arbitrary, we see that S can be approximated in point-norm by maps $S' : D \to M_{n^2}$, where S' is of the form

$$S'(x) = \sum_{i,j=1}^{n^2} f_{ij} \otimes \kappa(d_j^* x d_i).$$

This finishes the proof, since for each map S' on the above form, we have that

$$T \circ S'(b) = \sum_{i,j=1}^{n^2} \kappa(d_j^* b d_i) c_j^* c_i.$$

□

Note that if D is a unital, simple and infinite-dimensional C^*-algebra, then any pure state κ on D will satisfy the requirements for an application of the above theorem.

Theorem 2.6.12. *Assume that B is a unital, simple, separable, infinite-dimensional and nuclear C^*-algebra, and that A is a unital, separable, simple and exact C^*-algebra with $T(A) \neq \emptyset$ and strict comparison. Then every ccp. map $\varphi : B \to B$ can be excised in small central sequences in A.*

In particular, A has property (SI) relative to $\pi(B)$.

Proof. This follows from the above section. Indeed, since B is nuclear, every ccp. map $\varphi : B \to B$ is nuclear, and it therefore follows from Proposition 2.6.11 that we may approximate φ in point-norm by one-step elementary maps. Since the latter maps can all be excised in small central sequences in A by Lemma 2.6.8, we obtain, as a consequence of Lemma 2.6.9, that φ can also be excised in small central sequences in A.

The second statement follows from the first, along with Proposition 2.6.3.

□

Remark 2.6.13. Note that if $A = B$ above, then the Theorem states that A has property (SI) under the given assumptions. This is the result originally proven by Kirchberg and Rørdam in [15] and it is sufficient for their result on Jiang-Su stability (and also for our weaker version, originally proved by Matui and Sato). However, in the section on finite decomposition rank, we will need the corollary as it stands above, and hence we have made the small extra effort to obtain it in this form.

CHAPTER 3

$\mathcal{Z}$-Stability And Non-Commutative Covering Dimension

3.1 Jiang-Su Stability

In this section we seek to introduce the Jiang-Su algebra and prove that a certain class of C^*-algebras absorb this algebra tensorially. We will treat the Jiang-Su algebra in headline form. That is, we indicate how the algebra may be constructed and discuss certain properties, but we do not give proofs. However, we will prove some consequences of the theory gathered, consequences that drop out, with virtually no effort, from the work we have already done on central sequence algebras.

The Jiang-Su Algebra $\mathcal{Z}$

The dimension drop algebras are important in the construction and study of the Jiang-Su algebra. We let $\mathbf{1}_k \in M_k$ denote the identity, for any $k \in \mathbb{N}$.

Definition 3.1.1. For each pair of natural numbers $k, m \in \mathbb{N}$ and each $n \in \mathbb{N}$ such that both k and m divides n, we let $I(k, n, m) \subseteq C([0, 1]) \otimes M_n$ be given by

$$I(k, n, m) := \{f : [0, 1] \to M_n \mid f(0) \in M_k \otimes \mathbf{1}_{n/k} \text{ and } f(1) \in \mathbf{1}_{n/m} \otimes M_m\}.$$

We refer to such a C^*-algebra as a **dimension drop algebra**. If $n = km$ and k and m are relatively prime, then we write $I(k, m) := I(k, km, m)$ and refer to $I(k, m)$ as a **prime dimension drop algebra**.

It follows easily from Proposition 1.4.9 that the dimension drop algebras are nuclear. Indeed, let $k, n, m \in \mathbb{N}$ be given such that both k and m divides n, and let $\mathrm{ev}_0 : I(k, n, m) \to M_k$ be the *-homomorphism given by

$$\mathrm{ev}_0(f) = f(0).$$

It is easy to that ev_0 is surjective, with $\ker(\mathrm{ev}_0) = C_0\big((0,1]\big) \otimes \mathbf{1}_{n/m} \otimes M_m$. Since both M_k and $\ker(\mathrm{ev}_0)$ are nuclear, it follows that $I(k,n,m)$ is also nuclear.

The following theorem was proven by Jiang and Su in their original paper [11, Theorem 2.9].

Theorem 3.1.2. *There exists a sequence*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots,$$

with unital connecting maps, where each A_j is a prime dimension drop algebra, such that the inductive limit $A := \varinjlim(A_j, \varphi_j)$ is unital, simple, infinite-dimensional, projectionless and has a unique tracial state.

For the moment we fix one particular inductive limit as in the theorem above and denote it $\mathcal{Z}$. However, as the next theorem states, this algebra is essentially unique (see, [11, Theorem 6.2]).

Theorem 3.1.3. *The inductive limit of the system*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots,$$

where each A_j is a prime dimension drop algebra and the connecting maps are unital, is isomorphic to $\mathcal{Z}$ if and only if it is simple and has a unique tracial state.

Naturally, $\mathcal{Z}$ is nuclear, being the inductive limit of nuclear C^*-algebras (see Proposition 1.4.9). For the statement of the next proposition we will need a generalized version of the dimension drop algebras in Definition 3.1.1. If $p, q \in \mathbb{N}$ are natural numbers let $\mathbf{p}$ and $\mathbf{q}$ denote the associated infinite super-natural numbers, i.e., $\mathbf{p} := p^\infty$ and $\mathbf{q} := q^\infty$, and $M_\mathbf{p}, M_\mathbf{q}$ denote the UHF-algebras of type $\mathbf{p}$ and $\mathbf{q}$ respectively. Then we let $Z_{p,q} \subseteq C([0,1]) \otimes M_\mathbf{p} \otimes M_\mathbf{q}$ denote the C^*-algebra

$$Z_{p,q} := \{f : [0,1] \to M_\mathbf{p} \otimes M_\mathbf{q} \mid \ f(0) \in M_\mathbf{p} \otimes \mathbf{1_q} \text{ and } f(1) \in \mathbf{1_p} \otimes M_\mathbf{q}\},$$

where $\mathbf{1_p} \in M_\mathbf{p}$ denotes the identity. The next proposition, along with the following dicussion, may be found in [27, Theorem 3.4]. We say that an endomorphism φ on a C^*-algebra A is **trace-collapsing** if $\tau \circ \varphi = \tau' \circ \varphi$ for arbitrary tracial states τ, τ' on A.

Theorem 3.1.4. *Let $p, q \in \mathbb{N}$ be relatively prime natural numbers.*

(i) There exists a trace-collapsing unital endomorphism on $Z_{p,q}$.

(ii) Let φ be any trace-collapsing unital endomorphism on $Z_{p,q}$. Then $\mathcal{Z}$ is isomorphic to the inductive limit of the stationary system

$$Z_{p,q} \xrightarrow{\varphi} Z_{p,q} \xrightarrow{\varphi} Z_{p,q} \xrightarrow{\varphi} \cdots$$

It is not difficult to see that each $Z_{p,q}$ is an inductive limit of prime dimension drop algebras. Namely, let for each $j \in \mathbb{N}$, let $\sigma_j : M_{p^j} \otimes M_{q^j} \to M_{p^{j+1}} \otimes M_{q^{j+1}}$ be a *-homomorphism such that $\sigma_j(M_{p^j} \otimes \mathbf{1}_{q^j}) \subseteq M_{p^{j+1}} \otimes \mathbf{1}_{q^{j+1}}$ and $\sigma_j(\mathbf{1}_{p^j} \otimes M_{q^j}) \subseteq \mathbf{1}_{p^{j+1}} \otimes M_{q^{j+1}}$. Then $Z_{p,q}$ is the inductive limit of the system

$$I(p,q) \xrightarrow{\rho_1} I(p^2,q^2) \xrightarrow{\rho_2} I(p^3,q^3) \xrightarrow{\rho_3} \cdots,$$

where $\rho_j(f) := \sigma_j \circ f$. Therefore, as a particular consequence of Theorem 3.1.4 we see that for any pair of relatively prime integers $n, m \in \mathbb{N}$ there exists a unital *-homomorphism $I(n,m) \to \mathcal{Z}$.

In their paper, Jiang and Su proved the following results. As a service to those readers who are unfamiliar with infinite tensor powers, we start by defining this concept. In the following definition $A \otimes^{\mu} A$ denotes either the minimal or the maximal tensor product.

Definition 3.1.5. Let A be a unital C^*-algebra. Then $\bigotimes_{n\in\mathbb{N}}^{\mu} A$ is the inductive limit of the system $\big(\bigotimes_{n\in J_k}^{\mu} A, \varphi_k\big)$, where $J_k := \{1, 2, \ldots, k\}$ and

$$\varphi_k \colon \bigotimes_{n\in J_k}{}^{\mu} A \to \left(\bigotimes_{n\in J_k}{}^{\mu} A\right) \otimes^{\mu} A,$$

is given by $\varphi(a) = a \otimes 1_A$.

For a general C^*-algebra (i.e., non-unital) one would have to specify a sequence of non-trivial projections, but for unital C^*-algebras we avoid this problem. Furthermore, if A is nuclear, then the infinite maximal and infinite minimal tensor product agrees, and in that case we simple write $\bigotimes_{n=1}^{\infty} A$.

Theorem 3.1.6. *The Jiang-Su algebra $\mathcal{Z}$ has the following properties.*

(i) The two unital *-homomorphisms $\psi_i : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$, given by

$$\psi_1(a) = a \otimes 1_{\mathcal{Z}}, \quad \psi_2(a) = 1_{\mathcal{Z}} \otimes a$$

are approximately unitarily equivalent.

(ii) $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \displaystyle\bigotimes_{n=1}^{\infty} \mathcal{Z}$.

$\mathcal{Z}$-stability

Now we are interested in which C^*-algebras absorb $\mathcal{Z}$ tensorially, and any such C^*-algebra is said to be $\mathcal{Z}$-stable - hence the title. Some of the statements here are formulated in a slightly more general form than what is needed to examine Jiang-Su stability. This is done to aid us in the remaining sections of this chapter (in particular in the part pertaining to Kirchberg algebras).

Proposition 3.1.7. *Suppose that $\mathcal{D}$ is a unital, nuclear C^*-algebra such that $\bigotimes_{n=1}^{\infty} \mathcal{D} \cong \mathcal{D}$. If A is a C^*-algebra such that $A \otimes \mathcal{D} \cong A$ then there exists a sequence of isomorphisms $\varphi_n : A \otimes \mathcal{D} \to A$ such that*

$$\lim_{n \to \infty} \|\varphi_n(a \otimes 1_{\mathcal{D}}) - a\| = 0$$

Proof. See [25, Lemma 4.4]. The lemma is stated specifically for $\mathcal{Z}$ but clearly the proof works in the above specified setting as well. □

By Theorem 3.1.6 (part (ii)) we see that $\mathcal{Z}$ satisfies the conditions for an application of the above theorem. Furthermore, upon combing part (i) of Theorem 3.1.6 with [24, Theorem 7.2.2] we obtain the following theorem.

Theorem 3.1.8. *Let A be a unital, separable C^*-algebra. Then the following are equivalent:*

(i) $A \cong A \otimes \mathcal{Z}$.

(ii) There exists a unital *-homomorphism $\mathcal{Z} \to F(A)$.

Actually the above theorem is also true when A is non-unital, upon replacing condition (ii) with a unital *-homomorphism $\mathcal{Z} \to \mathcal{M}(A)_\omega \cap A'$, where $\mathcal{M}(A)$ denotes the multiplier algebra of A. At first glance, the above theorem does not solve any problems, since it is, in general, very difficult to prove or disprove the existence of a unital *-homomorphism $B \to F(A)$. However, in the case of the Jiang-Su algebra, we have general structure theorems, which makes the above theorem genuinely helpful. Amongst these deep structure theorems is the following, due to Andrew S. Toms and Marius Dadarlat

Theorem 3.1.9. *Suppose that A is a unital, separable C^*-algebra such that $\bigotimes_{n \in \mathbb{N}}^{(\min)} A$ contains, unitally, a subhomogeneous algebra with no characters. Then A is $\mathcal{Z}$-stable. In particular, there exists a unital *-homomorphism $\mathcal{Z} \to \bigotimes_{n \in \mathbb{N}}^{(\min)} A$.*

Proof. See [8, Theorem 1.1]. □

This theorem becomes particularly powerful when combined with the following observation due to Kirchberg (see [13, Corollary 1.13]) (note however, the difference in which tensor product is considered).

Theorem 3.1.10. *Suppose that A and B are unital, separable C^*-algebras and that there exists a unital *-homomorphism $A \to F(B)$. Then there exists a unital *-homomorphism $\bigotimes_{n \in \mathbb{N}}^{(\max)} A \to F(B)$.*

As a particular consequence of these two theorems, along with Theorem 3.1.8, we see that, if there exist a unital, subhomogeneous C^*-algebra D without characters and a unital *-homomorphism $D \to F(A)$ then $A \otimes \mathcal{Z} \cong \mathcal{Z}$ (since subhomogeneous C^*-algebras are nuclear). The dimension drop algebras $I(k, k+1)$ where $k \geq 2$, are important examples of such algebras;

Proposition 3.1.11. *For any $k \geq 2$ the C^*-algebra $I(k, k+1)$ has no characters.*

Proof. Fix $k \geq 1$ and assume that $\varphi : I(k, k+1) \to \mathbb{C}$ is a character (i.e., a unital and multiplicative linear functional). It follows from basic C^*-algebra theory that φ is automatically a *-homomorphism. We show that this implies that $I(k, k+1)/\ker\varphi$ must be full matrix algebra.

Let $\psi : C([0,1]) \to I(k, k+1)$ denote the embedding given by $\psi(f) = f \cdot \mathbf{1}_{k(k+1)}$, where $(f \cdot \mathbf{1}_{k(k+1)})(t) = f(t) \cdot \mathbf{1}_{k(k+1)}$. Then $\varphi \circ \psi$ is a character on $C([0,1])$ whence there exists $t_0 \in [0,1]$ such that $\varphi \circ \psi(f) = f(t_0)$ for all $f \in C([0,1])$. As a consequence we see that $Z_{t_0} := \{f \cdot \mathbf{1}_{k(k+1)} \mid f(t_0) = 0\} \subseteq \ker\varphi$. This is essentially the information we require, but a small amount of work is still required. Let $m(t) \in \mathbb{N}$ be given by

$$m(t) = \begin{cases} k & \text{if } t = 0 \\ k(k+1) & \text{if } 0 < t < 1 \\ k+1 & \text{if } t = 1 \end{cases}$$

and $\mathrm{ev}_t : I(k, k+1) \to M_{m(t)}$ be given by $\mathrm{ev}_t(f) = f(t)$. It is not difficult to see that ev_t is surjective, for each $t \in [0,1]$, and as consequence we obtain that $I_t := \ker(\mathrm{ev}_t) = \{f \in I(k, k+1) \mid f(t) = 0\}$ is a maximal ideal in $I(k, k+1)$ for each $t \in [0,1]$. We show that $I_{t_0} \subseteq \ker\varphi$ which will yield the desired result.

For each $\varepsilon > 0$ such that $0 < t_0 - \varepsilon, t_0 + \varepsilon < 1$, let $g_\varepsilon : [0,1] \to [0,1]$ be given by

$$g_\varepsilon(t) = \begin{cases} 1 & \text{if } t \leq t_0 - \varepsilon \text{ or } t_0 - \varepsilon \leq t, \\ (t_0 - t)\varepsilon^{-1} & \text{if } t_0 - \varepsilon \leq t \leq t_0, \\ (t - t_0)\varepsilon^{-1} & \text{if } t_0 \leq t \leq t_0 + \varepsilon. \end{cases}$$

Basic functional analysis will show that the elements $(g_\varepsilon \cdot \mathbf{1}_{k(k+1)})_\varepsilon \subseteq Z_{t_0}$ is an approximate unit for $\ker(\mathrm{ev}_{t_0})$ and since each element in Z_{t_0} is central in $I(k, k+1)$, the previous considerations show that $\ker(\mathrm{ev}_{t_0}) \subseteq \ker\varphi$. As $\ker(\mathrm{ev}_{t_0})$ is a maximal ideal and $\ker\varphi$ is non-trivial, we therefore obtain the desired result. □

Obviously, $I(k, k+1)$ is a subhomogeneous algebra, being a subalgebra of $C([0,1]) \otimes M_k \otimes M_{k+1}$. Thus, for a unital, separable C^*-algebra A it follows that if there exists a unital *-homomorphism $I(k, k+1) \to F(A)$, then $A \otimes \mathcal{Z} \cong A$, which is a quite remarkable reduction in the complexity of the task. Nonetheless, verifying the existence of such a *-homomorphism remains highly non-trivial. Another major help in this quest comes from Winter and Rørdam, namely [27, Proposition 5.1]. Recall that two positive elements $a, b \in A_+$ are said to be **equivalent** if there exists an element $x \in A$ such that $a = x^*x$ and $b = xx^*$.

Proposition 3.1.12. *Let A be a unital C^*-algebra and suppose that there exist $\varepsilon > 0$ and mutually equivalent and orthogonal positive elements $a_1, \dots, a_k \in A_+$ such that*

$$1_A - (a_1 + a_2 + \cdots + a_k) \precsim (a_1 - \varepsilon)_+. \tag{3.1}$$

Then there exists a unital *-homomorphism $I(k, k+1) \to A$.

Now we have the necessary general machinery in place, and focus on proving that strict comparison, along with certain other regularity properties, implies $\mathcal{Z}$-stability. Recall that A^ω denotes the tracial ultrapower of A for a unital C^*-algebra A with $T(A) \neq \emptyset$ (see Definition 2.2.6). We have already seen that for a unital, separable, nuclear C^*-algebra A with $T(A) \neq \emptyset$ the restriction of the quotient map $\Phi : F(A) \to A^\omega \cap A'$ is surjective, and, if A is furthermore simple, has strict comparison and only finitely many extremal tracial states, we may find a unital embedding $M_k \to A^\omega \cap A'$ for any $k \geq 2$ (we only proved this for $k = 2$, but the proof generalizes easily to any $k \geq 2$). Thus the question arises whether this embedding can be lifted to a unital *-homomorphism $I(k, k+1) \to F(A)$ (the proof of Theorem 3.1.13 shows why this is not an unreasonable thing to expect). In general, this may not be the case, but with the aid of property (SI) (see Definition 2.6.2), we can show that such a lift is indeed possible, and that is essentially the content of the next theorem.

Theorem 3.1.13. *Let A be a unital, simple, separable, infinite-dimensional and nuclear C^*-algebra with $0 < |\partial_e T(A)| < \infty$ and strict comparison. Then $A \otimes \mathcal{Z} \cong A$.*

Proof. It follows from Proposition 2.5.3 that M_2 embeds unitally in $A^\omega \cap A'$ and from Theorem 2.6.12 that A has property SI. We aim to show that these two properties combine to yield a unital *-homomorphism $I(2,3) \to F(A)$, which will complete the proof.

Since the cone over M_2 is projective (see Theorem 1.4.16), it follows from Corollary 1.6.3 that the unital *-homomorphism $M_2 \to A^\omega \cap A'$ lifts to a ccp. order zero map $\varphi : M_2 \to F(A)$ (this map need not be unital). Let $(e_{ij})_{i,j=1,2}$ be the standard set of matrix units in M_2 and $a_j := \varphi(e_{jj})$, $j = 1, 2$. Then a_1 and a_2 are orthogonal and equivalent positive contractions in $F(A)$. Indeed, letting π_φ be given as in Theorem 1.6.2 and $x := \varphi(\mathbf{1}_2)^{1/2}\pi_\varphi(e_{21})$, we see that $a_1 = x^*x$ and $a_2 = xx^*$.

Let $e = \mathbf{1} - (a_1 + a_2)$. Then we have that $e \in J_A$ and $\|\mathbf{1} - a_1^m\|_{1,\omega} = \frac{1}{2}$ for each $m \in \mathbb{N}$. Indeed, let $(a_{1,n})_n \subseteq A$ be a lift of a_1 consisting of positive contractions and $\varphi_n : M_2 \to A$ be a sequence of ccp. order zero maps such that $\pi_\omega\big((\varphi_n(x))_n\big) = \varphi(x)$ for all $x \in M_2$. Then, since M_2 has a unique tracial state, which we denote tr_2, and a_1 is a positive contraction we see that for any $m \in \mathbb{N}$;

$$\begin{aligned}\|\mathbf{1} - a_1^m\|_{1,\omega} &= \lim_{n\to\omega} \max_{\tau\in T(A)} \tau(\mathbf{1} - a_{1,n}^m) \\ &= 1 - \lim_{n\to\omega} \min_{\tau\in T(A)} \tau\big(\varphi_n(e_{11})^m\big) \\ &= 1 - \mathrm{tr}_2(e_{11})\big(\lim_{n\to\omega} \min_{\tau\in T(A)} \tau(\varphi_n(\mathbf{1}_2)^m)\big).\end{aligned}$$

Since $\|\mathbf{1} - \varphi(\mathbf{1}_2)^m\|_{1,\omega} = 0$ the desired result follows.

It follows from Lemma 2.6.5 (i) that there exists $f \in F(A)$ such that $\sup_m \|1 - f^m\|_{1,\omega} = \sup_m \|1 - a_1^m\|_{1,\omega} = \frac{1}{2} < 1$ and $fa_1 = f$. Then property (SI) implies that $e \precsim f$ in $F(A)$, and since $fa_1 = f$ we also have that $f \precsim (a_1 - 1/2)_+$, thus transitivity of $\precsim$ implies that $e \precsim (a_1 - 1/2)_+$. It therefore follows from Proposition 3.1.12 that there exists a unital *-homomorphism $I(2,3) \to F(A)$. □

Remark 3.1.14. Note the similarity between the above theorem and Dusa McDuff's theorem that a $\|\cdot\|_2$ separable II_1-factor $\mathscr{M}$ absorbs the hyperfinite II_1-factor $\mathscr{R}$ if and only if there exists a unital embedding $M_2 \hookrightarrow \mathscr{M}^\omega \cap \mathscr{M}'$. The relation between McDuff von Neumann algebras and $\mathcal{Z}$-stable C^*-algebras is not superficial (see for instance [15, Theorem 7.8]).

Furthermore, note that the restrictions placed on the trace simplex in the above theorem are there to ensure the existence of unital embedding $M_2 \hookrightarrow A^\omega \cap A'$, the rest of the proof works in full generality.

Corollary 3.1.15. *The implication (iii)⇒(i) in Theorem 0.0.1 is true.*

3.2 Decomposition Rank

In this part of the thesis we aim to prove that if A has strict comparison, and possess certain other reguarity properties, we can bound the decomposition rank of A, thus proving the implication (iii)$\Rightarrow$(ii) in Theorem 0.0.1. Note that Lemma 3.2.5 represents a significantly more general statement than [19, Lemma 4.3]. Unfortunately, this does not lead to a more general statement in Theorem 3.2.9. However, it is felt by the author that the lemma deserves to be proven in as wide generality as possible.

Preliminary Investigation

Recall that A^ω denotes the tracial ultrapower of A (see Definition 2.2.6) and that $\Phi : A_\omega \to A^\omega$ denotes the quotient map. The reader may also wish to reacquaint herself with the definition of strict comparison for projections (Definition 1.9.13). Using the results previously obtained, we get the following proposition.

Proposition 3.2.1. *Suppose that B is a unital, separabel, simple and nuclear C^*-algebra and that A is a unital, separable, simple and exact C^*-algebras with strict comparison and $T(A) \neq \emptyset$. Suppose furthermore, that there exists a unital embedding $\pi : B \to A_\omega$. Then the following hold:*

(i) The map $T\big(A^\omega \cap \Phi(\pi(B))'\big) \to T\big(F\big(A, \pi(B)\big)\big)$ induced by the restriction of the quotient map $\Phi : F\big(A, \pi(B)\big) \to A^\omega \cap \Phi\big(\pi(B)\big)'$ is surjective.

(ii) If A also satisfies $|\partial_e T(A)| < \infty$, then $F\big(A, \pi(B)\big)$ has strict comparison for projections.

Proof. (i): It suffices to prove that $\tau(x) = 0$ for any $\tau \in T\big(F\big(A, \pi(B)\big)\big)$ and $x \in F\big(A, \pi(B)\big) \cap \ker \Phi$. We may assume that x is a positive contraction. Note that it follows from Theorem 2.6.12 that A has property (SI) relative to $\pi(B)$. Since

$$\|1 - (1 - x)^m\|_{1,\omega} = \Big\| \sum_{k=1}^{m} \lambda_k x^k \Big\|_{1,\omega} = 0$$

for some suitable scalars $\lambda_k \in \mathbb{R}$, it follows that we may find $s_1 \in F\big(A, \pi(B)\big)$ such that $s_1^* s_1 = x$ and $(1 - x)s_1 = s_1$. Clearly, this implies that $x + s_1 s_1^*$ is a positive contraction in $F\big(A, \pi(B)\big) \cap \ker \Phi$, whence the same argument yields an element $s_2 \in F\big(A, \pi(B)\big)$ such that $s_2^* s_2 = x$ and $(1 - x - s_1 s_1^*)s_2 = s_2$. Thus $x + s_1 s_1^* + s_2 s_2^*$ is again a positive contraction in $F\big(A, \pi(B)\big) \cap \ker \Phi$. Iterating this argument, we may for any $n \in \mathbb{N}$ find elements $s_1, \ldots, s_n \in F\big(A, \pi(B)\big)$ such that $\tau(s_i s_i^*) = \tau(x)$ for any $\tau \in T\big(F\big(A, \pi(B)\big)\big)$ and

$$0 \leq x + s_1 s_1^* + s_2 s_2^* + \cdots + s_n s_n^* \leq 1.$$

It therefore follows that $\tau(x) = 0$ for any $\tau \in T\big(F\big(A, \pi(B)\big)\big)$, and the desired result has been proven.

(ii): Let $\pi^{(n)} : B \to A_\omega \otimes M_n$ be given by $\pi^{(n)}(b) = b \otimes \mathbf{1}_n$, where $\mathbf{1}_n \in M_n$ denotes the identity. It is easy to see that $A_\omega \otimes M_n \cong (A \otimes M_n)_\omega$ for each $n \in \mathbb{N}$ and that under this identification we have that $F\big(A, \pi(B)\big) \otimes M_n \cong F\big(A \otimes M_n, \pi^{(n)}(B)\big)$. As $A \otimes M_n$ and $\pi^{(n)}$ satisfies the same assumptions as A and π, it is sufficient to consider projections in A.

Let $p, q \in A$ be projections such that $\tau(p) < \tau(q)$ for all tracial states τ on $T\big(F\big(A, \pi(B)\big)\big)$. By (i) it follows that $\overline{\tau} \circ \Phi(p) < \overline{\tau} \circ \Phi(q)$ for all tracial states $\overline{\tau}$ on $T\big(A^\omega \cap \Phi(\pi(B))'\big)$, and therefore Proposition 2.5.4 implies the existence of $\tilde{r} \in A^\omega \cap \Phi\big(\pi(B)\big)'$ such that $\tilde{r}^*\tilde{r} = \Phi(p)$ and $\tilde{r}\tilde{r}^* \leq \Phi(q)$. It follows from Proposition 1.4.10 that we can find a contraction $r \in F\big(A, \pi(B)\big)$ such that $\Phi(r) = \tilde{r}$. Let $w := qrp$ and note that $\Phi(w^*w) = \Phi(p)$. Let $(w_n)_n \in \ell^\infty(A)$ be a lift of w consisting of contractions and $(q_n)_n \in \ell^\infty(A)$ be a lift of q consisting of projections. As $qw = w$, we may assume that $q_n \geq w_n w_n^*$ for all $n \in \mathbb{N}$ (otherwise, replace w_n with $q_n w_n$).

Since $\tau(ww^*) = \tau(pr^*qrp) \leq \tau(p)$ we see that $\tau(q - ww^*) \geq \tau(q - p) > 0$ for all $\tau \in T\big(F\big(A, \pi(B)\big)\big)$. In particular

$$\tau'_\omega(q - ww^*) = \lim_{n \to \omega} \tau'(q_n - w_n w_n^*) > 0,$$

for all $\tau' \in T(A)$. Note that it follows from Proposition 2.4.3, along with the fact that $|\partial_e T(A)| < \infty$, that for any bounded sequence $(a_n)_n \in \ell^\infty(A)$ we have

$$\lim_{n \to \omega} \min_{\tau \in T(A)} \tau(a_n) = 1 - \lim_{n \to \omega} \max_{\tau \in \partial_e T(A)} \tau(1 - a_n) = \min_{\tau \in \partial_e T(A)} \lim_{n \to \omega} \tau(a_n).$$

As $\Phi(q - ww^*)$ is a projection in $A^\omega \cap \Phi\big(\pi(B)\big)'$ we see that both $(q_n - w_n w_n^*)_n, ((q_n - w_n w_n^*)^m)_n \in \ell^\infty(A)$ are lifts of $q - ww^*$ for any $m \in \mathbb{N}$, whence we obtain that

$$\lim_{n \to \omega} \min_{\tau \in T(A)} \tau\big((q_n - w_n w_n^*)^m\big) = \lim_{n \to \omega} \min_{\tau \in T(A)} \tau(q_n - w_n w_n^*) > 0,$$

for any $m \in \mathbb{N}$. Since $p - w^*w \in F\big(A, \pi(B)\big) \cap \ker \Phi$ and A has property (SI) relative to $\pi(B)$, we may find $s \in F\big(A, \pi(B)\big)$ such that $s^*s = p - w^*w$ and $(q - ww^*)s = s$. Upon letting $v = w + s$, we conclude that $v^*v = w^*w + s^*s = p$ and $qv = v$, which completes the proof. □

Lemma 3.2.2. *Let A be a unital C^*-algebra with $T(A) \neq \emptyset$, and $p, q \in A$ be projections. If A has strict comparison for projections and $\tau(p) = \tau(q)$ for all $\tau \in T(A)$, then for each $n \in \mathbb{N}$ there exist two contractions $v_i \in A \otimes M_n$, $i = 1, 2$, such that*

$$\Big\| \sum_{i=1,2} v_i^* v_i - p \otimes \mathbf{1}_n \Big\| \leq \frac{4}{n}, \quad \Big\| \sum_{i=1,2} v_i v_i^* - q \otimes \mathbf{1}_n \Big\| \leq \frac{4}{n},$$

and

$$\operatorname{dist}(v_i^* v_i, \{p \otimes x \mid x \in M_n\}) \leq \frac{2}{n}, \quad i = 1, 2.$$

Proof. Note that the statement is trivial for $n = 1, 2, 3$ and thus we may assume that $n \geq 4$. Let $(e_{ij})_{i,j=1}^n$ be the standard set of matrix units for M_n and set $e_{jj} =: e_j$. Since A has strict comparison for projections and $\tau\big(p \otimes e_1\big) < \tau\big(p \otimes (e_1 + e_2)\big)$, we may find a partial isometry $r_1 \in A \otimes M_n$ such that

$$r_1^* r_1 = p \otimes e_1, \quad r_1 r_1^* \leq q \otimes (e_1 + e_2).$$

The same line of reasoning shows that we may find partial isometries $r_2, r_3 \in A \otimes M_n$ such that

$$\begin{aligned} r_2^* r_2 &\leq p \otimes (e_2 + e_3); & r_2 r_2^* &= q \otimes (e_1 + e_2) - r_1 r_1^*; \\ r_3^* r_3 &= p \otimes (e_2 + e_3) - r_2^* r_2; & r_3 r_3^* &\leq q \otimes (e_3 + e_4). \end{aligned}$$

Repeating this argument as many times as needed, we obtain partial isometries $r_k \in A \otimes M_n$ such that

$$\begin{aligned} r_{k-1}^* r_{k-1} &\leq p \otimes (e_{k-1} + e_k); \\ r_k^* r_k &= p \otimes (e_{k-1} + e_k) - r_{k-1} r_{k-1}^*, \end{aligned}$$

and

$$\begin{aligned} r_k r_k^* &\leq q \otimes (e_k + e_{k+1}); \\ r_{k+1} r_{k+1}^* &= q \otimes (e_k + e_{k+1}) - r_k r_k^*, \end{aligned}$$

for each odd $1 \leq k \leq n-1$, with the convention that $e_0 = r_0 = r_n = 0$. Set

$$f = \sum_{j=1}^{n-1} \frac{n-j}{n} e_j, \quad v_1 = \sum_{j=1}^{n-1} \sqrt{\frac{n-j}{n}} r_j.$$

Since $\{r_j^* r_j \mid j = 1, \ldots, n-1\}$ and $\{r_j r_j^* \mid j = 1, \ldots, n-1\}$ are sets of mutually orthogonal projections we see that

$$\begin{aligned} \|v_1^* v_1 - p \otimes f\| &= \Big\| \sum_{j=1}^{n-1} \frac{n-j}{n} r_j^* r_j - p \otimes f \Big\| \\ &= \Big\| \sum_{j=1}^{n-1} \frac{n-j}{n} r_j^* r_j - \sum_{j=1}^{n-1} \frac{n-j}{n} p \otimes e_j \Big\| \\ &\leq \frac{1}{n} \Big\| \sum_{k=1}^{\infty} (p \otimes e_{2k} - r_{2k}^* r_{2k}) \Big\| \\ &\leq \frac{2}{n}, \end{aligned}$$

with the convention that $r_j = e_j = 0$ when $j \geq n$. Some computations are hidden from the reader in the equations above, but they are rather tedious and completely straightforward. In a similar fashion we see that

$$\|v_1 v_1^* - q \otimes f\| \leq \frac{2}{n}.$$

Similarly, upon letting

$$v_2 = \sum_{j=1}^{n-1} \sqrt{\frac{j}{n}} r_j$$

we find that

$$\|v_2^* v_2 - p \otimes (\mathbf{1}_n - f)\| \leq \frac{2}{n}$$
$$\|v_2 v_2^* - q \otimes (\mathbf{1}_n - f)\| \leq \frac{2}{n}$$

which completes the proof. □

Lemma 3.2.3. *Let A be a C^*-algebra and $\mathcal{S} \subseteq A$ be a subset consisting of positive contractions. If $f \in C_0((0,1])$ is positive then the induced map $f : \mathcal{S} \to A$ is uniformly continuous, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|f(a) - f(b)\| < \varepsilon$ whenever $\|a - b\| < \delta$.*

Proof. First, observe that for any $n \in \mathbb{N}$ we have that

$$\|a^n - b^n\| \leq \|a^{n-1} - b^{n-1}\| + \|a - b\|,$$

whenever a, b are contractions. Thus an easy induction argument shows that the map $\mathcal{S} \to A$ given by $a \mapsto a^n$ is uniformly continuous for each $n \in \mathbb{N}$. For a polynomial $f : [0,1] \to \mathbb{R}$, given by $f(z) = \sum_{i=0}^{n} \lambda_i z^i$ we therefore see that

$$\|f(a) - f(b)\| \leq \sum_{i=1}^{n} |\lambda_i| \cdot \|a^i - b^i\|,$$

and thus $f : \mathcal{S} \to A$ will be uniformly continuous. The general case then follows from Stone-Weierstrass along with a standard $\varepsilon/3$ argument. □

The following lemma is a slight modification of a result from another article of Kirchberg and Rørdam ([14, Lemma 2.2]). The proof is rather technical, but on the plus side, it relies on no heavy machinery (compare with the proof of [19, Lemma 4.3]).

Lemma 3.2.4. *Let A be a C^*-algebra, $a, b \in A_+$ and $\varepsilon > \|a - b\|$ be given. Then there exists a contraction $d \in A$ such that $d^* b d = (a - \varepsilon)_+$.*

Moreover, for each $\delta > 0$ there exists $\delta > \varepsilon > 0$ such that if $a, b \in A_+$ are contractions satisfying $\varepsilon > \|a - b\|$, then there exists a contraction $d \in A$ such that $dbd^* = (a - \varepsilon)_+$ and $\|b^{1/2} d^* - b^{1/2}\| < \delta$.

Proof. Let $\delta > 0$ be given. It follows from Lemma 3.2.3 that we may choose $\varepsilon > 0$ such that if $h, k \in A$ are positive contractions satisfying $\|h-k\| \leq 2\varepsilon$ then $\|h^{1/2} - k^{1/2}\| \leq \delta/3$. We may further assume that $\varepsilon < \min\{\delta^2/18, \delta\}$.

For any $r > 1$ let $g_r : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $g_r(t) = \min\{t, t^r\}$. Let $a, b \in A$ be positive contractions and $\varepsilon_1 > 0$ be given such that $\|a - b\| < \varepsilon$ and $\|a - b\| < \varepsilon_1 < \varepsilon$. Choose some $r > 1$ such that $\|a - g_r(b)\| < \varepsilon_1$ and $\|g_r(b)^{1/2} - b^{1/2}\| \leq \delta/3$. Put $b_0 := g_r(b)$ and note that $b_0 \leq b$, $b_0 \leq b^r$ and that $a - \varepsilon_1 \leq b_0$ (in the minimal unitization $\tilde{A}$). Furthermore, choose a positive contraction $e \in C^*(a) \subseteq C^*(a, 1_{\tilde{A}})$ such that $e(a - \varepsilon_1)e = (a - \varepsilon)_+$. A straigtforward calculation shows that $\|(1-e)a(1-e)\| \leq \varepsilon$.

Let $x = b_0^{1/2} e$ and $x = v|x|$ be the polar decomposition of x, with $v \in A^{**}$. Then, since $(a - \varepsilon)_+ = e(a - \varepsilon_1)e \leq x^*x$, we see that $(a - \varepsilon)_+^{1/2} \in \overline{x^*Ax}$. As $v^*v|x| = |x|$ it follows that $v^*vz = z$ for all $z \in \overline{x^*Ax}$ and therefore we obtain that $y := v(a - \varepsilon)_+^{1/2} \in A$ and $y^*y = (a - \varepsilon)_+$. Furthermore;

$$yy^* = v(a-\varepsilon)_+ v^* \leq vx^*xv^* = xx^* = b_0^{1/2} e^2 b_0^{1/2} \leq b_0.$$

Recalling Lemma 1.9.4 and that $b_0 \leq b^r$, we see that the sequence $(d_n)_n$, where $d_n = y^*(\frac{1}{n} + b^r)^{-1/2} b^{(r-1)/2}$, is convergent. Let d denote the limit and note that

$$\begin{aligned}
\|y^* - db^{1/2}\|^2 &= \lim_{n\to\infty} \|y^* - d_n b^{1/2}\|^2 \\
&= \lim_{n\to\infty} \Big\| y^* \Big(1 - \Big(\frac{1}{n} + b^r\Big)^{-1/2} b^{r/2}\Big) \Big\|^2 \\
&\leq \lim_{n\to\infty} \Big\| b^{1/2} \Big(1 - \Big(\frac{1}{n} + b^r\Big)^{-1/2} b^{r/2}\Big) \Big\|^2 = 0,
\end{aligned}$$

where we have used Dini's theorem (see Proposition 1.9.3) for the last equality. Thus we see that $db^{1/2} = y^*$ whence $dbd^* = y^*y = (a - \varepsilon)_+$, and since $yy^* \leq b$, we also find for each $n \in \mathbb{N}$ that (in the unitization of A)

$$d_n^* d_n \leq b^{(r-1)/2} \Big(\frac{1}{n} + b^r\Big)^{-1/2} b \Big(\frac{1}{n} + b^r\Big)^{-1/2} b^{(r-1)/2} \leq 1,$$

and hence $d \in A$ is a contraction.

Now we seek to bound $\|b^{1/2}d^* - b^{1/2}\|$. First note that

$$\|x^*x - (a-\varepsilon)_+\| = \|eb_0e - e(a - \varepsilon_1)e\| \leq \|b_0 - (a - \varepsilon_1)\| \leq 2\varepsilon_1,$$

which implies that $\|(x^*x)^{1/2} - (a - \varepsilon)_+^{1/2}\| \leq \delta/3$. Since $b^{1/2}d^* = y =$

$v(a-\varepsilon)_+^{1/2}$ and $x = v(x^*x)^{1/2}$, we therefore obtain;

$$\begin{aligned}
\|y - b^{1/2}\| &\leq \|y - x\| + \|x - b^{1/2}\| \\
&\leq \delta/3 + \|(b_0^{1/2} - b^{1/2})e\| + \|b^{1/2}e - b^{1/2}\| \\
&\leq \delta/3 + \|b_0^{1/2} - b^{1/2}\| + \|(1-e)b(1-e)\|^{1/2} \\
&\leq (2\delta)/3 + (\|(1-e)(b-a)(1-e)\| + \|(1-e)a(1-e)\|)^{1/2} \\
&\leq (2\delta)/3 + (2\varepsilon)^{1/2} < \delta,
\end{aligned}$$

which proves the second part of the lemma.

Notice however that we only needed the assumption that $a, b \in A$ were contractions to obtain the estimate on $\|b^{1/2}d^* - b^{1/2}\|$. Therefore, if this estimate holds no interest, we may ignore that part of the proof and thereby obtain the first part of the lemma. □

We are now ready to prove the lemma that will actually be used in the proof of Theorem 3.2.9.

Lemma 3.2.5. *Let A be a C^*-algebra and $B \subseteq A$ a subalgebra. If $v \in A_\omega$ is a contraction such that $v^*v \in B_\omega \subseteq A_\omega$, then there exists a sequence of contractions $(v_n)_n \subseteq A$ such that $v_n^*v_n \in B$ for every $n \in \mathbb{N}$ and $\pi_\omega\big((v_n)_n\big) = v$.*

Proof. Let $(w_n)_n \in \ell^\infty(A)$ be a lift of v consisting of contractions. By assumption, there exists a sequence of contractions $(h_n)_n \subseteq B$ such that

$$\lim_{n\to\omega} \|h_n - w_n^*w_n\| = 0.$$

Let $(\delta_n)_n \subseteq \mathbb{R}_+$ be a decreasing sequence of numbers such that $\delta_n > 0$ for every $n \in \mathbb{N}$ and $\delta_n \to 0$, and for each $n \in \mathbb{N}$ let $0 < \varepsilon_n < \delta_n$ be the corresponding positive number from Lemma 3.2.4. We may assume that the sequence $(\varepsilon_n)_n$ is decreasing. Furthermore, let

$$X_n := \{k \in \mathbb{N} \mid \|w_k^*w_k - h_k\| < \varepsilon_n\}.$$

By assumption, $(X_n)_n$ forms a decreasing sequence of non-empty sets in ω and it follows from Lemma 3.2.4 that for each $k \in X_n$ there exists a contraction $d_k \in A$ such that $d_k^*w_k^*w_kd_k = (h_k - \varepsilon_n)_+$ and $\|w_k^*w_kd - w_k^*w_k\| < \delta_n$. Let $v_k = w_kd_k$, when $k \in X_n \backslash X_{n+1}$, and otherwise $v_k = h_k$. Let $X_0 = \mathbb{N}$. Since ω is free, we may assume that $\cap_{n\in\mathbb{N}} X_n = \emptyset$. We therefore have that

$$\mathbb{N} = \bigsqcup_{n=0}^{\infty} (X_n \backslash X_{n+1}),$$

and thus the described recipe yields a sequence of contractions $(v_n)_n \in \ell^\infty(A)$. Furthermore, we have $v_k^*v_k = d_k^*w_k^*w_kd_k = (h_k - \varepsilon_n)_+ \in B$ when

$k \in X_n \backslash X_{n+1}$ and otherwise $v_k^* v_k = h_k^2 \in B$. Finally, we need to show that $\pi_\omega\big((v_n)_n\big) = v$.

To this end, let $\varepsilon > 0$ be given and choose $k \in \mathbb{N}$ such that $2\delta_k^{1/2} < \varepsilon$. Consider $l \in X_k$. There exists some $n \geq k$ such that $l \in X_n \backslash X_{n+1}$ and therefore

$$\begin{aligned} \|(v_l - w_l)^*(v_l - w_l)\| &= \|(d_l^* w_l^* - w_l^*)(w_l d_l - w_l)\| \\ &= \|(h_l - \varepsilon_n)_+ - w_l^* w_l d_l - d_l^* w_l^* w_l - w_l^* w_l\| \\ &\leq 2\varepsilon_n + 2\|w_l^* w_l d_l - w_l^* w_l\| \\ &\leq 4\delta_n \leq 4\delta_k. \end{aligned}$$

It follows that for all $l \in X_k$ we have that $\|v_l - w_l\| \leq 2\delta_k^{1/2} < \varepsilon$, and this completes the proof. □

$\mathcal{Z}$-Absorbing Algebras

The largest step towards proving that strict comparison implies finite decomposition rank (under certain extra assumptions) is represented by Theorem 3.2.9. Before venturing into the proof we need a few details.

Recall that **the universal UHF-algebra** $\mathcal{U}$ is defined to be the UHF-algebra of type $2^\infty \cdot 3^\infty \cdot 5^\infty \cdots$ (the readers who are unfamiliar with UHF-algebras may wish to consult [26, Section 7.4] for a friendly introduction to the subject). It is not difficult to deduce from well-known results on UHF-algebras that if $M_{\mathbf{p}}$ and $M_{\mathbf{q}}$ are UHF-algebras, of type $\mathbf{p}$ and $\mathbf{q}$ respectively, then $M_{\mathbf{p}} \otimes M_{\mathbf{q}}$ is agian a UHF-algebra of type $\mathbf{p} \cdot \mathbf{q}$. As a consequence, $\mathcal{U} \otimes \mathcal{U} \cong \mathcal{U}$. Furthermore, we note that in their original article (see [11, Corollary 6.3]) Jiang and Su proved the following theorem:

Theorem 3.2.6. *Let A be a unital, separable, simple and infinite-dimensional AF-algbra. Then $A \otimes \mathcal{Z} \cong A$.*

This theorem, in conjunction with Theorem 1.9.12 immediately yields the following result. It was originally proven by Rørdam, who later extended the result to Theorem 1.9.12.

Theorem 3.2.7. *If A is a unital, simple and exact C^*-algebra, then $A \otimes \mathcal{U}$ has strict comparison.*

The last major result we will import before proving Theorem 3.2.9 is the following. It was originally proven by Alain Connes (see [7, Theorem 5.1]).

Theorem 3.2.8. *Let $\mathcal{N}$ be a II_1-factor acting on a separable Hilbert space. Then the flip automorphism $\mathcal{N}\overline{\otimes}\mathcal{N} \to \mathcal{N}\overline{\otimes}\mathcal{N}$ given by $x \otimes y \mapsto y \otimes x$ is approximately inner in the strong operator topology if and only if $\mathcal{N}$ is isomorphic to the hyperfinite II_1-factor.*

See [12, Section 11.2] for an introduction to tensor products of von Neumann algebras. At last we are ready to prove theorems, but let us first introduce some notation, as we will need it in the remaining chapters as well: Given C^*-algebras $A_1, \dots, A_n, B$ and cp. maps $\varphi_i : A_i \to B$ we let $\sum_{i=1}^n \varphi_i : \bigoplus_{i=1}^n A_i \to B$ denote the induced cp. map and write $\left(\sum_{i=1}^n \varphi_i\right)(x_1 \oplus \cdots \oplus x_n) = \sum_{i=1}^n \varphi_i(x_i)$.

Theorem 3.2.9. *Let A be a unital, separable, simple and nuclear C^*-algebra with $T(A) = \{\tau_A\}$. If A is quasidiagonal then $dr(A \otimes \mathcal{U}) \leq 1$, where $\mathcal{U}$ denotes the universal UHF-algebra.*

Proof. Before we start the proof, we fix some notation. For any element $a \in A$ we let $\mathrm{Ad}_a : A \to A$ be given by $\mathrm{Ad}_a(b) = aba^*$. As (almost) all the C^*-algebras we will encounter in this proof are equipped with a unique tracial state, we will usually omit the reference to the trace and simply let $\|\cdot\|_2$ be the 2-norm associated with the unique tracial state (of course this agrees with the notation established in Definition 2.2.1, so this is all by way of saying that the ambient C^*-algebra will be implied by the context). Finally, we let $\iota : A \to A \otimes \mathcal{U}$ be the canonical embedding, i.e., $\iota(a) = a \otimes 1_{\mathcal{U}}$.

Before forging ahead, note that since $A \otimes M_n$ satisfies the conditions of the theorem for any $n \in \mathbb{N}$, it is sufficient to find ccp. maps $\tilde{\psi}_n \to M_{k_n} \oplus M_{k_n}$ and order zero cp. maps $\tilde{\varphi}_{i,n} : M_{k_n} \to A \otimes \mathcal{U}$ for $i = 1, 2$ such that $\sum_{i=1,2} \tilde{\varphi}_{i,n} : M_{k_n} \oplus M_{k_n} \to A \otimes \mathcal{U}$ is contractive and

$$\lim_{n\to\infty} \Big\| \Big(\sum_{i=1,2} \tilde{\varphi}_{i,n} \Big) \circ \tilde{\psi}_n(a) - \iota(a) \Big\| = 0$$

for all $a \in A$.

Since $\mathcal{U} \otimes \mathcal{U} \cong \mathcal{U}$, we may choose unital, commuting subalgebras $\mathcal{U}_i \subseteq \mathcal{U}$ for $i = 1, 2$ such that $\mathcal{U}_1 \cong \mathcal{U}_2 \cong \mathcal{U}$ and $\mathcal{U} = C^*(\mathcal{U}_1 \cup \mathcal{U}_2)$. In particular, letting $\iota_i : \mathcal{U}_i \to \mathcal{U}$ denote the inclusion, it follows that $\iota_1 \times \iota_2 : \mathcal{U}_1 \otimes \mathcal{U}_2 \to \mathcal{U}$ is an isomorphism.

As A is quasidiagonal, there exists a sequence of integers $(k_n)_n \subseteq \mathbb{N}$ and ucp. maps $\psi_n : A \to M_{k_n}$ such that for all $a, b \in A$:

$$\lim_{n\to\infty} \|\psi_n(a)\psi(b) - \psi_n(ab)\| = 0, \quad \lim_{n\to\infty} \|\psi_n(a)\| = \|a\|. \tag{3.2}$$

For each $n \in \mathbb{N}$, let $\varphi_n : M_{k_n} \to \mathcal{U}_1$ be a unital embedding. Thus, we may define maps $\sigma_n : A \to \mathcal{U}_1$ and $\sigma : A \to (\mathcal{U}_1)_\omega$ by

$$\sigma_n(a) = \varphi_n \circ \psi_n;$$
$$\sigma(a) = \pi_\omega^{\mathcal{U}_1}\big((\sigma_n(a))_n\big).$$

Note that it follows from (3.2) that σ is a unital embedding. The rest of the proof will be spent modifying the maps φ_n suitably, so as to obtain a sequence of maps with the desired property (as it turns out, essentially no modification is required on the ψ_n's).

By the canonical identifications of $A \otimes \mathcal{U}_1$ and $(\mathcal{U}_1)_\omega$ with subalgebras of $(A \otimes \mathcal{U}_1)_\omega$, we may consider ι and σ as unital embeddings in $(A \otimes \mathcal{U}_1)_\omega$. Under this identification, it is clear that $[\iota(a), \sigma(a)] = 0$ for all $a \in A$ and since A is nuclear, we find a *-homomorphism $\iota \times \sigma : A \otimes A \to (A \otimes \mathcal{U}_1)_\omega$ such that $(\iota \times \sigma)(a \otimes b) = \iota(a)\sigma(b)$.

Since A is nuclear and $T(A) = \{\tau_A\}$, we see that $\mathscr{N} := \pi_{\tau_A}(A)'' \subseteq B(\mathcal{H}_{\tau_A})$ is isomorphic to the hyperfinite II_1-factor and therefore the flip automorphism $\mathscr{N} \overline{\otimes} \mathscr{N} \to \mathscr{N} \overline{\otimes} \mathscr{N}$ is approximately inner in the strong operator topology (by Theorem 3.2.8). Furthermore, since τ_A is the unique tracial state on A, the map $\tau_A \otimes \tau_A$ is the unique tracial state on $A \otimes A$, whence an application of Kaplansky's Density Theorem, along with Proposition 2.3.6, yields a sequence of unitaries $(w_n)_n \subseteq A \otimes A$ such that

$$\lim_{n\to\infty} \|w_n(a \otimes b)w_n^* - b \otimes a\|_2 = 0$$

for all $a, b \in A$. Let $U_n = (\iota \times \sigma)(w_n) \in (A \otimes \mathcal{U}_1)_\omega$. Note that since $\tau_A \otimes \tau_A$ is the unique tracial state on $A \otimes A$, we have that $(\tau_A \otimes \tau_{\mathcal{U}_1})_\omega \circ (\iota \times \sigma) = \tau_A \otimes \tau_A$ and we therefore find that

$$\begin{aligned} &\lim_{n\to\infty} \|U_n \sigma(a) U_n^* - \iota(a)\|_{2,(\tau_A \otimes \tau_{\mathcal{U}_1})_\omega} \\ &= \lim_{n\to\infty} \|w_n(1_A \otimes a)w_n^* - a \otimes 1_A\|_2 = 0. \end{aligned}$$

An easy application of the ε-test yields a sequence of unitaries $(u_n)_n \subseteq A \otimes \mathcal{U}_1$ such that

$$\lim_{n\to\infty} \|u_n \sigma_n(a) u_n^* - \iota(a)\|_2 = 0 \tag{3.3}$$

for all $a \in A$.

To obtain a cleaner aestetic in what follows, we introduce the following notation:

$$\begin{aligned} \mathcal{A} &:= (A \otimes \mathcal{U} \otimes M_2)_\omega; \\ \mathcal{A}_1 &:= (A \otimes \mathcal{U}_1 \otimes M_2)_\omega \cong (A \otimes \mathcal{U}_1)_\omega \otimes M_2; \\ \mathscr{M} &= (A \otimes \mathcal{U}_1 \otimes M_2)^\omega, \end{aligned}$$

and we let $\Phi : \mathcal{A}_1 \to \mathscr{M}$ denote the quotient map. Let $\pi : A \to \mathcal{A}_1$ and $u \in \mathcal{A}_1$ be given by

$$\pi(a) = \begin{bmatrix} \sigma(a) & 0 \\ 0 & \iota(a) \end{bmatrix}, \quad u = \pi_\omega\left(\left(\begin{bmatrix} 0 & 0 \\ u_n & 0 \end{bmatrix}\right)_n\right).$$

Obviously π is a unital embedding. Furthermore, it follows from (3.3) that $\lim_{n\to\omega} \|u_n \sigma_n(a) - \iota(a) u_n\|_2 = 0$, whence $\Phi(u) \in \mathscr{M} \cap \Phi\big(\pi(A)\big)'$. Let $p, q \in$

$\mathcal{A}_1$ be the projections

$$p = \begin{bmatrix} 1_{A\otimes\mathcal{U}_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_{A\otimes\mathcal{U}_1} \end{bmatrix}.$$

Thus we easily see that $\Phi(u), \Phi(p), \Phi(q) \in \mathscr{M} \cap \Phi\big(\pi(A)\big)'$, and since $u^*u = p$, $uu^* = q$ we find that

$$\tilde{\tau} \circ \Phi(p) = \tilde{\tau} \circ \Phi(q)$$

for all tracial states $\tilde{\tau}$ on $\mathscr{M} \cap \Phi\big(\pi(A)\big)$. It therefore follows from Proposition 3.2.1 part (i) (this is applicable by Theorem 3.2.7), that $\tau(p) = \tau(q)$ for all tracial states τ on $\mathcal{A}_1 \cap \pi(A)'$.

Another application of Proposition 3.2.1 (this time part (ii)), and Lemma 3.2.2 yields for each $n \in \mathbb{N}$ contractions $s_{i,n} \in \big(\mathcal{A}_1 \cap \pi(A)'\big) \otimes M_n$ $i = 1, 2$, such that

$$\Big\| \sum_{i=1,2} s_{i,n}^* s_{i,n} - p \otimes 1_n \Big\| \leq \frac{4}{n}, \quad \Big\| \sum_{i=1,2} s_{i,n} s_{i,n}^* - q \otimes 1_n \Big\| \leq \frac{4}{n}$$

and

$$\mathrm{dist}(s_{i,n}^* s_{i,n}, \{p \otimes x \mid x \in M_n\}) \leq \frac{2}{n}, \quad i = 1, 2.$$

Now, for each $n \in \mathbb{N}$ we may consider M_n as a unital subalgebra of $\mathcal{U}_2$ and therefore, we see that we have a unital inclusion

$$\mathcal{A}_1 \otimes M_n \cong (A \otimes \mathcal{U}_1 \otimes M_n \otimes M_2)_\omega \hookrightarrow (A \otimes \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes M_2)_\omega \cong \mathcal{A},$$

where the last isomorphism is induced by the obvious isomorphism $A \otimes \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes M_2 \to A \otimes \mathcal{U} \otimes M_2$. Furthermore, it is clear that composing π with the above indicated inclusion, amounts to changing the co-domain of π to $\mathcal{A}$, and that $\{p \otimes x \mid x \in M_n\}$ is carried into the set $p(1_A \otimes \mathcal{U}_2 \otimes 1_2)_\omega$. Therefore, we obtain that

$$\big(\mathcal{A}_1 \cap \pi(A)'\big) \otimes M_n \hookrightarrow \mathcal{A} \cap \pi(A)',$$

and the earlier considerations therefore imply the existence of contractions $s_{i,n} \in \mathcal{A} \cap \pi(A)'$ such that

$$\Big\| \sum_{i=1,2} s_{i,n}^* s_{i,n} - p \Big\| \leq \frac{4}{n}, \quad \Big\| \sum_{i=1,2} s_{i,n} s_{i,n}^* - q \Big\| \leq \frac{4}{n}$$

and

$$\mathrm{dist}(s_{i,n}^* s_{i,n}, p(1_A \otimes \mathcal{U}_2 \otimes 1_2)_\omega) \leq \frac{2}{n}.$$

Another application of the ε-test yields two contractions $\tilde{v}_i \in \mathcal{A} \cap \pi(A)'$ such that

$$\sum_{i=1,2} \tilde{v}_i^* \tilde{v}_i = p, \quad \sum_{i=1,2} \tilde{v}_i \tilde{v}_i^* = q$$

and $\tilde{v}_i^*\tilde{v}_i \in p(1_A \otimes \mathcal{U}_2 \otimes 1_2)_\omega \subseteq p\mathcal{A}p$. We may therefore find contractions $v_i \in (A \otimes \mathcal{U})_\omega$ such that

$$\begin{bmatrix} 0 & 0 \\ v_i & 0 \end{bmatrix} = \tilde{v}_i, \quad i = 1, 2,$$

and hence

$$\sum_{i=1,2} v_i^* v_i = \sum_{i=1,2} v_i v_i^* = 1_{A\otimes\mathcal{U}}.$$

Additionally, the fact that $\tilde{v}_i \in \mathcal{A} \cap \pi(A)'$ implies that $v_i\sigma(a) = \iota(a)v_i$ for all $a \in A$, from which it follows that

$$\iota(a) = \sum_{i=1,2} v_i \sigma(a) v_i^*. \tag{3.4}$$

We are now ready to pull everything together and modify the φ_n's as promised earlier in the proof. First, for $i = 1, 2$ let $(v_{i,n}) \in \ell^\infty(A)$ be a lift of v_i consisting of contractions. As $\tilde{v}_i^*\tilde{v}_i \in p(1_A \otimes \mathcal{U}_2 \otimes 1_2)_\omega$, we see that $v_i^* v_i \in (1_A \otimes \mathcal{U}_2)_\omega$ and therefore, by Lemma 3.2.5, we may assume that $v_{i,n}^* v_{i,n} \in 1_A \otimes \mathcal{U}_2 \subseteq A \otimes \mathcal{U} \cap (1_A \otimes \mathcal{U}_1)'$. Thus we may, for each $n \in \mathbb{N}$ define a completely positive order zero map $\tilde{\varphi}_{i,n} : M_{k_n} \to A \otimes \mathcal{U}$, $i = 1, 2$, by $\tilde{\varphi}_{i,n} = \mathrm{Ad}_{v_i} \circ \varphi_n$, and completely positive maps $\tilde{\psi}_n : A \to M_{k_n} \oplus M_{k_n}$ by $\tilde{\psi}_n(a) = \psi_n(a) \oplus \psi_n(a)$. Translating (3.4) appropriately we see that

$$\lim_{n\to\omega} \Big\| \Big(\sum_{i=1,2} \tilde{\varphi}_{i,n} \Big) \circ \tilde{\psi}_n(a) - \iota(a) \Big\| = 0 \tag{3.5}$$

for all $a \in A$. Although the maps $\sum_{i=1,2} \tilde{\varphi}_{i,n}$ may not be contractions at the outset, the fact that each ψ_n is unital immediately implies;

$$\lim_{n\to\omega} \Big\| \sum_{i=1,2} \tilde{\varphi}_{i,n}(\mathbf{1}_{n_k}) \Big\| = \lim_{n\to\omega} \Big\| \sum_{i=1,2} v_{i,n} v_{i,n}^* \Big\| = \|1_{A\otimes\mathcal{U}}\| = 1.$$

Thus we may rescale, without affecting the assymptotic behaviour of the $\tilde{\varphi}_{i,n}$'s. Finally, the eagle-eyed reader may have spotted that the convergence in (3.5) is along a free ultrafilter, rather than convergence in the usual sense as required in the definition. However, since A is separable, we can – if necessary – switch to a subsequence in order to obtain the desired convergence, thus completing the proof. □

Theorem 3.2.10. *Let A be a unital, simple, separable and nuclear C^*-algebra. If A is quasidiagonal, has strict comparison and a unique tracial state, then* $\mathrm{dr}(A) \leq 3$.

Proof. We may assume that A is infinite-dimensional, since otherwise the conclusion is trivial. It follows from Theorem 3.1.13 that $A \cong A \otimes \mathcal{Z}$.

Therefore, by Proposition 3.1.7, it suffices to find, for each $\varepsilon > 0$ and finite subset $F \subseteq A$, finite dimensional C^*-algebras F_i, $i = 0, 1, 2, 3$ and cp. maps $\rho : A \to \bigoplus_{i=0}^{3} F_i$, and $\sigma_i : F_i \to A \otimes \mathcal{Z}$, such that each σ_i is order zero, $\sum_{i=0}^{3} \sigma_i$ is contractive and

$$\Big\| \Big(\sum_{i=0}^{3} \sigma_i \Big) \circ \rho - a \otimes 1_{\mathcal{Z}} \Big\| < \varepsilon$$

for all $a \in F$.

Hence, let $\varepsilon > 0$ be arbitrary and $F \subseteq A$ be any finte subset. It follows from Theorem 3.2.9 that we may find finite dimensional C^*-algebras E_0, E_1 and ccp. maps $\psi : A \to E_0 \oplus E_1$ and cp. order zero maps $\varphi_i : E_i \to A \otimes \mathcal{U}$ such that $\sum_{i=0,1} \varphi_i$ is contractive and

$$\Big\| \Big(\sum_{i=0,1} \varphi_i \Big) \circ \psi - a \otimes 1_{\mathcal{U}} \Big\| < \varepsilon/2$$

for all $a \in F$. We may assume, by perturbing each φ_i slightly, that $\varphi_i(E_i) \subseteq A \otimes M_l$ for some $l \in \mathbb{N}$ (see Proposition 1.6.5).

Choose $m \in \mathbb{N}$ such that $(m-1)^{-1} < \varepsilon/2$, and let $e_1, e_2, \ldots, e_{lm} \in M_{lm}$, $f_0, f_1, \ldots, f_{lm} \in M_{lm+1}$ be mutually orthogonal minimal projections. For $j = 1, 2, \ldots, m$, let

$$I_j := \{n \in \mathbb{N} \mid (j-1)l + 1 \leq n \leq jl\}.$$

For each $j = 1, 2, \ldots, m$ let $\alpha_j : M_l \to M_{lm}$, $\beta_j : M_l \to M_{lm+1}$ and $\gamma_j : M_l \to M_{lm+1}$ be embeddings such that

$$\alpha_j(\mathbf{1}_l) = \sum_{n \in I_j} e_n; \quad \beta_j(\mathbf{1}_l) = \sum_{n \in I_j} f_n; \quad \gamma_j(\mathbf{1}_l) = \sum_{n \in I_j} f_{n-1}.$$

Note that the image of the α_j's are mutually orthogonal sets, and the same is true for the β_j's and γ_j's. Furthermore $f_0 + \sum_{j=1}^{m} \beta_j(\mathbf{1}_l) = \mathbf{1}_{lm+1}$.

Let $w \in M_l \otimes M_l$ be a unitary such that $w(x \otimes y)w^* = y \otimes x$, for all $x, y \in M_l$, and let $\tilde{u} : [0,1] \to M_l \otimes M_l$ be a continuous path of unitaries such that $\tilde{u}(0) = \mathbf{1}_l \otimes \mathbf{1}_l$ and $\tilde{u}(1) = w$. Define a continuous path of unitaries $u : [0,1] \to M_{lm} \otimes M_{lm+1}$ by

$$u(t) = \mathbf{1}_{lm} \otimes f_0 + \sum_{j,k=1}^{m} (\alpha_j \otimes \beta_k)(u(t)).$$

It follows easily from earlier remarks that this is indeed a path of unitaries and it is trivially continuous. For $i = 0, 1$ we now define order zero cp. maps

$\lambda_i : M_l \to C([0,1]) \otimes M_{lm} \otimes M_{lm+1}$ by

$$\begin{aligned}\lambda_0(x)(t) =& u(t)\Big(\sum_{j=1}^{m}(1-t)\alpha_j(x)\otimes f_0 \\ &+ \sum_{j,k=1}^{m}\Big(1-\frac{(m-k)t}{m-1}\Big)\alpha_j(x)\otimes\beta_k(\mathbf{1}_l)\Big)u(t)^*\end{aligned}$$

and

$$\lambda_1(x)(t) = \sum_{k=1}^{m}\frac{(m-k)t}{m-1}\mathbf{1}_{lm}\otimes\gamma_k(x).$$

Some rather tedious, but straightforward, computations show that for any $t \in [0,1]$ we have

$$\mathbf{1} - \frac{t}{m-1} \leq \lambda_0(\mathbf{1}_l)(t) + \lambda_1(\mathbf{1}_l)(t) \leq \mathbf{1}. \tag{3.6}$$

In particular we have that $\lambda_0 + \lambda_1$ is contractive. Moreover, we easily see that

$$\lambda_0(x)(0) = \sum_{j=1}^{m}\alpha_j(x)\otimes\mathbf{1}_{lm+1}, \quad \lambda_0(x)(1) = \sum_{k=1}^{m}\Big(1-\frac{m-k}{m-1}\Big)\mathbf{1}_{lm}\otimes\beta_k(x),$$

and

$$\lambda_1(x)(0) = 0, \quad \lambda_1(x)(1) = \sum_{k=1}^{m}\frac{m-k}{m-1}\mathbf{1}_{lm}\otimes\gamma_k(x).$$

Hence we may regard each λ_i as a map into the dimension drop algebra $I(lm, lm+1)$, which we, in the interest of brevity, denote Z. The situation at this point will become clearer after considering the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\iota\quad} & A \otimes Z \\ \Big\downarrow{\scriptstyle \psi\oplus\psi} & & \Big\uparrow{\scriptstyle \sum_{i=0,1}\mathrm{id}_A\otimes\lambda_i} \\ (E_0 \oplus E_1)\oplus(E_0\oplus E_1) & \xrightarrow[(\varphi_0+\varphi_1)\oplus(\varphi_0+\varphi_1)]{} & (A\otimes M_l)\oplus(A\otimes M_l). \end{array}$$

Letting $\theta : (A \otimes M_l) \oplus (A \otimes M_l) \to A \otimes (M_l \oplus M_l)$ denote the canonical isomorphism we see that $\sum_{i=0,1}\mathrm{id}_A\otimes\lambda_i = \mathrm{id}_A \otimes (\sum_{i=0,1}\lambda_i)\circ\theta$. As a result, $\sum_{i,j=0,1}(\mathrm{id}_A\otimes\lambda_j)\circ\varphi_j$ is a ccp. map and it is obviously 3-decomposable.

Furthermore, it follows from (3.6) that

$$\begin{aligned}
&\Big\| \Big(\sum_{i,j=0,1} (\mathrm{id}_A \otimes \lambda_i) \circ \varphi_j \Big) \circ (\psi \oplus \psi)(a) - a \otimes 1_Z \Big\| \\
&< \Big\| \sum_{i=0,1} (\mathrm{id}_A \otimes \lambda_i)(a \otimes 1_l) - a \otimes 1_Z \Big\| + \frac{\varepsilon}{2} \\
&\leq \frac{1}{m-1} + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

for every $a \in F$. Since Z embeds unitally in $\mathcal{Z}$, we have hereby reached the desired destination. □

Remark 3.2.11. Note that the assumptions placed on A in the preceding theorem are there to ensure that $\mathrm{dr}(A \otimes \mathcal{U}) \leq 1$ and $A \otimes \mathcal{Z} \cong A$ and the latter pair of assumptions are in fact precisely what is needed for the above proof to work. Hence one may replace the theorem with the statement: If A is a separable C^*-algebra such that $\mathrm{dr}(A \otimes \mathcal{U}) \leq 1$ and $A \otimes \mathcal{Z} \cong A$, then $\mathrm{dr}A \leq 3$. Thus, restrictions of Theorem 3.2.10 may be viewed as a set of sufficient conditions for this 'new' theorem to be true. It is stated as is, since it is felt by the author that it is more informative in this form.

3.3 Nuclear Dimension

We seek to bound the nuclear dimension (see Section 1.8) of Kirchberg Algebras, and in particular this will show that decomposition rank and nuclear dimension are entirely different concepts, despite the apparent similarity in the definitions.

Kirchberg Algebras

We will take a quick tour through the world of Kirchberg algebras in order to remind the reader of the most important results. Proofs will not be given but the reader may consult [24] or [6] for proofs of (most of) the statements made here.

Definition 3.3.1 (Simple, Purely Infinite)**.** A simple C^*-algebra A is said to be **purely infinite** if $A \ncong \mathbb{C}$ and for every pair of non-zero positive elements $a, b \in A$ there exists an element $x \in A$ such that $xax^* = b$.

There are many ways to define simple, purely infinite C^*-algebras (note that any C^*-algebra satisfying the above definition is necessarily simple), and one statement we shall be needing later on is the following. Recall that a projection $p \in A$ is said to be **properly infinite** if there exists projections $p_1, p_2 \in A$ such that $p \simeq p_1 \simeq p_2$ and $p_1 + p_2 \leq p$.

Theorem 3.3.2. *A simple C^*-algebra A is purely infinite if and only if every self-adjoint element in A is the norm limit of self-adjoint elements with finite spectrum and every non-zero projection is properly infinite.*

The above theorem implies that if A is a simple, purely infinite C^*-algebra, then the projections span a dense subset in A. Hence they all have lots of projections, in a certain sense.

Definition 3.3.3 (Kirchberg Algebras)**.** A **Kirchberg algebra** is a C^*-algebra which is separable, simple, purely infinite and nuclear.

The most important examples of Kirchberg algebras are the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$.

Definition 3.3.4 (Cuntz Algebras)**.** For every $2 \leq n < \infty$ the **Cuntz algebra** $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries $s_1, \dots, s_n$ such that $\sum_{i=1}^n s_i s_i^* = \mathbf{1}$. Similarly, $\mathcal{O}_\infty$ is the universal C^*-algebra generated by a sequence of isometries $(s_k)_{k\in\mathbb{N}}$ with orthogonal range projections.

The importance of $\mathcal{O}_2$ and $\mathcal{O}_\infty$ is expressed in the following three theorems.

Theorem 3.3.5. *Any separable, exact C^*-algebra A embeds in $\mathcal{O}_2$. Furthermore, if A is unital, then the embedding can be chosen to be unital.*

Theorem 3.3.6. *Let A be a C^*-algebra. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ if and only if A is unital, simple, separable and nuclear.*

Theorem 3.3.7. *Let A be a unital, simple, separable and nuclear C^*-algebra. Then $A \otimes \mathcal{O}_\infty \cong A$ if and only if A is purely infinite.*

The above theorems are collectively referred to as the Kirchberg-Phillips theorems in the litterature (or even collectively as *the* Kirchberg-Phillp theorem). As a particular consequence of these theorems we see that if A is a unital Kirchberg algebra, then A embeds unitally in $\mathcal{O}_2$, $\mathcal{O}_2$ absorbs A, i.e., $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and A absorbs $\mathcal{O}_\infty$. One last observation before we start proving some results:

Proposition 3.3.8. *Let A, B be unital Kirchberg algebras and assume there exists a unital embedding*

$$\pi : B \to A_\omega$$

for some free ultrafilter ω on $\mathbb{N}$. Then $A_\omega \cap \pi(B)'$ is a unital, simple and purely infinite C^*-algebra.

Proof. This is a straightforward modification of the proof of [24, Theorem 7.1.1]. Those familiar with Kirchberg algebras should experience no trouble in modifying the proof, and those who are unfamiliar, will not gain any significant insight by seeing a sketch of the proof anyway and therefore it is omitted. □

We aim to prove that any Kirchberg algebra has nuclear dimension at most 3. The proof roughly follows the outline of Theorem 3.2.10, but with $\mathcal{O}_\infty$ in place of $\mathcal{Z}$ and $\mathcal{O}_2$ in place of $\mathcal{U}$. Obviously, different techniques are required in the process, but we more or less seek to import the proof to this setting as directly as possible. Thus, the first step is proving that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ has nuclear dimension at most 1 (in fact it will equal 1 as $\mathcal{O}_2$ is not an AF-algebra. See Proposition 1.8.6).

The Cuntz Algebra $\mathcal{O}_2$.

It turns out that it will help us to discuss the Cuntz-Toeplitz algebras before getting into the proof of the main theorem. We will discuss this in connection with $\mathcal{O}_n$ for any $2 \leq n < \infty$, as the restriction to the case $n = 2$ does not lead to a significant reduction in complexity (at least not until the end). Readers should be advised that there is a lot of notation to keep track of and unfortunately the proof comes off as rather technical, although the basic idea is relatively straightforward. The notation introduced in this section will not be carried into the next section.

Let $2 \leq n < \infty$ be fixed. Recall that the Cuntz-Toeplitz algebra $\mathcal{T}_n$ is the universal C^*-algebra generated by n isometries with orthogonal range projections. We outline a particular realization of both the Cuntz-Toeplitz algebra $\mathcal{T}_n$ and the Cuntz algebra $\mathcal{O}_n$ in one fell swoop. Let $\mathcal{H}$ denote any n-dimensional Hilbert space and consider the full Fock space

$$\Gamma(n) := \bigoplus_{l=0}^{\infty} \mathcal{H}^{\otimes l},$$

where $\mathcal{H}^{\otimes l} := \bigotimes_{k=1}^{l} \mathcal{H}$ when $l \geq 1$ and $\mathcal{H}^{\otimes 0}$ is the span of a norm one vector Ω. It is common to refer to $\mathcal{H}^{\otimes 0}$ as "the vacuum". Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis for $\mathcal{H}$. It is well-known that the representation of $\mathcal{T}_n$ on $B(\Gamma(n))$ given by the isometries $T_i\xi = e_i \otimes \xi$, is faithful.

Let $I := \{1, 2, \dots, n\}$ and $W_n := \bigcup_{k=0}^{\infty} I^k$, i.e., the set of words in the letters $\{1, 2, \dots, n\}$, with the convention that $I^0 = \{0\}$. When $\mu = i_1 i_2 \dots i_k \in W_n$ we set $e_\mu := e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$ and $e_0 := \Omega$. Similarly, for $\mu = i_1 i_2 \cdots i_k \in W_n$ we let $T_\mu := T_{i_1} T_{i_2} \cdots T_{i_k}$ and $T_0 := 1_{\mathcal{T}_n}$. Then $\{e_\mu \mid \mu \in W_n\}$ is an orthonormal basis for $\Gamma(n)$, i.e., the set $\{e_\nu \mid |\nu| = k\}$ is an orthonormal basis for $\mathcal{H}^{\otimes k}$, and the subspace spanned by $\{T_\mu T_\nu^* \mid \mu, \nu \in W_n\}$ is dense in $\mathcal{T}_n$. For each pair of words $\mu, \nu \in W_n$ we define rank 1 maps $e_{\mu,\nu}$ by setting

$$e_{\mu,\nu}(\xi) := \langle \xi, e_\nu \rangle e_\mu.$$

For obvious reasons, we refer to the maps $e_{\mu,\nu}$ as matrix units. Let M_∞ denote the *-algebra spanned by $\{e_{\mu,\nu} \mid \mu, \nu \in W_n\}$. Obviously, $M_\infty \subseteq \overline{M_\infty} = \mathbb{K}(\Gamma(n)) \subseteq B(\Gamma(n))$. Note that $\mathcal{T}_n$ contains all the matrix units, namely

$$e_{\mu,\nu} = T_\mu \Big(\mathbf{1} - \sum_{i=1}^{n} T_i T_i^* \Big) T_\nu^* \in \mathcal{T}_n,$$

where $\mathbf{1} := \mathbf{1}_{\Gamma(n)}$ denotes the identity. This is easy to see, once it has been noted that $\mathbf{1} - \sum_{i=1}^{n} T_i T_i^*$ is the projection on to the vacuum. As a consequence, we find that $\mathbb{K}(\Gamma(n)) \subseteq \mathcal{T}_n$, and, letting $q \colon B(\Gamma(n)) \to B(\Gamma(n))/\mathbb{K}(\Gamma(n)) =: Q(\Gamma(n))$ denote the quotient map, we have the short exact sequence;

$$0 \longrightarrow \mathbb{K}(\Gamma(n)) \hookrightarrow \mathcal{T}_n \xrightarrow{q} \mathcal{O}_n \longrightarrow 0.$$

This is easy to see once it has been noted that the elements $S_i := q(T_i)$ satisfies the Cuntz relation, since $\mathbf{1} - \sum_{i=1}^{n} T_i T_i^*$ is a rank one projection. Let $\mathbf{1}_i := \mathbf{1}_{\mathcal{H}^{\otimes i}}$. We may regard $e_{\mu,\nu} \otimes \mathbf{1}_i$ as a partial isometry on $\Gamma(n)$ by setting $e_{\mu,\nu} \otimes \mathbf{1}_i(\xi) = 0$ when $\xi \in \big(\mathcal{H}^{\otimes (i+|\nu|)}\big)^{\perp}$, and we infer that

$$T_\mu T_\nu^* = \sum_{i=0}^{\infty} e_{\mu,\nu} \otimes \mathbf{1}_i,$$

where the sum is to be taken in the strong operator topology (it is convergent since each of the partial isometries $e_{\mu,\nu}\otimes\mathbf{1}_i$ have orthogonal range and source projections).

For fixed integers $r, k \in \mathbb{N}$ we define the cut-off spaces

$$\Gamma_{r,r+k}(n) := \bigoplus_{l=r}^{r+k-1} \mathcal{H}^{\otimes l};$$
$$\Gamma_k(n) := \Gamma_{0,k}(n).$$

We will frequently omit the reference to n when no confusion should arise, and simply write $\Gamma_{r,r+k}$. Furthermore, we let $\Lambda_k : M_\infty \to B(\Gamma(n))$ be given by

$$\Lambda_k(x) = \sum_{l=0}^{\infty} x \otimes \mathbf{1}_{kl}.$$

Once again, it is easily checked that this is well-defined on matrix units and we can therefore extend linearly.

The reader may verify that

$$\Gamma(n) \cong \Gamma_k(n) \otimes \Gamma(n^k),$$

via the map $e_\mu \mapsto e_{\mu_1}\otimes e_{\mu_2}$, where $\mu = \mu_1\mu_2$ is the unique decomposition such that $|\mu_1| < k$ and $|\mu_2|$ is a multiple of k. Corresponding to this factorization we consider

$$\mathcal{A}_k := B\big(\Gamma_k(n)\big) \otimes \mathcal{T}_{n^k} \subseteq B\big(\Gamma(n)\big).$$

Since dim $\Gamma_k(n) = 1 + n + n^2 + \cdots + n^{k-1} = \frac{n^k-1}{n-1} := d_k$, we see that $\mathcal{A}_k \cong M_{d_k} \otimes \mathcal{T}_{n^k}$. We immediately find that $\mathcal{A}_k$ (when considered as operators on $\Gamma(n)$) contains $\mathcal{T}_n$. Indeed, letting the generators $\hat{T}_v$ of $\mathcal{T}_{n^k} \subseteq B\big(\Gamma(n^k)\big)$ be indexed by the words $v \in W_n$ such that $|v| = k$, we see that the generators T_i of $\mathcal{T}_n$ have the following matrix representation:

$$\begin{aligned} T_i &= \sum_{j=0}^{\infty} e_{i,0} \otimes \mathbf{1}_j, \quad \big(\text{on } \Gamma(n)\big) \\ &= \Big(\sum_{j=0}^{k-2} e_{i,0} \otimes \mathbf{1}_j\Big) \otimes \mathbf{1}_{\Gamma(n^k)} + \sum_{|w|=k-1} e_{0,w} \otimes \hat{T}_{iw} \quad \big(\text{on } \Gamma_k(n) \otimes \Gamma(n^k)\big). \end{aligned}$$

This identity is readily checked on basis vectors.

We embark on the quest to determine the nuclear dimension of $\mathcal{O}_2$ with the following proposition.

Proposition 3.3.9. *With the notation established so far we have:*

(i) $\Lambda_k(M_\infty) \subseteq \mathcal{A}_k \cong M_{d_k} \otimes \mathcal{T}_{n^k}$.

(ii) *For any non-negative integer $r \in \mathbb{N}_0$, the restriction $\Lambda_k|_{B(\Gamma_{r,r+k})}$ is a *-homomorphism.*

Proof. (i): Given $\mu, \nu \in W_n$ we let $\mu = u\overline{\mu}$ and $\nu = v\overline{\nu}$ be the unqiue decompositions such that $|u|, |v| < k$ and $|\overline{\mu}|, |\overline{\nu}|$ are multiples of k. Letting $\hat{T}_\gamma$ be the generators of $\mathcal{T}_{n^k}$ indexed by the words in $\gamma \in W_n$ of length k, we find that

$$\begin{aligned}\Lambda_k(e_{\mu,\nu}) &= \sum_{l=0}^{\infty} e_{\mu,\nu} \otimes \mathbf{1}_{kl} \\ &= e_{u,v} \otimes \left(\sum_{l=0}^{\infty} e_{\overline{\mu},\overline{\nu}} \otimes \mathbf{1}_{kl} \right) \\ &= e_{u,v} \otimes \hat{T}_{\overline{\mu}} \hat{T}^*_{\overline{\nu}},\end{aligned}$$

which clearly completes the proof of (i).

(ii): This follows easily, once it has been observed that for each of the orthogonal subspaces $\Gamma_{r+kl,r+k(l+1)}$, the maps $B(\Gamma_{r,r+k}) \to B(\Gamma_{r+kl,r+k(l+1)})$ given by

$$x \mapsto x \otimes \mathbf{1}_{kl},$$

are *-homomorphisms. The details are left to the reader. □

Since the C^*-algebras $B(\Gamma_{r,r+k})$ are all finite-dimensional (full matrix algebras in fact), we now have an order zero ccp. map $B(\Gamma_{r,r+k}) \to \mathcal{A}_k$ (a *-homomorphism in fact), and therefore a reasonable starting point for obtaining the desired filtrations through finite-dimensional algebras. The next step is finding reasonable ccp. maps $\psi_k : \mathcal{T}_n \to B(\Gamma_{r(k),r(k)+k})$ for some non-negative integers $r(k)$. Of course, $\mathcal{A}_k$ is in general different from $\mathcal{T}_n$, but this is not a serious obstacle, as it turns out.

Let $P_{r,r+k} \in B(\Gamma(n))$ denote the projection onto $\Gamma_{r,r+k}$, let $P_k := P_{k,2k}$ and let $Q_k := P_{\lceil k/2 \rceil + k, \lceil k/2 \rceil + 2k}$. Let $l := \lceil k/2 \rceil$, and consider the following $k \times k$ matrices: when k is even, let

$$\kappa_k := \frac{1}{l+1} \begin{pmatrix} 1 & 1 & 1 & \cdots & & & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & & & \cdots & 2 & 2 & 1 \\ 1 & 2 & 3 & & & & & 3 & 2 & 1 \\ \vdots & \vdots & & & & & & & \vdots & \vdots \\ & & & & l & l & & & & \\ & & & & l & l & & & & \\ \vdots & \vdots & & & & & & & \vdots & \vdots \\ 1 & 2 & 3 & & & & & 3 & 2 & 1 \\ 1 & 2 & 2 & \cdots & & & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & & & \cdots & 1 & 1 & 1 \end{pmatrix}$$

and when k is odd let

$$\kappa_k = \frac{1}{l+1}\begin{pmatrix}
1 & 1 & \cdots & & & & \cdots & 1 & 1 \\
1 & 2 & & & & & & 2 & 1 \\
\vdots & & & & & & & & \vdots \\
 & & & l-1 & l-1 & l-1 & & & \\
 & & & l-1 & l & l-1 & & & \\
 & & & l-1 & l-1 & l-1 & & & \\
\vdots & & & & & & & & \vdots \\
1 & 2 & & & & & & 2 & 1 \\
1 & 1 & \cdots & & & & \cdots & 1 & 1
\end{pmatrix}$$

We check that in both cases κ_k is a positive matrix. For each $m \in \mathbb{N}$ let E_m denote the $m \times m$ matrix with 1 in each entry, and similarly 0_m denote the $m \times m$ matrix with 0 in each entry. Then we see (with the notation established in Section 1.9) that when k is even

$$\kappa_k = \frac{1}{l+1}\sum_{m=0}^{l-1} 0_m \oplus E_{2(l-m)} \oplus 0_m$$

and when k is odd

$$\kappa_k = \frac{1}{l+1}\sum_{m=0}^{l-1} 0_m \oplus E_{2(l-m)-1} \oplus 0_m.$$

Since $(0_m \oplus E_p \oplus 0_m)^2 = p(0_m \oplus E_p \oplus 0_m)$, for each $m, p \in \mathbb{N}$, it is easily deduced that $\frac{1}{p}(0_m \oplus E_p \oplus 0_m)$ is a projection, whence $0_m \oplus E_p \oplus E_m$ is a positive matrix. As κ_k can be written as a sum of positive elements, it is therefore positive. The next lemma, although a slight detour, is so easily proven that it may as well be included. First we define the Schur multiplier of matrices. This is a more general concept, and the general definition does not quite agree with the following, since our concept is dependent on a particular decomposition of the Hilbert space in question. However, we will only need the lemma in the specified setting, so no serious harm is done.

Definition 3.3.10. Let $A = [A_{i,j}]_{i,j=1}^k \in M_k$ and $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i$ be a finite direct sum of Hilbert spaces. Regard every operator $x \in B(\mathcal{H})$ as an operator matrix $[x_{i,j}]_{i,j=1}^k$ where $x_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$. Then the **Schur multiplier $\mathbf{m_A}$** with respect to the decomposition $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i$ associated with A is the map $m_A : B(\mathcal{H}) \to B(\mathcal{H})$ given by

$$m_A([x_{i,j}]) = [A_{i,j} \cdot x_{i,j}].$$

Note that in the context of the above definition, the Schur multiplier m_A is always well-defined.

Lemma 3.3.11. *Let $A \in M_k$ be a positive matrix and $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i$ be a separable Hilbert space. Then the Schur multiplier m_A associated with A is a completely positive map.*

Proof. Let $\mathcal{K} := \mathbb{C}^k$ and for each $1 \leq i \leq k$ let $(\eta_n^{(i)})_{n\in J_i}$ be an orthonormal basis for $\mathcal{H}_i$. Furthermore, let $(e_i)_{i=1}^k$ be the standard basis for $\mathcal{K}$, and $A = [A_{i,j}]_{i,j=1}^k$. Since A is positive, we see that

$$A_{i,j} = \langle Ae_j, e_i\rangle = \langle A^{1/2}e_j, A^{1/2}e_i\rangle.$$

For each $1 \leq i \leq k$ let $\xi_i = A^{1/2}e_i$. Let $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ be given by $V(\eta_n^{(i)}) = \eta_n^{(i)} \otimes \xi_i$ for each $n \in J_i$ and $1 \leq i \leq k$ and $x = [x_{i,j}]_{i,j} \in B(\mathcal{H})$ be arbitrary. Then we see that

$$\begin{aligned}\langle V^*(x \otimes \mathbf{1}_{\mathcal{K}})V(\eta_l^{(j)}), \eta_m^{(i)}\rangle &= \langle (x \otimes \mathbf{1}_{\mathcal{K}})(\eta_l^{(j)} \otimes \xi_j), \eta_m^{(i)} \otimes \xi_i\rangle \\ &= \langle x_{i,j}(\eta_l^{(j)}), \eta_m^{(i)}\rangle\langle \xi_j, \xi_i\rangle \\ &= \langle A_{i,j}x_{i,j}(\eta_l^{(j)}), \eta_m^{(i)}\rangle,\end{aligned}$$

for any $1 \leq i, j \leq k$, $l \in J_j$ and $m \in J_m$. It clearly follows that $V^*(x \otimes \mathbf{1}_{\mathcal{K}})V = m_A(x)$, and since $x \in B(\mathcal{H})$ was arbitrary we see that m_A is completely positive. □

The above lemma, when applied to κ_k, shows that $m_{\kappa_k} : B(\Gamma_{r,r+k}) \to B(\Gamma_{r,r+k})$ is a ccp. map for any $r \in \mathbb{N}_0$. Indeed, it follows from the lemma that m_{κ_k} is completely positive, and, upon denoting $\kappa_k = [\kappa_k(i,j)]_{i,j=1}^k$ and recalling that $\mathbf{1}_i$ is the identity operator on $\mathcal{H}^{\otimes i}$, we see that

$$\begin{aligned}\| m_{\kappa_k}(\mathbf{1}_{\Gamma_{r,r+k}}) \| &= \left\| m_{\kappa_k}\left(\sum_{l=1}^k \mathbf{1}_{l+r-1}\right)\right\| \\ &= \left\| \sum_{l=1}^k \kappa_k(l,l)\mathbf{1}_{l+r-1}\right\| \leq 1,\end{aligned}$$

since $|\kappa_k(i,i)| \leq 1$ for all $1 \leq i \leq k$.

With this at hand, we define maps $\psi\colon \mathcal{T}_n \to B(\Gamma_{k,2k}) \oplus B(\Gamma_{k+l,2k+l})$ (recall that $l := \lceil k/2 \rceil$) by

$$\psi_k(x) := m_{\kappa_k}(P_k x P_k) \oplus m_{\kappa_k}(Q_k x Q_k),$$

and $\varphi_k\colon B(\Gamma_{k,2k}) \oplus B(\Gamma_{k+l,2k+l}) \to \mathcal{A}_k$ by

$$\varphi_k(x \oplus y) := \Lambda_k(x) + \Lambda_k(y).$$

Evidently, it follows from the previous considerations that $\|\psi_k\| \leq 1$ and $\|\varphi_k\| \leq 2$. Clearly, Proposition 3.3.9 implies that φ_k is 1-decomposable for each $k \in \mathbb{N}$. Before proving the main theorem we consider the composition $q \circ \varphi_k \circ \psi_k : \mathcal{T}_n \to Q(\Gamma(n))$.

Lemma 3.3.12. *For fixed $\mu, \nu \in W_n$ we have*

$$\lim_{k\to\infty} \big\| q\big(\varphi_k \circ \psi_k(T_\mu T_\nu^*) - T_\mu T_\nu^*\big) \big\| = 0.$$

Proof. A detailed proof would involve some rather tedious, but straightforward, computations, which will be skipped here. Let A_k, B_k denote the $\mathbb{N}_0 \times \mathbb{N}_0$ matrices

$$A_k := 0_k \oplus \kappa_k \oplus \kappa_k \oplus \cdots = 0_k \oplus \bigoplus_{n=1}^{\infty} \kappa_k$$
$$B_k := 0_k \oplus 0_l \oplus \kappa_k \oplus \kappa_k \oplus \cdots = 0_k \oplus 0_l \oplus \bigoplus_{n=1}^{\infty} \kappa_k.$$

We let $A_k + B_k = [\sigma_{i,j}]_{i,j\in\mathbb{N}_0}$ and the reader may check that $|1 - \sigma_{i,i+p}| \leq \frac{2(2+2|p|)}{k}$, when $i > 2k + l$ and $|p| \leq l$. Furthermore, for sufficiently large $k \in \mathbb{N}$, we have that

$$\varphi_k \circ \psi_k(T_\mu T_\nu^*) = \sum_{r=0}^{\infty} \sigma_{|\mu|+r,|\nu|+r} e_{\mu,\nu} \otimes \mathbf{1}_r.$$

The above is certainly true when $|\mu|, |\nu| < 2k$ and $\big||\nu| - |\mu|\big| \leq l$. In this case, once it has been observed that

$$q\left(\sum_{r=0}^{\infty} \sigma_{|\mu|+r,|\nu|+r} e_{\mu,\nu} \otimes \mathbf{1}_r\right) = q\left(\sum_{r=m}^{\infty} \sigma_{|\mu|+r,|\nu|+r} e_{\mu,\nu} \otimes \mathbf{1}_r\right)$$

for any $m \in \mathbb{N}$, we find that

$$\big\| q\big(\varphi_k \circ \psi_k(T_\mu T_\nu^*) - T_\mu T_\nu^*\big) \big\| \leq \frac{2\big(2 + 2\big||\mu| - |\nu|\big|\big)}{k},$$

whence the desired result follows. $\square$

We are now close to a proof that $\dim_{\text{nuc}}\mathcal{O}_2 = 1$, but we need to modify the maps $q \circ \varphi_k$ in order to ensure that the range is contained in $\mathcal{O}_2 \subseteq Q(\Gamma(n))$. The contents of Theorem 3.3.14 is essentially that we can do just that. During the proof of that theorem we will need the following proposition. In the present context we could have applied Theorem 3.3.6 to obtain the desired result, but it seems unnecessary to apply that level of sophistication to a result which is this simple.

Proposition 3.3.13. *Let A be a unital C^*-algebra and assume that A contains a set of matrix units $(f_{ij})_{i,j=1}^n$ such that $\sum_{i=1}^n f_{ii} = 1_A$. Then $A \cong M_n \otimes (f_{11} A f_{11})$.*

In particular, $M_n \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for any $n \in \mathbb{N}$.

Proof. Let $A_0 := f_{11}Af_{11}$ and define a map $\varphi : A \to M_n \otimes A_0$ by

$$\varphi(a) = \sum_{i,j=1}^{n} e_{ij} \otimes f_{1i}af_{j1},$$

where $(e_{ij})_{i,j=1}^n \subseteq M_n$ denotes the standard set of matrix units. Note that since $f_{1i}af_{j1} = f_{11}f_{1i}af_{j1}f_{11}$, the map is well-defined. It is easily checked that φ is linear and self-adjoint. Furthermore, since $\sum_{k=1}^n f_{kk} = 1_A$, we find that

$$\begin{aligned}\varphi(ab) &= \sum_{i,j=1}^{n} e_{ij} \otimes f_{1i}abf_{j1} \\ &= \sum_{i,j=1}^{n} e_{i,j} \otimes \Big(\sum_{k=1}^{n} f_{1i}af_{k1}f_{1k}bf_{j1}\Big) \\ &= \varphi(a)\varphi(b),\end{aligned}$$

which shows that φ is a *-homomorphism. Assuming that $\varphi(a) = 0$ for some $a \in A$ we find that $f_{ii}af_{jj} = f_{i1}f_{1i}af_{j1}f_{1j} = 0$ for all $1 \leq i,j \leq n$ which implies that

$$\sum_{i,j=1}^{n} f_{ii}af_{jj} = 1_A \cdot a \cdot 1_A = 0,$$

and hence φ is injective. Finally, if $\sum_{i,j} e_{ij} \otimes a_{ij} \in M_n \otimes A_0$ is given, we see that

$$\varphi\Big(\sum_{i,j=1}^{n} f_{i1}a_{ij}f_{1j}\Big) = \sum_{i,j=1} e_{ij} \otimes a_{ij},$$

and therefore φ is also surjective, thereby completing the proof of the first statement.

For the second statement let $s_1, s_2 \in \mathcal{O}_2$ denote the generating isometries and $n \geq 2$ be given. Then we define isometries t_j for $1 \leq j \leq n$ by

$$t_j := \begin{cases} s_2^{j-1}s_1 & \text{when } 1 \leq j < n; \\ s_2^n & \text{when } j = n, \end{cases}$$

with the convention that $s_2^0 = 1_{\mathcal{O}_2}$. The reader may check that these are isometries and that $(t_it_j^*)_{i,j=1}^n$ defines a set of matrix units in $\mathcal{O}_2$. Furthermore, $t_1t_1^* \simeq t_1^*t_1 = 1_{\mathcal{O}_2}$, whence it follows that $(t_1t_1^*)\mathcal{O}_2(t_1t_1^*) \cong \mathcal{O}_2$ and therefore the first statement implies that $M_n \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

□

Note that in the proof of the following theorem we apply the preceding results to the context where $n = 2$.

Theorem 3.3.14. *We have* $\dim_{\mathrm{nuc}}\mathcal{O}_2 = 1$.

Proof. Let $\{x_1, x_2, \ldots, x_n\} \subset \mathcal{O}_2$ and let $\varepsilon > 0$ be given. We seek to find a finite-dimensional C^*-algebra $F = F^{(0)} \oplus F^{(1)}$ and cp. maps $\psi : \mathcal{O}_2 \to F$, $\varphi : F \to \mathcal{O}_2$ such that ψ is contractive, $\varphi|_{F^{(i)}}$ is an order zero ccp. map for $i = 0, 1$ and $\|x_i - \varphi \circ \psi(x_i)\| < \varepsilon$ for all $1 \le i \le n$.

To that end, let $F_k^{(0)} := B(\Gamma_{k,2k}(2))$ and $F_k^{(1)} := B(\Gamma_{k+\lceil k/2\rceil, 2k+\lceil k/2\rceil})$ and $F_k := F_k^{(0)} \oplus F_k^{(1)}$. Furthermore, let $\rho : \mathcal{O}_2 \to \mathcal{T}_2$ be a ucp. lift of the restriction of the quotient map $q : \mathcal{T}_2 \to \mathcal{O}_2$ (see Corollary 1.4.13).

Since $\Gamma(2) \cong \Gamma_k(2) \otimes \Gamma(2^k)$ for any k and $\Gamma_k(2)$ is finite-dimensional, we see that $\mathbb{K}\big(\Gamma(2)\big) \cong B\big(\Gamma_k(2)\big) \otimes \mathbb{K}\big(\Gamma(2^k)\big)$ and hence $q(\mathcal{A}_k) = B\big(\Gamma_k(2)\big) \otimes \mathcal{O}_{2^k} \cong M_{d_k} \otimes \mathcal{O}_{2^k}$. Since $\mathcal{T}_2 \subseteq \mathcal{A}_k$ we find that $M_{d_k} \otimes \mathcal{O}_{2^k}$ contains a copy of $\mathcal{O}_2$ and we may think of $\{x_1, \ldots, x_n\}$ as a subset of this copy. Since $\varphi_k \circ \psi_k(K) \to 0$ as $k \to \infty$ for any compact operator $K \in B(\Gamma(2))$, it follows from Lemma 3.3.12 that we may choose $n \in \mathbb{N}$ such that

$$\|x_i - q \circ \varphi_n \circ \psi_n \circ \rho(x_i)\| < \varepsilon/2.$$

Furthermore, letting s_1, s_2 denote the generators of $\mathcal{O}_2$ we see that $\{s_\mu \mid \mu \in W_2 \text{ and } |\mu| = n\} \subseteq \mathcal{O}_2$ generates a unital subalgebra of $\mathcal{O}_2$ which is isomorphic to $\mathcal{O}_{2^n}$. Hence we obtain the following sequence of unital embeddings;

$$\mathcal{O}_2 \hookrightarrow M_{d_n} \otimes \mathcal{O}_{2^n} \hookrightarrow M_{d_n} \otimes \mathcal{O}_2 \cong \mathcal{O}_2,$$

where the last isomorphism follows from Proposition 3.3.13 and the first embedding is simply inclusion, as we are currently considering a copy of $\mathcal{O}_2$ inside $M_{d_n} \otimes \mathcal{O}_{2^n}$. Let σ denote the composition of the inclusions

$$M_{d_n} \otimes \mathcal{O}_{2^n} \hookrightarrow \mathcal{O}_2.$$

Since any unital endomorphism on $\mathcal{O}_2$ is approximately inner (see [24, Theorem 5.1.1]), we may find a unitary $u \in \mathcal{O}_2$ such that

$$\|x_i - u^*\sigma(x_i)u\| < \varepsilon/2$$

for all $1 \le i \le n$. Let $F := F_n$, $\psi : \mathcal{O}_2 \to F$ be $\psi_n \circ \rho$ and $\varphi : F \to \mathcal{O}_2$ be given by $\varphi(x) = u^*(\sigma \circ q \circ \varphi_n(x))u$. Then we find that

$$\|x_i - \varphi \circ \psi(x_i)\| \le \|x_i - u^*\sigma(x_i)u\| + \varepsilon/2 < \varepsilon,$$

and therefore the triple (F, ψ, φ) is as desired. $\square$

Passing to General Kirchberg Algebras

In this section we seek to pass to general Kirchberg algebras using essentially the same method as in Theorems 3.2.9 and 3.2.10. First, we need to straighten a few details out.

Proposition 3.3.15. *Let A be a Kirchberg algebra. Then pAp is a unital Kirchberg algebra for any non-zero projection $p \in A$.*

Proof. Clearly pAp is unital (with unit p), and it is simple and nuclear since $pAp \subseteq A$ is a hereditary subalgebra (see Propositions 1.4.2 and 1.4.9). Thus, it remains to be seen that pAp is purely infinite, but this is easy: let $a, b \in pAp$ be non-zero positive elements. By definition (see Definition 3.3.1) we may choose $\tilde{x} \in A$ such that $\tilde{x}a\tilde{x}^* = b$. Upon letting $x = p\tilde{x}p$ and recalling that $pa = a$ and $pb = b$ we find that $x \in pAp$ and

$$xax^* = p\tilde{x}a\tilde{x}^*p = pbp = b,$$

which completes the proof. □

It follows from the above proposition, along with Proposition 1.8.4, that if we can bound the nuclear dimension of unital Kirchberg algebras, the same bound will hold for all Kirchberg algebras. Next we need to transport Lemma 3.2.2 to the current context. However, this turns out to be relatively easy, and we leave the details to the reader. Recall that $\mathbf{1}_n \in M_n$ denotes the identity.

Lemma 3.3.16. *Let A be a unital, simple and purely infinite C^*-algebra and $p, q \in A$ be non-zero projections. Then, for every $n \in \mathbb{N}$ there exist contractions $v_1, v_2 \in A \otimes (M_n \oplus M_{n+1})$ such that*

$$\Big\| \sum_{i=1,2} v_i^* v_i - p \otimes (\mathbf{1}_n \oplus \mathbf{1}_{n+1}) \Big\| \leq \frac{4}{n}; \quad \Big\| \sum_{i=1,2} v_i v_i^* - q \otimes (\mathbf{1}_n \oplus \mathbf{1}_{n+1}) \Big\| \leq \frac{4}{n};$$

and

$$\operatorname{dist}(v_i^* v_i, \{p \otimes x \mid x \in M_n \oplus M_{n+1}\}) \leq \frac{2}{n}.$$

Proof. Since $A \otimes M_n$ is simple and purely infinite for any $n \in \mathbb{N}$ we see that for any pair of non-zero projections $e, f \in A \otimes M_n$ we have that

$$e \preceq f \quad \text{and} \quad f \preceq e.$$

See [9, Lemma V.5.4] for a proof of this fact. Hence we can apply the proof of Lemma 3.2.2 unchanged, to obtain contractions $w_j^{(1)} \in A \otimes M_n$, $w_j^{(2)} \in A \otimes M_{n+1}$ for $, j = 1, 2$ and elements $f_1 \in M_n$, $f_2 \in M_{n+1}$ such that

$$\Big\| \sum_{i=1,2} \big(w_i^{(j)}\big)^* w_i^{(j)} - p \otimes f_j \Big\| \leq \frac{2}{n}$$
$$\Big\| \sum_{i=1,2} w_i^{(j)} \big(w_i^{(j)}\big)^* - q \otimes f_j \Big\| \leq \frac{2}{n},$$

and

$$\Big\| \sum_{i=1,2} \big(w_i^{(j)}\big)^* w_i^{(j)} - p \otimes (\mathbf{1}_{n-1+j} - f_j) \Big\| \leq \frac{2}{n}$$
$$\Big\| \sum_{i=1,2} w_i^{(j)} \big(w_i^{(j)}\big)^* - q \otimes (\mathbf{1}_{n-1+j} - f_j) \Big\| \leq \frac{2}{n},$$

which (essentially) completes the proof. □

The proofs of the next results rely on K-theory, although we do not delve into this part. We only really need (a particular consequence of) [24, Proposition 4.1.4], which is stated below for the readers convience.

Proposition 3.3.17. *Let A be a simple, purely infinite C^*-algebra. Then*

$$K_0(A) = \{[p]_0 \mid p \in A \text{ is a non-zero projection}\}.$$

Furthermore, if p, q are non-zero projections such that $[p]_0 = [q]_0$, then $p \simeq q$.

Remark 3.3.18. It is not difficult to see that we can sharpen the above statement to the following; if $q \in A$ is a non-zero projection then

$$K_0(A) = \{[p]_0 \mid p \text{ is a non-zero projection and } p \leq q\}.$$

Indeed, since $q \in A$ is non-zero then $p \preceq q$ for any projection $p \in A$, and since equivalent projections define the same element in $K_0(A)$, the desired result follows.

With this at hand we can prove the following proposition. We use basic properties of K_0 without mention (see for instance [26, Proposition 3.1.7]).

Proposition 3.3.19. *Let A be a simple, purely infinite C^*-algebra and $p \in A$ be a non-zero projection. If $[p]_0 = 0$ then there exists a unital embedding*

$$\mathcal{O}_2 \hookrightarrow pAp.$$

Proof. Since $p \in A$ is a non-zero projection, it is properly infinite (see Theorem 3.3.2) and we can therefore find a projection $p_1 \in A$ such that $p \simeq p_1$ and $0 \neq (p - p_1)$. Hence both p_1 and $p_2 := p - p_1$ are non-zero projections, and since

$$[p_2]_0 = [p]_0 - [p_1]_0 = [p]_0 - [p]_0 = 0 = [p_1]_0,$$

it follows from Proposition 3.3.17 that $p \simeq p_1 \simeq p_2$. Letting $s_1, s_2 \in A$ be partial isometries such that $s_i^* s_i = p$ and $s_i s_i^* = p_i$ for $i = 1, 2$, we find that $s_1, s_2 \in pAp$ satisfies the required relations for $\mathcal{O}_2$ and therefore obtain a unital embedding as desired. □

Proposition 3.3.20. *For any $n \in \mathbb{N}$ there exists a unital embedding $M_n \oplus M_{n+1} \hookrightarrow \mathcal{O}_\infty$.*

Proof. We remind the reader that there is an isomorphism $K_0(\mathcal{O}_\infty) \to \mathbb{Z}$ such that $[1_{\mathcal{O}_\infty}]_0 = 1$.

Let $\{s_i\}_{i=1}^\infty$ be the generating isometries for $\mathcal{O}_\infty$, and $e_j = s_j s_j^*$ for each $1 \leq j \leq n+1$. Since the projections $e_1, \dots e_{n+1}$ are pairwise orthogonal and pairwise equivalent, they give rise to a (non-unital) embedding $M_{n+1} \hookrightarrow \mathcal{O}_\infty$. We aim to show that there exist pairwise orthogonal, pairwise equivalent projections $f_1, \dots, f_n \in \mathcal{O}_\infty$ such that f_i is orthogonal to e_j for each i, j and

$$e_1 + e_2 + \cdots + e_{n+1} + f_1 + \cdots + f_n \simeq 1_{\mathcal{O}_\infty},$$

which will finish the proof (recall that for any C^*-algebra and projections $p, q \in A$ the corners pAp and qAq are isomorphic if $p \simeq q$).

With the aid of Proposition 3.3.17 (along with Remark 3.3.18) this is not too difficult. Indeed, letting $p := 1_{\mathcal{O}_\infty} - \sum_{j=1}^{n+1} e_j$ we may find $f_1 \leq p$ such that $[f_1]_0 = -1$. Similarly, we can find $f_2 \leq p - f_1$ such that $[f_2]_0 = -1$. Repeating this process we obtain pairwise orthogonal projections $f_1, \dots, f_n$ such that $[f_j]_0 = -1$ for each $1 \leq j \leq n$. Therefore, we see that the projections $f_1, \dots, f_n$ are also pairwise equivalent and that

$$\Big[\sum_{j=1}^{n+1} e_j + \sum_{k=1}^{n} f_k\Big]_0 = n + 1 - n = 1 = [1_{\mathcal{O}_\infty}]_0.$$

Thus, the desired result follows from Proposition 3.3.17. □

Remark 3.3.21. The above proposition explains the change in Lemma 3.3.16 compared with Lemma 3.2.2. Indeed, although we *can* embed $M_n \oplus M_{n+1}$ unitally in $\mathcal{O}_\infty$, there is no hope of embedding M_n unitally in $\mathcal{O}_\infty$ for any $n \geq 2$. This follows from the above proof, since the equation $nx = 1$ does not have a solution $x \in \mathbb{Z}$.

Theorem 3.3.22. *Any Kirchberg algebra has nuclear dimension at most three.*

Proof. Let A be any Kirchberg algebra. It follows from Proposition 3.3.15, along with the comments below, that we may assume A to be unital. Furthermore, it is well-known that $\bigotimes_{n=1}^\infty \mathcal{O}_\infty \cong \mathcal{O}_\infty$. It therefore follows from Proposition 3.1.7 that it suffices, for each finite set $\chi \subseteq A$ and $\varepsilon > 0$, to prove the existence of cp. maps $\varphi : A \to \bigoplus_{i=0}^3 F_i$, $\psi : \bigoplus_{i=0}^3 F_i \to A \otimes \mathcal{O}_\infty$, such that each F_i is finite-dimensional, φ is contractive, the restrictions $\psi_i := \psi|_{F_i}$ are order zero ccp. maps and

$$\Big\| \Big(\sum_{i=0}^{3} \psi_i \Big) \circ \varphi(a) - a \otimes 1_{\mathcal{O}_\infty} \Big\| < \varepsilon$$

for every $a \in \chi$.

To that en, let $B_1, B_2 \subseteq \mathcal{O}_\infty$ be unital, commuting subalgebras such that $B_1 \cong B_2 \cong \mathcal{O}_\infty$ and $C^*(B_1 \cup B_2) = \mathcal{O}_\infty$ (see Theorem 3.3.7). Let $s \in B_1$ be a non-unitary isometry and let $q := ss^*$, $p = 1_{B_1} - ss^*$. Let $\iota : A \to A \otimes qB_1q$ be given by $\iota(a) = a \otimes q$. It obviously follows from Proposition 3.3.19 that we may find a unital embedding $\rho : \mathcal{O}_2 \to 1_A \otimes pB_1p$ and Theorem 3.3.6 tells us that there exists a unital embedding $\sigma : A \to \mathcal{O}_2$. Hence, we obtain a unital embedding $\varphi := \rho \circ \sigma : A \to 1_A \otimes pB_1p$.

From here on, the proof is as close to a copy of the proof of Theorems 3.2.9 and 3.2.10 as possible. Namely, we let

$$\mathcal{A} := (A \otimes \mathcal{O}_\infty)_\omega, \quad \mathcal{A}_1 := (A \otimes B_1)_\omega.$$

Since $p+q = 1_{B_1}$, we may define a unital *-homomorphism $\pi : A \to \mathcal{A}_1 \subseteq \mathcal{A}$ by

$$\pi(a) = \varphi(a) + \iota(a).$$

Note that as an immediate consequence of Theorem 3.3.7 we have that $A \cong A \otimes B_1$. In particular $A \otimes B_1$ is simple and purely infinite, whence it follows from Proposition 3.3.8 that $\mathcal{A}_1 \cap \pi(A)'$ is simple and purely infinite. We may therefore apply Lemma 3.3.16 on the projections $1_A \otimes p$ and $1_A \otimes q$ to obtain contractions $v_{i,n} \in \big(\mathcal{A}_1 \cap \pi(A)'\big) \otimes (M_n \oplus M_{n+1})$ for each $n \in \mathbb{N}$ and $i = 0, 1$ such that

$$\Big\| \sum_{i=0,1} v_{i,n}^* v_{i,n} - 1_A \otimes p \otimes (\mathbf{1}_n \oplus \mathbf{1}_{n+1}) \Big\| \leq \frac{4}{n};$$
$$\Big\| \sum_{i=0,1} v_{i,n} v_{i,n}^* - 1_A \otimes q \otimes (\mathbf{1}_n \oplus \mathbf{1}_{n+1}) \Big\| \leq \frac{4}{n};$$

and

$$\operatorname{dist}(v_{i,n}^* v_{i,n}, \{1_A \otimes p \otimes x \mid x \in (M_n \oplus M_{n+1})\}) \leq \frac{2}{n}.$$

Since $M_n \oplus M_{n+1}$ can be unitally embedded in $\mathcal{O}_\infty \cong B_1$, we obtain, in the same way as in the proof of Theorem 3.2.9, contractions $v_0, v_1 \in \mathcal{A} \cap \pi(A)'$ such that $\sum_{i=0,1} v_i v_i^* = 1_A \otimes q$, $\sum_{i=0,1} v_i^* v_i = 1_A \otimes p$ and $v_i^* v_i \in (1_A \otimes pB_1)_\omega$. The fastidious reader may check that the first two relations in combination implies that $(1_A \otimes q) v_i (1_A \otimes p) = v_i$ for $i = 0, 1$. Therefore, since $v_i \in \mathcal{A} \cap \pi(A)'$ we find that

$$v_i \varphi(a) = \iota(a) v_i,$$

for $i = 0, 1$, which of course implies the validity of the following computation:

$$\begin{aligned} a \otimes 1_{\mathcal{O}_\infty} &= (1_A \otimes s^*)\iota(a)(1_A \otimes s) \\ &= (1_A \otimes s^*)\Big(\sum_{i=0,1} v_i \varphi(a) v_i^*\Big)(1_A \otimes s) \\ &= \sum_{i=0,1} \mathrm{Ad}_{1_A \otimes s^*} \circ \mathrm{Ad}_{v_i} \circ \rho \circ \sigma(a), \end{aligned}$$

for any $a \in A$.

Let $\rho_i : \mathcal{O}_2 \to \mathcal{A}$ be given by $\mathrm{Ad}_{1_A \otimes s^*} \circ \mathrm{Ad}_{v_i} \circ \rho$, for $i = 0, 1$. Since $v_i v_i^*(1_A \otimes ss^*) = v_i v_i^*$ we see that $x(1_A \otimes ss^*) = x$ for all $x \in \overline{v_i A v_i^*}$. Furthermore $v_i^* v_i \in (1_A \otimes pB_1p)_\omega$, which implies that ρ_0, ρ_1 are both ccp. order zero maps.

We have already shown that $\dim_{\mathrm{nuc}}\mathcal{O}_2 = 1$. Hence, for any finite set $F \subseteq A$ and any $\varepsilon > 0$ we may choose ccp. maps $\alpha : \mathcal{O}_2 \to E_0 \oplus E_1$ and $\beta_j : E_j \to \mathcal{O}_2$ such that each E_j is finite-dimensional, β_j is order zero and

$$\Big\| \Big(\sum_{j=0,1} \beta_j \Big) \circ \alpha(\sigma(a)) - \sigma(a) \Big\| < \varepsilon,$$

for all $a \in F$. The maps $\rho_i \circ \beta_j : E_j \to \mathcal{A}$ are order zero ccp. maps for each $i, j = 0, 1$. Thus, as in the proof of Theorem 3.2.10, upon letting $\tilde{\sigma} : A \to E_0 \oplus E_1 \oplus E_0 \oplus E_1$ be the ccp. map given by $\tilde{\sigma}(a) = \alpha(\rho(a)) \oplus \alpha(\rho(a))$, we have that

$$\Big\| \Big(\sum_{i,j=0,1} \rho_i \circ \beta_j \Big) \circ \tilde{\sigma}(a) - a \otimes 1_{\mathcal{O}_\infty} \Big\| < \varepsilon,$$

for any $a \in F$. Each of the maps $\rho_i \circ \beta_j$ lifts to a ccp. order zero map $\mathcal{O}_2 \to \ell^\infty(A \otimes \mathcal{O}_\infty)$ (see Theorem 1.4.16 and Corollary 1.6.3) which completes the proof. □

Remark 3.3.23. The above theorem bounds the nuclear dimension of $\mathcal{O}_n$ for each $2 \le n \le \infty$, but this bound is not optimal. Indeed, one may prove that $\dim_{\mathrm{nuc}}\mathcal{O}_n = 1$ for each $2 \le n < \infty$ and $\dim_{\mathrm{nuc}}\mathcal{O}_\infty \le 2$ (see [33, Theorem 7.4]). The proof is very similar to the proof that $\dim_{\mathrm{nuc}}\mathcal{O}_2 = 1$ but relies on heavier classification results. Thus we have elected to bound the nuclear dimension of all Kirchberg algebras in one fell swoop, at the cost of not obtaining the optimal bound.

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