Abstract. *This project aims to present the answers given by the work of C. Schafhauser in [30], to the following two main questions. 1) If A is a separable, unital, exact C^*-algebra, satisfying the UCT, with a faithful, amenable trace, then does A admit a unital, trace-preserving embedding into a simple, unital, AF-algebra B with unique trace and divisible K_0-group ?*
2) If two such trace-preserving embeddings exist and have identical K_0-behaviour, then can they be classified in terms of approximate unitary equivalence ?

Contents

Introduction

The major influence of this thesis is from the work of C. Schafhauser about C^*-subalgebras of simple, unital AF-algebras in [30]. AF-algebras have been in the center of interest and have been studied thoroughly since introduced and classified by Bratteli in 1972. One of the most remarkable results about AF-algebras, was proved by Elliott in 1976, showing that all unital AF-algebras can be classified by their ordered K_0-group. Now, in this project we first emphasize on whether a separable, unital, exact C^*-algebra satisfying the universal coefficient theorem (UCT) can be embedded in a trace-preserving way into a simple, unital AF-algebra with unique trace and divisible K_0-group, and then we are asking if any two such embeddings can be classified by their induced K_0-group homomorphisms.

However, instead of developing our theory for simple, unital AF-algebras with unique trace, we move our focus to the more general setting of a simple, unital, $\mathscr{Q}$-stable C^*-algebra, with unique trace, trivial K_1-group and where every quasi-trace is a trace; $\mathscr{Q}$ is the universal UHF-algebra. Here comes the purpose of Chapter 1, where these conditions are examined, and necessary arguments for addressing the embeddability problem are presented. The preliminary essence of the first chapter, is that in the first three sections we take over a brief survey on different classes of C^*-algebras, staying focused on the simplicity and $\mathscr{Q}$-stability conditions, while section 1.4 consists of methods for reducing a non-separable C^*-algebraic setting to a separable one.

On the other hand, a major aspect in the approach of Schafhauser in [30], is the employment of KK-theory and in particular the Cuntz picture of KK-theory. A central motivation for this, is the universal coefficient theorem (UCT) for C^*-algebras, and the strong relevance of KK-theory with asymptotic and proper asymptotic unitary equivalence relations between *-homomorphisms, as it will be evident in Chapter 2. Nevertheless, instead of embarking immediately to presenting Cuntz picture of KK-theory, we choose first to introduce Hilbert C^*-module theory, that enables a broader understanding of the subject and facilitates the exposition in the following sections. Finally, section 2.4 contains a brief expository about admissible kernels and trace-kernel extensions, while offers a first glimpse at the utility of Chapter 1. This section also serves as a forerunner to the main core of this project.

Lastly, in Chapter 3, all the necessary steps for addressing the major claims at stake are collected, and it mainly consists of a more detailed proof-presentation of the results in sections 4 and 5 in [30]. The primary idea of this chapter is that it starts by giving affirmative answers to the main questions, when some mild extra conditions are satisfied, but for ultrapowers of the C^*-algebras under consideration. Then, it proceeds to reduce these results to the original C^*-algebras by using the local notions of K_0-triples and $(\mathscr{G}, \delta)$-multiplicative maps. Throughout this last chapter we keep working in the general case of simple, unital C^*-algebras with all the conditions that we state above, and the result concerning unital, simple AF-algebras with unique trace, will come up beautifully as a straightforward consequence.

This project cannot be considered as self-contained, since it assumes that the reader is familiar with basic theory of C^*-algebras and von Neumman Algebras, as demonstrated in [22] and [34], for instance. Meanwhile, a familiarization with basic K-theory as presented for example in [20], is considered as a prerequisite. However, an effort to have self-reliable proofs and to keep a detailed track of the ideas and arguments that are used throughout the project, has been made.

Acknowledgements. First of all, I would like to thank my advisor, Mikael Rørdam, for his help during these last four months. Also, I would like to thank the people in my life for their enduring support while writing this project.

1 Preliminaries on C*-algebras

The main purpose of this chapter is to present part of the purely C^*-algebraic backround material needed in the proofs of the main results of this project. In the first three sections, results regarding the structure of a $\mathscr{Q}$-stable C^*-algerba are developed, where $\mathscr{Q}$ is the universal UHF-algbera. The last section is devoted in introducing methods for addressing issues that arise when a C^*-algebra is non-separable.

1.1 Strongly Self-Absorbing C*-algebras

In this section we will examine the class of strongly self-absorbing C^*-algebras, mainly aiming to harvest criteria for a C^*-algebra being stable with respect to a strongly self-absorbing C^*-algebra. We start by introducing the notions of essential ideal, multiplier algebra and inner automorphism, that they will all be needed in the rest of the chapter. Throughout this section, we denote the minimal tensor product of C^*-algebras using just $\otimes$, except otherwise is mentioned. The material exposed in the sequel is mainly from [33].

Definition 1.1.1. *Let A be a C^*-algebra. An ideal $I \subseteq A$ is called essential if it has "no orthogonal complement" i.e*

$$I^{\perp} = \{a \in A \colon ax = xa = 0, \ \ \textit{for all} \ \ x \in I\} = 0$$

Definition 1.1.2. *Let I be a C^*-algebra. Let $I \subseteq B(H)$ be any non-degenerate(cyclic) representation and define $M(I) = \{T \in B(H) \colon Tx \in I \quad \text{and} \quad xT \in I, \ \text{for all } x \in I\}$. We call $M(I)$ the multiplier algebra of I.*

Note 1.1.3. $M(I)$ is a unital C^*-algebra and if I is unital then $M(I) = I$. Moreover, it is a fact that $M(I)$ has the property of being the largest C^*-algebra containing I as an essential ideal and that it is the unique (up to isomorphism) algebra with this property.

Definition 1.1.4. *Let A be a C^*-algebra and $u \in M(A)$. Then, the automorphism of the form*

$$\mathrm{Ad}_u(a) = u^* a u$$

is called inner automorphism. Moreover, we say that an automorphism α is approximately inner if it is the point-norm limit of inner automorphisms.

Having these new notions in mind, we proceed to define the approximate unitary equivalence between c.c.p. maps.

Definition 1.1.5. *For i=0,1, let $\phi_i \colon A \to B$ be a c.c.p map between separable C^*- algebras. We say that ϕ_1 and ϕ_2 are approximately unitarily equivalent, $\phi_1 \sim_{a.u} \phi_2$, if there is a sequence of unitaries $(u_n)_n \subset M(B)$ such that*

$$\|u_n^* \phi_1(a) u_n - \phi_2(a)\| \to 0, \ \ \textit{as} \ \ n \to \infty, \ \ \textit{for all} \ \ a \in A$$

.

The following proposition is an immediate and rather useful result about approximate unitary equivalence. The proof is omitted.

Proposition 1.1.6. *Let A, B, C and D be separable C^*-algebras, and C, D be unital. Suppose $\phi\colon A \to B$, $\alpha, \beta, \gamma\colon B \to C$ and $\psi\colon C \to D$ are *-homomorphisms, ψ unital. Then*

i) If $\alpha \sim_{a.u} \beta$ and $\beta \sim_{a.u} \gamma$, then $\alpha \sim_{a.u} \gamma$, i.e $\sim_{a.u}$ is a transitive relation.

ii) If $\alpha \sim_{a.u} \beta$ then, $\psi \circ \alpha \sim_{a.u} \psi \circ \beta$ and $\alpha \circ \phi \sim_{a.u} \beta \circ \phi$.

iii) Suppose that α, β are pointwise limits of sequences of *-homomorphisms $\alpha_n, \beta_n\colon B \to C$.If $\alpha_n \sim_{a.u} \beta_n$ for each n, then $\alpha \sim_{a.u} \beta$.

In order to start working with strongly self-absorbing C^*-algebras, we need first to coin the terms of flip, approximately inner flip and approximately inner half flip, which we do in the definition below.

Definition 1.1.7. *Let D be a separable, unital C^*-algebra.*

i) By the flip on the minimal tensor product $D \otimes D$ we mean the automorphism σ_D of $D \otimes D$ given by $\sigma_D(a \otimes b) = b \otimes a, \quad a, b \in D$

ii) D is said to have approximately inner flip, if $\sigma_D \sim_{a.u} id_{D\otimes D}$

iii) D is said to have approximately inner half flip, if $id_D \otimes 1_D \sim_{a.u} 1_D \otimes id_D$

iv) D is strongly self-absorbing, if $D \neq \mathbb{C}$ and there is an isomorphism $\phi\colon D \to D \otimes D$, satisfying $\phi \sim_{a.u} id_D \otimes 1_D$

Example 1.1.8. In this project, the only strongly self-absorbing C^*-algebra that we need is the universal UHF-algebra $\mathscr{Q}$. For this reason, let us now make a brief review on this special class of AF-algebras, the UHF-algebras, and finally argue that $\mathscr{Q}$ is indeed strongly self-absorbing.
A UHF-algebra is an inductive limit of

$$M_{k_1}(\mathbb{C}) \xrightarrow{\phi_1} M_{k_2}(\mathbb{C}) \xrightarrow{\phi_2} M_{k_3}(\mathbb{C}) \xrightarrow{\phi_3} \cdots \qquad (1)$$

where ϕ_n are unital *-homomorphisms. It is fact that there is a unital *-homomorphism $M_n(\mathbb{C}) \to M_m(\mathbb{C})$ iff m is a multiple of n. An interesting fact about UHF-algebras is that they can be classified using supernatural numbers. A supernatural number is a a sequence of numbers $n = (n_j)_{j=1}^{\infty}$ in $\{0, 1, 2, ...\infty\}$, where each n_j can be interpreted as a power in a generalized infinite prime factorization, i.e $n = (n_j)_{j=1}^{\infty} = \prod_{j=1}^{\infty} p_j^{n_j}$, where $\{p_j\colon j \geq 1\}$ is an increasing order enumeration of the prime numbers. Now, we associate to the sequence $(k_j)_{j=1}^{\infty}$ of (1) a supernatural number $n = (n_j)_{j=1}^{\infty}$ in the following way

$$n_j = \sup\{r\colon \ p_j^r | k_i \ \text{ for some i}\}$$

On the other way around if $n = (n_j)_{j=1}^{\infty}$ is a supernatural number, then if we associate to each j the element $k_j = \prod_{i=1}^{j} p_i^{\min\{j, n_i\}}$, it is straightforward to see that the supernatural number associated to $(k_j)_{j=1}^{\infty}$ is $n = (n_j)_{j=1}^{\infty}$. Also we say that a natural or a supernatural number $m = \prod_{j=1}^{\infty} p_j^{m_j}$ divides a supernatural number $n = (n_j)_{j=1}^{\infty}$ if $m_j \leq n_j$ for all j and moreover we associate to each supernatural $n = (n_j)_{j=1}^{\infty}$ a subgroup of $\mathbb{Q}$ by

$$Q(n) = \{z/m\colon z \in \mathbb{Z}, m \in \mathbb{N}, z|n\}$$

Under these observations and taking motivation from Elliot's classification of AF-algebras, the first step in showing that UHF-algebras are classified by supernatural numbers is showing that $K_0(M_n) \cong Q(n)$, where M_n denotes the UHF-algebra associated to the supernatural number n as above.

Now, the universal UHF-algebra, $\mathscr{Q}$, is the one associated to the supernatural number $n = (n_j)_{j=1}^{\infty}$, where $n_j = \infty$ for all j. Thus, $Q(n) = \mathbb{Q}$, and furthermore $\mathscr{Q}$ can be viewed as the inductive limit of $(M_{(n-1)!}(\mathbb{C}), \phi_n)$, where ϕ_n is a unital *-homorphism with multiplicity n, for each $n \geq 1$. So, using the fact that the tensor product of two UHF-algebras M_n, M_m is given by $M_n \otimes M_m = M_{nm}$, we immediately see that $\mathscr{Q} \otimes \mathscr{Q} \cong \mathscr{Q}$. Furthermore, employing that the flip automorphism on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is inner, for any $n \geq 1$, and that $\mathscr{Q} \otimes \mathscr{Q} = \overline{\cup_n (M_{(n-1)!}(\mathbb{C}) \otimes M_{(n-1)!}(\mathbb{C}))}$, we see that the flip automorphism on $\mathscr{Q} \otimes \mathscr{Q}$ is approximately inner. Hence, *Proposition* 1.1.15 below asserts that $\mathscr{Q}^{\otimes\infty}$ is strongly self-absorbing. But, as $\mathscr{Q} \cong \mathscr{Q}^{\otimes\infty}$, we conclude that $\mathscr{Q}$ is a strongly self-absorbing C^*-algebra.

Proposition 1.1.9. *Any separable, unital and strongly self-absorbing C^*-algebra D has approximately inner half flip.*

Proof. Let $\phi \colon D \to D \otimes D$ be the given *-isomorphism satisfying that $\phi \sim_{a.u} id_D \otimes 1_D$. Then, define a *-homomorphism $\psi \colon D \to D$ by $\psi := \phi^{-1} \circ (1_D \otimes id_D)$. Now, using repetitively Proposition 1.1.6 (ii) we firstly get

$$\begin{aligned} 1_D \otimes id_D &= \phi \circ \phi^{-1} \circ (1_D \otimes id_D) \\ &= \phi \circ \psi \\ &\sim_{a.u} (id_D \otimes 1_D) \circ \psi \\ &= \psi \otimes 1_D \end{aligned}$$

Also,

$$\begin{aligned} id_D \otimes 1_D &= \sigma_D \circ (1_D \otimes id_D) \\ &\sim_{a.u} (\sigma_D \circ (\psi \otimes 1_D) \\ &= 1_D \otimes \psi \end{aligned}$$

Now, we are ready to get the conclusion, but first note that if $(u_n)_n \subset D \otimes D$ are the unitaries implementing the relation $\psi \otimes 1_D \sim_{a.u} 1_D \otimes id_D$, then $(u_n \otimes 1_D)_n \subseteq D \otimes D \otimes D$ are unitaries that exhibit the relation $\psi \otimes 1_D \otimes 1_D \sim_{a.u} 1_D \otimes id_D \otimes 1_D$. So,

$$\begin{aligned} \psi \otimes 1_D &= (id_D \otimes \phi^{-1}) \circ (\psi \otimes 1_D \otimes 1_D) \\ &\sim_{a.u} (id_D \otimes \phi^{-1}) \circ (1_D \otimes id_D \otimes 1_D) \\ &\sim_{a.u} (id_D \otimes \phi^{-1}) \circ (1_D \otimes 1_D \otimes \psi) \\ &\sim_{a.u} (id_D \otimes \phi^{-1}) \circ (id_D \otimes 1_D \otimes 1_D) \\ &= id_D \otimes 1_D \end{aligned}$$

Therefore, by transitivity of $\sim_{a.u}$ we obtain that $id_D \otimes 1_D \sim_{a.u} 1_D \otimes id_D$. Hence D has approximately inner half flip.

□

Using this result, the following theorem asserts that strongly self-absorbing C^*-algebras have a rather interesting structure.

Theorem 1.1.10. *If a separable, unital C^*-algebra D has approximate inner half flip then it is simple and nuclear.*

Proof. For showing that is simple, let J be a proper closed two sided ideal in D, and let $J_1 = J \otimes D$, $J_2 = D \otimes J$ be the corresponding closed two-sided ideals in $D \otimes D$. That D

has approximately inner half flip, i.e $id_D \otimes 1_D \sim_{a.u} 1_D \otimes id_D$, implies that $d \otimes 1_D \in J_2$, for any $d \in J$, and therefore $J_1 = J_2$. But we claim that this is not the case. Let $d_1 \in J$ and $d_2 \notin J$, then by Hahn-Banach Theorem find $f, g\colon D \to \mathbb{C}$ bounded linear functionals such that $f(d_1) \neq 0$, $g|_J = 0$ and $g(d_2) \neq 0$. Then by proposition 5.1 in [16], $f \otimes g\colon D \otimes D \to \mathbb{C}$ is a bounded linear functional and it satisfies that

$$f \otimes g(J_2) = 0$$

and

$$f \otimes g(d_1 \otimes d_2) \neq 0$$

Hence, $d_1 \otimes d_2 \in J_1 \setminus J_2$, which proves the claim. Thus, D does not contain any proper two-sided closed ideal, hence is simple. For nuclearity see Proposition 2.8 in [11] $\square$

Now, we are going to focus our interest on D-stable C^*-algebras, and our main goal is to show that the following statement holds.

Theorem 1.1.11. *Let A, D be separable C^*-algebras, and D unital, strongly self-absorbing. If there is an isomorphism $\phi\colon A \to D \otimes A$ then, there exists a $*$-homomorphism*

$$\sigma\colon A \otimes D \to \mathscr{Q}(A)$$

satisfying

$$\sigma(a \otimes 1_D) = a, \quad \textit{for all} \quad a \in A$$

where, $\mathscr{Q}(A) = \prod_{\mathbb{N}} A / \sum_{\mathbb{N}} A$.

Note 1.1.12. In the above statement, if we further assume that D is K_1-injective, i.e the canonical map $U(D)/U_0(D) \to K_1(D)$ is injective, then it is a fact that the converse is also true. Also, note that since $M_n(\mathbb{C})$ is K_1-injective, for any $n \in \mathbb{N}$, then any UHF-algebra is K_1-injective.

There is also an another version of the above result that will be useful in the following sections.

Theorem 1.1.13. *Let A, D be separable C^*-algebras and suppose that D is moreover unital and strongly self-absorbing. Then, there exists a *-isomorphism $\phi\colon A \to A \otimes D$ if and only if there is a unital *-homomorphism*

$$\sigma\colon D \to \mathscr{Q}(M(A)) \cap A'$$

where $\mathscr{Q}(M(A)) \cap A'$ is the relative commutant of A in $\mathscr{Q}(M(A))$
Moreover, in this case $\phi \sim_{a.u} id_A \otimes 1_D$.

Before embarking into proving Theorem 1.1.11, some preparation is required. Firstly we show that the tensor product of two separable, unital C^*- algebras with approximate inner (half) flip, has again an approximate inner (half) flip. Secondly, we prove a rather technical proposition that will eventually provide us the main machinery for showing Theorem 1.1.11.

Proposition 1.1.14. *Let A, B be separable, unital C^*- algebras with approximately inner (half) flip. Then $A \otimes B$ has approximately inner (half) flip.*

Proof. . Suppose first that A, B have approximately inner flip. Then, there exist $(w_n)_n \in A \otimes A$, $(v_n)_n \in B \otimes B$ sequences of unitaries such that for all $\;x \in A \otimes A, y \in B \otimes B$

$$\|\sigma_A(x) - w_n^* x w_n\| \xrightarrow{n\to\infty} 0$$
$$\|\sigma_B(y) - v_n^* y v_n\| \xrightarrow{n\to\infty} 0$$

We aim to show that there is a sequence of unitaries $(u_n)_n \in (A \otimes B) \otimes (A \otimes B)$ such that

$$\|\sigma_{A\otimes B}(X) - u_n^* X u_n\| \xrightarrow{n\to\infty} 0$$

for all $X \in (A\otimes B)\otimes(A\otimes B)$. But, since both $\sigma_{A\otimes B}, \mathrm{Ad}_{u_n}$ are linear isometries, under the natural identification $(A \otimes B) \otimes (A \otimes B) \cong (A \otimes A) \otimes (B \otimes B)$ we may suppose that X is a simple tensor of the form $X = X_1 \otimes X_2$, where $X_1 \in A \otimes A$ and $X_2 \in B \otimes B$, while the same identification says that $\sigma_{A\otimes B}$ corresponds to $\sigma_A \otimes \sigma_B$.

So, let $u_n = w_n \otimes v_n$ which are unitaries in $(A \otimes A) \otimes (B \otimes B)$, then

$$\|\sigma_{A\otimes B}(X) - u_n^* X u_n\| = \|\sigma_A(X_1) \otimes \sigma_B(X_2) - u_n^*(X_1 \otimes X_2)u_n\| =$$

$$\|\sigma_A(X_1) \otimes \sigma_B(X_2) - \sigma_A(X_1) \otimes v_n^* X_2 v_n + \sigma_A(X_1) \otimes v_n^* X_2 v_n - u_n^*(X_1 \otimes X_2)u_n\| \leq$$

$$\|\sigma_A(X_1)\| \, \|\sigma_B(X_2) - v_n^* X_2 v_n\| + \|\sigma_A(X_1) - w_n^* X_1 w_n\| \, \|v_n^* X_2 v_n\| \xrightarrow{n\to\infty} 0$$

which implies that $\sigma_{A\otimes B} \sim_{a.u} id_{(A\otimes B)\otimes(A\otimes B)}$, i.e $A \otimes B$ has approximate inner flip.

To show that the same holds for the approximately inner half flips, we use exactly the same argumentation, but instead of flips, we work with the (unital) embeddings of A and B, to $A \otimes A$ and $B \otimes B$, respectively. □

Proposition 1.1.15. *Let D be a separable, unital C^*-algebra with approximately inner half flip. Then*

i) $D^{\otimes\infty}$ has approximately inner flip

ii) $D^{\otimes\infty}$ is strongly self-absorbing

iii) There is a sequence of *-homomorphisms

$$\phi_n \colon D^{\otimes\infty} \otimes D^{\otimes\infty} \to D^{\otimes\infty}$$

satisfying

$$\|\phi_n(d \otimes 1_{D^{\otimes\infty}}) - d\| \to 0, \quad as \quad n \to \infty$$

Proof. *i*) We view $D^{\otimes\infty}$ as the inductive limit of

$$D \xrightarrow{\mu_1} D^{\otimes 2} \xrightarrow{\mu_2} D^{\otimes 4} \xrightarrow{\mu_4} \ldots$$

where

$$\mu_1 \colon D \to D^{\otimes 2}, \quad \mu_{2n} \colon D^{\otimes 2n} \to D^{\otimes 4n}, \quad n \in \mathbb{N}$$

are the unital-preserving *-homomorphisms $\mathrm{id}_D \otimes 1_D$, $\mathrm{id}_{D^{\otimes 2n}} \otimes 1_{D^{\otimes 2n}}$ respectively, and the maps $\iota_1 \colon D \to D^{\otimes\infty}$, $\iota_{2n} \colon D^{\otimes 2n} \to D^{\otimes\infty}$ are the inclusions (see [2], II.9.8). In a similar manner we can view $D^{\otimes\infty} \otimes D^{\otimes\infty}$ as an inductive limit with connecting maps

$$\lambda_1 \colon D \otimes D \to D^{\otimes 2} \otimes D^{\otimes 2}, \quad \lambda_{2n} \colon D^{\otimes 2n} \otimes D^{\otimes 2n} \to D^{\otimes 4n} \otimes D^{\otimes 4n}$$

given by

$$\lambda_1 = (id_D \otimes 1_D) \otimes (id_D \otimes 1_D)$$

$$\lambda_{2n} = (id_{D^{\otimes 2n}} \otimes 1_{D^{\otimes 2n}}) \otimes (id_{D^{\otimes 2n}} \otimes 1_{D^{\otimes 2n}})$$

and ι_1, ι_{2n} be again the inclusions into $D^{\otimes\infty} \otimes D^{\otimes\infty}$.

Now, since we want to show that

$$\sigma_{D^{\otimes\infty}} \sim_{a.u} id_{D^{\otimes\infty} \otimes D^{\otimes\infty}}$$

using the inductive limits expressions, it suffices to show that $\lambda_k \circ \sigma_{D^{\otimes k}} \sim_{a.u} \lambda_k$, for all $k \in \mathbb{N}$, where $\sigma_{D^{\otimes\infty}}$ is the flip on $D^{\otimes\infty} \otimes D^{\otimes\infty}$.

We denote the embedding of $D^{\otimes k}$ into $(D^{\otimes k})^{\otimes 4}$ at the i-th factor as $\iota_k^{(i)}$, and similarly we get the *- homomorphisms

$$\iota_k^{(i,j)} \colon (D^{\otimes k})^{\otimes 2} \to (D^{\otimes k})^{\otimes 4} \quad i \neq j \in \{1,2,3,4\}$$

where

$$\iota_k^{(i,j)}|_{D^{\otimes k} \otimes 1_{D^{\otimes k}}} = \iota_k^{(i)}$$

and we note that they are well defined since $\iota_k^{(i)}(D^{\otimes k})$ and $\iota_k^{(j)}(D^{\otimes k})$ commute. By identifying $(D^{\otimes k})^{\otimes 2}$ with $D^{\otimes k} \otimes D^{\otimes k}$ in the obvious way, we see that

$$\lambda_k = \iota_k^{(1,3)} \quad \text{and} \quad \lambda_k \circ \sigma_{D^{\otimes k}} = \iota_k^{(3,1)}$$

Now, we use that since D has approximately inner half flip then $D^{\otimes k}$ has also approximately inner half flip by Proposition 1.1.14, and so there is a sequence of unitaries

$$(u_m)_m \subset D^{\otimes k} \otimes D^{\otimes k} (\cong (D^{\otimes k})^{\otimes 2})$$

such that

$$\|u_m^*(d \otimes 1_{D^{\otimes k}})u_m - (1_{D^{\otimes k}} \otimes d)\| \xrightarrow{n \to \infty} 0$$

and let $i,j,k,l \in \{1,2,3,4\}$ be pairwise distinct. Then, the unitaries

$$(\iota_k^{(j,l)}(u_m))_m \subseteq (D^{\otimes k})^{\otimes 4}$$

satisfy

$$\left\| \iota_k^{(j,l)}(u_m^*)\iota_k^{(j)}(d)\iota_k^{(j,l)}(u_m) - \iota_k^{l}(d) \right\| \leq \|u_m^*(d \otimes 1_{D^{\otimes k}})u_m - (1_{D^{\otimes k}} \otimes d)\| \xrightarrow{m \to \infty} 0$$

which shows that $\iota_k^{(j)} \sim_{a.u} \iota_k^{(l)}$, and in turn that $\iota_k^{(i,j)} \sim_{a.u} \iota_k^{(i,l)}$, since $\iota_k^{(j,l)}(u_m)$ commute with $\iota_k^{(i)}(D^{\otimes k})$. So, in particular we have

$$\iota_k^{(1,3)} \sim_{a.u} \iota_k^{(1,2)} \sim_{a.u} \iota_k^{(3,2)} \sim_{a.u} \iota_k^{(3,1)}$$

which shows that

$$\lambda_k \circ \sigma_{D^{\otimes k}} \sim_{a.u} \lambda_k, \quad \text{for all } k \in \mathbb{N}$$

ii) For $k \in \mathbb{N}$ define

$$\alpha_k \colon D^{\otimes k} \to D^{\otimes k+1}$$
$$d \mapsto d \otimes 1_D$$

Then,

$$D^{\otimes \infty} = \varinjlim (D^{\otimes k}, a_k) \qquad (1)$$

$$D^{\otimes \infty} = \varinjlim (D^{\otimes 2k}, a_{2k+1} \circ a_{2k}) \qquad (2)$$

$$D^{\otimes \infty} \otimes D^{\otimes \infty} = \varinjlim (D^{\otimes k} \otimes D^{\otimes k}, a_k \otimes a_k) \qquad (3)$$

Also, by (i) we have that

$$D^{\otimes \infty} = \varinjlim (D^{\otimes 2^m}, id_{D^{\otimes 2^m}} \otimes 1_{D^{\otimes 2^m}}) \qquad (4)$$

$$D^{\otimes \infty} \otimes D^{\otimes \infty} = \varinjlim (D^{\otimes 2^m} \otimes D^{\otimes 2^m}, \lambda_{2^m}) \qquad (5)$$

Now, consider the *-isomorphisms $\psi_k \colon D^{\otimes 2k} \to D^{\otimes k} \otimes D^{\otimes k}$ given by

$$\psi_k(d_1 \otimes d_1', ..., d_k \otimes d_k') = d_1 \otimes d_2 \otimes ... \otimes d_k \otimes d_1' \otimes d_2' ... \otimes d_k'$$

which satisfy $\psi_{k+1} \circ \alpha_{2k+1} \circ \alpha_{2k} = (a_k \otimes a_k) \circ \psi_k$. Thus, by the universal property of (2) there is a *-homomorphism

$$\psi \colon D^{\otimes \infty} \to D^{\otimes \infty} \otimes D^{\otimes \infty}$$

such the diagram

$$\begin{tikzcd} D^{\otimes\infty} \arrow[r, "\psi"] & D^{\otimes\infty}\otimes D^{\otimes\infty} \\ D^{\otimes 2k} \arrow[u, "\iota_{2k}\circ\psi_k"] \arrow[ur, "\iota'_{2k}"'] & \end{tikzcd}$$

commutes, where ι_{2k}, ι'_{2k} are the inclusions to $D^{\otimes\infty}$ and $D^{\otimes\infty}\otimes D^{\otimes\infty}$, respectively. Since ψ_k are isomorphisms, ψ is an isomorphism as well.

Now, it remains to show that

$$\psi \sim_{a.u} id_{D^{\otimes\infty}} \otimes 1_{D^{\otimes\infty}}$$

But arguing as in (i), it suffices to show that

$$\lambda_k \circ \psi_k \circ (id_{D^{\otimes k}} \otimes 1_{D^{\otimes k}}) \sim_{a.u} \lambda_k \circ (id_{D^{\otimes k}} \otimes 1_{D^{\otimes k}}), \quad \text{for all} \quad k \in \mathbb{N}$$

First, define *-homomorphisms $\beta_k \colon (D^{\otimes k})^4 \to (D^{\otimes k})^4$ by $\beta_k = id_{D^{\otimes k}} \otimes \sigma_{D^{\otimes k}} \otimes id_{D^{\otimes k}}$, where $\sigma_{D^{\otimes k}}$ is the flip on $D^{\otimes k}$ and employing again the i-th factor embeddings $\iota_k^{(j)}$ from (i) we get that

$$\beta_k \circ (\psi_k \otimes id_{D^{\otimes 2k}}) \circ \iota_k^{(1)} = \lambda_k \circ \psi_k \circ (id_{D^{\otimes k}} \otimes id_{D^{\otimes k}})$$

and

$$\beta_k \circ (\psi_k \otimes id_{D^{\otimes 2k}}) \circ \iota_k^{(3)} = \iota_k^{(2)}$$

Since by (i) we have that $\iota_k^{(i)} \sim_{a.u} \iota_k^{(j)}$, $i, j \in \{1,2,3,4\}$, then by using Proposition 1.1.6 we obtain that

$$\begin{aligned} \lambda_k \circ \psi_k \circ (id_{D^{\otimes k}} \otimes 1_{D^{\otimes k}}) &= \beta_k \circ (\psi_k \otimes id_{D^{\otimes 2k}}) \circ \iota_k^{(1)} \\ &\sim_{a.u} \beta_k \circ (\psi_k \otimes id_{D^{\otimes 2k}}) \circ \iota_k^{(3)} \\ &= \iota_k^{(2)} \\ &\sim_{a.u} \iota_k^{(1)} \\ &= \lambda_k \end{aligned}$$

as required.

$iii)$ Suppose that D has approximately inner half flip, then in (ii) we showed that $D^{\otimes\infty}$ is strongly self-absorbing, so let $\phi \colon D^{\otimes\infty} \to D^{\otimes\infty} \otimes D^{\otimes\infty}$ be the given isomorphism satifying that $\phi \sim_{a.u} id_{D^{\otimes\infty}} \otimes 1_{D^{\otimes\infty}}$. Hence, there is a sequence of unitaries $(u_n)_n \subset D^{\otimes\infty} \otimes D^{\otimes\infty}$, such that

$$\|\phi(d) - u_n(d \otimes 1_{D^{\otimes\infty}})u_n^*\| \xrightarrow{n\to\infty} 0$$

thus,

$$\left\|\phi^{-1}(\phi(d) - u_n(d \otimes 1_{D^{\otimes\infty}})u_n^*)\right\| = \left\|d - \phi^{-1}(u_n(d \otimes 1_{D^{\otimes\infty}})u_n^*)\right\| \xrightarrow{n\to\infty} 0$$

So, if we let $\phi_n \colon D^{\otimes\infty} \otimes D^{\otimes\infty} \to D^{\otimes\infty}$ be a sequence of *-homomorphisms given by $\phi_n(d_1 \otimes d_2) = \phi^{-1}(u_n(d_1 \otimes d_2)u_n^*)$, then the above norm considerations give the desired result. □

Now, following the construction of sequences of *- homomorphisms in the proof of iii) above, we are ready to harvest Theorem 1.1.11. The proof follows.

Proof. (Theorem 1.1.11) Suppose that A is D-stable, $A \otimes D \cong A$, and consider the *- homomorphisms

$$\psi_n \colon A \otimes D \otimes D \to A \otimes D$$

given as $\psi_n = \mathrm{id}_A \otimes \phi_n$, where ϕ_n are *-homomorphisms by Proposition 1.1.15 (iii). Now, take the induced *-homomorphism $\psi\colon (A \otimes D) \otimes D \to \mathscr{Q}(A \otimes D)$ given by

$$(a \otimes d_1) \otimes d_2 \longmapsto (\psi_n(a \otimes d_1 \otimes d_2))_n$$

then,

$$\|\psi_n(a \otimes d \otimes 1_D) - a \otimes d\| = \|a \otimes (\phi_n(d \otimes 1_D) - d)\| \xrightarrow{n\to\infty} 0$$

which shows that $(\psi_n(a \otimes d \otimes 1_D))_n = (a \otimes d)_n$ in $\mathscr{Q}(A \otimes D)$, or equivalently, that

$$\psi(a \otimes d \otimes 1_D) = a \otimes d \quad \text{for all} \quad a \in A,\ d \in D$$

Therefore, using D-stability of A we obtain a *-homomorphism $\psi'\colon A \otimes D \to \mathscr{Q}(A)$ satisfying

$$\psi'(a \otimes 1_D) = a, \quad \text{for all} \quad a \in A$$

as desired. □

In the rest of this section, we continue working in the setting of strongly self-absorbing C^*-algebras, say D, but we move our focus to examining some permanence properties of D-stability. Namely, how D-stability behaves with respect to *hereditary subalgebras* (see Definition 1.4.9), *quotients*, *inductive limits* and *extensions* (see Definition 2.4.1).

In the following propositions, we assume that D is a separable, unital, strongly self-absorbing and K_1-injective C^*-algebra

Lemma 1.1.16. *Let A be a separable, D-stable C^*-algebra. If $B \subset_{her} A$, then B is D-stable.*

Proof. Let $B \subset_{\text{her}} A$ and let $(h_n)_n$ be an approximate unit of positive contractions for B. Also, let $\iota\colon B \hookrightarrow A$ be the inclusion and h be the image of $(h_n)_n$ in $\mathscr{Q}(B) \subset \mathscr{Q}(A)$. Now, define $\beta\colon \mathscr{Q}(A) \to \mathscr{Q}(B)$ by $\beta(x) = hxh$ and we claim that it is a c.c.p map.

For this purpose, firstly let $x \in \mathscr{Q}(A)^+$, then $x = y^*y$ for some $y \in \mathscr{Q}(A)$ and so $\beta(x) = \beta(y^*y) = (yh)^*yh \in \mathscr{Q}(B)^+$. For contractivity, since $\|h\| \leq \sup\{\|h_n\| : n \in \mathbb{N}\} \leq 1$ we get that $\|\beta(x)\| = \|hxh\| \leq \|x\|$. So, it remains to show that β is completely positive. To this end, let $X \in M_n(\mathscr{Q}(A))^+$, then $X = [x_i^* x_j]_{i,j}$ for some $x_i, x_j \in \mathscr{Q}(A)$. So, $\beta_n(X) = [hx_i^*x_jh](:= [Y_{ij}]_{i,j})$ and it suffices to show that $\sum_{i,j=1}^n b_i^* Y_{ij} b_j \geq 0$ for all $b_1, b_2, ...b_n \in \mathscr{Q}(B)$. So,

$$\sum_{i,j=1}^n b_i^* Y_{ij} b_j = \sum_{i,j=1}^n b_i^* h x_i^* x_j h b_j = \sum_{i,j=1}^n (x_i h b_i)^* x_j h b_j =$$
$$\sum_{i=1}^n (x_i h b_i)^* \sum_{j=1}^n x_j h b_j = (\sum_{i=1}^n x_i h b_i)^* \sum_{i=1}^n x_i h b_i \geq 0$$

hence, β is a c.c.p map.

Since, A is D-stable by Theorem 1.1.11 there is a *-homomorphism $\sigma\colon A \otimes D \to \mathscr{Q}(A)$ such that $\sigma(a \otimes 1_D) = a$ for all $a \in A$, and define a c.c.p map $\hat{\sigma}\colon B \otimes D \to \mathscr{Q}(B)$, as $\hat{\sigma} := \beta \circ \sigma \circ (\iota \otimes id_D)$. Then, for $b \in B$ we have that

$$\hat{\sigma}(b \otimes 1_D) = \beta \circ \sigma(b \otimes 1_D) = \beta(b) = hbh = b$$

and if $b \in B^+$, $d \in D^+$ then

$$\hat{\sigma}(b \otimes d) = \beta \circ \sigma(b \otimes d) = \beta(\sigma(b^{1/4} \otimes 1_D)\sigma(b^{1/2} \otimes d)\sigma(b^{1/4} \otimes 1_D))$$
$$= \beta(b^{1/4}\sigma(b^{1/2} \otimes d)\beta^{1/4}) = hb^{1/4}\sigma(b^{1/2} \otimes d)\beta^{1/4}h = b^{1/4}\sigma(b^{1/2} \otimes d)\beta^{1/4} = \sigma(b \otimes d)$$

where the next to last equality holds because $b^{1/4}\sigma(b^{1/2} \otimes d)\beta^{1/4} (= x) \in B$ and $h_n x h_n \xrightarrow{\|\cdot\|} x$, thus $hxh = x$, in $\mathscr{Q}(B)$. Hence, we showed that $\hat{\sigma}$ is multiplicative and in turn a *-homomorphism. Thus, by Note 1.1.12, B is D-stable. □

Note 1.1.17. Since closed two-sided ideals of C^*-algebras are hereditary subalgebras (see Theorem 1.4.13), we obtain that D-stability is preserved by ideals.

Lemma 1.1.18. *If A is a separable, D-stable C^*-algebra, and $J \lhd A$ a closed ideal, then A/J is $D-$stable.*

Proof. Let $\pi\colon A \to A/J$ be the quotient map, and consider the well defined *-homomorphism $\hat{\pi}\colon \mathscr{Q}(A) \to \mathscr{Q}(A/J)$, which is given by $\hat{\pi}(a_1, a_2, ...) = (\pi(a_1), \pi(a_2), ...)$. Since, A is D-stable, there exists a *- homomorphism $\sigma\colon A \otimes D \to \mathscr{Q}(A)$ such that $\sigma(a \otimes 1_D) = a$, for all $a \in A$. Then, $\hat{\pi} \circ \sigma(a \otimes 1_D) = \hat{\pi}(a) = \pi(a)$, hence $\hat{\pi} \circ \sigma$ induces a map $\hat{\sigma}\colon A/J \otimes D \to \mathscr{Q}(A/J)$, which satisfies $\hat{\sigma}(\pi(a) \otimes d) = \hat{\pi}(\sigma(a \otimes d))$. Moreover, for $a \in J$, $\hat{\pi}(\sigma(a \otimes 1_D) = \hat{\pi}(a) = \pi(a) = 0$, thus $\hat{\pi} \circ \sigma(J \otimes D) = 0$, which implies that $\hat{\sigma}$ is well defined. Finally, by surjectivity of π, we get that $\hat{\sigma}$ is a well defined *-homomorphism, and $\hat{\sigma}(x \otimes 1_D) = x$, for all $x \in A/J$, by Note 1.1.12, A/J is D-stable. $\square$

Next, we see that D-stability is also preserved by inductive limits.

Lemma 1.1.19. *If $A = \varinjlim A_i$ is an inductive limit of separable D-stable C^*-algebras, then A is D-stable.*

Proof. Firstly, we note that if (A, ψ_i) is the inductive limit of (A_i, ϕ_i) , then it is a standard fact for inductive limits of C^*-algebras that $\varinjlim A_i \cong \varinjlim A_i/Ker\psi_i$. So, knowing from the previous lemma that D-stability is preserved by quotients, , we may assume that $(A_i)_i$ is an increasing sequence of separable, D-stable C^*-algebras, and that $A = \overline{\cup_i A_i}$. Now, since A_i D-stable, the construction in the proof of Theorem 1.1.11, provides a sequence of *-homomorphisms $\sigma_{i,n}\colon A_i \otimes D \to A_i$, satisfying that

$$\|\sigma_{i,n}(a \otimes 1_D) - a\| \xrightarrow{n\to\infty} 0, \qquad \forall a \in A_i, \forall i \in \mathbb{N}$$

If $F = \{a_j : j \in \mathbb{N}\}$ is a countable dense subset of A, then for any $k \in \mathbb{N}$, we can find (i, n) such that $\{a_1, ..., a_k\} \subset A_i$ and $\|\sigma_{i,n}(a_j \otimes 1_D) - a_j\| \leq 1/k, \quad j \in \{1, 2, .., k\}$. Thus, if we set $\sigma_k = \sigma_{i,n}$, we get that

$$\|\sigma_k(a_j \otimes 1_D) - a_j\| \xrightarrow{k\to\infty} 0$$

for all $a_j \in F$, but since F is dense in A, it follows that

$$\|\sigma_k(a \otimes 1_D) - a\| \xrightarrow{k\to\infty} 0$$

for all $a \in A$. Since, σ_k is a *-homomorphism for each $k \in \mathbb{N}$, we obtain the induced *-homomorphism $\sigma\colon A \otimes D \to \prod A/\sum_n A$, which satisfies

$$\sigma(a \otimes 1_D) = a$$

for all $a \in A$. Hence, by Theorem 1.1.11 and Note 1.1.12 it follows that A is D-stable. $\square$

As the last bit of permanence properties of D-stability in this section, we state that D-stability is preserved by extensions. The proof is omitted.

Theorem 1.1.20. *Let a short exact sequence of separable C^*-algebras*

$$0 \longrightarrow I \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0$$

If I and B are D-stable, then A is D-stable.

1.2 Strict comparison of positive elements in a C^*-algebra

The central purpose of this section, is to give an introduction to strict comparison of positive elements in a C^*-algebra, and finally to prove that the tensor product of a simple, unital, C^*-algebra with any UHF algebra, has the property of strict comparison. The material in this section is from [28], except otherwise is mentioned.

In the following, A will always be a unital C^*-algebra and A^+ its positive cone. Also, for $\epsilon > 0$, we define $f_\epsilon\colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f_\epsilon(t) = \begin{cases} 0, & t \leq \epsilon \\ \epsilon^{-1}(t-\epsilon), & \epsilon \leq t \leq 2\epsilon \\ 1, & t \geq 2\epsilon \end{cases}$$

and

$$(t-\epsilon)_+ = \max\{t-\epsilon, 0\}$$

Now, let us start by creating the setting that we are going to work with.

Definition 1.2.1. *Let $x, y \in A^+$, we write $x \lessapprox y$ if $\exists\, r \in A$, such that $x \leq ryr^*$. Also, we write $x \lesssim y$, if $f_\epsilon(x) \lessapprox y$, for all $\epsilon > 0$. Finally, we write $x \sim y$ iff $x \lesssim y$ and $y \lesssim x$.*

Note 1.2.2. Both $\lessapprox$ and $\lesssim$ define preorderings on A^+, $x \lessapprox y$ implies $x \lesssim y$, $\sim$ is an equivalence relation and $f_\epsilon(x) \sim (x-\epsilon)_+$, for any $x \in A^+$ and $\epsilon > 0$. For reflexivity of $\lesssim$, let $a \in A^+$. For $\epsilon > 0$, we have that $f_\epsilon(a) \leq (\epsilon 1_A)^{-1/2} a (\epsilon 1_A)^{-1/2}$, hence $f_\epsilon(a) \lessapprox a$, for any $\epsilon > 0$ and so in turn we get that $a \lesssim a$. Now, let us show that $\lessapprox$ is a transitive relation. To this end, let $x, y, z \in A^+$ such that $x \lessapprox y$ and $y \lessapprox z$, then $\exists\, r_1, r_2 \in A$ satisfying that $x \leq r_1 y r_1^*$ and $y \leq r_2 z r_2^*$. Hence, we obtain that $x \leq r_1 y r_1^* \leq r_1 r_2 z r_2^* r_1^* = (r_1 r_2) z (r_1 r_2)^*$, which implies that $x \lessapprox z$, as desired. The reflexivity of $\lessapprox$ is obvious. Moreover, let us now show that $\lessapprox$ is a stronger relation than $\lesssim$. Let $x, y \in A^+$ such that $x \lessapprox y$. As we already showed, $f_\epsilon(x) \lessapprox x$, $\forall \epsilon > 0$, so by transitivity of $\lessapprox$ we get that $f_\epsilon(x) \lessapprox y$, $\forall \epsilon > 0$, showing that $x \lesssim y$, as required.

Using these first assertions, we procced to show that $f_\epsilon(x) \sim (x-\epsilon)_+$, for any $x \in A^+$ and $\epsilon > 0$. We may suppose that $x \in A_1^+$, $0 < \epsilon < 1$ and $\|x\| > \epsilon$. Then, as $\epsilon^{-1}(t-\epsilon) > (t-\epsilon)$, for all $t \in \sigma(x)$ and $(x-\epsilon)_+ \leq \|(x-\epsilon)_+\| \leq \|x\| \leq 1$, we see that $(x-\epsilon)_+ \leq f_\epsilon(x)$. Hence, $(x-\epsilon)_+ \lessapprox f_\epsilon(x)$, which implies that $(x-\epsilon)_+ \lesssim f_\epsilon(x)$. For the other direction, observe that $f_\epsilon(x) \leq (x-\epsilon)_+^{-1/2}(x-\epsilon)_+(x-\epsilon)_+^{-1/2}$, which shows that $f_\epsilon(x) \lessapprox (x-\epsilon)_+$ and therefore $f_\epsilon(x) \lesssim (x-\epsilon)_+$. Also, note that since $(t-\epsilon)_+ \xrightarrow{\epsilon \to 0^+} t$, then $(x-\epsilon)_+ \xrightarrow{\epsilon \to 0^+} x$, by standard functional calculus arguments.

Now, for transitivity of $\lesssim$, let $x, y, z \in A^+$, such that $x \lesssim y$ and $y \lesssim z$. Then, for any $\epsilon > 0$, $f_\epsilon(x) \lessapprox y$ and $f_\epsilon(y) \lessapprox z$, and so $(x-\epsilon)_+ \lessapprox y$ and $(y-\epsilon)_+ \lessapprox z$. Hence, $x \lessapprox y$ and $y \lessapprox z$, using that $(x-\epsilon)_+ \xrightarrow{\epsilon \to 0^+} x$. By transitivity of $\lessapprox$, the desired result follows.

As a first result we show that the comparison theory defined above, when applied to projections, is identical to the usual von Neumman-Murray comparison theory.

Proposition 1.2.3. *Let $p, q \in A$ projections. Then $p \lesssim q$ iff exists $u \in A$ such that $p = uu^*$ and $u^*u \leq q$.*

Proof. Suppose that $p \lesssim q$, then for any $\epsilon > 0$ there is $w \in A$, such that $f_\epsilon(p) \leq wqw^*$. Let $0 < \epsilon < \frac{1}{2}$, then $f_\epsilon(0) = 0$ and $f_\epsilon(1) = 1$. Now, consider the function $\iota(t) = t$ and we see that $\iota|_{\sigma(p)} = f_\epsilon|_{\sigma(p)}$, hence $f_\epsilon(p) = \iota(p) = p$. Thus, if we set $u = pwq$, we get that $uu^* = pwqw^*p = p$, and $u^*u = qw^*pwq \leq q$. For the other direction, since for any ϵ, there is $w \in A$ such that $f_\epsilon(p) \leq wpw^*$ and by hypothesis $p \leq uqu^*$, it follows that $f_\epsilon(p) \leq uwqw^*u^*$. So, for $x = uw$, we have that $f_\epsilon(p) \leq xqx^*$, and since ϵ was arbitrary, we conclude that $p \lesssim q$. $\square$

Proposition 1.2.4. *Let $x, y \in A^+$ and $\delta_0 = \|x - y\|$. Then, for $\delta > \delta_0$, $f_\delta(x) \lessapprox y$.*

Proof. First note that $x - y \leq \|x - y\| = \delta_0$, implies that $x - \delta_0 1_A \leq y$. Moreover, observe that $f_\delta(x)^{\frac{1}{2}}(x - \delta)f_\delta(x)^{\frac{1}{2}} \geq 0$, and so we have

$$f_\delta(x)^{\frac{1}{2}}(\delta - \delta_0)f_\delta(x)^{\frac{1}{2}} \leq f_\delta(x)^{\frac{1}{2}}(x - \delta_0)f_\delta(x)^{\frac{1}{2}} \leq f_\delta(x)^{\frac{1}{2}} y f_\delta(x)^{\frac{1}{2}}$$

Hence, $f_\delta(x) \leq ryr^*$, where $r = (\delta - \delta_0)^{-\frac{1}{2}} f_\delta(x)^{\frac{1}{2}}$. $\square$

Moreover, it is the case that the preordering $\lesssim$ generalizes the usual order $\leq$ on A^+. To show this fact, we need first some preliminary results that we present right away.

Proposition 1.2.5 ([24], Proposition 1.4.4)**.** *Let $x, y \in A$, $a \in A^+$, such that $x^*x \leq a^\alpha$ and $yy^* \leq a^\beta$, where $\alpha + \beta > 1$. Then, the sequence $(u_n)_n = (x(\frac{1}{n} + a)^{-\frac{1}{2}}y)_n$ is convergent to some $u \in A$, such that $\|u\| \leq \left\|a^{(\alpha+\beta-1)/2}\right\|$*

Proof. Firstly, set $d_{nm} = (\frac{1}{n} + a)^{-\frac{1}{2}} - (\frac{1}{m} + a)^{-\frac{1}{2}}$, and see that,

$$\|u_n - u_m\|^2 = \left\|x[(1/n) + a]^{-\frac{1}{2}} - [(1/m) + a]^{-\frac{1}{2}}]y\right\|^2 = \|xd_{nm}y\|^2$$

$$= \|y^*d_{nm}x^*xd_{nm}y\| \leq \|y^*d_{nm}a^\alpha d_{nm}y\| = \left\|a^{\frac{\alpha}{2}}d_{nm}y\right\|^2$$

$$= \left\|a^{\frac{\alpha}{2}}d_{nm}yy^*d_{nm}a^{\frac{\alpha}{2}}\right\| \leq \left\|a^{\frac{\alpha}{2}}d_{nm}a^\beta d_{nm}a^{\frac{\alpha}{2}}\right\| = \left\|d_{nm}a^{\frac{\alpha+\beta}{2}}\right\|^2$$

Now, the sequence $a^{\frac{\alpha+\beta}{2}}(1/n + a)^{-\frac{1}{2}}$ is increasing, and so by spectral theory is (uniformly) convergent to $a^{\frac{\alpha+\beta-1}{2}}$. Hence, $(d_{nm}a^{\frac{\alpha+\beta}{2}})$ converges to 0, when $n, m \to \infty$, which implies that $(u_n)_n$ is a convergent sequence. Finally, using the same norm calculations as above, it follows that

$$\|u_n\| \leq \left\|a^{\frac{\alpha}{2}}a^{\frac{\beta}{2}}((1/n) + a)^{-\frac{1}{2}}\right\| \leq \left\|a^{\frac{\alpha}{2}}a^{\frac{\beta}{2}}a^{-\frac{1}{2}}\right\| = \left\|a^{(\alpha+\beta-1)/2}\right\|, \quad \forall \quad n \in \mathbb{N}$$

thus,

$$\|u\| \leq \left\|a^{(\alpha+\beta-1)/2}\right\|.$$

$\square$

As a consequence, we can now obtain a rather useful proposition, that will be critical in showing that $\lesssim$ generalizes $\leq$.

Proposition 1.2.6 ([24], Proposition 1.4.5)**.** *Let $x \in A$ and $a \in A^+$, such that $x^*x \leq a$. Then, for $0 < \beta < \frac{1}{2}$, there exists $u \in A$ satisfying that $\|u\| \leq \left\|a^{\frac{1}{2}-\beta}\right\|$, and $x = ua^\beta$.*

Proof. Let $u_n = x[(1/n) + a]^{-\frac{1}{2}}a^{\frac{1}{2}-\beta}$ and apply Proposition 1.2.5, to get that u_n is convergent to some u, such that $\|u\| \leq \left\|a^{\frac{1}{2}-\beta}\right\|$. To show that $x = ua^\beta$, observe

$$\left\|x - u_na^\beta\right\| = \left\|x[1 - ((1/n) + a)^{-\frac{1}{2}}a^{\frac{1}{2}}]\right\|$$
$$\leq \left\|a^{\frac{1}{2}}[1 - ((1/n) + a)^{-\frac{1}{2}}a^{\frac{1}{2}}]\right\| \xrightarrow{n\to\infty} 0$$

by spectral theory. Thus, $x = ua^\beta$, as desired. $\square$

Lemma 1.2.7. *Let $x, y \in A^+$, such that $x \leq y$. Then*

$$i) \ \exists\, r \in A, \ \ \textit{such that } x = ry^{\frac{1}{2}}r^*$$
$$ii) \ \exists\, (r_n)_n \subset A, \ \ \textit{such that } r_n y r_n^* \to x$$

Proof. *i)* Since, $x^{\frac{1}{2}}x^{\frac{1}{2}} \leq y$, let $\beta = \frac{1}{4}$, and apply Proposition 1.2.6 to find $r \in A$ such that $x^{\frac{1}{2}} = ry^{\frac{1}{4}}$, and in turn $x = ry^{\frac{1}{2}}r^*$.
ii) Let $\delta > 0$, and define $g_\delta(t) = \min\{t^{-1}, \delta^{-1}\}$. Moreover, if we define $r_\delta = x^{\frac{1}{2}}g_\delta(y)^{\frac{1}{2}}$, and $s_\delta = x^{\frac{1}{2}}(1 - g_\delta(y)y)^{\frac{1}{2}}$, then it follows that

$$\begin{aligned} s_\delta s_\delta^* &= x^{\frac{1}{2}}(1-g_\delta(y)y)^{\frac{1}{2}}(1-g_\delta(y)y)^{\frac{1}{2}}x^{\frac{1}{2}} = x - x^{\frac{1}{2}}g_\delta(y)yx^{\frac{1}{2}} \\ &= x - r_\delta y r_\delta^* \end{aligned}$$

and

$$s_\delta^* s_\delta = (1-g_\delta(y)y)^{\frac{1}{2}}x(1-g_\delta(y)y)^{\frac{1}{2}} \leq (1-g_\delta(y)y)^{\frac{1}{2}}y(1-g_\delta(y)y)^{\frac{1}{2}} = (1-g_\delta(y)y)y$$

Now, if $\delta \to 0$, then $g_\delta(y) \to y^{-1}$, which shows that $\|x - r_\delta y r_\delta^*\| = \|s_\delta s_\delta^*\| = \|s_\delta^* s_\delta\| \to 0$, as required. □

Note 1.2.8. We claim that if $f, g \in C(\sigma(x))^+$, such that $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$, then $f(x) \lesssim g(x)$. To do so, first observe that, for $\epsilon > 0$, it suffices to find $r \in C(\sigma(x))$ such that $\|f - rgr^*\|_\infty < \epsilon$, since this implies that there is $y \in A$ satisfying, $f(x) \leq yg(x)y^*$, hence $f(x) \lesssim g(x)$. Now, to find such an r, we proceed as follows. Let $\epsilon > 0$, and set $K = \{t \in \sigma(x) : f(t) \geq \epsilon\}$, a compact set, and note that $K \subseteq \mathrm{supp}(g)$. Since, g is strictly positive, there is $\delta > 0$ such that $g(t) > \delta$, for all $t \in K$, hence $K \subseteq \{t \in \sigma(x) : g(t) > \delta/2\} = U$. Moreover, U is an open set, and so by Uryshon's Lemma, we find $s \colon \sigma(x) \to [0,1]$ continuous function, such that $s = 1$ in K, and $s = 0$ in U^c. Now, define $h \colon \sigma(x) \to \mathbb{R}_+$ by

$$h(t) = \begin{cases} \frac{f(t)}{g(t)}s(t), & t \in U \\ 0, & t \notin U \end{cases}$$

and we immediately see that $\|f - hg\|_\infty < \epsilon$. So, $r = h^{\frac{1}{2}}$, has the desired properties.

Note 1.2.9. Let $\alpha \in \mathbb{R}^+$, and take $f(t) = t$, $g(t) = t^\alpha$ defined on $\sigma(x)$, for some $x \in A^+$. Since, $\mathrm{supp}(f) = \mathrm{supp}(g)$, we obtain in a straightforward way from 1), that $x \sim x^\alpha$.

Having these initial insights into the preordering $\lesssim$, we can now show that there are various (equivalent) ways of reformulating the condition $x \lesssim y$. This result will be of great importance throughout the rest of this section.

Proposition 1.2.10. *Let $x, y \in A^+$. The following are equivalent*

$$i) \ x \lesssim y$$
$$ii) \ \exists\, r_n \in A : r_n y r_n^* \to x$$
$$iii) \ \exists\, r_n, s_n \in A : r_n y s_n \to x$$
$$iv) \ \forall \epsilon > 0, \ \ \exists \delta > 0, \ \ \exists r \in A \ \ \textit{such that } f_\epsilon(x) = rf_\delta(y)r^*$$

Proof. *i)* $\implies$ *ii)* For $\epsilon > 0$, find $h \in A$ such that $f_\epsilon(x) \leq hyh^*$, therefore, by Lemma 1.2.7 there is $s \in A$, so that

$$\|shyh^*s^* - f_\epsilon(x)\| < \epsilon$$

Now, set $K = \{t \in \mathbb{R}_+ \colon t \geq \epsilon\}$ and $U = \{t \in \mathbb{R}_+ \colon t > \epsilon/2\}$. Then, K is a closed set, U is an open set, and $K \subset U$, thus by Uryshon's Lemma we find a continuous function $\rho \colon \mathbb{R}^+ \to [0,1]$, such that $\rho|_K = 1$, and $\rho|_{U^c} = 0$. Moreover, set $g' \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$g'(t) = \begin{cases} \frac{i(t)}{f_\epsilon(t)}\rho(t), & t \in U \\ 0, & t \notin U \end{cases}$$

where i is the identity map. Then g is a continuous map and if we set $g = (g')^{1/2}$, then g satisfies that

$$\|x - g(x)f_\epsilon(x)g(x)\| < \epsilon$$

Finally, set $r = g(x)sh$, and the conclusion follows.

ii) $\implies$ *iii)* Obvious

iii) $\implies$ *i)* Set $a_n = r_n y^{\frac{1}{2}}$, $b_n = y^{\frac{1}{2}} s_n$, and since $b_n^* a_n^* a_n b_n \xrightarrow{\|\cdot\|} x^* x = x^2$, for $\epsilon > 0$, we can find large enough n such that

$$\left\|b_n^* a_n^* a_n b_n - x^2\right\| < \epsilon$$

Thus, by Proposition 1.2.4, we get

$$f_\epsilon(x^2) \precsim b_n^* a_n^* a_n b_n \leq \|a_n\|^2 b_n^* b_n \precsim y$$

hence, $x^2 \precsim y$. But, Note 1.2.9 asserts that $x^2 \sim x$, concluding that $x \precsim y$.

ii) $\implies$ *iv)* Firstly, let $\epsilon > 0$ and find $s_1 \in A$ such that

$$\|x - s_1 y s_1^*\| < \epsilon$$

Put, $h_\delta(t) = \max\{t - \delta, 0\}$, then by Note 1.2.2, we know that $h_\delta(y) \sim f_\delta(y)$ and moreover that $h_\delta(y) \xrightarrow{\delta \to 0} y$. Hence,

$$\|x - s_1 h_{2\delta}(y) s_1^*\| < \epsilon$$

and $s_1 h_{2\delta}(y) s_1^* = s_2 f_{2\delta}(y) s_2^*$, for some $\delta > 0$ and $s_2 \in A$. Combining these two observations and employing Proposition 1.2.4, we can find $s_3 \in A$ such that

$$f_\epsilon(x) \leq s_3 f_{2\delta}(y) s_3^*$$

Set $z = s_3 f_{2\delta}(y)^{\frac{1}{2}} \in A$, and let $z = u|z|$, be its polar decomposition. If $s_4 = u|z|^{\frac{1}{2}}$, then $s_4 \in A$ and since $f_{2\delta}(y) f_\delta(y) = f_{2\delta}(y)$, we get that $s_4 f_\delta(y) = s_4$, so it follows that

$$(s_4 f_\delta(y) s_4^*)^2 = (s_4 s_4^*)^2 = zz^* = s_3 f_{2\delta}(y) s_3^* \geq f_\epsilon(x)$$

Now, by Lemma 1.2.7, there exists $r \in A$ satisfying

$$f_\epsilon(x) = r f_\delta(y) r^*$$

as desired.

iv) $\implies$ *i)* Let $\epsilon > 0$, and take $r_1 \in A$, $\delta > 0$ such that $f_\epsilon(x) \leq r_1 f_\delta(y) r_1^*$. But, $f_\delta(y) \precsim y$ implies that we can find $r_2 \in A$, such that $f_\delta(y) \leq r_2 y r_2^*$, hence $f_\epsilon(x) \leq r y r^*$, where $r = r_2 r_1$. ☐

We finalize this first discussion about strict comparison of positive elements, with a useful standard lemma in spectral theory of C^*-algebras, that we will employ critically in the coming up, and a proposition that will enable us to define the so called Cuntz Semigroup in the following pages. The proof of the proposition is omitted.

Lemma 1.2.11 ([20], Lemma 1.2.5). *Let K be a non-empty compact subset of $\mathbb{R}$, and $f \colon K \to \mathbb{C}$ be a continuous function. Let A be a unital C^*-algebra and Ω_K be the set of all self-adjoint elements in A, whose spectra is contained in K. Then, the induced map*

$$f \colon \Omega_K \to A, \quad a \mapsto f(a)$$

is continuous.

Proof. First note that since the multiplication is continuous in any C^*-algebra, then the map

$a \to a^n$ is continuous, for any $n \in \mathbb{N}$, obtaining that any polynomial induces a continuous map $A \to A$

Now, let $f\colon K \to \mathbb{C}$ be a continuous function and $\epsilon > 0$, then by Stone-Weierstrass theorem we find polynomial p such that

$$|f(z) - p(z)| < \epsilon/3, \quad \forall z \in K$$

Also, for $a \in A$, find $\delta > 0$ such that

$$\|p(a) - p(b)\| < \epsilon/3$$

when $\|a - b\| < \delta$. Moreover, we observe that for $c \in \Omega_K$

$$\|f(c) - g(c)\| = \|(f - g)(c)\| = \sup\{|(f - g)(z)| : z \in \sigma(c)\} < \epsilon/3$$

hence, for $a, b \in \Omega_K$, such that $\|a - b\| < \delta$ it follows that

$$\|f(a) - f(b)\| = \|f(a) - p(a) + p(a) - p(b) + p(b) - f(b)\|$$

$$\leq \|f(a) - p(a)\| + \|p(a) - p(b)\| + \|p(b) - f(b)\| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

concluding that f is continuous. □

Note 1.2.12. Let $a, b \in A^+$, and suppose that $\|a\| \leq \|b\|$ and consider $f'_\epsilon = f_\epsilon|_{[0,\|b\|]}$, the restriction of the continuous function that we defined in the introduction. Then, by the lemma above we deduce that for any $\epsilon > 0$, we can find $\delta > 0$ so that, whenever $\|a - b\| < \delta$, then $\|f'_\epsilon(a) - f'_\epsilon(b)\| < \epsilon$.

Proposition 1.2.13. *Let $x, x', y, y' \in A^+$ such that $x \precsim y$, $x' \precsim y'$ and $y'y = 0$. Then, $x + x' \precsim y + y'$.*

As we mentioned in the beginning of this section, our main goal is to prove that the tensor product of a simple, unital, C^*-algebra with any UHF algebra, has the strict comparison property. To do so, we have to introduce few new notions, starting by the almost unperforated partially ordered abelian semigroup.

Definition 1.2.14. *A partially ordered abelian semigroup $(S, \leq)$ is an abelian semigroup equipped with a partial order satisfying $t_1 + t_2 \leq s_1 + s_2$, when $t_i, s_i \in S$, and $t_i \leq s_i$. We further assume that S has zero element 0, such that $0 \leq s$, for all $s \in S$. Also, an element $t \in S$ is called strong order unit, if for all $s \in S$, there is $n \in \mathbb{N}$ such that $s \leq nt$.*

In this setting, the term *state* is reserved for an order preserving, additive map $d\colon S \to R_+$. Meanwhile, we say that a partially ordered abelian semigroup $(S, \leq)$ is almost unperforated if, whenever $k, k' \in \mathbb{N}$, $s, t \in S$, such that $ks \leq k't$ and $k' < k$, then $s \leq t$.

The following propositions aim to show that if $(S, \leq)$ is an almost unperforated partially ordered abelian semigroup, then the state space on S, determines the order structure.

Proposition 1.2.15. *Let $(S, \leq)$ be a partially ordered abelian semigroup. If $t, t' \in S$, t is a strong order unit and $d(t') < d(t)$ for all states d on S , then there is $n \in \mathbb{N}$ and $u \in S$ satisfying that $nt' + u \leq nt + u$.*

Proof. Let $G(S)$ be the Grothendieck group associated to S and let $\gamma\colon S \to G(S)$ be the Grothendieck map, which is additive and $[s] = \gamma(s) = \gamma(t) = [t]$ if and only if $s + u = t + u$ for some $u \in S$. Also, let

$$G(S)^+ = \{[s] - [t] \colon \; s, t \in S, \; s + u \geq t + u, \text{ for some } u \in S\}$$

and it is straightforward to see that $(G(S), G(S)^+)$ is a partially ordered abelian group and that

$[t]$ is a strong order unit.

Now, suppose that $nt' + u \not\leq nt + u$, for all $u \in S$, $n \in \mathbb{N}$, then it follows that $n[t'] \not\leq n[t]$, for all $n \in \mathbb{N}$. Moreover, set

$$f^*(t') = \inf\{n/m : \ n, m > 0, \ mt' \leq nt\}$$

and

$$f_*(t') = \sup\{n/m : \ n, m > 0, \ nt \leq mt'\}$$

and observe that $f^*(t') > 1$ and $f_*(t') < 1$, hence by Lemma 4.1 in [15], there is a state $\overline{d}$ on $G(S)$ such that $\overline{d}([t']) \geq 1 = \overline{d}([t])$. Since the map defined as $d(t) = \overline{d}([t])$, for $t \in S$, is a state on S, and $d(t') \geq d(t)$, we get a contradiction. □

Proposition 1.2.16. *Let $(S, \leq)$ be an almost unperforated partially ordered abelian semigroup. If $t, t' \in S$, t is a strong order unit and $d(t') < d(t)$ for all states d on S, then $t' \leq t$*

Proof. Let Σ be the set of all states d on S such that $d(t) = 1$ and we claim that Σ is compact. To see this, note that for any $s \in S$, $\exists\, n_s \in \mathbb{N}$ such that $s \leq n_s t$, which implies that $d(s) \leq n_s$, for all $d \in \Sigma$. Let $(d_n)_n$ be a sequence in Σ, then for each $s \in \Sigma$, $(d_n(s))_n) \subseteq [0, n_s]$ and since $[0, n_s]$ is compact there is $(d_{k_n}(s))_n$ convergent subsequence to some x_s. If we define d by $s \mapsto x_s$, we see that $d \in \Sigma$ and that $d_{k_n} \to d$, concluding that Σ is compact.

Now, we claim that due to compactness there is $c < 1$, such that $d(t') \leq c$, for all $d \in \Sigma$. If otherwise, we could find for any $n \in \mathbb{N}$, $d_n \in \Sigma$, such that $d_n(t') > 1 - \frac{1}{n}$. Since Σ is compact, the sequence $(d_n)_n$ has a subsequence $(d_{k_n})_n$ converging to some $d \in \Sigma$. So, we can find $n_o \in \mathbb{N}$, such that $|d(t') - d(t)| < \frac{1}{k_n}$, for all $n \geq n_0$ and by arranging k_n, so that $k_n \geq k_m$, when $n \geq m$, we get that $d(t')$ is arbitrarily close to $d(t)(= 1)$, which contradicts that $d(t') < d(t)$.

Furthermore, find $m, m' \in \mathbb{N}$, $m' \geq m$, such that $\frac{m}{m'} > c$. Then, $d(t') < \frac{m}{m'} \implies d(m't') < m = md(t) = d(mt)$, for all $d \in \Sigma$, and by observing that if τ is any state on S, where $\tau(t) = k \in \mathbb{R}_+$, then $d = \frac{1}{k}\tau$ is again a state and $d \in \Sigma$, we obtain that $d(m't') < d(mt)$, for any state d on S.

Now, let any $k \in \mathbb{N}$, then $d(km't') < d(kmt)$, for all states on S, and so, by Proposition 1.2.15, there exists $n \in \mathbb{N}$ and $u \in S$ satisfying,

$$nkm't' + u \leq nkmt + u$$

Since t is a strong order unit, we can find $l \in \mathbb{N}$, such that $u \leq lt$, which shows that

$$knm't' \leq knm't' + u \leq knmt + u \leq (knm + l)t$$

and if k is large enough in order that $knm' > knm + l$, the hypothesis that S is almost unperforated, yields that $t' \leq t$, as required. □

Now, we are ready to present the so called Cuntz Semigroup. Also, it is essential for our purposes to make a brief introduction to dimension functions and quasi-traces, that will be central ingredients in the final results of this section.

Let A be a unital C^*-algebra and denote by $M_\infty(A)$ the union of $M_n(A)$, for all $n \in \mathbb{N}$, with inclusions $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. For $x \in M_\infty(A)^+$, we set $\langle x \rangle = \{y \in M_\infty(A)\colon x \sim y\}$, where $\sim$ is the equivalence relation defined earlier. Moreover, we define addition by $\langle x \rangle + \langle y \rangle = \langle x' + y' \rangle$, where $x' \sim x$, $y' \sim y$ and $x' \perp y'$(i.e $x'y' = 0$). By Proposition 1.2.13, it is evident that this is well defined. Finally, we define a partial order $\leq$, such that $\langle x \rangle \leq \langle y \rangle$ if $x \precsim y$.

Definition 1.2.17. *For any unital C^*-algebra A, the associated Cuntz semigroup is the abelian semigroup*

$$S(A) = \{\langle x\rangle \colon x \in M_\infty(A)^+\}$$

which equipped with the partial order $\leq$ becomes a partialy oredered abelian semigroup.

Definition 1.2.18. *A state d on $S(A)$, is called dimension function if $d(\langle 1_A\rangle) = 1$, and the set of all such states is denoted by $DF(A)$. Moreover, if $d \in DF(A)$ and $d(\langle x\rangle) \leq \liminf_n d(\langle x_n\rangle)$, whenever $x_n \to x$, then d is called lower semicontinuous. The set of all lower semicontinuous states is denoted $LDF(A)$.*

Proposition 1.2.19. *Let $d \in DF(A)$ and set*

$$\overline{d}(\langle x\rangle) = \lim_{\epsilon\to 0} d(\langle f_\epsilon(x)\rangle)$$

Then, $\overline{d} \in LDF(A)$, $\overline{d} \leq d$, and $d = \overline{d}$ if $d \in LDF(A)$

Proof. First we show that $\overline{d}$ is additive. Let $x \in M_n(A)^+$, $y \in M_m(A)^+$, then observe that $x \oplus 0_m \sim x$, $0_n \oplus y \sim y$ and $(x \oplus 0_m)(0_n \oplus y) = 0$. Thus, we have that $\langle x\rangle + \langle y\rangle = \langle (x \oplus 0_m) + (0_n \oplus y)\rangle$ and so it suffices to show that $f_\epsilon((x \oplus 0_m) + (0_n \oplus y)) = f_\epsilon(x \oplus 0_m) + f_\epsilon(0_n \oplus y)$, since then, using that $f_\epsilon(0_n \oplus y)f_\epsilon(x \oplus 0_m) = 0$, we get $\langle f_\epsilon(x \oplus 0_m) + f_\epsilon(0_n \oplus y)\rangle = \langle f_\epsilon(x \oplus 0_m)\rangle + \langle f_\epsilon(0_n \oplus y)\rangle$ and the conclusion follows by additivity of d.
To this end, we claim that for any real valued continuous function defined on some $[0, c]$, $c \in \mathbb{R}_+$, such that $f(0) = 0$, we have $f(x + y) = f(x) + f(y)$, whenever $xy = 0$. But, for any polynomial p, we have that $p(x + y) = p(x) + p(y)$, since $xy = 0$, which implies that $(x + y)^k = x^k + y^k$ for any $k \in \mathbb{N}$. So, using Weirstrass approximation theorem we get that for $\epsilon > 0$ there exists polynomial p such that the following holds

$$\begin{aligned}|f(x+y) - f(x) + f(y)| &= |f(x+y) - p(x+y) + p(x) + p(y) - f(x) - f(y)| \\ &\leq |f(x+y) - p(x+y)| + |f(x) - p(x)| + |f(y) - p(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

since ϵ was arbitrary, we conclude that $f(x + y) = f(x) + f(y)$.

We now proceed to show that $\overline{d}(\langle x\rangle) \leq \overline{d}(\langle y\rangle)$, whenever $\langle x\rangle \leq \langle y\rangle$. Since $x \lesssim y$, Proposition 1.2.10 (iv) shows that $\forall \epsilon > 0$ $\exists \delta$ such that $f_\epsilon(x) \lesssim f_\delta(y)$, hence

$$d(\langle f_\epsilon(x)\rangle) \leq d(\langle f_\delta(y)\rangle) \leq \overline{d}(\langle y\rangle), \quad \forall \epsilon > 0$$

This implies that $\overline{d}(\langle x\rangle) \leq \overline{d}(\langle y\rangle)$, as desired.

To show that $\overline{d} \in LDF(A)$, let $x_n \to x$. Then for $\epsilon > 0$, find $n_0 \in \mathbb{N}$, such that $\|x_n - x\| < \frac{\epsilon}{2}$, for all $n \geq n_0$, and by Proposition 1.2.4 we get that $f_{\frac{\epsilon}{2}}(x) \lesssim x_n$. Moreover, by Proposition 1.2.10 (iv), there are δ_n for all $n \geq n_0$, so that

$$f_\epsilon(x) \lesssim f_{\frac{\epsilon}{2}}(f_{\frac{\epsilon}{2}}(x)) \lesssim f_{\delta_n}(x_n)$$

Hence,

$$d(\langle f_\epsilon(x)\rangle) \leq d(\langle f_{\delta_n}(x_n)\rangle) \leq \overline{d}(\langle x_n\rangle)$$

Since ϵ was arbitrary, we conclude that

$$\overline{d}(\langle x\rangle) \leq \liminf_n \overline{d}(\langle x_n\rangle)$$

thus $\overline{d} \in LDF(A)$.

Now, since $f_\epsilon(x) \lesssim x$, for all $\epsilon > 0$, it follows that

$$d(\langle f_\epsilon(x)\rangle) \leq d(\langle x\rangle), \quad \forall \epsilon > 0$$

and in turn that, $\overline{d}(\langle x\rangle) \leq d(\langle x\rangle)$.

Finally, suppose that $d \in LDF(A)$. For $\epsilon > 0$, consider as previously, the continuous function

$(t-\epsilon)_+ = \max\{t-\epsilon, 0\}$, which satisfies that $(x-\epsilon)_+ \sim f_\epsilon(x)$, and $(x-\epsilon)_+ \xrightarrow{\epsilon\to 0} x$. Then, using that $d \in LDF(A)$ we obtain

$$d(\langle x\rangle) \leq \liminf_n d(\langle f_{\frac{1}{n}}(x)\rangle) = \lim_{\epsilon\to 0} d(f_\epsilon(\langle x\rangle)) = \overline{d}(\langle x\rangle)$$

showing that $d = \overline{d}$. $\square$

Definition 1.2.20. *A function $\tau\colon A \to \mathbb{C}$ is called (normalised) quasi-trace if*

i) $\tau(1_A) = 1$

ii) $0 \leq \tau(x^*x) = \tau(xx^*)$

iii) $\tau(a+ib) = \tau(a) + i\tau(b)$ *for all* $a,b \in A_{s.a}$

iv) τ *is linear on abelian C^*-subalgebras of A*

v) τ *extends to a function from $M_n(A)$ to $\mathbb{C}$ satisfying all the above conditions.*

Moreover, the set of all quasi-traces on A is denoted $QT(A)$.

Note 1.2.21. Let $\tau \in QT(A)$ and define $d_\tau(\langle x\rangle) = \lim_{\epsilon\to 0} \tau(f_\epsilon(x))$. It is a fact that, for any $\tau \in QT(A)$, we can define $d_\tau(\langle x\rangle) = \lim_{n\to\infty} \tau(x^{1/n})$, and that these two definitions are equivalent. Blackadar and Handelman showed that d_τ is well defined and $d_\tau \in LDF(A)$. In fact, there is an isomorphism between $QT(A)$ and $LDF(A)$, which follows by the theorem below. Using these observations, we can now rigorously say that a C^*-algebra has strict comparison of positive elements with respect to its traces if for any $\langle x\rangle, \langle y\rangle \in S(A)$, $\langle x\rangle \leq \langle y\rangle$ whenever $d_\tau(\langle x\rangle) < d_\tau(\langle y\rangle)$ for all $d_\tau \in LDF(A)$.

Theorem 1.2.22. *If $d \in LDF(A)$, then there is $\tau \in QT(A)$ such that $d = d_\tau$.*

With these tools in our disposal, we are just one step behind the main goal of this section. But firstly, we make the following observations and we prove a rather useful lemma that will be heavily employed in the process of this project. We observe, as in the proof of additivity in Proposition 1.2.19, that whenever $x \in A^+$ then $\langle x\rangle + \langle x\rangle = \langle x' + x''\rangle$, where $x' = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} (= x \oplus 0)$ and $x'' = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} (= 0 \oplus x)$, since $x'x'' = 0$, and $\begin{pmatrix} 1 & 0 \end{pmatrix} x' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x = \begin{pmatrix} 0 & 1 \end{pmatrix} x'' \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which implies that $x' \sim x$ and $x'' \sim x$. Hence, $\langle x\rangle + \langle x\rangle = \langle x \otimes 1_2\rangle$, where $x \otimes 1_2 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Using this observation, let $k, k' \in \mathbb{N}$, and $x, y \in A^+$, then it follows that,

$$k\langle x\rangle \leq k'\langle y\rangle \iff \langle x \otimes 1_k\rangle \leq \langle y \otimes 1_{k'}\rangle \iff x \otimes 1_k \precsim y \otimes 1_{k'}$$

and by Proposition 1.2.10 this is equivalent to the existence of a sequence $(r_j)_j \subseteq M_{k,k'}(A)$ such that $r_j(y \otimes 1_{k'})r_j^* \to x \otimes 1_k$.

Lemma 1.2.23 ([32], Lemma 2.2)**.** *Let A be a C^*-algebra which admits strict comparison of positive elements by bounded traces. Then, if $a \in A_+^1$ and there exists $m \in \mathbb{N}$ such that $\tau(a) > \frac{2}{m}$ for all $\tau \in T(A)$, there are m^2 contractions $b_1, ..., b_{m^2}$ such that*

$$1_A = \sum_{j=1}^{m^2} b_j a b_j^*$$

.

Proof. Let $\delta = \frac{1}{2}\min_{\tau\in T(A)}\tau(a)$. Then $m\delta > 1$ and for $\tau \in T(A)$ we have

$$d_\tau(\langle(a-\delta)_+^2\rangle) = \lim_{n\to\infty}\tau((a-\delta)_+^{2/n}) > \tau((a-\delta)_+) \geq \tau(a-\delta) = \tau(a) - \delta \geq \delta$$

where the first inequality holds because $(a-\delta)_+ < (a-\delta)_+^{2/n}$ for all $n \geq 2$, since $\|(a-\delta)_+\| \leq 1$. So, we get that

$$d_\tau(\langle(a-\delta)_+^{2\ \oplus m}\rangle) > m\delta > 1 = d_\tau(\langle 1_A \oplus 0^{(m-1)}\rangle)$$

and by strict comparison of A, $1_A \oplus 0^{(m-1)} \precsim (a-\delta)_+^2)^{\oplus m}$. Now, as $1_A \oplus 0^{(m-1)} \sim 1_A$, , for any $\epsilon < 1/2$ we find $b_1, ..., b_m \in A$, by Proposition 1.2.10 (iv) such that

$$(1_A =) f_\epsilon(1_A) = \sum_{j=1}^m b_j((a-\delta)_+^2)b_j^*$$

In fact, $(b_j(a-\delta)_+)(b_j(a-\delta)_+)^* \leq \sum_{j=1}^m (b_j(a-\delta)_+)(b_j(a-\delta)_+)^* = 1_A$, hence $\|(b_j(a-\delta)_+)\| \leq 1$ Now, define $h \in C([0,1])$ by

$$h(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \delta \leq t \\ \frac{t}{\delta^{\frac{3}{2}}}, & 0 \leq t \leq \delta \end{cases}$$

and note that $h(t)^2 t = 1$ for $t \geq \delta$. Hence, $(a-\delta)_+ h(a)^2 a = (a-\delta)_+$ and if we set $c_j = m^{-1/2} b_j (a-\delta)_+ h(a)$, then $\|c_j\| \leq m^{-1/2}\|h(a)\| \leq (m\delta)^{-1/2} < 1$, since $\sigma(h(a)) \subseteq [0, \delta^{-1/2}]$. This shows that c_j is a contraction for all $j = 1, ..., m$. Moreover, c_j satisfy that

$$c_j a c_j^* = m^{-1} b_j (a-\delta)_+ h(a)^2 a (a-\delta)_+ b_j^* = m^{-1} b_j (a-\delta)_+^2 b_j^*$$

so

$$m\sum_{j=1}^m c_j a c_j^* = \sum_{j=1}^m b_j(a-\delta)_+^2 b_j^* = 1_A$$

Therefore, if we set $d_1, ..., d_{m^2} \in A$ such that $c_1 = d_1 = ... = d_m$,...... , $c_m = d_{(m-1)m} = ... = d_{m^2}$, the desired result follows. ☐

After this slight digression, we are ready to exhibit the last two rather important results of this section. Firstly, we show that all Cuntz semigroups associated to a tensor product of a unital C^*-algebra with any UHF-algebra, are almost unperforated.

Lemma 1.2.24. *Let B be a UHF algebra and D a unital C^*-algebra. Then, $S(B \otimes D)$ is an almost unperforated partially ordered abelian semigroup.*

Proof. Set $A = B \otimes D$ a unital C^*-algebra. Then, $A = \varinjlim(A_n, \rho_n)$, where $A_n = M_{k_n}(D)$, and $\rho_n \colon M_{k_n}(D) \to M_{k_{n+1}}(D)$ are unital *-homomorphisms, i.e $k_n \mid k_{n+1}$. Also, denote by $\lambda_n \colon A_n \to A$ the canonical embeddings, whence $\cup_n \lambda_n(A_n)$ is dense in A. We aim to show that if $k, k' \in \mathbb{N}$, $x, y \in M_n(A)^+$, $n \in \mathbb{N}$, such that $k' < k$ and $k\langle x\rangle \leq k'\langle y\rangle$, then $\langle x\rangle \leq \langle y\rangle$. Upon changing B to $M_n(B)$, we may assume that $x, y \in A^+$.

Assume first that $x, y \in \lambda_{n_0}(A_{n_0})$, for some $n_0 \in \mathbb{N}$ and take $\epsilon > 0$. Then, using the observation above, $k\langle x\rangle \leq k'\langle y\rangle$ implies that there is sequence $(r_j)_j \subseteq M_{k,k'}(A)$ such that

$$r_j(y \otimes 1_{k'})r_j^* \to x \otimes 1_k$$

So, find j_0 such that $\left\|r_{j_0}(y\otimes 1_{k'})r_{j_0}^* - x \otimes 1_k\right\| < \epsilon/2$, and moreover by density of $\cup_n\lambda_n(A_n)$ in A, find $n \geq n_0$ and $r \in M_{k,k'}(\lambda_n(A_n))$ such that $\|r - r_{j_0}\| < \min\{\epsilon/4\|yr^*\|, \epsilon/4\|r_{j_0}y\|\}$. Then, we get that,

$$\begin{aligned}\|r(y\otimes 1_{k'})r^* - x\otimes 1_k\| &= \left\|r(y\otimes 1_{k'})r^* - r_{j_0}(y\otimes 1_{k'})r_{j_0}^* + r_{j_0}(y\otimes 1_{k'})r_{j_0}^* - x\otimes 1_k\right\| \\ &\leq \left\|r(y\otimes 1_{k'})r^* - r_{j_0}(y\otimes 1_{k'})r_{j_0}^*\right\| + \left\|r_{j_0}(y\otimes 1_{k'})r_{j_0}^* - x\otimes 1_k\right\| < \epsilon\end{aligned}$$

By Proposition 1.2.4, the above consideration implies that

$$f_\epsilon(x\otimes 1_k) = f_\epsilon(x)\otimes 1_k \precsim r(y\otimes 1_{k'})r^* \precsim y\otimes 1_{k'}$$

in $M_k(\lambda_n(A_n))$, hence $k\langle f_\epsilon(x)\rangle \le k'\langle y\rangle$ in $S(\lambda_n(A_n))$. Furthermore, take $m\ge n$, such that $l = k_m/k_n \ge (\frac{1}{k'}-\frac{1}{k})^{-1}$, then $dk'\le l\le dk$ for some $d\in\mathbb{N}$ from which it follows that

$$l\langle f_\epsilon(x)\rangle \le dk\langle f_\epsilon(x)\rangle \le dk'\langle y\rangle \le l\langle y\rangle$$

Thus, $f_\epsilon(x)\otimes 1_l \precsim y\otimes 1_l$, which in turn means that $f_\epsilon(x)\precsim y$ in $\lambda_m(A_m)$. Since ϵ was arbitrary and $f_\epsilon(x)\sim (x-\epsilon)_+ \xrightarrow{\epsilon\to 0} x$, we conclude that $x\precsim y$.

For the general case, let $x,y\in A^+$ and $\epsilon>0$. Again, $k\langle x\rangle \le k'\langle y\rangle$, implies that $x\otimes 1_k \precsim y\otimes 1_{k'}$ and by Proposition 1.2.10 there is $\delta>0$ and $r\in M_{k,k'}(A)$ such that

$$f_{\epsilon/4}(x)\otimes 1_k = r(f_\delta(y)\otimes 1_{k'})r^*$$

Using once more the density of $\cup_n\lambda_n(A_n)$ in A, and the continuity of $f_{\epsilon/2}$, f_δ in the sense of Lemma 1.2.11, we find $n\in\mathbb{N}$ and $x',y'\in\lambda_n(A_n)^+$ such that

$$\|y-y'\|<\delta \qquad (1)$$

$$\left\|f_{\epsilon/2}(x)-f_{\epsilon/2}(x')\right\|<1/2 \qquad (2)$$

and

$$\left\|f_{\epsilon/4}(x')\otimes 1_k - r(f_\delta(y')\otimes 1_{k'})r^*\right\|<1/2 \qquad (3)$$

Before continuing, let us note that for any $\epsilon>0$

$$f_{1/2}(f_{\epsilon/2}(t)) = \begin{cases} 0, & t\le 3\epsilon/4 \\ 2(f_{\epsilon/2}(t)-1/2), & 3\epsilon/4\le t\le\epsilon \\ 1, & t\ge\epsilon \end{cases}$$

which clearly implies that $f_\epsilon(x)\le f_{1/2}(f_{\epsilon/2}(x))$ for any $x\in A^+$, and in a similar way, we also get that $f_{\epsilon/2}(x)\le f_{1/2}(f_{\epsilon/4}(x))$. So, using this observation and by applying Proposition 2.3 to the inequality (3), it follows that

$$f_{\epsilon/2}(x')\otimes 1_k \le f_{1/2}(f_{\epsilon/4}(x')\otimes 1_k) \precsim r(f_\delta(y')\otimes 1_{k'})r^* \precsim f_\delta(y')\otimes 1_{k'}$$

Hence, from the first part of the proof, we obtain that

$$f_{\epsilon/2}(x')\precsim f_\delta(y')$$

Now, Proposition 1.2.4 applied to inequalities (1) and (2), yields the following

$$f_\epsilon(x)\le f_{1/2}(f_{\epsilon/2}(x))\precsim f_{\epsilon/2}(x')\precsim f_\delta(y')\precsim y$$

and since ϵ was arbitrary, we conclude that

$$x\precsim y$$

as required. □

At this point, we are ready to harvest the total goal of this section. The proof goes as follows.

Theorem 1.2.25. *Let D be a simple, unital C^*-algebra and B a UHF algebra. Set $A=B\otimes D$, then*

i) If $x,y\in M_\infty(A)^+$, $y\ne 0$ and $d(\langle x\rangle)<d(\langle y\rangle)$ for all $d\in LDF(A)$, then $x\precsim y$.

ii) If $p,q\in A\otimes\mathbb{K}$, $q\ne 0$ are projections and $\tau(p)<\tau(q)$ for all $\tau\in QT(A)$, then $p\precsim q$.

Proof. $i)$ Firstly, for $\epsilon>0$ and $d\in DF(A)$, there is $\overline{d}\in LDF(A)$ by Proposition 1.2.19 satisfying that

$$d(\langle f_\epsilon(x)\rangle)\le\overline{d}(\langle x\rangle)<\overline{d}(\langle y\rangle)\le d(\langle y\rangle) \qquad (1)$$

Now, we claim that $\langle 1_A \rangle$ is a stong order unit for $S(A)$. To see this, let $x \in M_n(A)^+$, then $x = z^*z$ for some $z \in M_n(A)$, and so $x = z^* 1_n z$. Thus, $\langle x \rangle \leq \langle 1_n \rangle = \langle 1_A \oplus 0_{n-1} + \cdots + 0_{n-1} \oplus 1_A \rangle$, and since $0_j \oplus 1_A \oplus 0_{n-j-1} \sim 1_A$ and are orthogonal to each other, for each j, it follows that $\langle 1_n \rangle = n\langle 1_A \rangle$. Hence, $\langle x \rangle \leq n\langle 1_A \rangle$, which proves the claim.
So, using that $\langle 1_A \rangle$ is a strong order unit for $S(A)$, and since $d(\langle 1_A \rangle) = 1$, for all $d \in DF(A)$, we get that any state on $S(A)$ is proportional to some dimension function. Therefore, (1) holds for all the states on $S(A)$. Moreover, since A is simple and $y \in A^+$, $y \neq 0$, there are $x_1, ..., x_n \in A$ such that $1_A = \sum_i^n x_i y x_i^*$ (see Exercise 4.9 in [20]), and because $\langle x_i y x_i^* \rangle \leq \langle y \rangle$, for any $i = 0, 1, ..., n$, we get that

$$\langle 1_A \rangle = \langle \sum_i^n x_i y x_i^* \rangle \leq n\langle y \rangle$$

But, for any $n \in \mathbb{N}$, $s \in M_n(A)^+$, there is $k \in \mathbb{N}$, such that $\langle s \rangle \leq k\langle 1_A \rangle$, hence $\langle s \rangle \leq kn\langle y \rangle$, which shows that $\langle y \rangle$ is a strong order unit for $S(A)$. Thus, due to Proposition 1.2.16 and Lemma 1.2.23, it follows that $\langle f_\epsilon(x) \rangle \leq \langle y \rangle$ or equivalently that $f_\epsilon(x) \precsim y$ for all $\epsilon > 0$, concluding that $x \precsim y$, as desired.

ii) Because each projection in $A \otimes \mathbb{K}$ is equivalent to a projection in $M_\infty(A)$, we may assume that $p, q \in M_\infty(A)$. By Note 1.2.21 and Theorem 1.2.22, to each $\tau \in QT(A)$, corresponds a $d_\tau \in LDF(A)$, defined by $d_t(\langle x \rangle) = \lim_{\epsilon \to 0} \tau(\langle f_\epsilon(x) \rangle)$ and for each $d \in LDF(A)$ there is $\tau \in QT(A)$, such that $d_\tau = d$. But, since for every projection p, and $0 \leq \epsilon < 1/2$, $f_\epsilon(p) = p$, we see that $d_\tau(\langle p \rangle) = \tau(\langle p \rangle)$ which in turn shows that

$$d(\langle p \rangle) < d(\langle q \rangle), \qquad \forall\, d \in LDF(A)$$

Hence, by (*i*) we obtain that $p \precsim q$, and in view of Proposition 1.2.3, $p \precsim q$ in the von Neumann-Murray sense, finalizing the proof. $\square$

1.3 Stable rank one C*-algebras

In this section, we focus on the class of C^*-algebras having dense invertible group or equivalently, C^*-algebras with stable rank one. After presenting some basic facts about approximation by invertible elements and constructing the essential background, we move forward to show that the tensor product of a simple, stably finite C^*-algebra with any UHF-algebra, has dense invertible group, i.e has stable rank one. We start by shaping the setting that we are going to work with. The material in this section is derived from [27].

Let A be a unital C^*-algebra, and denote by $GL(A)$ and $U(A)$ the group of invertible and unitary elements in A, respectively. Let, $A \subseteq B(H)$ be a faithful representation of A on some separable Hilbert space H, then each $x \in A$ has a polar decomposition $x = u|x|$, where $u \in B(H)$ is a partial isometry, and $|x| = (x^*x)^{\frac{1}{2}}$. Moreover, set $x_\epsilon = u(|x| - \epsilon)_+ (= (|x| - \epsilon)_+ u^*)$ and note that $x_\epsilon \in A$, since for any $h \in \overline{|x|A|x|}$, $uh \in A$ and $(|x| - \epsilon)_+ \in \overline{|x|A|x|}$. Lastly, for $x \in A$, we set $\alpha(x) = \mathrm{dist}(x, GL(A))$.

As a first result, we exhibit some natural obstructions, preventing $x \in A$ from belonging to the norm closure of $GL(A)$.

Proposition 1.3.1. *Let $\phi\colon A \to B$ be a surjective *-homomorphism between unital C^*-algebras and let $x \in A$, $u \in B$ such that $\phi(x) = u$. Then,* *i) If $u \in GL(B)$, but $u \notin \phi(GL(A))$, we get that $\alpha(x) \geq \left\|u^{-1}\right\|^{-1}$*

ii) If u is left (or right) invertible but not two sided invertible, then there is $w \in B$ such that $wu = 1_B$(or $uw = 1_B$) and $\alpha(x) \geq \|w\|^{-1}$.

Proof. $i)$ Suppose in contradiction, that there is $z \in GL(A)$ such that

$$\|z - x\| < \left\|u^{-1}\right\|^{-1}$$

Then,

$$\left\|1_B - u^{-1}\phi(z)\right\| = \left\|u^{-1}(u - \phi(z)\right\| \leq \left\|u^{-1}\right\| \|u - \phi(z)\| < \left\|u^{-1}\right\| \left\|u^{-1}\right\|^{-1} = 1$$

Hence, $u^{-1}\phi(z) \in GL_0(B))$, i.e $u^{-1}\phi(z)$ belongs to the connected component of the identity in $GL(B)$, and we claim that $u^{-1}\phi(z) \in \phi(GL(A))$.

To see this, let $w = u^{-1}\phi(z)$, and take its polar decomposition $w = v|w|$, where $v \in U(B)$ and $v \sim_h w \sim_h 1_B$. Hence, $v \in \phi(U_0(A))$, and so let $v' \in U_0(A)$, such that $\phi(v') = v$. Now, it suffices to find $r \in GL(A)$, such that $\phi(r) = |w|$, since then $w = \phi(v'r)$ and $v'r \in GL(A)$. Since $|w|$ is self-adjoint, we find $r' \in A_{s.a}$, such that $\phi(r') = |w|$. Moreover, $|w|$ is positive and invertible, implying that $\sigma(|w|) \subseteq [\lambda_1, \lambda_2]$, for some $0 < \lambda_1 \leq \lambda_2$. Define a continuous map $f \colon \mathbb{R} \to \mathbb{R}_+$, by

$$f(t) = \begin{cases} \lambda_1, & t \leq \lambda_1 \\ t, & \lambda_1 \leq t \leq \lambda_2 \\ \lambda_2, & t \geq \lambda_2 \end{cases}$$

and observe that $\sigma(f(r')) \subseteq [\lambda_1, \lambda_2]$. Then,

$$\phi(f(r')) = f(\phi(r')) = f(|w|) = i(|w|) = |w|$$

where $i \colon \mathbb{R}+ \to \mathbb{R}_+$ is the identity map. Hence, $|w| \in \phi(GL(A))$, which proves the claim.

So, $u^{-1}\phi(z) = \phi(r)$, for some $r \in GL(A)$, which implies that $u = \phi(zr^{-1})$, where $zr^{-1} \in GL(A)$, reaching a contradiction.

$ii)$ Suppose in contrary that there is $z \in GL(A)$, such that

$$\|x - z\| < \|w\|^{-1}$$

Then,

$$\|1_B - w\phi(z)\| = \|w(u - \phi(z)\| \leq \|w\| \|x - z\| < \|w\| \|w\|^{-1} = 1$$

Hence, $w\phi(z) \in GL(B)$, which in turn implies that $w \in GL(B)$, having a contradiction. The case of u being right invertible is identical. □

Note 1.3.2. If A in the proposition above is simple, then the obstruction (i), cannot occur. This is because, simplicity of A implies that ϕ is injective and therefore, there is no such $u \in GL(B) \setminus \phi(GL(A))$. Additionally, if A is also finite then there are no one sided invertibles, hence the obstruction (ii) cannot occur as well. To see this, take $a \in A$ be left invertible and find $b \in A$ such that $ba = 1_A$. Since, $(ba)^*ba$ is a self-adjoint element we have that $1_A = (ba)^*ba \leq \|b\|^2 a^*a$, so $\sigma(a^*a) \subseteq [\|b\|^{-2}, \infty)$, which in turn implies that a^*a is invertible. Set $u = a(a^*a)^{-1/2}$, then $s^*s = 1_A$, and since A is finite, u is a unitary. But this clearly shows that a is invertible, since $a = u(a^*a)^{-1/2}$.

Now, for A unital C^*-algebra, denote $GL_n(A)$, the invertible group of $M_n(A)$. Also, as previously, denote $x \otimes 1_n$, the matrix $(x_{ij})_{ij} \in M_n(A)$, where $x_{ii} = x$ and $x_{ij} = 0$, when $i \neq j$. Finally, for each $x \in A$ set

$$a_s(x) = \limsup_n \operatorname{dist}(x \otimes 1_n, GL_n(A))$$

and under these considerations, we strive to show that the obstructions that keep $a(x)$ away from zero, keep $a_s(x)$ away from zero as well.

Proposition 1.3.3. *Let $\phi\colon A \to B$ be a surjective *-homomorphism between unital C^*-algebras and for $x \in A$, set $\phi(x) = u$. Then,*

i) If $u \in GL(B)$ but $[u] \notin Im(K_1(\phi))$, then $a_s(x) \geq \left\|u^{-1}\right\|^{-1}$

ii) If u is left(or right) invertible but not two-sided invertible, then there is $w \in B$ such that $wu = 1_B$ (or $uw = 1_B$), and $a_s(x) \geq \|w\|^{-1}$.

Proof. $i)$ First note that ϕ induces a surjective *-homomorphism $\tilde{\phi}\colon M_n(A) \to M_n(B)$, for all $n \in \mathbb{N}$, satisfying that $\tilde{\phi}(x \otimes 1_n) = u \otimes 1_n$. Now, assume that $a_s(x) < \left\|u^{-1}\right\|^{-1}$, then $\exists\ n \in \mathbb{N}$, such that $\operatorname{dist}(x \otimes 1_m, GL_m(B)) < \left\|u^{-1}\right\|^{-1}$, $\forall\ m \geq n$. Since,

$$\left\|(u \otimes 1_m)^{-1}\right\|^{-1} = \left\|u^{-1} \otimes 1_n\right\|^{-1} = \left\|u^{-1}\right\|^{-1}$$

we employ Proposition 1.3.1, to find $w_m \in GL_m(A)$, such that $\tilde{\phi}(w_m) = u \otimes 1_m$, for all $m \geq n$. Set $v = w_{n+1} diag(1_A, w_n^*)$. Then, $v \in GL_{n+1}(A)$ and $\tilde{\phi}(v) = diag(u, 1_n)$. But, in $K_1(A)$, $[diag(u, 1_n] = [u]$, which shows that

$$[u] = [\tilde{\phi}(v)] = K_1(\phi)(v)$$

fact that contradicts the assumption $u \notin Im(K_1(\phi))$.

$ii)$ If u is left invertible, and $w \in B$ such that $wu = 1_B$, then $u \otimes 1_n$ is left invertible and $(w \otimes 1_n)(u \otimes 1_n) = 1_n$. Hence, again by Proposition 1.3.1 we get that

$$\operatorname{dist}(x \otimes 1_n, GL_n(A)) \geq \|w\|^{-1}$$

for all $n \in \mathbb{N}$. Thus,

$$a_s(x) \geq \|w\|^{-1}$$

as desired. The case of right invertibility is identical. □

As already stated in Note 1.3.2, such obstructions can occur but not always, thus it is natural to ask under which conditions the invertible group can be dense. In the following lines, we will try to prove that for a unital C^*-algebra A, the condition $a_s(x) = 0$, for all $x \in M_n(A)$ and for all $n \in \mathbb{N}$ is necessary and sufficient for having dense invertible group in any tensor product C^*-algebra $A \otimes B$, where B is a UHF-algebra. For this purpose, as usual, we need some preliminary work.

Lemma 1.3.4. *For any $\delta > 0$ the sets*

$$\Gamma(x, \delta) = \{n \in \mathbb{N}\colon \operatorname{dist}(x \otimes 1_n, GL_n(A)) \leq \delta\}$$
$$\Gamma_0(x, \delta) = \{n \in \mathbb{N}\colon \operatorname{dist}(x \otimes 1_n, GL_n(A)) < \delta\}$$

are additive.

Proof. Let $n_1, n_2 \in \Gamma_0(x, \delta)$ and find $z_1 \in GL_{n_1}(A)$, $z_2 \in GL_{n_2}(A)$ such that

$$\|x \otimes 1_{n_1} - z_1\| < \delta$$

and

$$\|x \otimes 1_{n_2} - z_2\| < \delta$$

Now, take $z = diag(z_1, z_2) \in GL_{n_1+n_2}(A)$, and we get that

$$\|x \otimes 1_{n_1+n_2} - z\| = \|diag(x \otimes 1_{n_1} - z_1, x \otimes 1_{n_2} - z_2)\|$$

$$= \max\{\|x \otimes 1_{n_1} - z_1\|, \|x \otimes 1_{n_2} - z_2\|\} < \delta$$

hence, $n_1 + n_2 \in \Gamma_0(x, \delta)$. But, since $\Gamma(x, \delta) = \bigcap_{\delta' > \delta} \Gamma_0(x, \delta')$, it follows that $\Gamma(x, \delta)$ is additive as well. □

Lemma 1.3.5. *For any Γ additive subset of $\mathbb{N}$, there is $k \in \mathbb{N}$ unique, such that $\Gamma \subseteq k\mathbb{N}$. Moreover, there is $n_0 \in \mathbb{N}$ such that*

$$\Gamma \cap \{n \in \mathbb{N} \colon n \geq n_0\} = k\mathbb{N} \cap \{n \in \mathbb{N} \colon n \geq n_0\}$$

Proof. Firstly, it is obvious that there is always $k \in \mathbb{N}$ such that $\Gamma \subseteq k\mathbb{N}$, since if there is common divisor of all the elements in Γ, say k, then $\Gamma \subseteq k\mathbb{N}$, where if Γ contains coprime elements then $\Gamma \subseteq \mathbb{N}$. Moreover, suppose that there are $k_1, k_2 \in \mathbb{N}$, such that $\Gamma \subseteq k_1\mathbb{N}$ and $\Gamma \subseteq k_2\mathbb{N}$, where k_1, k_2 are coprime. Then for any $n \in \Gamma$, $k_1 \mid n$ and $k_2 \mid n$, hence $k_1k_2 \mid n$, which implies that $\Gamma \subseteq k_1k_2\mathbb{N}$. Since we can have only finitely many common divisors of the elements in Γ, there is always a maximum $k \in \mathbb{N}$ such that $\Gamma \subseteq k\mathbb{N}$. If k_1, $k_2 \in \mathbb{N}$ such that $\Gamma \subseteq k_1\mathbb{N}$, $\Gamma \subseteq k_2\mathbb{N}$ and k_1, k_2 are not coprime, then the conclusion is straightforward.

Now, suppose that there exist coprime elements in Γ, say $a_{j_1}, a_{j_2}, ...a_{j_m}$, then by Bezout's Lemma, there are $n_{j_1}, n_{j_2}, ..., n_{j_m} \in \mathbb{Z}$, such that

$$\sum_{i=1}^{m} a_{j_i} n_{j_i} = 1$$

Moreover, suppose that a_1 is the smallest element in Γ and set $b = a_1 \sum_{i=1}^{m} a_{j_i} |n_{j_i}| \in \Gamma$. Then, $b+1 = \sum_{i=1}^{m} a_{j_i}(a_1|n_{j_i}|+n_{j_i}) \in \Gamma$ and in fact $b+k = \sum_{i=1}^{m} a_{j_i}(a_1|n_{j_i}|+kn_{j_i}) \in \Gamma$, for all $k = 1, ..., a_1$. Using this fact, it is now easily verified that there exists $n_0 \in \mathbb{N}$ such that Γ coincides with $\mathbb{N}$ for all $n \geq n_0$. On the other hand, if the greatest common divisor of the elements in Γ is $k \neq 1$, then consider the subset $\Gamma' = \{a/k \colon a \in \Gamma\}$. Since the greatest common divisor of the elements in Γ' is 1, by the argument above we find $n_0 \in \mathbb{N}$ such that Γ' coincides with $\mathbb{N}$ for all $n \geq n_0$. Hence $\Gamma = k\Gamma'$ coincides with with $k\mathbb{N}$ for all $n \geq kn_0$, as desired. □

Proposition 1.3.6. *Let A be a unital C^*-algebra, and $x \in A$. Then,*

i) $a_s(x) \leq a(x)$

ii) There is $k \in \mathbb{N}$, $k \geq 2$, such that $\operatorname{dist}(x \otimes 1_n, GL_n(A)) \geq a_s(x)$, $\forall n \in \mathbb{N} \setminus k\mathbb{N}$

Proof. *i*) Observe that $1 \in \Gamma(x, a(x))$, which in turn implies that $\Gamma(x, a(x)) = \mathbb{N}$, thus $a_s(x) \leq a(x)$.

ii) Consider the additive sets $\Gamma(x, a_s(x) - \frac{1}{n})$, $\Gamma(x, a_s(x))$, and take the corresponding $k_n, k \in \mathbb{N}$ by Lemma 6.5. Since,

$$\Gamma(x, a_s(x) - \frac{1}{m}) \subseteq \Gamma(x, a_s(x) - \frac{1}{n})$$

whenever $m \leq n$, the sequence k_n is decreasing, and so for large n, $k_n = k$. Moreover, $1 \notin \Gamma(x, a_s(x) - \frac{1}{n})$, because if so, then $a(x) < a_s(x)$ which contradicts (i), thus $k \geq 2$ as desired. □

Since the result that we are after incorporates UHF algebras, it is logical to develop some arguments about inductive limits of C^*-algebras, too. The following proposition fills this gap.

Proposition 1.3.7. *Let $A = \varinjlim A_n$, where A_n are unital C^*-algebras, with unital, injective connecting *-homomorphisms $\phi_{m,n} \colon A_n \to A_m$. Then, $GL(A)$ is dense in A if, and only if, for each $n \in \mathbb{N}$ and $x \in A_n$,*

$$\lim_{k \to \infty} \operatorname{dist}(\phi_{n+k,n}(x), GL(A_{n+k})) = 0 \qquad (*)$$

Proof. Let $\lambda_n\colon A_n \to A$ be the canonical embeddings, satisfying $\lambda_n = \lambda_{n+k}\circ\phi_{n+k,n}$, for all $n, k \in \mathbb{N}$ and $\cup_n \lambda_n(A_n)$ dense in A. Observe that $\lambda_1(1_{A_1}) = \lambda_n(1_{A_n})$, for all $n \in \mathbb{N}$, implies that λ_n are unital for all $n \in \mathbb{N}$, which in turn shows that $\lambda_n(GL(A_n)) \subseteq GL(A)$, $\forall n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $x \in A_n$ and take any $k \in \mathbb{N}$ and $y \in GL(A_{n+k})$. Then,

$$\begin{aligned}\|\phi_{n+k,n}(x) - y\| = \|\lambda_{n+k}\circ\phi_{n+k,n}(x) - \lambda_{n+k}(y)\| &\geq \operatorname{dist}(\lambda_{n+k}\circ\phi_{n+k,n}(x), \lambda_{n+k}(GL(A_{n+k})) \\ &\geq \operatorname{dist}(\lambda_{n+k}\circ\phi_{n+k,n}(x), GL(A)) = \operatorname{dist}(\lambda_n(x), GL(A))\end{aligned}$$

Since y was arbitrary, we get that

$$\operatorname{dist}(\phi_{n+k,n}(x), GL(A_{n+k})) \geq \operatorname{dist}(\lambda_n(x), GL(A)), \quad \forall k \in \mathbb{N}$$

So, if $(*)$ holds, then $\lambda_n(x) \in \overline{GL(A)}$, for all $n \in \mathbb{N}$ and $x \in A_n$ i.e $\lambda_n(A_n) \subseteq \overline{GL(A)}$, for all $n \in \mathbb{N}$, and so by density of $\cup_n\lambda_n(A_n)$ it follows that $A = \overline{GL(A)}$.

On the other hand, suppose that $GL(A)$ is dense in A. For $\epsilon > 0$, take any $n \in \mathbb{N}$ and $x \in A_n$, and find $z \in GL(A)$ such that $\|\lambda_n(x) - z\| < \epsilon/2$. Also, by density of $\cup_n\lambda_n(A_n)$, there is $m \in \mathbb{N}$ and $w \in A_m$ such that $\|\lambda_m(w) - z\| < \min\{\|z^{-1}\|^{-1}, \epsilon/2\}$. Hence, $\lambda_m(w)$ is invertible and since λ_m is injective we have that $0 \notin \sigma(\lambda_m(w)) = \sigma(w)$, showing that w is invertible as well. Therefore, for $k \geq n, m$, we get

$$\begin{aligned}\operatorname{dist}(\phi_{k,n}(x), GL(A_k)) \leq \|\phi_{k,n}(x) - \phi_{k,m}(w)\| &= \|\lambda_k(\phi_{k,n}(x) - \phi_{k,m}(w))\| \\ &= \|\lambda_n(x) - \lambda_m(w)\| < \epsilon\end{aligned}$$

showing that $(*)$ holds. ☐

Theorem 1.3.8. *Let A be a unital C^*-algebra. Then, $a_s(x) = 0$ for all $n \in \mathbb{N}$ and $x \in M_n(A)$ if and only if $GL(A \otimes B)$ is dense in $A \otimes B$ for any B UHF-algebra.*

Proof. As B is a UHF-algebra is isomorphic to some $\varinjlim B_l$, where $B_l = M_{n_l}(\mathbb{C})$, with connecting *-homomorphisms $\phi_{m,l}\colon B_l \to B_m$, given by $\phi(x) = x \otimes 1_{n_{m,l}}$, $n_{m,l} = n_m/n_l$. So, for A unital C^*-algebra, $A \otimes B \cong \varinjlim A \otimes B_l$, where $A \otimes B_l = M_{n_l}(A)$, and connecting *-homomorphisms, $\psi_{m,l}\colon A \otimes B_l \to A \otimes B_m$, which are given by $\psi_{m,l}(x) = x \otimes 1_{n_{m,l}}$.

Now, assume that $a_s(x) = 0$ for any $n \in \mathbb{N}$ and $x \in M_n(A)$. Then, in fact $a_s(x) = 0$ for all $x \in M_{n_m}(A)$, and $m \in \mathbb{N}$. Hence,

$$\operatorname{dist}(\psi_{m,l}(x), GL(A \otimes B_m)) = \operatorname{dist}(x \otimes 1_{n_{m,l}}, GL_{n_{m,l}}(A \otimes B_l)) \xrightarrow{b\to\infty} 0$$

which shows that $GL(A \otimes B)$ is dense in $A \otimes B$, by Proposition 1.3.7.

On the other hand, assume that $a_s(x) > 0$, for some $x \in M_n(A)$ and $n \in \mathbb{N}$. From Proposition 1.3.6, we find $k \in \mathbb{N}$, $k \geq 2$, such that

$$\operatorname{dist}(x \otimes 1_l, GL_l(A)) \geq a_s(x) > 0, \quad \forall l \in \mathbb{N} \setminus k\mathbb{N}$$

Now, choose $p \in \mathbb{N}$, $p \geq 2$, such that $p^m \in \mathbb{N} \setminus k\mathbb{N}$, for all $m \in \mathbb{N}$ and set $n_l = np^{l-1}$. Then, the corresponding UHF-algebra $\varinjlim B_l$ is given by $B_l = M_{n_l}(\mathbb{C})$, with connecting *-homomorphisms $\phi_{m,l}\colon B_l \to B_m$, defined as $x \mapsto x \otimes 1_{n_{m,l}}$, $n_{m,l} = n_m/n_l$. As in the first part of the proof, we obtain a unital C^*-algebra $A \otimes B = \varinjlim A \otimes B_l$, with connecting *-homomorphisms $\psi_{m,l}\colon A \otimes B_l \to A \otimes B_m$, given by $\psi_{m,l}(x) = x \otimes 1_{n_{m,l}}$. Since $p^m \in \mathbb{N} \setminus k\mathbb{N}$, for all $m \in \mathbb{N}$ and $n_{m,1} = p^{m-1}$, we see that

$$\operatorname{dist}(\psi_{m,1}(x), GL(A \otimes B_m)) = \operatorname{dist}(x \otimes 1_{p^{m-1}}, GL_{p^{m-1}}(M_n(A))) \geq a_s(x) > 0$$

Thus, again by Proposition 1.3.7, we conclude that $GL(A \otimes B)$ is not dense in $A \otimes B$, as required. ☐

Before embarking for the last portion of this section and proving our main goal, we briefly present the notion of two-sided zero divisors and we present a theorem that gives an affirmative answer to the question: If A is a simple, unital C^*-algebra and x is a two-sided zero divisor can we find m_0 such that for all $m \geq m_0$, $u(x \otimes 1_m)$ is nilpotent, i.e $\exists\ n \in \mathbb{N}$ such that $[u(x \otimes 1_m)]^n = 0$, where $u \in U_m(A)$. The proof of this result is ommited and the reader is encouraged to see Theorem 6.4 in [28] for a detailed exposition.

Definition 1.3.9. *Let A be a C^*-algebra. Then, an element $x \in A$ is called two sided zero divisor if $ax = 0 = xb$ for some non zero elements $a, b \in A$. The set of all two-sided zero divisors in A is denoted by $ZD(A)$.*

Proposition 1.3.10. *For each unital C^*-algebra A, $\overline{ZD(A)}$ consists of all the elements in A that are not one-sided invertible.*

Proof. Firstly, let $x \in ZD(A)$ and suppose that it is left invertible. Then, there are $y \in A$ and $0 \neq b \in A$ such that $yxb = 0$ or $b = 0$, while the case of right invertibility is identical. Hence there is no one-sided invertible element in $ZD(A)$. Now, if the set of one-sided invertible elements is open, then it would be disjoint from $\overline{ZD(A)}$, as required. To show that the set of one-sided invertible elements is open, let $x \in A$ be left invertible and find $y \in A$ such that $yx = 1_A$.
Then, set

$$U_x = \{z \in A : \|z - x\| < \|y\|^{-1}\}$$

and if we show that U_x is contained in the set of left invertible elements, we are done. But, if $z \in U_x$, then we get

$$\begin{aligned} \|z - x\| < \|y\|^{-1} &\implies \|y\|\,\|z - x\| < 1 \implies \|y(z - x)\| < 1 \\ &\implies \|yz - 1_A\| < 1 \end{aligned}$$

hence, yz is invertible, which in turn means that z is left invertible.

On the other hand, take $x \in A$ to be neither left nor right invertible. Then, the same holds for $|x|$ and $|x^*|$, and for $\epsilon > 0$ let $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, such that $g(0) = 1$ and $\operatorname{supp}(g) \subseteq [0, \epsilon]$. Furthermore, set $a = g(|x^*|)$ and $b = g(|x|)$ and see that $ax_\epsilon = 0 = x_\epsilon b$. Since $0 \in \sigma(|x|)$ and $0 \in \sigma(|x^*|)$, a and b are non zero, hence $x_\epsilon \in ZD(A)$. But, owing to the fact that $\|x_\epsilon - x\| < \epsilon$, we obtain that $x \in \overline{ZD(A)}$, as desired. □

Theorem 1.3.11. *Let A be a unital, simple C^*-algebra, and $x \in ZD(A)$. Then, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $u(x \otimes 1_m)$ is nilpotent, for some $u \in U_m(A)$. In particular, $(x \otimes 1_m) \in \overline{GL_m(A)}$, for $m \geq m_0$.*

Corollary 1.3.12. *i) Let A be a simple, finite C^*-algebra, then $a_s(x) = 0$ for all $x \in A$.*

ii) Let A be an infinite, simple C^*-algebra, then $a_s(x) > 0$ if and only if x is one-sided but not two-sided invertible.

Proof. *i*) By Theorem 1.3.11, $a_s(x) = 0$, for all $x \in ZD(A)$. Moreover, since the map $x \mapsto a_s(x)$ is continuous, $a_s(x) = 0$ for all $x \in \overline{ZD(A)}$, and by Proposition 3.10, it follows that $a_s(x) = 0$, for all non one-sided invertible elements. But, A is finite, therefore has no one-sided invertible elements, concluding that $a_s(x) = 0$, for all $x \in A$.

ii) If $a_s(x) > 0$, then by the same reasoning as in (*i*), $x \notin \overline{ZD(A)}$, hence by Proposition 1.3.10, x is one-sided invertible. On the other hand, if x is one-sided invertible, then by taking the identity map on A and applying Proposition 1.3.3 (ii), we get that $a_s(x) > 0$, as desired. □

Now, we have reached the point where the main goal of this section, can be immediately deduced by combining aforementioned results.

Corollary 1.3.13. *If A is a simple, stably finite C^*-algebra, and B any UHF-algebra, then $GL(A\otimes B)$ is dense in $A\otimes B$.*

Proof. Since $M_n(A)$ is a simple and finite C^*-algebra, for all $n\in\mathbb{N}$, we get by Corollary 1.3.12 that $a_s(x)=0$, for any $x\in M_n(A)$ and $n\in\mathbb{N}$. Therefore, by Theorem 1.3.8, we conclude that $GL(A\otimes B)$ is dense in $A\otimes B$, for any B UHF-algebra. □

1.4 Separability Issues

In the following of this project, a lot of the C^*-algebras that will be used, are non-separable. Hence, we need to demonstrate a way of addressing this issue. In this section, we are going to present some general methods for reducing problems to the separable setting. The main source of this section is [30]

Definition 1.4.1 ([2], Section II.8.5)**.** *A property (P) of C^*-algebras is called separably inheritable if*
i) whenever A is a C^*-algebra satisfying (P) and A_0 is a separable C^*-subalgebra of A, there is a separable C^*-subalgebra $\hat{A}$ of A which satisfies (P) and contains A_0
ii) whenever $A_1\hookrightarrow A_2\hookrightarrow A_3\hookrightarrow\cdots$ is an inductive system of separable C^*-algebras with injective connecting maps, if each A_n satisfies (P), then $\varinjlim A_n$ satisfies (P)

Also, a slightly different definition of separable inheritability will be useful.

Definition 1.4.2. *Let (P) be a property of separable C^*-algebras. A C^*-algebra A separably satisfies (P) if whenever A_0 is a separable C^*-subalgebra of A, there is a separable C^*-subalgebra $\hat{A}$ of A which satisfies (P) and contains A_0.*

Note 1.4.3. If (P) is a separably inheritable property and A is a C^*-algebra satifying (P), then A separably satifies (P). Moreover, if (P) is a property for separable C^*-algebras, which is preserved under inductive limits with injective connecting maps, then separably (P) is a separably inheritable property.

Proposition 1.4.4. *Let (P_k) be a countable family of properties of separable C^*-algebras preserved under sequential inductive limits with injective connecting maps. If A is a C^*-algebra separably satisfying (P_k) for each i, then A separably satisfies the meet of the (P_k).*

Proof. Let B be a separable C^*-subalgebra of A and using that A separably satisfies P_k, for all k, find a sequence

$$B\subseteq A_{1,1}\subseteq A_{2,1}\subseteq A_{2,2}\subseteq A_{3,1}\subseteq\cdots\subseteq A_{n,1}\subseteq\cdots\subseteq A_{n,n}\subseteq A_{n+1,1}\subseteq\cdots$$

where, $A_{n,k}$ are C^*-subalgebras of A satisfying P_k. Now, set

$$\Gamma=\overline{\cup_k\cup_{n\geq k}A_{n,k}}$$

then, Γ is a separable C^*-subalgebra of A and observe that $\Gamma=\overline{\cup_{n\geq k}A_{n,k}}$ for any k. Thus, Γ can be seen as the inductive limit, $\varinjlim_{n\geq k} A_{n,k}$ for any k, with the corresponding inclusions as connecting maps. But, since the properties (P_k) respect inductive limits with injective connecting maps, we get that Γ satisfies (P_k) for all k, hence satisfies also the meet of all (P_k). As B was an arbitrary C^*-subalgebra of A, we conclude that A separably satisfies the meet of all P_k. $\square$

Note 1.4.5. As it might be already evident, in the exact same fashion, it is shown that if (P_k) is a countable sequence of separably inheritable properties, then the meet of (P_k) is separably inheritable as well.

Many properties of C^*-algebras are sebarably inheritable such as exactness, nuclearity, stable rank one, among others. Below, we show that "stable rank one" is a separably inheritable property.

Proposition 1.4.6 ([2], Proposition II.8.5.4)**.** *The property "A has stable rank one" is separably inheritable.*

Proof. Assume that A is a unital C^*-algebra, while the case of non-unitality is treated in the same way. Suppose that A has stable rank one and let $B \subseteq A$ be a separable C^*-subalgebra. Let $\{x_n : n \in \mathbb{N}\}$ be countable dense in B, and for each x_n, find $(x_{n_k})_k \subset GL(A)$ converging to x_n. If A_1 is the C^*-subalgebra of A generated by $(x_{n_k})_k$ for all $n \in \mathbb{N}$, then $B \subseteq A_1$ and A_1 is separable. In fact, B is contained in the norm closure of $GL(A_1)$. Now, proceed in the same way to find A_2 separable C^*-subalgebra of A containing A_1 and such that the norm closure of $GL(A_2)$ contains A_1. Continuing inductively, a sequence (A_n) of separable C^*-subalgebras of A is obtained satisfying the above conditions. Set $\Gamma = \overline{\cup_n A_n}$, then Γ is a separable C^*-subalgebra of A and we claim that has stable rank one. But this is evident, because if $x \in \cup_n A_n$, then $x \in A_{n_0}$ for some $n_0 \in \mathbb{N}$, hence $x \in \overline{GL(A_{n_0+1})} \subseteq \overline{GL(\Gamma)}$, which in turn shows $\Gamma \subseteq \overline{GL(\Gamma)}$, and the conclusion follows.

Finally, we argue that the property "stable rank one" is preserved under inductive limits with injective connecting maps. Let

$$A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$$

where A_n is a stable rank one C^*-algebra, for each $n \in \mathbb{N}$ and let A be its inductive limit. Also, let $\lambda_n : A_n \to A$ be the *- homomorphisms, such that $A = \overline{\cup_n \lambda_n(A_n)}$. Then for $x \in A$ and $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ and $x_{n_0} \in A_{n_0}$, such that

$$\|\lambda_{n_0}(x_{n_0}) - x\| < \epsilon/2$$

Moreover, find $y \in GL(A_{n_0})$ such that $\|\lambda_{n_0}(x_{n_0}) - \lambda_{n_0}(y)\| < \epsilon/2$, then we have that $\lambda_{n_0}(y) \in GL(A)$ and

$$\|x - \lambda_{n_0}(y)\| \leq \|x - \lambda_{n_0}(x_{n_0})\| + \|\lambda_{n_0}(x_{n_0}) - \lambda_{n_0}(y)\| < \epsilon$$

Since ϵ was arbitrary, $GL(A)$ is dense A as required. $\square$

As short exact sequences of C^*-algebras have central role in chapter 3, the following result will be crucial. Roughly speaking, the proposition below provides a way to pass from a non-separable short exact sequence, whose parts separably satisfying some property, to a separable one, whose parts satisfy the same property.

Proposition 1.4.7. *Consider the short exact sequence of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{\ j\ } E \xrightarrow{\ q\ } D \longrightarrow 0$$

and suppose that for each $X \in \{I, E, D\}$, (P_X) is a property of separable C^*-algebras preserved under inductive limits with injective connecting maps and that X separably satisfies (P_X). If for each $X \in \{I, E, D\}$, a separable C^*-subalgebra X_0 is given, then there exist $\hat{X}$ separable C^*-subalgebra of X that *contains X_0, satisfies (P_X) and there is the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{I} & \longrightarrow & \hat{E} & \longrightarrow & \hat{D} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{j} & E & \xrightarrow{q} & D & \longrightarrow & 0 \end{array}$$

where the vertical rows are the inclusions.

Proof. For $X \in \{I, E, D\}$, and X_0 separable C^*-subalgebra of X, we construct an increasing sequence (X_n) of separable C^*-subalgebras of X, that contain X_0, satisfy (P_X) and

$$D_n \subseteq q(E_n) \subseteq D_{n+1}, \quad I_{n-1} \subseteq j^{-1}(E_n) \subseteq I_n \qquad (1)$$

Then, $\hat{X} = \overline{\cup_n X_n}$, is a separable C^*-subalgebra of X, contains X_0 and satisfies (P_X) since it can be seen as the inductive limit of the sequence $X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$. Moreover, since *-homomorphisms always have closed range ([34], Theorem 11.1), it is straightforward to see that $q(\hat{E}) = \hat{D}$, and $j^{-1}(\hat{E}) = \hat{I}$, which immediately show that the first row of the diagram above is indeed a short exact sequence, and in turn that the whole diagram is commutative.

We construct the desired sequence inductively. Assume that X_{n-1}, $n \geq 1$ has been already constructed, and we procceed to show that there exist X_n, satisfying all the aforementioned conditions. Firstly, consider the C^*-algebra generated by $q(E_{n-1}), D_{n-1}$. This C^*-algebra is a separable C^*-subalgebra of D, hence there exists D_n separable C^*-subalgebra of D, containing $q(E_{n-1}), D_{n-1}$ and satisfying (P_D). Let T_n be the countable dense subset of D_n, and find $S_n \subset E$ countable such that $q(S_n) = T_n$. Then, consider the C^*-algebra generated by $j(I_{n-1}), E_{n-1}$ and S_n, which is a separable C^*-subalgebra of E, thus there is E_n separable C^*-subalgebra of E, containing $j(I_{n-1}), E_{n-1}$ and S_n, and satisfying (P_E). Now, since j is injective, $j(I_{n-1}) \subseteq E_n$ implies that $I_{n-1} \subseteq j^{-1}(E_n)$, and since q has closed range, T_n dense in D_n and $T_n = q(S_n) \subseteq q(E_n)$, we obtain that $D_n \subseteq q(E_n)$. Finally, as $j^{-1}(E_n)$ is a separable C^*-subalgebra of I, we find I_n separable C^*-subalgebra of I, containing $j^{-1}(E_n)$ and satisfying (P_I).

We have constructed I_n, E_n, D_n separable C^*-subalgebras of I, E and D, respectively, such that all the required conditions are satisfied, hence by induction the proof is finished. ☐

One more useful observation is that the properties (P) and separably (P) tend to have the same permanence properties. Here, we emphasize on the case of ideals, quotients and extensions (see Definition 2.4.1).

Corollary 1.4.8. *Let (P) be a property for separable C^*-algebras, preserved under inducive limits with injective connecting maps. If (P) is preserved by ideals, quotients or extensions of separable C^*-algebras, then separably (P) has the same permanennce properties among all C^*-algebras.*

Proof. we consider first the case of extensions. Let A be a C^*-algebra, $I \lhd A$ be a closed to sided ideal and A_0 a separable C^*-subalgebra of A. Moreover, suppose that I and A/I separably satisfy (P), where (P) a property given by hypothesis, and we aim to show that A separably satisfies (P). By the previous proposition, there are $\hat{I}, \hat{A}, \Gamma$, separable C^*-subalgebras of I, A and A/I respectively, where $\hat{I}, \Gamma$ satisfy (P), $A_0 \subseteq \hat{A}$, and the following diagram with inclusions as vertical maps, commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{I} & \longrightarrow & \hat{A} & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{j} & A & \xrightarrow{q} & A/I & \longrightarrow & 0 \end{array}$$

Since, (P) is preserved by extensions, $\hat{A}$ satisfies (P), which in turn shows that A separably satisfies (P).

Now, the case of ideals and quotients is treated similarly, by employing again the commutative diagram given in Proposition 1.4.7. Therefore, the proof is omitted. $\square$

Another instance of similar behaviour between separably (P) and (P), can be found in the case of hereditary subalgebras. Since this class of subalgebras has its own particular interest, we first prove some useful facts about it, before establishing the desired permanence property. For the shake of completion we start by the definition.

Definition 1.4.9. *A C^*-subalgebra B of a C^*-algebra A is said to be hereditary, if whenever $a \in A^+$, $b \in B^+$, such that $a \leq b$, then $a \in B$*

Note 1.4.10. Observe that if A is a unital C^*-algebra and $p \in A$ is a projection, then pAp is a hereditary subalgebra. Fisrtly, pAp is a C^*-subalgebra of A, and let $a \in A^+$, $pbp \in (pAp)^+$, such that $a \leq pbp$. Then, $0 \leq (1-p)a(1-p) \leq (1-p)pbp(1-p) = 0$, which implies that $\left\|a^{1/2}(1-p)\right\|^2 = 0$ and in turn that $a(1-p) = 0$. Hence, $a = pap \in pAp$

Theorem 1.4.11 ([22], Theorem 3.2.1)**.** *Let A be a unital C^*-algebra. Then,*

i) If I is a closed left ideal in a unital C^*-algebra A, then $B = I \cap I^*$ is a hereditary subalgebra of A. Moreover, the map

$$I \mapsto I \cap I^*$$

from the closed left ideals in A, to the hereditary subalgebras of A, is a bijection.

ii) If I_1, I_2 closed left ideals in A, then $I_1 \subseteq I_2$ if and only if $I_1 \cap I_1^* \subseteq I_2 \cap I_2^*$

iii) If B is a hereditary subalgebra of A, then the set

$$I(B) = \{a \in A \colon a^*a \in B\}$$

is the unique closed left ideal corresponding to B.

Proof. i) Firstly, $B = I \cap I^*$ is clearly a C^*-subalgebra of A, so let $a \in A^+$ and $b \in B^+$ such that $a \leq b$. Since I is a closed left ideal , we can find an approximate unit $(u_\lambda)_\lambda$, satisfying $x = \lim_\lambda x u_\lambda$, for all $x \in I$. Now, by the inequality, $0 \leq (1-u_\lambda)a(1-u_\lambda) \leq (1-u_\lambda)b(1-u_\lambda)$, we get that $\left\|a^{1/2} - a^{1/2}u_\lambda\right\|^2 = \|(1-u_\lambda)a(1-u_\lambda)\| \leq \|(1-u_\lambda)b(1-u_\lambda)\| \leq \|b - bu_\lambda\|$. Hence, $a^{1/2} = \lim a^{1/2}u_\lambda$, implies that $a^{1/2} \in I$, and so $a \in B$.

ii) If $I_1 \subseteq I_2$, then also $I_1^* \subseteq I_2^*$, hence $I_1 \cap I_1^* \subseteq I_2 \cap I_2^*$. For the other direction, suppose that $I_1 \cap I_1^* \subseteq I_2 \cap I_2^*$, and take $a \in I_1$. As a C^*-algebra, $I_1 \cap I_1^*$ admits an approximate unit, say $(u_\lambda)_\lambda$, and observe that

$$\lim_\lambda \|a - au_\lambda\|^2 = \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \leq \lim_\lambda \|a^*a(1-u_\lambda)\| = 0$$

since $a^*a \in I_1 \cap I_1^*$. Hence, $a = \lim_\lambda au_\lambda$, and using that $u_\lambda \in I_1 \cap I_1^* \subseteq I_2$, it follows, $au_\lambda \in I_2$, but I_2 is closed, concluding that $a \in I_2$, as desired.

iii) Let $a, b \in I(B)$, then $(a+b)^*(a+b) \leq (a+b)^*(a+b) + (a-b)^*(a-b) = 2a^*a + 2b^*b \in B$, but B is hereditary subalgebra, hence $(a+b)^*(a+b) \in B$, so $(a+b) \in I(B)$. Also, for $x \in A$, $a \in I(B)$, we have that $(xa)^*(xa) = a^*x^*xa \leq \|x\|^2 a^*a \in B$, thus $xa \in I(B)$. Moreover, $I(B)$ is obviously closed under multiplication by scalars, and is closed since B is closed, concluding that $I(B)$ is a closed left ideal in A.

To show that, $B = I(B)^* \cap I(B)$, first consider an element $b \in B$, then $b^*b \in B$, hence $B \subseteq I(B)$, and similarly $B = B^* \subseteq I(B)^*$, so $B \subseteq I(B)^* \cap I(B)$. If $0 \leq b \in I(B)^* \cap I(B)$, then $b^2 = b^*b \in B$, and by applying continuous functional calculus with $f(t) = t^{1/2}$ to b^2, we get that $b \in B$. Since, $I(B)^* \cap I(B)$ is spanned by its positive elements, we get that $I(B)^* \cap I(B) \subseteq B$, hence $B = I(B)^* \cap I(B)$. $\square$

Note 1.4.12. Actually, in the hypothesis of the theorem above, the condition for A to be unital can be lifted, and instead work with the unitization $\tilde{A}$ of A. This detail does not change anything in the proof process.

As an easy consequence of Theorem 1.4.11, we can now harvest the following rather useful theorem, which provides a versatile criterion for a C^*-subalgebra to be hereditary.

Theorem 1.4.13 ([22], Theorem 3.2.2)**.** *Let B be a C^*-subalgebra of a C^*-algebra A. Then B is hereditary if and only if $bab' \in B$, for any $b, b' \in B$ and $a \in A$*

Proof. Suppose first that B is hereditary and take $a \in A$ and $b, b' \in B$. If $I(B)$ is the (unique) left closed ideal in A corresponding to B, then $b, b' \in I(B) \cap I(B)^*$, so $bab' \in I(B) \cap I(B)^* = B$.

On the other hand, suppose that for any $b, b' \in B$ and $a \in A$, $bab' \in B$. Take $a \in A^+$ and $b \in B^+$ such that $a \leq b$ and let $(u_\lambda)_\lambda$ to be an approximate unit for B. Then, $(1_{\tilde{A}} - u_\lambda)a(1_{\tilde{A}} - u_\lambda) \leq (1_{\tilde{A}} - u_\lambda)b(1_{\tilde{A}} - u_\lambda)$ and so, $\|a^{1/2} - a^{1/2}u_\lambda\| \leq \|b^{1/2} - b^{1/2}u_\lambda\| \to 0$. Thus, $a^{1/2} = \lim_\lambda a^{1/2}u_\lambda$, and therefore, $a = \lim_\lambda u_\lambda a u_\lambda$. But, $u_\lambda a u_\lambda \in B$ by hypothesis and B is closed, hence $a \in B$, which implies that B is hereditary. □

Proposition 1.4.14. *If (P) is a property for separable C^*-algebras preserved by hereditary subalgebras, then separably (P) is preserved by hereditary subalgebras as well.*

Proof. Let A be a C^*-algebra and $B \subset A$ a hereditary subalgebra. Suppose that A separably satisfies (P) and let $B_0 \subset B$ be a separable subalgebra. Now, find $\hat{A} \subset A$ separable subalgebra satisfying (P) and containing B_0. Set $\hat{B} = B_0\hat{A}B_0$. Then, $\hat{B}$ is clearly a separable C^*-subalgebra of A and if $a \in \hat{A}$, b_1xb_1', $b_2yb_2' \in \hat{B}$, we see that $b_1xb_1'ab_2yb_2' \in \hat{B}$, since $xb_1'ab_2y \in \hat{A}$, hence by Theorem 1.4.13, $\hat{B}$ is a hereditary subalgebra of $\hat{A}$.

Furthermore, since B is a hereditary subalgebra of A, using Theorem 1.4.13, for any $a \in \hat{A} \subseteq A$ and $b, b' \in B_0 \subseteq B$, it follows that $bab' \in B$, hence $B_0\hat{A}B_0 \subseteq B$.

Finally, let $b \in B_0^+$, then $b = b^{1/3}b^{1/3}b^{1/3} \in \hat{B}$, thus $B_0^+ \subseteq \hat{B}$. Since B_0 is a C^*-algebra, it is spanned by its positive elements, hence $B_0 \subseteq \hat{B}$.

So, by combining these three facts about $\hat{B}$ and using that the property (P) passes to hereditary subalgebras, we conclude that separably (P) is preserved by hereditary subalgebras, as desired. □

We end this section by proving one lemma and one proposition that combine facts from the section devoted to strongly self-absorbing C^*-algebras and the present section. Both results will be critical in the process of this project.

Lemma 1.4.15. *For any D separable, unital, strongly self-absorbing C^*-algebra a unital C^*-algebra A, is separably D-stable iff for any $\epsilon > 0$, $F \subset A$, $G \subset D$ finite sets there is a u.c.p map $\phi\colon D \to A$ satisfying the following*

$$\|\phi(dd') - \phi(d'd)\| < \epsilon, \quad \|a\phi(d) - \phi(d)a\| < \epsilon \qquad (*)$$

for every $d, d' \in D$, $a \in A$.

Proof. For the fisrt implication, let $\epsilon > 0$ and $F \subset A$, $G \subset D$ finite sets, and denote by A_0 the C^*-algebra generated by 1_A anf F. Since A is separably D-stable, there is a unital, separable, D-stable C^*-subalgebra of A, say $\hat{A}$, such that $A_0 \subseteq \hat{A}$. Since $\hat{A}$ is D-stable, Theorem 1.1.13 tells us that there is a unital embedding $\psi\colon D \to \mathscr{L}(\mathscr{A}) \cap A'$, while nuclearity of D (see Theorem 1.1.10) allows to employ Choi-Effros Lifting Theorem, to get a u.c.p map $\hat{\phi}\colon D \to \prod \hat{A}$, lifting ψ. Now, consider the u.c.p maps $\phi_n\colon D \to \hat{A}$, defined as $\hat{\phi}(d) = (\phi_n(d))_n$ and observe that $\pi \circ \hat{\phi}(dd') =$

$\pi \circ \hat{\phi}(d'd)$ and $\pi(\hat{\phi}(d)a) = \pi(a\hat{\phi}(d))$, where $\pi\colon \prod \hat{A} \to \prod \hat{A}/\sum \hat{A}$ the quotient map, and $d, d' \in D$, $a \in \hat{A}$. Hence,

$$\lim_n \|\phi_n(dd') - \phi(d'd)\| = \lim_n \|a\phi_n(d) - \phi_n(d)a\| = 0$$

So, if we set $\phi = \phi_n$ for sufficiently large n, the first direction follows.

Now, suppose that the conditions $(*)$ hold and let A_0 be a separable, unital C^*-subalgebra of A. Since, A_0 and D separable, we can find increasing sequences of finite subsets $(F_{0,n})_n$ and $(G_n)_n$, whose union is dense in A_0 and D, respectively. Using hypothesis, there is a u.c.p map $\phi_{0,n}\colon D \to A$, for each n, satisfying

$$\|\phi_{0,n}(dd') - \phi_{0,n}(d'd)\| < 1/n, \quad \|a\phi_{0,n}(d) - \phi_{0,n}(d)a\| < 1/n$$

for all $d, d' \in G_n$, $a \in F_{0,n}$. If we denote by A_1 the C^*-algebra generated by A_0 and $\phi_{0,n}(D)$, for each $n \in \mathbb{N}$, then A_1 is unital and separable C^*-algebra, since A_0 and D are unital and separable. Now, following the same procedure, we construct an increasing sequence $(A_k)_k$ of unital, separable C^*-subalgebras of A, where for each k there is a sequence of u.c.p maps $\phi_{k,n}\colon D \to A$ satisfying

$$\lim_n \|\phi_{k,n}(dd') - \phi_{k,n}(d'd)\| = 0 = \lim_n \|a\phi_{k,n}(d) - \phi_{k,n}(d)a\|$$

for all $d, d' \in D$, $a \in A_k$. Denote by $\hat{A}$ the separable, unital C^*-subalgebra of A, defined as the closed union of all A_k and note that for sufficiently large k, we can pick u.c.p maps from the sequence $(\phi_{k,n})_{k,n}$, to construct a new sequence $\psi_m\colon A \to \hat{A}$ of u.c.p maps satisfying

$$\lim_m \|\psi_m(dd') - \psi_m(d'd)\| = 0 = \lim_m \|a\psi_m(d) - \psi_m(d)a\|$$

for all $d, d' \in D$, $a \in \hat{A}$. So, if $\psi\colon D \to \prod \hat{A}$ is the u.c.p map defined by $\psi(d) = (\psi_n(d))_n$, we get that $\pi \circ \psi$ is a *-homomorphism from D into $\prod \hat{A}/\sum \hat{A}$ whose image commutes with $\hat{A}$. Thus, $\pi \circ \psi\colon D \to \mathscr{Q}(\hat{A}) \cap \hat{A}'$, is a *- homomorphism, and by Theorem 1.1.13, we obtain that $\hat{A}$ is D-stable. Since $A_0 \subseteq \hat{A}$, and $\hat{A}$ separable, it follows that A is separably D-stable, as desired. $\square$

Proposition 1.4.16. *If D is a strongly self-absorbing C^*-algebra then, hereditary subalgebras, quotients and extensions of separably D-stable C^*-algebras, are separably D-stable. Moreover, l^∞-products and ultraproducts of unital, separably D-stable C^*-algebras, are separably D-stable.*

Proof. From Lemma 1.1.16, Lemma 1.1.18 and Theorem 1.1.20, we know that D-stability is preserved by hereditary subalgebras, quotients and extensions. Also, by Lemma 1.1.19, D-stability is preserved by inductive limits, and so we can apply Corollary 1.4.8 and Proposition 1.4.14, showing that separably D-stability is preserved by hereditary subalgebras, quotients and extensions.

Now, let $(A_m)_m$ be a family of unital, separably D-stable C^*-algebras, and take $\epsilon > 0$, and $F \subset \prod_m A_m$, $G \subseteq D$ finite sets. Then, $F = \{(a_m)^1_m, (a_m)^2_m, ..., (a_m)^k_m\}$, for some $k \in \mathbb{N}$ and consider the corresponding finite sets in A_m

$$F^m = \{a^1_m, a^2_m, ..., a^k_m\}, \qquad m \in \mathbb{N}$$

Since, for each $m \in \mathbb{N}$, A_m is D-stable, by the previous lemma we find $(\phi^m_n)_n\colon D \to A_m$ sequence of u.c.p maps satisfying

$$\|\phi^m_n(dd') - \phi^m_n(d'd)\| \xrightarrow{n\to\infty} 0, \quad \|a\phi^m_n(d) - \phi^m_n(d)a\| \xrightarrow{n\to\infty} 0$$

for all $d, d' \in G$, $a \in F^m$ $m \in \mathbb{N}$. Next, define the u.c.p maps $\Phi_n\colon D \to \prod_m A_m$ by,

$$\Phi_n(d) = (\phi^1_n(d), \phi^2_n(d), ...)$$

It follows that,

$$\|\Phi_n(dd') - \Phi_n(d'd)\| = \sup_m \|\phi^m_n(dd') - \phi^m_n(d'd)\| \xrightarrow{n\to\infty} 0$$

and

$$\|a\Phi_n(d) - \Phi_n(d)a\| = \sup_m \|a_m\phi_n^m(d) - \phi_n^m(d)a_m\| \xrightarrow{n\to\infty} 0$$

for all $d, d' \in G$, $a \in F$ hence, Lemma 1.4.15 implies that $\prod_m A_m$ is separably D-stable.

Finally, since separably D-stability is preserved by products and quotients, it follows that ultraproducts of unital separably D-stable C^*-algebras is again unital, separably D-stable. $\square$

2 Hilbert C*-modules and the Cuntz picture of KK-theory

Since K_0-behaviour of C^*-algebras and *-homomorphisms plays a central role in both the existence and the classification result that we are after, the following sections aim to present KK-theoretical arguments that will facilitate the analysis towards this direction. As KK-theory is a vast area of independent interest, which admits several interpretations, except for establishing the specific results that we need, an effort to give some insights into some preliminary concepts and constructions concerning this subject, has been made.

2.1 Hilbert C*-modules

We start this chapter by establishing the fundamental notions in Hilbert C^*-module theory. Although in the following sections we will emphasize on specific Hilbert C^*-modules, the exposition here has a generic essence, aiming to achieve a broader understanding of the subject. Our starting point is the definition of a pre-Hilbert C^*-module. Throughout this section let B be a C^*-algebra. The main source of the following material is [18].

Definition 2.1.1. *A pre-Hilbert B-module E, is a complex vector space and a right B-module equipped with a map $\langle \cdot, \cdot \rangle \colon E \times E \to B$ which is linear in the second variable and satisfies the following conditions for any $b \in B$, $x, y \in E$:*

i) $\langle x, yb \rangle = \langle x, y \rangle b$

ii) $\langle x, y \rangle^* = \langle y, x \rangle$

iii) $\langle x, x \rangle \geq 0$

iv) $x \neq 0 \implies \langle x, x \rangle \neq 0$

Note 2.1.2. It is implied by Definition 1.1, that the scalar multiplication and the B-module structure on E are compatible, in the sense that $(\lambda x)b = \lambda(xb) = x(\lambda b)$, for all $\lambda \in \mathbb{C}, x \in E, b \in B$

In order to turn the pre-Hilbert B-module E into a Hilbert B-module, a norm structure is needed. This work is done by the following lemma.

Lemma 2.1.3. *Let E be a pre-Hilbert B-module, and define $\|e\| = \|\langle e, e \rangle\|^{1/2}$. Then E equipped with this norm, becomes a normed vector space, and the following inequalities hold,*

i) $\|eb\| \leq \|e\| \, \|b\|, \quad e \in E, b \in B$

ii) $\|\langle e, f \rangle\| \leq \|e\| \, \|f\|, \quad e, f \in E$

Proof. Let us start by proving the inequalities. For the first one, it is straightforward to see that

$$\begin{aligned} \|eb\|^2 &= \|\langle eb, eb \rangle\| = \|\langle eb, e \rangle b\| = \|b^* \langle e, e \rangle b\| \\ &\leq \|\langle e, e \rangle\| \, \|b\|^2 = \|e\|^2 \, \|b\|^2 \end{aligned}$$

For the second inequality, observe that the map ϕ given by $\phi(\langle e, f \rangle^* \langle e, f \rangle) = \|\langle e, f \rangle\|^2$, $e, f \in E$ is a state on B. Also, we may assume that $\langle e, f \rangle \neq 0$, fact that allows us to set $\alpha = \langle e, f \rangle^* \|\langle e, f \rangle\|^{-1}$. Note that $\|\alpha\| = 1$ and that $\phi(\langle e, f \rangle \alpha) = \|\langle e, f \rangle\|$. So, under these considerations we get

$$\|\langle e, f \rangle\|^2 = \phi(\langle e, f \rangle \alpha)^2 = \phi(\langle e, f\alpha \rangle)^2$$

but as ϕ composed with the "inner product" defined above, yields to an inner product on E, we can apply Cauchy-Swartz Inequality to obtain that

$$\phi(\langle e, f\alpha\rangle)^2 \leq \phi(\langle e, e\rangle)\phi(\langle f\alpha, f\alpha\rangle) \leq \|\langle e, e\rangle\| \, \|\langle f\alpha, f\alpha\rangle\| \leq \|e\|^2 \, \|f\|^2 \, \|\alpha\|^2 = \|e\|^2 \, \|f\|^2$$

where in the second inequality it is employed that ϕ is a state, hence contractive.

Now, it remains to show that the norm at hand satisfies the triangular inequality, since the other three norm conditions are satisfied by conditions (iii), (iv) in Definition 1.1, and by compatibility of scalar multiplication. So, let $e, f \in E$, and get the following

$$\begin{aligned} \|e+f\|^2 &= \|\langle e+f, e+f\rangle\| = \|\langle e, e\rangle + \langle e, f\rangle + \langle f, e\rangle + \langle f, f\rangle\| \\ &\leq \|e\|^2 + \|f\|^2 + 2\|e\| \, \|f\| = (\|e\| + \|f\|)^2 \end{aligned}$$

as desired.

□

Definition 2.1.4. *A Hilbert B-module is a pre-Hilbert B-module E which is complete with respect to the norm*

$$\|e\| = \|\langle e, e\rangle\|^{1/2}, \qquad e \in E$$

Note 2.1.5. The first inequality given in Lemma 1.3 guarantees that the B-module structure on E extends by continuity to the completion of E, while the second inequality asserts that the "inner product" on E extends also by continuity, to turn the completion of E into a Hilbert B-module.

As a first result about Hilbert B-modules, we show that an approximate unit of a suitable ideal in B can be seen as an "approximate identity" for E itself. But firstly, for E Hilbert B-module, define $\langle E, E\rangle = \overline{span\{\langle e, f\rangle \colon e, f \in E\}}$ and note that $\langle E, E\rangle$ is a closed two-sided ideal in B.

Lemma 2.1.6. *Let E be a Hilbert B-module and let $(u_\lambda)_\lambda$ be an approximate unit of positive contractions for $\langle E, E\rangle$. Then, $\lim_\lambda eu_\lambda = e$, for all $e \in E$.*

Proof. Let $e \in E$, then

$$\begin{aligned} \|eu_\lambda - e\|^2 &= \|\langle eu_\lambda - e, eu_\lambda - e\rangle\| = \|\langle eu_\lambda, eu_\lambda\rangle - \langle e, eu_\lambda\rangle - \langle eu_\lambda, e\rangle + \langle e, e\rangle\| \\ = \|u_\lambda^*\langle e, e\rangle u_\lambda - \langle e, e\rangle u_\lambda - u_\lambda^*\langle e, e\rangle + \langle e, e\rangle\| &\leq \|u_\lambda^*(\langle e, e\rangle u_\lambda - \langle e, e\rangle)\| + \|\langle e, e\rangle - \langle e, e\rangle u_\lambda\| \longrightarrow 0 \end{aligned}$$

Hence, $\lim_\lambda eu_\lambda = e$, for any $e \in E$ □

Note 2.1.7. There is a an elegant way to the change the module structure on a Hilbert B-module E, while keeping the same "inner product" structure. For this, let A be a C^*-algebra which contains $\langle E, E\rangle$ as a closed two sided ideal and for $a \in A$, $e \in E$ and $(u_\lambda)_\lambda$ approximate unit for $\langle E, E\rangle$, observe the following

$$\begin{aligned} &\langle eu_\lambda a - eu_{\lambda'} a, eu_\lambda a - eu_{\lambda'} a\rangle \\ &= a^* u_\lambda \langle e, e\rangle u_\lambda a + a^* u_{\lambda'} \langle e, e\rangle u_{\lambda'} a - a^* u_\lambda \langle e, e\rangle u_{\lambda'} a - a^* u_{\lambda'} \langle e, e\rangle u_\lambda a \xrightarrow{\|\cdot\|} 0 \end{aligned}$$

which shows that the sequence $(eu_\lambda a)_\lambda$ is Cauchy, hence convergent in E, for any $e \in E$, $a \in A$. So, we may define the A-module structure on E by $ea = \lim_\lambda eu_\lambda a$ and we see that condition (i) in Definition 1.1 is satisfied, since

$$\begin{aligned} \langle e, fa\rangle &= \langle e, \lim_\lambda fu_\lambda a = \lim_\lambda \langle e, fu_\lambda a\rangle \\ &= \lim_\lambda \langle e, f\rangle u_\lambda a = \langle e, f\rangle a \end{aligned}$$

while the rest of the conditions are satisfied a priori. Thus, by completing this pre-Hilbert

A-module using Lemma 1.3, E turns into a Hilbert A-module with the same "inner product" structure.

Let us now see some first examples of Hilbert B-modules.

Example 2.1.8. i) B is a Hilbert B-module with "inner product" given by $\langle b_1 b_2\rangle = b_1^* b_2$. Also, note that any closed two sided ideal in B is a Hilbert B-module with the same "inner product".

ii) Let, $E = \sum_n B$, be the C^*-algebra of sequences in B converging to 0, and define the "inner product" on E by $\langle (b_1, b_2, ...), (b_1', b_2', ...)\rangle = \sum_n b_n^* b_n'$. The completion of E with respect to the norm on $\sum_n B$, is denoted H_B. Note that if $B = \mathbb{C}$, then $H_B = l^2(\mathbb{N})$ which we will regularly denote it by H.

After defining Hilbert B-modules, it is natural to ask how the morphisms between Hilbert B-modules look like. In the following lines, an effort to address this question appears.

Let E_1, E_2 be two Hilbert B-modules and denote by $L_B(E_1, E_2)$ the space of all maps $T\colon E_1 \to E_2$ for which exist an adjoint counterpart $T^*\colon E_2 \to E_1$, satisfying $\langle Tx, y\rangle = \langle x, T^* y\rangle$, and trivially we see that $T^* \in L_B(E_2, E_1)$, and $T^{**} = T$. Now, this condition on T implies that T is a linear and B-module map. Let us see first that T is linear:

Let $e_1, f_1 \in E_1$ and $e_2 \in E_2$ then,

$$\begin{aligned}\langle T(e_1 + f_1), e_2\rangle &= \langle e_1 + f_1, T^* e_2\rangle = \langle e_1, T^* e_2 + \langle f_1, T^* e_2\rangle \\ &= \langle Te_1, e_2\rangle + \langle Tf_1, e_2\rangle = \langle Te_1 + Tf_1, e_2\rangle\end{aligned}$$

concluding that $T(e_1 + f_1) = Te_1 - Tf_1$

T is a B-module map: let $b \in B$, $e_1 \in E_1$, $e_2 \in E_2$ then,

$$\begin{aligned}\langle T(e_1 b), e_2\rangle &= \langle e_1 b, T^* e_2\rangle = b^* \langle e_1, T^* e_2\rangle = b^* \langle Te_1, e_2\rangle \\ &= \langle (Te_1) b, e_2\rangle\end{aligned}$$

hence, $T(e_1 b) = (Te_1) b$. In a similar one can show that $T^* \in L_B(E_2, E_1)$ is a linear, B-module map.

Moreover, let any $f \in E_2$ and $T \in L_B(E_1, E_2)$, and consider the set $B' = \{\langle Te, f\rangle \colon \|e\| \leq 1\}$. Since,

$$\|\langle Te, f\rangle\| = \|\langle e, T^* f\rangle\| \leq \|e\| \, \|T^* f\| \leq \|T^* f\| < \infty$$

we see that B' is a bounded subset of B. In fact, $\|\langle Te, Te\rangle\| = \|Te\|^2 < \infty$, for any $T \in L_B(E_1, E_2)$ and $e \in E_1$. Hence, $\sup_{T \in L_B(E_1, E_2)} \|Te\| < \infty$, for any $e \in E_1$, and by Uniform Boundedness Principle we get that

$$\sup_{T \in L_B(E_1, E_2)} \|T\| < \infty$$

which shows that any $T \in L_B(E_1, E_2)$ is bounded. Thus, $L_B(E_1, E_2)$ is a linear subspace of the Banach space of bounded linear maps from E_1 to E_2. Therefore, $L_B(E_1, E_2)$ inherits the operator norm from the ambient Banach space, and it is evident that $\|T^*\| = \|T\|$. In particular, if $(T_n)_n \in L_B(E_1, E_2)$, a convergent sequence, say $T_n \xrightarrow{\|\cdot\|} T$, then $(T_n^*)_n$ is also a convergent sequence, say $T_n^* \xrightarrow{\|\cdot\|} S$, and so

$$\langle Te, f\rangle = \lim_n \langle T_n e, f\rangle = \lim_n \langle e, T_n^* f\rangle = \langle e, Sf\rangle, \qquad e \in E_1, f \in E_2$$

which implies that $T \in L_B(E_1, E_2)$. Hence, $L_B(E_1, E_2)$ is a closed linear space.

Lemma 2.1.9. *For any E Hilbert B-module, $L_B(E) (= L_B(E, E))$ is a C^*-algebra.*

Proof. Using the discusion above, it is straightforward to see that $L_B(E)$ is a closed *-algebra and that $\|T \circ S\| \leq \|T\| \|S\|$, for any $T, S \in L_B(E)$. So it remains to show that $\|T\|^2 = \|T^*T\|$. For this, let $e \in E$, $\|e\| \leq 1$. Then,

$$\|\langle Te, Te\rangle\| = \|\langle e, T^*Te\| \leq \|T^*Te\| \leq \|T^*T\|$$

hence $\|T\|^2 \leq \|T^*T\|$, and the other direction is clear since the *-operation is isometric. $\square$

Taking motivation by Example 2.1.8, we aim to construct a space of maps between Hilbert B-modules which generalizes the concept of compact operators on a Hilbert space.

Let E_1, E_2, be two Hilbert B-modules. Take $e_1 \in E_1$, $e_2 \in E_2$, and define a map $\Theta_{e_2,e_1} \colon E_1 \to E_2$, by $\Theta_{e_2,e_1}(x) = e_2\langle e_1, x\rangle$. Firstly, we see that $\Theta_{e_2,e_1} \in L_B(E_1, E_2)$: Let $y \in E_2$, then

$$\langle \Theta_{e_2,e_1}(x), y\rangle = \langle e_2\langle e_1, x\rangle, y\rangle = \langle x, e_1\rangle\langle e_2, y\rangle = \langle x, e_1\langle e_2, y\rangle\rangle$$

thus, the map $\Theta_{e_1,e_2} \colon E_2 \to E_1$ is the adjoint map of Θ_{e_2,e_1}, hence $\Theta_{e_2,e_1} \in L_B(E_1, E_2)$.

Now, denote by $K_B(E_1, E_2)$, the closed linear span of $\{\Theta_{e_2,e_1} \colon e_1 \in E_1, e_2 \in E_2\}$. As a first result, we show that $K_B(E)$ is a closed two sided ideal in $L_B(E)$.

Lemma 2.1.10. *For any E Hilbert B-module, $K_B(E)$ is a closed two-sided ideal in $L_B(E)$.*

Proof. Firstly, $K_B(E)$ is closed by construction. Now, let $T \in L_B(E)$ and $e, f \in E$. Then

$$T \circ \Theta_{e,f}(x) = T(e\langle f, x\rangle) = \Theta_{Te,f}(x), \qquad x \in E$$

Hence, $T \circ \Theta_{e,f} \in K_B(E)$.

That $K_B(E)$ is a right ideal of $L_B(E)$, is clear. $\square$

Lemma 2.1.11. $K_B(E_1, E_2) = \{T \in L_B(E_1, E_2) \colon TT^* \in K_B(E_2)\}$

Proof. For showing that $K_B(E_1, E_2) \subseteq \{T \in L_B(E_1, E_2) \colon TT^* \in K_B(E_2)\}$ it suffices to show it for the generators of $K_B(E_1, E_2)$. So, let $\Theta_{e_2,e_1} \in K_B(E_1, E_2)$, $x, e_1 \in E_1, e_2 \in E_2$, then

$$\begin{aligned}\Theta_{e_2,e_1} \circ \Theta_{e_1,e_2}(x) &= \Theta_{e_2,e_1}(e_1\langle e_2, x\rangle) = \Theta_{e_2,e_1}(e_1)\langle e_2, x\rangle \\ &= e_2\langle e_1, e_1\rangle\langle e_2, x\rangle = e_2\langle\langle e_1, e_1\rangle e_2, x\rangle = \Theta_{e_2,\langle e_1,e_1\rangle e_2}(x)\end{aligned}$$

which shows that $\Theta_{e_2,e_1} \circ \Theta^*_{e_2,e_1} \in K_B(E_2)$.

For the other direction, first note that for any $T \in L_B(E_1, E_2)$, $\|TT^*\| = \|T\|^2$ and similarly to Lemma 2.1.10, we get that $K_B(E_2)L_B(E_1, E_2) \subseteq K_B(E_1, E_2)$. Now, since $K_B(E_2)$ is a separable, closed two-sided ideal in $L_B(E)$, admits an approximate unit $(u_n)_n$ and we we observe that

$$\|u_nT - T\|^2 = \|(u_nT - T)(u_nT - T)^*\| = \|u_nTT^*u_n - u_nTT^* - TT^*u_n + TT^*\| \longrightarrow 0$$

hence, $T = \lim_n u_nT$, and since $u_nT \in K_B(E_1, E_2)$ for all $n \in \mathbb{N}$, we conclude that $T \in K_B(E_1, E_2)$, as desired. $\square$

Note 2.1.12. In continuation of Example 2.1.8, consider the Hilbert $\mathbb{C}$-module $H = H_{\mathbb{C}} = l^2(\mathbb{N})$, and let $\Theta_{x,y} \in K_{\mathbb{C}}(H)$. Then, for any $z \in H$, $\Theta_{x,y}(z) = x\langle y, z\rangle$, hence a rank one operator. So, $span\{\Theta_{x,y} \colon x, y \in H\}$ consists of all finite rank operators, which implies that $K_{\mathbb{C}}(H) = \overline{span\{\Theta_{x,y} \colon x, y \in H\}} = \mathbb{K}$, where $\mathbb{K}$ are the compact operators on H.

Lemma 2.1.13. *Let B be a C^*-algebra. Then, if we consider B as a Hilbert B-module, then the map*

$$\Gamma \colon \Theta_{x,y} \longmapsto xy^*$$

is a *-isomorphism from $K_B(B)$ onto B.

Proof. We start by showing that Γ is a *-homomorphism.

Let $x_1, x_2, y_1, y_2 \in B$, then for any $b \in B$ we have

$$\Theta_{x_1,y_1} \circ \Theta_{x_2,y_2} = \Theta_{x_1,y_2\langle y_1,x_2\rangle^*}$$

hence,

$$\Gamma(\Theta_{x_1,y_1} \circ \Theta_{x_2,y_2}) = \Gamma(\Theta_{x_1,y_2\langle y_1,x_2\rangle^*})$$

$$= x_1\langle y_1, x_2\rangle y_2^* = x_1 y_1^* x_2 y_2^* = \Gamma(\Theta_{x_1,y_1})\Gamma(\Theta_{x_2,y_2})$$

Now, let $(u_\lambda)_\lambda$ be an approximate unit for B and let $x_1, x_2, y_1, y_2 \in B$. Set $u = \lim_\lambda u_\lambda$, and obtain that,

$$\Theta_{x_1,y_1} + \Theta_{x_2,y_2}(z) = \langle y_1 x_1^* + y_2 x_2^*, z\rangle$$

$$= u\langle y_1 x_1^* + y_2 x_2^*, z\rangle = \Theta_{u,y_1x_1^*+y_2x_2^*}(z)$$

thus,

$$\Gamma(\Theta_{x_1,y_1} + \Theta_{x_2,y_2}) = \Gamma(\Theta_{u,y_1x_1^*+y_2x_2^*})$$

$$= u(y_1x_1^* + y_2x_2^*)^* = (y_1x_1^* + y_2x_2^*)^* = x_1y_1^* + x_2y_2^*$$

$$= \Gamma(\Theta_{x_1,y_1}) + \Gamma(\Theta_{x_2,y_2})$$

Moreover, for any $x, y, z \in B$,

$$\|\Theta_{x,y}(z)\| = \|xy^*z\| \leq \|xy^*\|\,\|z\|$$

and since $\|\Theta_{x,y}(u)\| = \|xy^*\|$, we get that $\|\Theta_{x,y}\| = \|xy^*\|$. Hence, Γ is contractive, which means that can be extended continuously to $K_B(B)$. To finalize that Γ is indeed a *-homomorphism, we have to show that respects the *-operation, but this is trivial.

Finally, let us argue that Γ is a bijection. For injectivity, let $x, y \in B$, and suppose that $\Gamma(\Theta_{x,y}) = xy^* = 0$. It follows that, $0 = \|xy^*\| = \|\Theta_{x,y}\|$, hence $\Theta_{x,y} = 0$. Now, let $x \in B$, then $\Gamma(\Theta_{u,x^*}) = \lim_\lambda u_\lambda x = x$, concluding that Γ is also surjective, and in turn a *-isomorphism as required. □

Note 2.1.14. Using Lemma 2.1.10 and Lemma 2.1.13, we can consider that a C^*-algebra B lies as a closed two-sided ideal inside $L_B(B)$. This amounts to the identification of an element $b \in B$ with the left multiplication map by b in $L_B(B)$. Under this identification $T(b) = Tb$, $T \in L_B(B)$, $b \in B$, where Tb is the usual composition.

In the following, we define the multiplier algebra of a C^*-algebra B, as $M(B) = L_B(B)$. It is a fact that $L_B(B)$ is actually isomorphic to the multiplier algebra of B as defined in Section 1/ Chapter 1.

Definition 2.1.15. *Let E be a Hilbert B-module. The semi-norms $\|\cdot\|_e$, $e \in E$ on $L_B(E)$, given by $\|T\|_e = \|Tx\| + \|T^*e\|$, $T \in L_B(E)$ define a locally convex topology on $L_B(E)$, which we call the strict topology.*

Note 2.1.16. It is not hard to see that for E Hilbert B-module, $L_B(E)$ is complete with respect to the strict topology. Let $T_n \in L_B(E)$ be a strictly Cauchy sequence. Then, for each $e \in E$, $\|T_n - T_m\|_e = \|T_ne - T_me\| + \|T_n^*e - T_m^*e\| \longrightarrow 0$. Thus, for each $e \in E$, $(T_ne)_n$, $(T_n^*e)_n$, are Cauchy sequences in E, and since E is complete, they are convergent. If we set T, S be the maps given by $e \mapsto \lim_n T_ne$, and $e \mapsto \lim_n T_n^*e$ and use the continuity of the "inner product" on E, we get that $\langle Te, f\rangle = \lim_n\langle T_ne, f\rangle = \lim_n\langle e, T_n^*f\rangle = \langle e, Sf\rangle$. Hence, $T \in L_B(E)$, and $\|T_n - T\|_x \longrightarrow 0$, for each $x \in E$.

Definition 2.1.17. *Let E be a Hilbert A-module and F a Hilbert B-module. Then, a map $\rho\colon L_A(E) \to L_B(F)$ is called strictly continuous, if it is continuous with respect to the strict topologies of $L_A(E)$ and $L_B(F)$.*

Proposition 2.1.18. *Let A be a C^*-algebra, E a Hilbert B-module, and $\phi\colon A \to L_B(E)$ a *-homomorphism. Then the following are equivalent:*

i) there is a projection $p \in L_B(E)$, such that $p(E) = \overline{span\{\phi(a)e\colon a \in A, e \in E\}}$

ii) there is a (unique)strictly continuous *-homomorphism $\underline{\phi}\colon M(A) \to L_B(E)$, extending ϕ

Proof. We argue first that if such an extension $\underline{\phi}$ exists, then is unique. For this, let $(u_\lambda)_\lambda$ be an approximate unit for A and take any $T \in M(A)$. Then, $(Tu_\lambda)_\lambda$ is a sequence in A satisfying that

$$Tu_\lambda(x) = T(u_\lambda x) \longrightarrow Tx, \qquad \forall x \in A$$

and,

$$(Tu_\lambda)^* x = u_\lambda T^* x \longrightarrow T^* x, \qquad \forall x \in A$$

Hence, $\|Tu_\lambda\|_x = \|Tu_\lambda x\| + \|(Tu_\lambda)^* x\| \longrightarrow \|Tx\| + \|T^*x\| = \|T\|_x$, for all $x \in A$. This shows that $Tu_\lambda \longrightarrow T$ in strict topology of $M(A)$. So, if $\underline{\phi}$ is the given extension of ϕ, then for any $T \in M(A)$, $\underline{\phi}(T) = \lim_\lambda \underline{\phi}(Tu_\lambda) = \lim_\lambda \phi(Tu_\lambda)$, which clearly means that $\underline{\phi}$ is unique.

$(i) \implies (ii)$: First note that $p\phi(a) = \phi(a)p = \phi(a)$, for all $a \in A$. Now, we claim that for any $T \in M(A)$, $e \in E$, the sequences $(\phi(Tu_\lambda)e)_\lambda$ and $(\phi(u_\lambda T)e)_\lambda$ converge to the same point in E. Let $\epsilon > 0$, $e \in E$, $T \in M(A)$ and find finite sequences $(a_k) \subset A$, $(e_k) \subset E$, such that

$$2\|T\| \left\| p(e) - \sum_k \phi(a_k)e_k \right\| < \epsilon$$

Then, using that,

$$\|\phi(Tu_\lambda)e - \phi(Tu_{\lambda'})e\|$$

$$= \left\| (\phi(Tu_\lambda) - \phi(Tu_{\lambda'}))p(e) - (\phi(Tu_\lambda) - \phi(Tu_{\lambda'}))\sum_k \phi(a_k)e_k + (\phi(Tu_\lambda) - \phi(Tu_{\lambda'}))\sum_k \phi(a_k)e_k \right\|$$

$$\leq \left\| \phi(Tu_\lambda) - \phi(Tu_{\lambda'})(p(e) - \sum_k \phi(a_k)e_k) \right\| + \left\| \sum_k \phi(Tu_\lambda a_k)e_k - \phi(Tu_{\lambda'} a_k)e_k \right\|$$

$$\leq 2\|T\| \left\| p(e) - \sum_k \phi(a_k)e_k \right\| + \sum_k \|\phi(Tu_\lambda a_k)e_k - \phi(Tu_{\lambda'} a_k)e_k\| < \epsilon$$

since $\phi(Tu_\lambda a_k) \longrightarrow \phi(Ta_k)$, for all k. So $(\phi(Tu_\lambda)e)_\lambda$ is Cauchy in E, hence convergent, while in exactly the same fashion we obtain that $((\phi(u_\lambda T)e)_\lambda$ is convergent in E. Moreover, note that using the same norm considerations as above, it follows that

$$\|\phi(Tu_\lambda)e - \phi(u_\lambda T)e\| \leq$$

$$\leq 2\|T\| \left\| p(e) - \sum_k \phi(a_k)e_k \right\| + \left\| \sum_k \phi(Tu_\lambda a_k)e_k - \phi(u_\lambda T a_k)e_k \right\| < \epsilon$$

concluding that, indeed $(\phi(Tu_\lambda)e)_\lambda$ and $(\phi(Tu_\lambda)e)_\lambda$ has the same limit point.

This fact, enables us to define for any $T \in M(A)$ a map $\underline{\phi}(T)\colon E \to E$, by $\underline{\phi}(T)e = \lim_\lambda \phi(Tu_\lambda)e = \lim_\lambda \phi(u_\lambda T)e$, for all $e \in E$. Since,

$$\langle \underline{\phi}(T)e, f\rangle = \lim_\lambda \langle \phi(Tu_\lambda e, f\rangle$$
$$= \lim_\lambda \langle e, \phi(u_\lambda T^*)f\rangle = \langle e, \underline{\phi}(T^*)f\rangle$$

it follows that $\underline{\phi}(T) \in L_B(E)$, and that $\underline{\phi}(T^*) = \underline{\phi}^*(T)$, for any $T \in M(A)$. Moreover, it obvious that $\underline{\phi}\colon M(A) \to L_B(E)$ is linear, and so it remains to show that it is multiplicative. Let $T, S \in M(A)$, $e \in E$, then,

$$\begin{aligned}\underline{\phi}(TS)e &= \lim_\lambda \phi(u_\lambda TS)e = \lim_\lambda \lim_{\lambda'} \phi(u_\lambda TSu_{\lambda'})e \\ &= \lim_\lambda \lim_{\lambda'} \phi(u_\lambda T)\phi(Su_{\lambda'})e = \underline{\phi}(T)\underline{\phi}(S)e\end{aligned}$$

hence $\underline{\phi}$ is *-homomorphism. To show that $\underline{\phi}$ extends ϕ, let $a \in A$, $e \in E$, then

$$\underline{\phi}(a)e = \lim_\lambda \phi(au_\lambda)e = \phi(a)e$$

which means that $\underline{\phi}$ extends ϕ.

Finally, we argue that $\underline{\phi}$ is strictly continuous. Let $(T_\lambda)_\lambda$ be a net in $M(A)$, such that $T_\lambda \longrightarrow T$ in the strict topology and we aim to show that $\underline{\phi}(T_\lambda) \longrightarrow \underline{\phi}(T)$ in the strict topology on $L_B(E)$. Since, $\sup_\lambda \|(T_\lambda - T)e\| < \infty$ for all $e \in E$, by Uniform Boundedness Principle there is $0 < M < \infty$, such that $\sup \|T_\lambda - T\| \leq M$. Moreover, for $\epsilon > 0$, $e \in E$, find $(a_k)_k \subset A$, $(e_k)_k \subset E$, so that

$$M \left\| p(e) - \sum_k \phi(a_k)e_k \right\| < \epsilon$$

Since,

$$\left\|(\underline{\phi}(T) - \underline{\phi}(T_\lambda))e\right\| = \lim_{\lambda'} \|(\phi(Tu_{\lambda'}) - \phi(T_\lambda u_{\lambda'}))p(e)\|$$

$$= \lim_{\lambda'} \left\|(\phi(Tu_{\lambda'}) - \phi(T_\lambda u_{\lambda'}))p(e) - (\phi(Tu_{\lambda'}) - \phi(T_\lambda u_{\lambda'}))\sum_k \phi(a_k)e_k + (\phi(Tu_{\lambda'}) - \phi(T_\lambda u_{\lambda'}))\sum_k \phi(a_k)e_k\right\|$$

$$\leq \|T - T_\lambda\| \left\|p(e) - \sum_k \phi(a_k)e_k\right\| + \sum_k \left\|\lim_{\lambda'}(\phi(Tu_{\lambda'}a_k) - \phi(T_\lambda u_{\lambda'}a_k))e_k\right\|$$

$$\leq M \left\|p(e) - \sum_k \phi(a_k)e_k\right\| + \sum_k \|(\phi(Ta_k) - \phi(T_\lambda a_k))e_k\| < \epsilon$$

because $\phi(T_\lambda a_k)e_k \longrightarrow \phi(Ta_k)e_k$, for all k. Hence, $\underline{\phi}(T_\lambda))e \longrightarrow \underline{\phi}(T)e$, for all $e \in E$, and similarly we show that $\underline{\phi}(T_\lambda)^*e \longrightarrow \underline{\phi}(T)^*e$, for all $e \in E$. Combining these two facts, we conclude that $\underline{\phi}(T_\lambda) \longrightarrow \underline{\phi}(T)$ in strict topology, which in turn implies that $\underline{\phi}$ is strictly continuous, as desired.

$(ii) \implies (i)$: Set $p = \underline{\phi}(1_{M(A)})$ and observe that if $(u_\lambda)_\lambda$ is an approximate unit for A, then $u_\lambda \longrightarrow 1_{M(A)}$ in strict topology. Thus, for each $a \in A$, $e \in E$, we have that $p(\phi(a)e) = \lim_\lambda \underline{\phi}(u_\lambda)\phi(a)e = \lim_\lambda \phi(u_\lambda a)e = \phi(a)e$, hence $\overline{span\{\phi(a)e\colon a \in A, e \in E\}} \subseteq p(E)$. On the other hand, if $e \in p(E)$, then $e = p(e) = \lim_\lambda \phi(u_\lambda)e \in \overline{span\{\phi(a)e\colon a \in A, e \in E\}}$, and the proof is complete. ☐

Taking as a starting point this rather useful proposition, we pursue to show that for any E Hilbert B-module, there is a *- isomorphism from $M(K_B(E))$ onto $L_B(E)$. Actually, we first show a more general result, from which the statement above will come up as a straightforward application. For the shake of convenience, in the following we use the overline to refer to "closed linear span".

Lemma 2.1.19. *Let A be C^*-algebra and E a Hilbert B-module. If there is a *-isomorphism $\phi\colon A \to K_B(E)$, then $\overline{\phi(A)(E)} = E$ and its strictly continuous extension $\underline{\phi}\colon M(A) \to L_B(E)$ is a *-isomorphism.*

Proof. Firstly, by the construction of $K_B(E)$, we get that $E\langle E, E\rangle \subseteq \overline{K_B(E)(E)}$. Also, by Lemma 1.6 it is straightforward that $E \subseteq \overline{E\langle E, E\rangle}$, hence $E = \overline{E\langle E, E\rangle}$. So, $\overline{\phi(A)(E)} = \overline{K_B(E)(E)} = E$.

To show that $\underline{\phi}$ is injective, let $T \in M(A)$ and suppose that $\underline{\phi}(T) = 0$. If $(u_\lambda)_\lambda$ is an approximate unit for A, then

$$\underline{\phi}(Tu_\lambda) = \phi(Tu_\lambda) = \underline{\phi}T\phi(u_\lambda) = 0$$

shows that $Tu_\lambda = 0$ for all λ. But $Tu_\lambda \longrightarrow T$ in strict topology on $M(A)$, hence $T = 0$.

For surjectivity, let $T \in L_B(E)$ and define a map $\psi\colon A \to A$ by $\psi(a) = \phi^{-1}(T\phi(a))$. Then ψ satisfies the following for all $a, a' \in A$,

$$\langle \psi(a), a' \rangle = \psi(a)^* a' = a^* \phi^{-1}(T^*) a' = a^* \phi^{-1}(T^* \phi(a')) = \langle a, \phi^{-1}(T^* \phi(a')) \rangle$$

hence, if $\sigma\colon A \to A$ is given by $\sigma(a) = \phi^{-1}(T^*\phi(a))$, then $\sigma = \psi^*$, showing that $\psi \in M(A)$. Now, since

$$\underline{\phi}(\psi)\phi(a) = \underline{\phi}(\psi(a)) = \phi(\phi^{-1}(T\phi(a))) = T\phi(a) \qquad \forall a \in A$$

and as $\overline{\phi(A)(E)} = E$, we conclude that $\underline{\phi}(\psi) = T$, showing that $\underline{\phi}$ is surjective. As we already know from Proposition 2.1.18 that $\underline{\phi}$ is *-homomorphism, the proof is over. □

Before showing that $M(K_B(E)) \cong L_B(E)$, for any E Hilbert B-module, we can't resist from applying the last two results to get the following interesting corollary.

Corollary 2.1.20. *Let A, B be C^*-algebras and $\phi\colon A \to M(B)$ a *-homomorphism. Then, the following are equivalent,*

i) there is a projection $p \in M(B)$, such that $p(B) = \overline{\phi(A)(B)}$,

ii) there exists a (unique)strictly continuous *-homomorphism $\underline{\phi}\colon M(A) \to M(B)$, extending ϕ.

In fact, when ϕ is *-isomorphism from A onto B, then $\underline{\phi}$ is *-isomorphism as well.

Proof. For the equivalence $(i) \iff (ii)$, we apply Proposition 2.1.18 to B seen as a Hilbert B-module. Now, let $\phi\colon A \to B$ be a *-isomorphism. Since Lemma 2.1.13 asserts that $B \cong K_B(B)$, then Lemma 2.1.19 applied again for B seen as a Hilbert B-module offers the desired *-isomorphism. □

Corollary 2.1.21. *Let E be a Hilbert B-module. Then, there is *-isomorphism $\psi\colon M(K_B(E)) \to L_B(E)$ such that the following diagram commutes*

$$\begin{tikzcd} M(K_B(E)) \arrow[r, "\psi"] & L_B(E) \\ K_B(E) \arrow[u, hook] \arrow[r, equal] & K_B(E) \arrow[u, hook] \end{tikzcd}$$

Proof. Consider the identity map $id\colon K_B(E) \to K_B(E)$ and apply Lemma 2.1.19. □

We now present a powerful criterion for a net in $L_B(E)$ to be strictly convergent. This is a fact that will emerge and be used critically in the rest of this project.

Lemma 2.1.22. *Let E be a Hilbert B-module. Then, a net $(T_\lambda)_\lambda \in L_B(E)$ is strictly convergent if and only if $(T_\lambda k)_\lambda$, and $(T_\lambda^* k)_\lambda$ are norm convergent sequences, for any $k \in K_B(E)$.*

Proof. Suppose that $(T_\lambda)_\lambda$ is a strictly convergent net. Then, there exists $T \in L_B(E)$ such that for any $e \in E$, $\sup_\lambda \|(T_\lambda - T)e\| < \infty$. Hence, Uniform Boundedness Principle provides a global boundary $0 < M < \infty$, satisfying $\sup_\lambda \|T_\lambda - T\| \leq M$. So, since any $k \in K_B(E)$ is a limit point of elements of the form $\sum_{k=1}^n \Theta_{x_k, y_k}$, where $x_k, y_k \in E$, in order to show that $((T_\lambda - T)k)_\lambda$, $((T_\lambda^* - T^*)k)_\lambda$ are norm convergent to zero, it suffices to show it only for $k = \Theta_{x,y}$, $x, y \in E$.

But, for any $e \in E$, $\|e\| \leq 1$, we have that

$$\|(T_\lambda - T)\Theta_{x,y}(e)\| = \|(T_\lambda - T)x\langle y, e\rangle\| \leq \|(T_\lambda - T)x\| \, \|y\| \longrightarrow 0$$

which implies that $\|(T_\lambda - T)\Theta_{x,y}\| \longrightarrow 0$. Similarly, we get that $\|(T_\lambda^* - T^*)\Theta_{x,y}\| \longrightarrow 0$ for any $x, y \in E$, so the conclusion follows.

For the other direction, assume that there is $T \in L_B(E)$ such that $T_\lambda k \longrightarrow Tk$ and $T_\lambda^* k \longrightarrow T^* k$, for any $k \in K_B(E)$, in the strict topology. Now, employ the *-isomorphism ψ from Corollary 1.21, then $\|T_\lambda - T\| = \left\|\psi^{-1}(T_\lambda - T)\right\|$, and $\psi^{-1}(T_\lambda - T) \in M(K_B(E))$. Since, by hypothesis $\sup_\lambda \|(T_\lambda - T)(k)\| < \infty$, for any $k \in K_B(E)$, we also have that $\sup_\lambda \left\|\psi^{-1}((T_\lambda - T)(k))\right\| = \sup_\lambda \left\|\psi^{-1}(T_\lambda - T)k\right\| < \infty$, for any $k \in K_B(E)$. Thus, employing again the Uniform Boundedness Principle we get that $\left\|\psi^{-1}(T_\lambda - T)\right\|$ is uniformly bounded, hence $\|T_\lambda - T\|$ is uniformly bounded. So, since $\overline{K_B(E)(E)} = E$, we get that $\|(T_\lambda - T)e\| \longrightarrow 0$ and $\|(T_\lambda^* - T^*)e\| \longrightarrow 0$, for all $e \in E$, hence $T_\lambda \longrightarrow T$ in the strict topology. □

We continue our wandering in Hilbert C^*-module theory, by showing some results that are very closely connected to the construction of Cuntz picture of KK-theory and hence they will be critical for the next section. One of the most interesting results in this last portion, will be that for any B stable C^*-algebra, H_B and B are isomorphic as Hilbert B-modules.

Definition 2.1.23. *Two Hilbert B-modules E, F are isomorphic if there is a linear bijection $\psi \colon E \to F$ such that,*

$$\langle \psi(e_1), \psi(e_2)\rangle = \langle e_1, e_2\rangle$$

for all $e_1, e_2 \in E$.

Lemma 2.1.24. *Let $\psi \colon E \to F$ be an isomorphism of Hilbert B-modules. Then, the map $\Gamma \colon L_B(E) \to L_B(F)$, given by $\Gamma(T) = \psi T \psi^{-1}$ is a *-isomorphism, which also maps $K_B(E)$ onto $K_B(F)$.*

Proof. That Γ is linear, multiplicative and continuous is obvious. Now, we argue that it is *-preserving. Let $f, g \in F$, and $T \in L_B(E)$ then

$$\begin{aligned}\langle \psi T \psi^{-1}(f), g\rangle &= \langle T\psi^{-1}(f), \psi^{-1} g\rangle = \langle \psi^{-1}(f), T^*\psi^{-1}(g)\rangle \\ &= \langle f, \psi T^* \psi^{-1}(g)\rangle\end{aligned}$$

hence, $(\psi T \psi^{-1})^* = \psi T^* \psi^{-1}$, which in turn implies that $\Gamma(T^*) = \Gamma(T)^*$.

Γ is injective: Let $T \in L_B(E)$ such that $\Gamma(T) = 0$, and some $e \in E$, where $\psi^{-1}(f) = e$ then

$$\langle Te, Te\rangle = \langle \psi Te, \psi Te\rangle = \langle \psi T \psi^{-1}(f), \psi T \psi^{-1}(f)\rangle = 0$$

hence, $Te = 0$ for any $e \in E$, concluding that $T = 0$.

For surjectivity, take $S \in L_B(F)$, and set $T = \psi^{-1} S \psi$, then $T \in L_B(E)$, and the result folllows.

Finally, to see that Γ maps $K_B(E)$ onto $K_B(F)$, it suffices to show it for the generators. So take $e_1, e_2 \in E$ and $f \in F$, then

$$\begin{aligned}\psi \Theta_{e_1,e_2} \psi^{-1}(f) &= \psi(e_1\langle e_2, \psi^{-1}(f)\rangle) = \psi(e_1)\langle e_2, \psi^{-1}(f)\rangle \\ &= \psi(e_1)\langle \psi(e_2), f\rangle = \Theta_{\psi(e_1),\psi(e_2)}(f)\end{aligned}$$

hence, $\Gamma(\Theta_{e_1,e_2}) = \Theta_{\psi(e_1),\psi(e_2)} \in K_B(F)$, while the other direction is clear. □

Now, we momentarily turn to a construction that is essential for the development of this exposistion. Namely, we devote the following lines to introduce the internal tensor product construction in the Hilbert modules setting.

Let A and B be C^*-algebras. Suppose that E is a Hilbert B-module and F a Hilbert A-module, and let $\phi\colon B \to L_A(F)$ be a *-homomorphism. Then, we can equip F with a left B-module structure by $bf = \phi(b)f,\ b \in B,\ f \in F$. Thus, we can form the algebraic tensor product $E \otimes_B F$, which we consider as a right A-module by $(e \otimes_B f)a := e \otimes fa$. Also, we equip $E \otimes_B F$ with a (unique) right-linear and left-conjugate linear map $\langle \cdot\,, \cdot \rangle\colon E \otimes_B F \times E \otimes_B F \to A$, satisfying $\langle e_1 \otimes_B f_1, e_2 \otimes_B f_2 \rangle := \langle f_1, \phi(\langle e_1, e_2 \rangle) f_2 \rangle$. So far so good, now set $N_{E,F} = \{x \in E \otimes_B F \colon \langle x, x \rangle = 0\}$, and it is straightforward to see that $N_{E,F}$ is an A-submodule of $E \otimes_B F$, therefore we can consider the quotient $E \otimes_B F / N_{E,F}$ and the corresponding quotient map q. Then, $E \otimes_B F / N_{E,F}$ has a right A-module structure by $q(x)a = q(xa),\ x \in E \otimes_B F,\ a \in A$ and an A-valued "inner product" is defined by $\langle q(x), q(y) \rangle = \langle x, y \rangle,\ x, y \in E \otimes_B F$. One easily checks that this map turns $E \otimes_B F / N_{E,F}$ into a pre-Hilbert A-module. So, after completing it, we get a Hilbert A-module, which we denote as $E \otimes_\phi F$. This Hilbert A-module is called the internal tensor product of E and F with respect to ϕ.

After this quite brief discussion about internal tensor product of Hilbert C^*-modules, we immediately start to use these preparation to get some rather useful and interesting results. We continue to regard B as a C^*-algebra in the following.

Lemma 2.1.25. *Let $s\colon \mathbb{C} \to M(B)$ be the *-homomorphism given by $s(z)b = zb$, $z \in \mathbb{C}$, $b \in B$. Then $H_B \cong H \otimes_s B$.*

Proof. Define $U\colon H \otimes_{\mathbb{C}} B \to H_B$, given on simple tensors by $U(z \otimes_{\mathbb{C}} b) = (z_1 b, z_2 b, ...)$, where $z = (z_1, z_2, ...) \in H (= l^2(\mathbb{N}),\ b \in B$. Let us see that U preserves the "inner products". Let $z^1, z^2 \in H,\ b_1, b_2 \in B$, then

$$\langle U(z^1 \otimes_{\mathbb{C}} b_1, z^2 \otimes_{\mathbb{C}} b_2) \rangle = \sum_i b_1^* \overline{z_i^1} z_i^2 b_2$$

while,

$$\begin{aligned} \langle z^1 \otimes_s b_1, z^2 \otimes_s b_2 \rangle &= \langle z^1 \otimes_{\mathbb{C}} b_1, z^2 \otimes_{\mathbb{C}} b_2 \rangle = \langle b_1, s(\langle z^1, z^2 \rangle) b_2 \rangle \\ &= \langle b_1, \langle z^1, z^2 \rangle b_2 \rangle = \langle b_1, \sum_i \overline{z_i^1} z_i^2 b_2 \rangle = \sum_i b_1^* \overline{z_i^1} z_i^2 b_2 \end{aligned}$$

hence, U indeed preserves the "inner products", and therefore we can extend it to a map, say $\overline{U}$, $\overline{U}\colon H \otimes_s B \to H_B$. Moreover, $\overline{U}$ is clearly linear and injective, and since U has already dense range, we find that $\overline{U}$ is the desired "inner product"-preserving linear bijection between H_B and $H \otimes_s B$. ☐

Lemma 2.1.26. *There is a *-isomorphism $L_B(H_B) \cong M(B \otimes \mathbb{K})$, mapping $K_B(H_B)$ onto $B \otimes \mathbb{K}$.*

Proof. By Lemma 2.1.19 it suffices to show that $K_B(H_B) \cong B \otimes \mathbb{K}$, and by additionally using Lemma 2.1.24 and Lemma 2.1.25, it suffices to show that $K_B(H \otimes_s B) \cong B \otimes \mathbb{K}$.

So, consider the *-homorphisms $\pi_1\colon B \to L_B(H \otimes_s B)$, $\pi_2\colon \mathbb{K} \to L_B(H \otimes_s B)$, given by

$$\pi_1(c)(z \otimes_s b) = z \otimes_s cb, \quad \pi_2(k)(z \otimes_s b) = k(z) \otimes_s b, \quad b, c \in B, z \in H, k \in \mathbb{K}$$

since their ranges commute, they induce a *-homomorphism $\lambda\colon B \otimes \mathbb{K} \to L_B(H \otimes_s B)$ which satisfies that $\lambda(\Theta_{z_1, z_2} \otimes bc^*) = \Theta_{z_1 \otimes b, z_2 \otimes c},\ z_1, z_2 \in H,\ b, c \in B$. Hence, λ maps onto $K_B(H \otimes_s B)$.

To see that λ is injective, let ϕ be a state on B, and consider the representation of $L_B(H \otimes_s B)$, $\pi_\phi\colon L_B(H \otimes_s B) \to B(H_\phi)$(see Remark 1.1.8 in [18]). Note that, if $\pi'_\phi\colon B \to B(H'_\phi)$ is the usual GNS-representation of B, then the linear map $H_\phi \to H \otimes H'_\phi$, given by $[z \otimes_s b] \mapsto z \otimes [b]$, is a Hilbert space isomorphism. Moreover, $\pi_\phi \circ \lambda = \pi'_\phi \otimes id_{\mathbb{K}}$, and if $\pi = \sum_\phi \pi_\phi$, $\pi' = \sum_\phi \pi'_\phi$, it follows that $\pi \circ \lambda$ unitary equivalent to $\pi' \otimes id_{\mathbb{K}}$. But, the latter is a faithful representation of $B \otimes \mathbb{K}$, implying that λ is injective. $\square$

Recall that a C^*-algebra B is called stable if $B \otimes \mathbb{K} \cong B$.

Lemma 2.1.27. *If B is stable then $H_B \cong B$ as Hilbert B-modules.*

Proof. Firstly, we consruct a sequence of isometries $(V_i)_i \subset L_B(H_B)$, satisfying that $\sum_i V_i V_i^* = 1$, and $V_i^* V_j = 0$, whenever $i \neq j$. To do so, let $(N_i)_i$ be a partition of $\mathbb{N}$ into infinite subsets and let the $\phi_i \colon N_i \to \mathbb{N}$ be linear bijections. Then define $V_i \in H_B \to H_B$ by

$$V_i(b_1, b_2, ...)_k = \begin{cases} b_{\phi_i(k)}, & k \in \mathbb{N}_i \\ 0, & else \end{cases}$$

Then $V_i \in L_B(H_B)$, with adjoint given by $V^i(b_1, b_2, ...)_k = b_{\phi^{-1}(k)}$, $k \in \mathbb{N}$, and it straightforward to check that $V_i^* V_i(b_1, b_2, ...)_k = b_k$, for all $k \in \mathbb{N}$, $V_i^* V_j = 0$ and moreover that,

$$V_i V_i^*(b_1, b_2, ...)_k = \begin{cases} b_k, & k \in \mathbb{N}_i \\ 0, & else \end{cases}$$

Hence, $(V_i)_i$ is the desired sequence of isometries.

Now, employing Lemma 2.1.22 and Lemma 2.1.26, we obtain a sequence of isometries $(U_i)_i \subset M(B \otimes \mathbb{K})$ satisfying the same conditions, while since B stable, using Corollary 2.1.20, we get a sequence of isometries $(W_i)_i \subset M(B)$ satisfying the same conditions. Define now $\rho \colon H_B \to B$ by $\rho(b_1, b_2, ...) = \sum_i W_i b_i$, and firstly we note that ρ is well defined since the sequence $(\sum_i^n W_i b_i)_n$ is square summable, hence convergent in B. Also, ρ preserves "inner products" since

$$\langle \rho(b_1, b_2, ...), \rho(b_1', b_2', ...) \rangle = (\sum_i W_i b_i)^* (\sum_j W_j b_j') = \sum_i b_i^* W_i^* W_i b_i'$$
$$= \sum_i b_i^* b_i' = \langle (b_1, b_2, ...), (b_1', b_2', ...) \rangle$$

This fact, also shows that ρ is injective, while for surjectivity, let $b \in B$, then the sequence $(W_1^* b, W_2^* b, ...)$ belongs to H_B, since

$$\sum_i (W_i^* b)^* (W_i^* b) = \sum_i b^* W_i W_i^* b = b^* b < \infty$$

and $\rho(W_1^* b, W_2^* b, ...) = b$. So, we conclude that ρ is a Hilbert B-module isomorphism, as required. $\square$

In order to be fully prepared for the next section, we state few more results and definitions. The proof of the first result consists of a combination of arguments that we have already presented in this section, while the second result is more involved. The last proof is omitted and can be found in the relevant reference.

Lemma 2.1.28. *Let B be a stable C^*-algebra, then there is a path of isometries $\{v_t \colon t \in (0, 1]\}$ in $M(B)$ such that*

i) $t \mapsto v_t$ *is a strictly continuous map*

ii) $v_1 = 1$

iii) $v_t v_t^* \xrightarrow{t \to 0} 0$ *in the strict topology.*

Lemma 2.1.29. *Let B be a stable C^*-algebra and let $w \in M(B)$ be an isometry. Then, there exists a strictly continuous path $(w_t)_t$ in $M(B)$, such that $w_0 = 1$ and $w_1 = w$.*

Proof. Let $(v_t)_t$ be the strictly continuous path given the previous lemma. Set $w_t = v_t w v_t^* + 1 - v_t v_t^*$ and $w_0 = 1$. Then, for $t \in (0,1]$, $t \mapsto w_t$ is a strictly continuous map, while if $b \in B$ observe that

$$\|v_t w v_t^* b\|^2 = \|b^* v_t w^* v_t^* v_t w u_t^*\| = \|b^* u_t u_t^* b\| \leq \|b^*\| \, \|u_t u_t^* b\| \xrightarrow{t \to 0} 0$$

hence, $w_t \xrightarrow{t \to 0} 1 = w_0$, which implies that $t \mapsto w_t$ is a strictly continuous maps for all $t \in [0,1]$. Moreover, trivially $w_1 = w$ and the proof is complete. □

Definition 2.1.30. *A Hilbert B-module E is called countably generated if there exists a countable set $\{e_n\}$ in E such that $span\{e_n b \colon b \in B\}$ is dense in E.*

Recall that a positive element h in a C^*-algebra B is strictly positive iff $\phi(h) > 0$ for all states ϕ on B.

Definition 2.1.31. *A C^*-algebra B is called σ-unital if it contains a strictly positive element*

Proposition 2.1.32 ([24], Propositions 3.10.4-3.10.5)**.** *Let B be a C^*-algebra. Then the following are equivalent.*

i) There is a is strictly positive element $h \in B$

ii) B has a countable approximate unit

2.2 Cuntz picture of KK-theory and absorbing representations

Let A be a C^*-algebra. Throughout the first results of this section we reserve the term representation for a *-homomorphism $A \to L_B(E)$, where B is a σ-unital C^*-algebra and E is a Hilbert B-module. We start by defining the equivalence relations $\sim_{a.u}$, $\sim_{asymp}$, in the context of Hilbert C^*-modules and we then proceed to give some interesting insights into these relations. Afterwards, by emphasizing on specific Hilbert C^*-modules and on weakly nuclear representations, which we define later, we introduce the notion of absorbing representations and the Cuntz picture of KK-theory, KK_{nuc}, for which we establish its main traits.

Definition 2.2.1. *Fix B a σ-unital C^*-algebra and let $\gamma \colon A \to L_B(E)$, $\gamma' \colon A \to L_B(E')$ be two representations. We say that γ, and γ' are approximately unitarily equivalent and write $\gamma \sim_{a.u} \gamma'$, if there exist a sequence of unitaries $(u_n)_n \subset L_B(E', E)$ such that for any $a \in A$:*

i) $\|\gamma(a) - u_n \gamma'(a) u_n^*\| \xrightarrow{n \to \infty} 0$

ii) $\gamma(a) - u_n \gamma'(a) u_n^* \in K_B(E)$, *for all* $n \in \mathbb{N}$

Moreover, we say that γ and γ' are asymptotically unitarily equivalent, $\gamma \sim_{asymp} \gamma'$, if there exists a norm-continuous path of unitaries $u \colon [0, \infty) \to L_B(E', E)$, $u = (u_t)_t$ such that for any $a \in A$:

iii) $\|\gamma(a) - u_t \gamma'(a) u_t^*\| \xrightarrow{t \to \infty} 0$

iv) $\gamma(a) - u_t \gamma'(a) u_t^* \in K_B(E)$, *for all* $t \in [0, \infty)$

Also, if $\sigma \colon A \to L_B(F)$, a representation, we define its "infinite repeat", $\sigma_\infty \colon A \to L_B(F_\infty)$, where $F_\infty = F \oplus F \oplus \cdots$. Moreover, define $w_\infty \colon F_\infty \to F \oplus F_\infty$, by $w_\infty(\xi_1, \xi_2, \xi_3, ...) = \xi_1 \oplus (\xi_2, \xi_3, ...)$

As a first result, we state the following lemma but without embarking into it's proof.

Lemma 2.2.2 ([9], Lemma 2.2)**.** *Let $\pi \colon A \to L_B(E)$ and $\sigma \colon A \to L_B(F)$ be two representations.*

Then for any isometry $v\colon F_\infty \to E$, the unitary $u = (1_F \oplus v)w_\infty v^* + 1_E - vv^* \in L_B(E, F \otimes E)$ satisfies

$$\|\sigma(a) \oplus \pi(a) - u\pi(a)u^*\| \leq 6\,\|v\sigma_\infty(a) - \pi(a)v\| + 4\,\|v\sigma_\infty(a^*) - \pi(a^*)v\|$$

Moreover, if $v\sigma_\infty(a) - \pi(a)v \in K_B(F_\infty, E)$, for all $a \in A$, then $\sigma(a) \oplus \pi(a) - u\pi(a)u^* \in K_B(F \oplus E)$.

Lemma 2.2.3 ([9], Lemma 2.3)**.** *Let $\pi\colon A \to L_B(E)$ and $\sigma\colon A \to L_B(F)$ be two representations. Suppose that there exists a sequence of isometries $v_i\colon F_\infty \to E$ satisfying for all $a \in A$,*

$$v_i\sigma_\infty(a) - \pi(a)v_i \in K_B(F_\infty, E), \;\; \|v_i\sigma_\infty(a) - \pi(a)v_i\| \xrightarrow{i\to\infty} 0$$

and $v_i^* v_j = 0$ whenever $i \neq j$. Then $\pi \oplus \sigma \sim_{asymp} \pi$

Proof. For $t \in [0,1]$, set $v_{i+t} = (1-t)^{1/2}v_i + t^{1/2}v_{i+1}\colon F_\infty \to E$. Then, $v_{i+t}^* v_{i+t} = 1$, and

$$v_{i+t}\sigma_\infty(a) - \pi(a)v_{i+t} \in K_B(F_\infty, E), \;\; \|v_{i+t}\sigma_\infty(a) - \pi(a)v_{i+t}\| \xrightarrow{i\to\infty} 0$$

for any $t \in [0,1]$, by hypothesis. Hence, we obtain a continuous path of isometries $v_t\colon F_\infty \to E$, $t \in [0,\infty)$, satisfying the above conditions, and so, by the previous lemma there is a continuous path of unitaries $(u_t)_t \subseteq L_B(E, F \oplus E)$, $t \in [0,\infty)$, such that

$$\|\sigma(a) \oplus \pi(a) - u_t\pi(a)u_t^*\| \leq 6\,\|v_t\sigma_\infty(a) - \pi(a)v_t\| + 4\,\|v_t\sigma_\infty(a^*) - \pi(a^*)v_t\|$$

and

$$\sigma(a) \oplus \pi(a) - u_t\pi(a)u_t^* \in K_B(F \oplus E) \quad \forall a \in A, t \in [0,\infty)$$

Hence,

$$\|\sigma(a) \oplus \pi(a) - u_t\pi(a)u_t^*\| \xrightarrow{t\to\infty} 0, \;\; \sigma(a) \oplus \pi(a) - u_t\pi(a)u_t^* \in K_B(F \oplus E)$$

for all $a \in A$, $t \in [0,\infty)$, which implies that $\sigma \oplus \pi \sim_{asymp} \pi$ ☐

Lemma 2.2.4 ([9], Lemma 2.4)**.** *Let $\pi\colon A \to L_B(E)$ and $\sigma\colon A \to L_B(F)$ be two representations. If $\sigma \oplus \pi \sim_{a.u} \pi$, then $\sigma \oplus \pi_\infty \sim_{asymp} \pi_\infty$.*

Proof. Suppose that $\sigma \oplus \pi \sim_{a.u} \pi$, and find a sequence of unitaries $(u_n)_n \subseteq L_B(F \oplus E, E)$ satisfying the conditions $(i), (ii)$ in Definition 2.2.1. Then, define $u_n^\infty = (u_n, u_n, ...)$ and it is straightforward that $u_n^\infty \in L_B(F_\infty \oplus E_\infty, E_\infty)$, are unitaries for each $n \in \mathbb{N}$, satisfying

$$u_n^\infty \sigma_\infty \oplus \pi_\infty(a) u_n^{\infty *} - \pi_\infty(a) \in K_B(E_\infty), \quad \|u_n^\infty \sigma_\infty \oplus \pi_\infty(a) u_n^{\infty *} - \pi_\infty(a)\| \xrightarrow{n\to\infty} 0$$

for all $a \in A$. Hence, $\sigma_\infty \oplus \pi_\infty \sim_{a.u} \pi_\infty$. Now, using that $(E_\infty)_\infty = E_\infty \oplus E_\infty \oplus \cdots = E_\infty$, we find out from (u_n^∞), isometries $v_n\colon F_\infty \oplus E_\infty \to E_\infty^n (= E_\infty)$, taking values in "disjoint" copies of E_∞(i.e $v_m^* v_n = 0$, when $n \neq m$) and satisfying

$$v_n\sigma_\infty \oplus \pi_\infty(a) - \pi_\infty(a)v_n \in K_B(E_\infty \oplus F_\infty, E_\infty), \quad \|v_n\sigma_\infty \oplus \pi_\infty(a) - \pi_\infty(a)v_n\| \xrightarrow{n\to\infty} 0$$

for all $a \in A$. Now, let $W\colon F_\infty \to F_\infty \oplus E_\infty$ be given by $f \mapsto f \oplus 0_{E_\infty}$. Then, W is a linear isometry, and $[\sigma_\infty \oplus \pi_\infty(a)]W = W\sigma_\infty(a)$, for all $a \in A$. Thus,

$$v_nW\sigma_\infty(a) - \pi_\infty(a)v_nW = (v_n\sigma_\infty \oplus \pi_\infty(a) - \pi_\infty(a)v_n)W \in K_B(F_\infty, E_\infty)$$

and

$$\|v_nW\sigma_\infty(a) - \pi_\infty(a)v_nW\| \leq \|v_n\sigma_\infty \oplus \pi_\infty(a) - \pi_\infty(a)v_n\| \xrightarrow{n\to\infty} 0$$

Thus, if we set $v_n' = v_nW$, then $v_n'\colon F_\infty \to E_\infty$ are isometries satisfying $v_n'^* v_m' = 0$, $n \neq m$, and by the previous considerations, v_n' finally satisfy all the conditions of Lemma 2.2.3 for the representations σ and π_∞. Hence, we conclude that $\sigma \oplus \pi_\infty \sim_{asymp} \pi_\infty$, as desired.

☐

Now, we move forward to introduce some related notions to approximately unitarily equivalent and asymptotically unitarily equivalent representations.

Definition 2.2.5. *Let A, B be C^*-algebras, A separable, B σ-unital, and let $\Phi\colon A \to L_B(E)$ be a representation and $\phi\colon A \to L_B(F)$ a c.p map, where E, F are countably generated Hilbert B-modules. We say that Φ approximately dominates ϕ if there is a bounded sequence $(u_n)_n \subset L_B(F,E)$ such that,*

$$i)\ \ u_n^*\Phi(a)u_n - \phi(a) \in K_B(E)$$
$$ii)\ \|u_n^*\Phi(a)u_n - \phi(a)\| \xrightarrow{n\to\infty} 0$$

Furthermore, we say that Φ strongly approximately dominates ϕ, if there is sequence $(u_n)_n$ as above, satisfying also that

$$iii)\ \|u_n^*Tu_n\| \xrightarrow{n\to\infty} 0, \qquad \forall\ T \in K_B(E)$$

Moreover, if we may find a norm-continuous bounded path $(u_t)_{t\in[0,\infty)}$ satisfying the obvious analogues of (i),(ii)(and (iii)) above, then we say that Φ (strongly) asymptotically dominates ϕ.

We are interested in these notions of dominant representations, mainly due to the next theorem, that we will use critically in the rest of this section.

Theorem 2.2.6 ([12], Theorem 3.4)**.** *Let A, B be C^*-algebras, A separable, unital and B σ-unital. Also, let $\Phi, \Psi\colon A \to L_B(H_B)$ be unital representations and $\Psi_\infty\colon A \to L_B(H_B^\infty)$ the "infinite repeat" of Ψ. The following are equivalent,*

$i)$ *Φ strongly approximately dominates $T^*\Psi(-)T$, for any $T \in K_B(H_B)$*

$ii)$ *Φ strongly approximately dominates Ψ*

$iii)$ *Φ strongly asymptotically dominates Ψ*

$iv)$ *There is a unitary $U \in L_B(H_B \oplus H_B^\infty, H_B)$ such that*
$U^*\Phi(a)U - \Phi \oplus \Psi_\infty(a) \in K_B(H_B \oplus H_B^\infty), \quad \forall\, a \in A$

$v)$ $\Phi \oplus \Psi_\infty \sim_{a.u} \Phi$

$vi)$ $\Phi \oplus \Psi_\infty \sim_{asymp} \Phi$

Proof. We will prove $v) \implies iv) \implies iii) \implies ii) \implies i) \implies v)$, and $iii) \implies vi) \implies v)$.

$v) \implies iv)$: Follows from the definition of $\sim_{a.u}$.

$iv) \implies iii)$: Let $U \in L_B(H_B \oplus H_B^\infty, H_B)$ be the unitary given in iv), and let $V_n\colon H_B \to H_B \oplus H_B^\infty$ be an isometry defined as the inclusion into the n-th coordinate. Now, define $V_t = (n+1-t)^{1/2}V_n + (t-n)^{1/2}V_{n+1}$, $t \in [n, n+1]$, and since $V_{n+1}^*V_n = 0$, we get that $V_t^*V_t = 1$, for all $t \in [n, n+1]$, $n \in \mathbb{N}$. Hence, $(V_t)_{t\in[0,\infty)}$ is a norm-continuous path of isometries and since for any $T \in K_B(H_B \oplus H_B^\infty)$, we have that $V_n^*T \xrightarrow{n\to\infty} 0$, we get that $V_t^*TV_t \xrightarrow{t\to\infty} 0$. Set $W_t = UV_t$, and since by the construction of V_t we have that $V_t^*(\Phi(a) \oplus \Psi_\infty(a))V_t = \Psi(a)$, for all $a \in A$, it follows

$$W_t^*(\Phi(a))W_t - \Psi(a) = V_t^*U^*\Phi(a)UV_t - \Psi(a)$$
$$= V_t^*U^*\Phi(a)UV_t - V_t^*(\Phi(a) \oplus \Psi_\infty(a))V_t$$
$$= V_t^*(U^*\Phi(a)U - \Phi(a) \oplus \Psi_\infty(a))V_t$$

where, $U^*\Phi(a)U - \Phi(a) \oplus \Psi_\infty(a) \in K_B(H_B \oplus H_B^\infty)$ by hypothesis, so we conclude that

$$W_t^*(\Phi(a))W_t - \Psi(a) \in K_B(H_B), \quad \forall\, a \in A$$

and

$$\|W_t^*(\Phi(a))W_t - \Psi(a)\| \xrightarrow{t\to\infty} 0$$

Hence, Φ strongly asymptotically dominates Ψ.

$iii) \implies ii)$: If there is a norm-continuous bounded path satisfying conditions $i), ii), iii)$ in the definition above, then trivially we find a bounded sequence that satisfies the same conditions. Hence, Φ strongly approximately dominates Ψ.

$ii) \implies i)$: Let $(u_n)_n \subset L_B(H_B)$ be a bounded sequence satisfying $i), ii), iii)$ in the definition above, and let $T \in K_B(H_B)$. Then, it is straightforward that $W_n = u_n T$, $n \in \mathbb{N}$, define a bounded sequence in $L_B(H_B)$, satisfying the same conditions, but now for Φ and $T^*\Psi(-)T$. Since T was arbitrary the conclusion follows.

$i) \implies v)$: This follows from Theorem 2.13 in [9].

$iii) \implies vi)$: Since $iii) \iff iv)$, we can find a unitary U by $iv)$, and follow the same procedure as in the direction $iv) \implies iii)$. But, using that $\Psi_\infty = (\Psi_\infty)_\infty$, this time we get $(W_t)_{t\in[0,\infty)}$ norm-continuous bounded path of isometries satisfying

$$W_t^*(\Phi(a))W_t - \Psi_\infty(a) \in K_B(H_B^\infty)$$

and

$$\|W_t^*(\Phi(a))W_t - \Psi_\infty(a)\| \xrightarrow{t\to\infty} 0$$

thus Φ strongly asymptotically dominates Ψ_∞. Now, let $V_t \in L_B(H_B^\infty, H_B)$ be a bounded continuous family of elements such that $V_t^*\Phi(a)V_t - \Psi_\infty(a)$ is compact and tends to zero for every $a \in A$. By a trick of Arveson (see proof of Corollary 1 in [1]), it follows that $V_t\Psi_\infty(a) - \Phi(a)V_t$ is compact and tends to zero for all $a \in A$. So, if we identify Ψ_∞ and $(\Psi_\infty)_\infty$, it follows from Lemma 2.16 in [9] that $\Phi \oplus \Psi_\infty \sim_{asymp} \Phi$.

$vi) \implies v)$: Similarly to $iii) \implies ii)$.

□

Another important aspect of the theory under examination is the concept of absorbing representations. In the following, we slightly modify our setting, as we start working with A separable C^*-algebra, B σ-unital C^*-algebra and we regard $B \otimes \mathbb{K}$ as a Hilbert $B \otimes \mathbb{K}$-module. By Lemma 2.1.26, $L_B(H_B) \cong M(B \otimes \mathbb{K})$ and $K_B(H_B) \cong B \otimes \mathbb{K}$ and in the following we reserve the term representation for any *-homomorphism $\phi\colon A \to M(B \otimes K)$. Moreover, we call such a representation, weakly nuclear, if the c.p map $A \to B \otimes \mathbb{K}$, given by $a \mapsto b^*\phi(a)b$ is nuclear, for all $b \in B \otimes \mathbb{K}$.

Definition 2.2.7. *A representation $\phi\colon A \to M(B \otimes \mathbb{K})$ is called absorbing if $\phi \oplus \psi \sim_{a.u} \phi$, for any representation $\psi\colon A \to M(B \otimes \mathbb{K})$. If moreover, A is unital, then a representation $\phi\colon A \to M(B \otimes \mathbb{K})$ is called unitally absorbing if $\phi \oplus \psi \sim_{a.u} \phi$, for any unital representation $\psi\colon A \to M(B \otimes \mathbb{K})$.*

Note 2.2.8. Suppose that $\phi\colon A \to M(B \otimes \mathbb{K})$ unital representation, $\psi\colon A \to M(B \otimes \mathbb{K})$ non unital representation, and assume that $\phi \oplus \psi \sim_{a.u} \phi$. Then, there exist $(u_n)_n \subset M(B \otimes \mathbb{K})$ such that $\|\phi \oplus \psi(a) - u_n\phi(a)u_n^*\| \xrightarrow{n\to\infty} 0$, for all $a \in A$. Hence,
$0 \neq \|\psi(1_A) - 1_A\| = \|\phi \oplus \psi(1_A) - u_n\phi(1_A)u_n^*\| \xrightarrow{n\to\infty} 0$, which is a contradiction, concluding that a unital representation cannot absorb a non-unital representation.

Definition 2.2.9. *A representation $\phi\colon A \to M(B \otimes \mathbb{K})$ is called (unitally) nuclearly absorbing if $\phi \oplus \psi \sim_{a.u} \phi$, for any (unital) weakly nuclear representation $\psi\colon A \to M(B \otimes \mathbb{K})$.*

An alternative definition of the "infinite repeat" of a representation, will be also rather useful in the process of this section.

Definition 2.2.10. *Let $\phi\colon A \to M(B \otimes \mathbb{K})$ be a representation and let a sequence of isometries*

$(s_n)_n \subseteq M(B \otimes \mathbb{K})$ *satisfying* $\sum_n s_n s_n^* = 1$, *and* $s_n^* s_m = 0$, *when* $n \neq m$. *Then, we define the "infinite repeat" of* ϕ *by*

$$\phi_\infty(a) = \sum_n s_n \phi(a) s_n^*, \quad a \in A$$

where the convergence is in the strict topology.

Note 2.2.11. It is straightforward to check that the "infinite repeat" ϕ_∞ of a representation $\phi\colon A \to M(B \otimes \mathbb{K})$ is linear and *-preserving. Using that $s_n^* s_m = 0$, whenever $n \neq m$, we also see that

$$\phi_\infty(x)\phi_\infty(y) = (\sum_n s_n \phi(x) s_n^*)(\sum_m s_m \phi(y) s_m^*) = \sum_n s_n \phi(x) s_n^* s_n \phi(y) s_n^* = \phi_\infty(xy)$$

hence multiplicative. Moreover, that the convergence is in strict topology, implies that for any $b \in B \otimes \mathbb{K}$ and $a \in A$,

$$\sup_n \left\| \sum_{k=1}^n s_k \phi(a) s_k^*(b) - \phi_\infty(a)(b) \right\| < \infty$$

and now Uniform Boundedness Principle shows that

$$\sup_n \left\| \sum_{k=1}^n s_k \phi(a) s_k^* - \phi_\infty(a) \right\| < \infty$$

which clearly implies that ϕ_∞ is continuous. Furthermore, if $(s_n)_n, (\lambda_n)_n \subset M(B \otimes \mathbb{K})$ are two sequences of isometries defining ϕ_∞, then $u = \sum_n s_n \lambda_n^*$ is a unitary in $M(B \otimes \mathbb{K})$ satisfying

$$u(\sum_n \lambda_n \phi(a) \lambda_n^*) u^* = \sum_n s_n \phi(a) s_n^*$$

This shows that ϕ_∞ (up to unitary equivalence) is independent from the selection of isometries. As a consequence, note that if we select our isometries to be the the n-th factor inclusions $s_n\colon B \otimes \mathbb{K} \to (B \otimes \mathbb{K})^\infty$, then this definition coincides with the definition of the "infinite repeat" at the beginning of this section. Another consequence is that $\phi_\infty \oplus \phi$ is unitarily equivalent to ψ_∞. To see this, take a suitable sequence of isometries $(s_n)_n \subseteq M(B \otimes \mathbb{K})$ such that $\phi_\infty(-) = \sum_{n=1}^\infty s_n \phi(-) s_n^*$ and let $v_1, v_2 \in M(B \otimes \mathbb{K})$ isometries such that $\phi_\infty \oplus \phi = v_1 \phi_\infty(-) v_1^* + v_2 \phi(-) v_2^*$. Then, if we set $w_1 = v_2$, and $w_n = v_1 s_{n-1}$, for $n \geq 2$, we easily verify that $(w_n)_n \subseteq M(B \otimes \mathbb{K})$ is a sequence of isometries, satisfying the conditions in the definition of ϕ_∞, while

$$\sum_{n=1}^\infty w_n \phi(-) w_n^* = v_1 (\sum_{n=1}^\infty s_n \phi(-) s_n^*) v_1^* + v_2 \phi(-) v_2^* = \phi_\infty \oplus \phi$$

Hence, $\phi_\infty \oplus \phi$ is indeed unitarily equivalent to ϕ_∞

Lastly, we claim that the "infinite repeat" is weakly nuclear whenever ϕ is weakly nuclear. To this end, assume that ϕ is weakly nuclear, then for any $b \in B \otimes \mathbb{K}$, there are c.c.p maps $\rho_n\colon A \to M_{k_n}(\mathbb{C})$, $\rho_n'\colon M_{k_n}(\mathbb{C}) \to B \otimes \mathbb{K}$ satisfying

$$\|b\phi(a)b^* - \rho_n' \circ \rho_n(a)\| \xrightarrow{n \to \infty} 0$$

Now, for each $n \in \mathbb{N}$ we find sufficiently large $K_n \in \mathbb{N}$ and we consider the c.c.p maps $P_n\colon A \to M_{K_n}(\mathbb{C})$, $P_n'\colon M_{K_n}(\mathbb{C}) \to B \otimes \mathbb{K}$ defined by

$$P_n(a) = \rho_n(a) \oplus \rho_n(a) \oplus \cdots, \quad P_n'(a) = \rho_n'(a) \oplus \rho_n'(a) \oplus \cdots$$

Then,

$$\|b\psi_\infty(a)b^* - P_n' \circ P_n(a)\| = \sup\{\|b\phi(a)b^* - \rho_n' \circ \rho_n(a)\|\} \xrightarrow{n \to \infty} 0$$

and since b was arbitrary, it follows that ϕ_∞ is a weakly nuclear representation.

Having these new notions in our possession, we proceed to prove the following interesting result, namely that nuclear absorbing representations satisfy an even stronger asymptotic absorption property.

Proposition 2.2.12 ([30], Proposition 2.4)**.** *Suppose that* $\phi\colon A \to M(B\otimes\mathbb{K})$ *is a nuclearly absorbing representation. If* $\psi\colon A \to M(B\otimes\mathbb{K})$ *is a weakly nuclear representation, then* $\phi\oplus\psi \sim_{asymp} \phi$*.*

Proof. Since ψ weakly nuclear, the previous note says that ψ_∞ is also weakly nuclear, hence $\phi\oplus\psi_\infty \sim_{a.u} \phi$. Note that in view of Lemma 2.1.26, we can employ Theorem 2.2.6 ((v) $\iff$ (vi)), to obtain that $\phi\oplus\psi_\infty \sim_{asymp} \phi$.
Now, since ψ_∞ is unitary equivalent to $\psi\oplus\psi_\infty$, we obtain

$$\phi\oplus\psi \sim_{asymp} \phi\oplus\psi_\infty\oplus\psi \sim_{asymp} \phi\oplus\psi_\infty \sim_{asymp} \phi$$

as desired. □

At this point, we are ready to introduce Cuntz picture of KK-theory. Keeping exactly the same setting as in the previous pages, we start by defining the notion of a Cuntz pair.

Definition 2.2.13. *A pair* (ϕ,ψ) *of representations* $\phi,\psi\colon A\to M(B\otimes\mathbb{K})$*, is called Cuntz pair if* ϕ,ψ *are weakly nuclear and* $\phi(a)-\psi(a)\in B\otimes\mathbb{K}$*, for all* $a\in A$*. The set of all Cuntz pairs is denoted by* $\mathscr{E}_{nuc}(A,B)$

The central relation in constructing the KK_{nuc}-group of $\mathscr{E}_{nuc}(A,B)$ is the homotopy equivalence between Cuntz pairs, which is defined right below.

Definition 2.2.14. *Two Cuntz pairs* $(\phi_0,\psi_0),(\phi_1,\psi_1)$ *are called homotopic if there is a path* $(\lambda_t,\lambda_t')\in\mathscr{E}_{nuc}(A,B)$ *such that*

i) the maps $t\mapsto\lambda_t(a)$, $t\mapsto\lambda_t'(a)$, *from* $[0,1]$ *to* $M(B\otimes\mathbb{K})$ *are strictly continuous, for all* $a\in A$

ii) the map $t\mapsto\lambda_t(a)-\lambda_t'(a)$ *from* $[0,1]$ *to* $B\otimes\mathbb{K}$ *is norm continuous, for all* $a\in A$.

iii) $(\lambda_0,\lambda_0')=(\phi_0,\psi_0)$, $(\lambda_1,\lambda_1')=(\phi_1,\psi_1)$

When these conditions are satisfied, we write $(\phi_0,\psi_0)\sim(\phi_1,\psi_1)$*. Furthermore, we denote by* $KK_{nuc}(A,B)$ *the set of homotopy classes of Cuntz pairs in* $\mathscr{E}(A,B)$*, and by* $[\phi,\psi]$ *the homotopy class of* (ϕ,ψ) *in* $KK_{nuc}(A,B)$

Lemma 2.2.15. *Let* $\phi\colon A\to M(B\otimes\mathbb{K})$ *be a representation, then* $(\phi,\phi)\sim(0,0)$*.*

Proof. Let $(s_t)_t$, $t\in(0,1]$ be the path of isometries in $M(B\otimes\mathbb{K})$ given in Lemma 2.1.28. Then, set $\lambda_t(-)=s_t\phi(-)s_t^*=\lambda_t'(-)$, $t\in(0,1]$ and $\lambda_0=0=\lambda_0'$. To see that $t\mapsto\lambda_t(a)$ strictly continuous at 0, for any $a\in A$, note that as for any $h\in B\otimes\mathbb{K}$, $\|s_ts_t^*(h)\|\xrightarrow{t\to0}0$, then

$$\|s_t\phi(a)s_t^*(h)\|=\|s_t\phi(a)s_t^*s_ts_t^*h\|\le\|s_t\phi(a)s_t^*\|\,\|s_ts_t^*h\|\xrightarrow{t\to0}0$$

Since, the rest of the conditions of Definition 2.2.14 are easily verified, we get that (λ_t,λ_t') is an homotopty path between $(\lambda_0,\lambda_0')=(0,0)$ and $(\lambda_1,\lambda_1')=(\phi,\phi)$. Hence, $(\phi,\phi)\sim(0,0)$ □

In order to turn $KK_{nuc}(A,B)$ into a group, we need a group operation. This operation is provided by the so called Cuntz sum.

Definition 2.2.16. *Let* $s_1,s_2\in M(B\otimes\mathbb{K})$ *be isometries such that* $s_1s_1^*+s_2s_2^*=1$ *and* $s_i^*s_j=0$*, when* $i\neq j$*. If* $\phi,\psi\colon A\to M(B\otimes\mathbb{K})$ *representations, then the representation* $\phi\oplus_{s_1,s_2}\psi=s_1\phi(-)s_1^*+s_2\psi(-)s_2^*$ *is called the Cuntz sum of* ϕ *and* ψ*, with respect to* s_1 *and* s_2*.*

Note 2.2.17. If $s_1, s_2, \lambda_1, \lambda_2 \in M(B \otimes \mathbb{C})$ are two pairs of isometries as in the definition above, then if we set $u = s_1\lambda_1^* + s_2\lambda_2^*$, $u \in M(B \otimes \mathbb{K})$ unitary satisfying that $u(\phi \oplus_{\lambda_1,\lambda_2} \psi)u^* = \phi \oplus_{s_1,s_2} \psi$ for any two representations $\phi, \psi \colon A \to M(B \otimes \mathbb{K})$. Hence, (up to unitary equivalence) the Cuntz sum is independent from the choice of isometries.

Another way of expressing the Cuntz sum will also be useful, and it will create an immediate connection with the first section. First we need the following.

Definition 2.2.18. *Let B be a C^*-algebra. Then, a *-isomorphism $\Theta\colon M_n(B) \to B$ is called inner if there are isometries $s_1, s_2, .., s_n$ in $M(B)$ such that $s_i^* s_j = 0$, $i \neq j$, $\sum_i s_i s_i^* = 1$ and*

$$\Theta((b_{ij})_{ij}) = \sum_{i,j} s_i b_{ij} s_j^*$$

Note 2.2.19. If $\Theta\colon M_n(B) \to B$ is an inner *-isomorphism given by some isometries $w_1, w_2, ..., w_n$ then the same formula of the definition above gives a *-isomorphism from $M_n(M(B))$ onto $M(B)$. Also note that, Lemma 2.1.27 always provides isometries in $M(B \otimes \mathbb{K})$ satisfying the conditions of Definition 2.2.18, and it is straightforward to check that the map given by these isometries as above, is a *-isomorphism from $M_n(M(B \otimes \mathbb{K}))$ onto $M(B \otimes \mathbb{K})$, $n \in \mathbb{N}$.

Now, if $\phi, \psi \colon A \to M(B \otimes \mathbb{K})$ two representations, find $s_1, s_2 \in M(B \otimes \mathbb{K})$ by the previous note and set Θ_{s_1,s_2} the inner *-isomorphism expressed by these isometries as in definiton above. Then, $\phi \oplus_{s_1,s_2} \psi = \Theta_{s_1,s_2} \circ \left(\begin{smallmatrix} \phi(-) & 0 \\ 0 & \psi(-) \end{smallmatrix}\right)$ and therefore we see that both ways of expressing Cuntz sum, coincide.

Now, we are ready to define the the group operation on $KK_{nuc}(A, B)$. Let $[\phi_1, \psi_1]$, $[\phi_2, \psi_2] \in KK_{nuc}(A, B)$ and define their addition as $[\phi_1, \psi_1] + [\phi_2, \psi_2] = [\phi_1 \oplus_{s_1,s_2} \phi_2, \psi_1 \oplus_{s_1,s_2} \psi_2] = [\Theta_{s_1,s_2} \circ \left(\begin{smallmatrix} \phi_1(-) & 0 \\ 0 & \phi_2(-) \end{smallmatrix}\right), \Theta_{s_1,s_2} \circ \left(\begin{smallmatrix} \psi_1(-) & 0 \\ 0 & \psi_2(-) \end{smallmatrix}\right)]$. It is a fact that the unitary group of $M(B \otimes \mathbb{K})$ is path connected in the operator norm topology (See [7]), hence the class $[\phi_1 \oplus_{s_1,s_2} \phi_2, \psi_1 \oplus_{s_1,s_2} \psi_2]$ in $KK_{nuc}(A, B)$ is independent of the choice of isometries, thus by abusing notation this element will be written as $[\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2]$. Also, for any $\phi\colon A \to B$ nuclear *-homomorphism and $p \in \mathbb{K}$ rank one projection, we define the representation $\phi_p \colon A \to M(B \otimes \mathbb{K})$ by $\phi_p(a) = \phi(a) \otimes p$, for all $a \in A$. Then, it is not hard to see that $(\phi_p, 0)$ is a Cuntz pair, and we denote it's class in $KK_{nuc}(A, B)$ by $[\phi]$. It is a fact that $[\phi]$ is independent from the choice of the rank one projection.

Now, we partly establish that $KK_{nuc}(A, B)$ is an abelian group.

Lemma 2.2.20. *$KK_{nuc}(A, B)$ is an abelian group, where $[(0, 0)]$ represents the zero element and $[\phi, \psi] = -[\psi, \phi]$.*

Proof. We see first that $[(0, 0)]$ is the zero element in $KK_{nuc}(A, B)$. Let (ϕ, ψ) be a Cuntz pair, then for any isometry $v \in M(B \otimes \mathbb{K})$ it suffices to show that $(\mathrm{Ad}_s \circ \phi, \mathrm{Ad}_s \circ \psi) \sim (\phi, \psi)$. But from Lemma 2.1.29 there is a strictly continuous path $t \mapsto v_t$ in $M(B \otimes \mathbb{K})$, such that $v_0 = 1$ and $v_1 = v$. Set $\lambda_t(-) = v_t \phi(-) v_t^*$ and $\lambda'_t(-) = v_t \psi(-) v_t^*$. Then, $t \mapsto \lambda_t(a)$, $t \mapsto \lambda'_t(a)$ are strictly continuous maps, $t \mapsto \lambda_t(a) - \lambda'_t(a)$ is in $B \otimes \mathbb{K}$ and is norm continuous for all $a \in A$. Finally, $(\lambda_0, \lambda'_0) = (\phi, \psi)$, $(\lambda_1, \lambda'_1) = (\mathrm{Ad}_s \circ \phi, \mathrm{Ad}_s \circ \psi)$, concluding that $[\phi, \psi] = [\phi, \psi] + [0, 0]$.

Now, we argue that the inverse of $[\phi, \psi]$ is $[\psi, \phi]$. First, consider the rotational matrix

$$R_t = \begin{pmatrix} cos\frac{\pi}{2}t & sin\frac{\pi}{2}t \\ -sin\frac{\pi}{2}t & cos\frac{\pi}{2}t \end{pmatrix} \in M_2(M(B)), \quad t \in [0, 1]$$

Then, set $\lambda_t = \Theta \circ \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}$, $\lambda'_t = \Theta \circ \mathrm{Ad}_{R_t} \circ \begin{pmatrix} \psi & 0 \\ 0 & \phi \end{pmatrix}$, and observe that all conditions of Definition

2.2.14 are satisfied and moreover $(\lambda_0, \lambda'_0) = (\phi \oplus \psi, \psi \oplus \phi)$, $(\lambda_1, \lambda'_0) = (\phi \oplus \psi, \phi \oplus \psi)$. Thus by Lemma 2.2.15 we obtain that $[\phi \oplus \psi, \psi \oplus \phi] = 0$, hence $[\phi, \psi] = -[\psi, \phi]$ as desired.

Commutativity of $KK_{nuc}(A, B)$ is proven in same manner and for associativity look up to Lemma 1.3.12 in [18]. ☐

We end this first discussion about KK_{nuc} with some functorial properties followed by a rather interesting result. Let $\rho\colon B \to C$ be a *-homomorphism between σ-unital C^*-algebras, then there is an induced group homomorphism

$$\rho_*\colon KK_{nuc}(A, B) \to KK_{nuc}(A, C)$$

In this way, $KK_{nuc}(A, -)$ becomes a covariant functor from the category of σ-unital C^*-algebras to the category of groups; see Chapter 4 in [18] for details. The proof of the following proposition is omitted.

Proposition 2.2.21 ([30], Proposition 2.1)**.** *Let A, E be separable C^*-algebras, I a closed two sided ideal in E, such that $I \otimes \mathbb{K} \cong I$ and $\phi, \psi\colon A \to E$ two nuclear $*$-homomorphisms. Also let $\lambda\colon E \to M(I)$ be the canonical $*$-homomorhphism and note that $(\lambda\phi, \lambda\psi) \in \mathscr{E}_{nuc}(A, I)$.*

If $j\colon I \to E$ is the inclusion, then $j_*[\lambda\phi, \lambda\psi] = [\phi] - [\psi]$ in $KK_{nuc}(A, E)$

2.3 Destabilizing KK-theory

The main concern of this section is to prove one theorem and two propositions that will be critical towards the end of this project. We keep in mind the theory already presented and we proceed to define a new equivalence relation for representations. Also, we briefly introduce a picture of KK-theory, which is equivalent to Cuntz picture, and offers some extra information, expanding our capacity of tackling KK-theoretical problems.

Again, throughout this section A is a separable C^*-algebra and B is a σ-unital C^*-algebra, except otherwise is mentioned.

Definition 2.3.1. *Two representations $\phi, \psi\colon A \to M(B \otimes \mathbb{K})$ are called properly asymptotically unitarily equivalent, written $\phi \approxeq \psi$, if there exists a norm-continuous path of unitaries $(u_t)_{t \geq 0}$ in $B \otimes \mathbb{K} + \mathbb{C}1_{M(B \otimes \mathbb{K})}$ satisfying that*

$$i)\ \|\phi(a) - u_t\psi(a)u_t^*\| \xrightarrow{t \to \infty} 0$$
$$ii)\ \phi(a) - u_t\psi(a)u_t^* \in B \otimes \mathbb{K}, \text{ for all } t \geq 0\ ,\ a \in A$$

Lemma 2.3.2 ([10], Lemma 3.2)**.** *Let $\phi, \psi\colon A \to M(B \otimes \mathbb{K})$ be representations such that $(\phi, \psi) \in \mathscr{E}_{nuc}(A, B)$. If $(u_t)_{t \geq 0}$ is a norm continuous path of unitaries such that $u_t\phi(a)u_t^* - \psi(a) \in B \otimes \mathbb{K}$, and $\|u_t\phi(a)u_t^* - \psi(a)\| \xrightarrow{t \to \infty} 0$, then $[\phi, \psi] = [\phi, u_1\phi u_1^*]$.*

Proof. Set

$$\lambda_t(a) = \phi(a), \quad \forall a \in A$$

and

$$\lambda'_t(a) = u_t\phi(a)u_t^*,\ \ t > 0,\ \ \forall a \in A$$

$$\lambda'_0(a) = \psi(a),\ \ \forall a \in A$$

Then $(\lambda_t, \lambda'_t) \in \mathscr{E}_{nuc}(A, B)$ for all $t \geq 0$, $t \mapsto \lambda_t(a)$, $t \mapsto \lambda'_t(a)$ are strictly continuous maps for all $a \in A$ and $t \mapsto \lambda_t(a) - \lambda'_t(a)$ is norm-continuous map in $B \otimes \mathbb{K}$. Since, $(\lambda_0, \lambda'_0) = (\phi, \psi)$,

$(\lambda_1, \lambda_1') = (\phi, u_1\phi u_1^*)$, we obtain that $[\phi, \psi] = [\phi, u_1\phi u_1^*]$. ☐

Now, as we promised, we introduce an another picture of KK-theory, namely the Fredholm picture, using general Hilbert modules, that will enable us to prove the following lemmas. The source of the following lines is [10].

Let A be a separable C^*-algebra, B a σ-unital C^*-algebra and E_0, E_1 some countably generated Hilbert B-modules. The Fredholm picture of KK-theory is described in terms of triples (ϕ, ψ, v), where $\phi\colon A \to L_B(E_0)$, $\psi\colon A \to L_B(E_1)$ are *-homomorphisms, $v \in L_B(E_0, E_1)$, satisfying for all $a \in A$ the following

$$v\phi(a) - \psi(a)v \in K_B(E_0, E_1)$$

and

$$\phi(a)(v^*v - 1) \in K_B(E_0),\ \psi(a)(vv^* - 1) \in K_B(E_1)$$

Moreover, a cycle (ϕ, ψ, v) is called degenerate if

$$v\phi(a) - \psi(a)v = 0,\ \phi(a)(v^*v - 1) = 0,\ \psi(a)(vv^* - 1) = 0$$

for all $a \in A$.

An operatorial homotopy of cycles is a homotopy (ϕ, ψ, v_t), where $(v_t)_{t\geq 0}$ is a norm continuous path. Now, if we denote by $\mathbb{E}(A, B)$ the set of these cycles, then it is fact shown by Kasparov that $KK(A, B)$ is isomorphic to the quotient of $\mathbb{E}(A, B)$ with respect to the equivalence relation generated by unitary equivalence, operatorial homotopy and addition of degenerate cycles. In order to become more acquainted with Fredholm picture, we state the fact that the map $[\phi, \psi] \to [\phi, \psi, 1_{M(B\otimes\mathbb{K})}]$ defines an isomorphism between Cuntz and Fredholm picture, where here $\phi, \psi\colon A \to M(B \otimes \mathbb{K})$ are some representations.

Lemma 2.3.3 ([10], Lemma 3.3)**.** *If $\phi, \psi\colon A \to M(B \otimes \mathbb{K})$ are representations such that $\phi \approxeq \psi$, then $\phi(a) - \psi(a) \in B \otimes \mathbb{K}$, for all $a \in A$, and moreover $[\phi, \psi, 1] = 0$.*

Proof. Let $(u_t)_{t\geq 0} \subseteq B \otimes \mathbb{K}\mathbb{C}1_{M(B\otimes\mathbb{K})}$ be the norm continuous path of unitaries witnessing the relation $\phi \approxeq \psi$. If $u_t = w_t + z_t 1_{M(B\otimes\mathbb{K})}$, then as

$$u_t\phi(a)u_t^* - \psi(a) \in B \otimes K$$

and $z_t z_t^* = 1$, we immediately see that $\phi(a) - \psi(a) \in B \otimes K$, for all $a \in A$. Moreover, using the isomorphism between the two pictures of KK-theory and Lemma 2.3.2, we get that

$$[\phi, \psi, 1_{M(B\otimes\mathbb{K})}] = [\phi, u_1\phi u_1^*, 1_{M(B\otimes\mathbb{K})}]$$

Now, it is straightforward to see that (ϕ, ϕ, u_1^*) satisfies all the conditions for being a cycle and moreover as u_1 is unitary the cycle $(\phi, u_1\phi u_1^*, 1_{M(B\otimes\mathbb{K})})$ is unitarily equivalent to the cycle (ϕ, ϕ, u_1^*). Also, as $u_1 = w_1 + z1_{M(B\otimes\mathbb{K})}$, we define a norm continous path by

$$\lambda_t\colon t \mapsto tw_1^* + z^*1_{M(B\otimes\mathbb{K})}, \quad t \in [0, 1]$$

and therefore (ϕ, ϕ, λ_t) is an operatorial homotopy between (ϕ, ϕ, u_1^*) and $(\phi, \phi, z^*1_{M(B\otimes\mathbb{K})})$. As, the latter cycle is unitarily equivalent to $(\phi, z\phi z^*, 1_{M(B\otimes\mathbb{K})})$ and $zz^* = 1$, we get that

$$[\phi, \phi, u_1^*] = [\phi, z\phi z^*, 1_{M(B\otimes\mathbb{K})}] = [\phi, \phi, 1_{M(B\otimes\mathbb{K})}]$$

and since we already showed that $[\phi, \psi, 1_{M(B\otimes\mathbb{K})}] = [\phi, u_1\phi u_1^*, 1_{M(B\otimes\mathbb{K})}]$, it follows that

$$[\phi, \psi, 1_{M(B\otimes\mathbb{K})}] = [\phi, \phi, 1_{M(B\otimes\mathbb{K})}]$$

By employing again the isomorphism between the two KK-pictures and Lemma 2.2.15, we obtain that $[\phi, \phi, 1_{M(B\otimes\mathbb{K})}] = 0$, thus $[\phi, \psi, 1_{M(B\otimes\mathbb{K})}] = 0$, as desired. ☐

Lemma 2.3.4 ([10], Lemma 3.4)**.** *Let $\phi, \psi, \gamma, \theta \colon A \to M(B \otimes \mathbb{K})$ be representations such that $\phi \oplus \gamma \approxeq \psi \oplus \gamma$ and $\gamma \sim_{assymp} \theta$. Then, $\phi \oplus \theta \approxeq \psi \oplus \theta$*

Proof. Let $(u_t)_{t \geq 0} \subseteq M_2(B \otimes \mathbb{K}) + \mathbb{C}1_{M_2(M(B \otimes \mathbb{K}))}$ be a norm continuous path of unitaries such that

$$u_t(\phi \oplus \gamma(a))u_t^* - \psi \oplus \gamma(a) \in M_2(B \otimes \mathbb{K}) \quad (1)$$
$$\|u_t(\phi \oplus \gamma(a))u_t^* - \psi \oplus \gamma(a)\| \xrightarrow{t \to \infty} 0$$

for all $a \in A$. Moreover, let $(w_t)_{t \geq 0} \subseteq M(B \otimes \mathbb{K})$ be a norm cotinuous path of unitaries satisfying that

$$w_t \gamma(a) w_t^* - \theta(a) \in B \otimes \mathbb{K} \quad (2)$$
$$\|w_t \gamma(a) w_t^* - \theta(a)\| \xrightarrow{t \to \infty} 0$$

for all $a \in A$. Now, if we set $v_t = (1 \oplus w_t)u_t(1 \oplus w_t^*)$, it is easily verified that $(v_t)_{t \geq 0}$ is a norm continuous path of unitaries in $M_2(B \otimes \mathbb{K}) + \mathbb{C}1_{M_2(M(B \otimes \mathbb{K}))}$ which satisfies the following

$$\|v_t(\phi \oplus \theta(a))v_t^* - \psi \oplus \theta(a)\| \leq \|u_t(\phi \oplus w_t^* \theta(a) w_t)u_t^* - \psi(a) \oplus w_t^* \theta(a) w_t\|$$
$$\leq \|u_t(\phi(a) \oplus \gamma(a))u_t^* - \psi \oplus \gamma(a)\| + 2\,\|w_t^* \theta(a) w_t - \gamma(a)\| \xrightarrow{t \to \infty} 0$$

Also, (1) and (2) imply that in $M(M_2(B \otimes \mathbb{K}))/M_2(B \otimes \mathbb{K})$ the following holds

$$\|u_t(\phi(a) \oplus \gamma(a))u_t^* - \psi \oplus \gamma(a)\| = 0 = \|w_t^* \theta(a) w_t - \gamma(a)\|$$

So, by the inequality above, we get that

$$\|v_t(\phi \oplus \theta(a))v_t^* - \psi \oplus \theta(a)\| = 0$$

in $M(M_2(B \otimes \mathbb{K}))/M_2(B \otimes \mathbb{K})$, which shows that

$$v_t(\phi \oplus \theta(a))v_t^* - \psi \oplus \theta(a) \in M_2(B \otimes \mathbb{K})$$

for all $a \in A$. Thus, $\phi \oplus \theta \approxeq \psi \oplus \theta$, as required. □

We now turn to the main results of this section. Taking motivation by Lemma 2.3.3, we firstly aim to show that the relevance of proper asymptotic unitary equivalence in KK-theory is even stronger than what we have already seen. All the following results are derived from [30].

Theorem 2.3.5. *Let $(\phi, \psi) \in \mathscr{E}_{nuc}(A, B)$. Then the following are equivalent*

i) $[\phi, \psi] = 0$

ii) there exists weakly nuclear representation $\theta \colon A \to M(B \otimes \mathbb{K})$ such that $\phi \oplus \theta \approxeq \psi \oplus \theta$

iii) for any weakly nuclear, nuclearly absorbing representation $\theta \colon A \to M(B \otimes \mathbb{K})$, $\phi \oplus \theta \approxeq \psi \oplus \theta$

Proof.

i) $\implies$ *ii)* ([10], Theorem 3.6)

ii) $\implies$ *i)* Since $\phi \oplus \theta \approxeq \psi \oplus \theta$, by *Lemma* 2.3.3 we get that $[\phi \oplus \theta, \psi \oplus \theta] = 0$, hence $[\phi, \psi] + [\theta, \theta] = 0$, and therefore *Lemma* 2.2.15 implies that $[\phi, \psi] = 0$.

Now, Theorem 3.8 in [10] asserts that *ii)* is equivalent to

iii′) for any weakly nuclear, nuclearly absorbing representation $\theta \colon A \to M(B \otimes \mathbb{K})$, $\phi \oplus \theta_\infty \approxeq \psi \oplus \theta_\infty$

hence it suffices to show that *iii)* $\iff$ *iii′)*. To this end, let $\theta \colon A \to M(B \otimes \mathbb{K})$ be a weakly nuclear, nuclearly absorbing representation, then by Note 2.2.11 θ_∞ is weakly nuclear, and therefore $\theta \oplus \theta_\infty \sim_{asymp} \theta$ by Proposition 2.2.12. So, $\theta_\infty \sim_{asymp} \theta$ and by Lemma 2.3.4 the desired result folllows. $\square$

Proposition 2.3.6. *Let A and B be C^*-algebras such that A is separable and B σ-unital. If $x \in KK_{nuc}(A,B)$ and $\psi\colon A \to M(B\otimes\mathbb{K})$ is a weakly nuclear, nuclearly absorbing representation, then there exists weakly nuclear, nuclearly absorbing representation $\phi\colon A \to M(B\otimes\mathbb{K})$ such that $x = [\phi,\psi]$*

Proof. Let $(\theta,\rho) \in \mathscr{E}_{nuc}(A,B)$ such that $x = [\theta,\rho]$. Then by Proposition 2.2.12 there is a norm continuous path of unitaries $(u_t)_{t\geq 0} \subset M(B\otimes\mathbb{K})$ witnessing $\rho\oplus\psi \sim_{asymp} \psi$. Now, since

$$u_t\rho\oplus\psi u_t^* - \psi(a) \in B\otimes\mathbb{K} \quad \text{for all } \ t\geq 0$$

it follows that

$$u_t\rho\oplus\psi u_t^* - u_0\rho\oplus\psi u_0^* \in B\otimes\mathbb{K} \quad \text{for all } \ t\geq 0$$

Moreover, since (θ,ρ) is a Cuntz pair we get that

$$\theta\oplus\psi(a) - \rho\oplus\psi(a) \in B\otimes\mathbb{K}, \quad \text{for all } \ a\in A$$

hence $\mathrm{Ad}_{u_0}(\theta\oplus\psi(a) - \rho\oplus\psi(a)) \in B\otimes\mathbb{K}$, for all $a\in A$. So, if we set $\lambda_t = \mathrm{Ad}_{u_0}(\theta\oplus\psi)$, $\lambda'_t = \mathrm{Ad}_{u_t}(\rho\oplus\psi)$, by the observations above we have that $\lambda_t(a) - \lambda'_t(a) \in B\otimes\mathbb{K}$, for all $a\in A$, $t\geq 0$, while both λ_t and λ'_t are weakly nuclear representations, for any $t\geq 0$, since θ,ρ,ψ are weakly nuclear representations. Thus, $(\lambda_t,\lambda'_t) \in \mathscr{E}_{nuc}(A,B)$ and since all the conditions of Definition 2.2.14 are satisfied, we obtain an homotopy(defined in $[0,\infty)$, instead of $[0,1]$)between $(\mathrm{Ad}_{u_0}(\theta\oplus\psi), \mathrm{Ad}_{u_0}(\rho\oplus\psi))$ and $(\mathrm{Ad}_{u_0}(\theta\oplus\psi),\psi)$. Hence,

$$[\mathrm{Ad}_{u_0}(\theta\oplus\psi), \mathrm{Ad}_{u_0}(\rho\oplus\psi)] = [\mathrm{Ad}_{u_0}(\theta\oplus\psi),\psi]$$

and using the fact that homotopy classes in $KK_{nuc}(A,B)$ are independent of the selection of isometries, it follows that

$$[\theta\oplus\psi,\rho\oplus\psi] = [\mathrm{Ad}_{u_0}(\theta\oplus\psi), \mathrm{Ad}_{u_0}(\rho\oplus\psi)] = [\mathrm{Ad}_{u_0}(\theta\oplus\psi),\psi]$$

and in turn that

$$[\theta,\rho] = [\mathrm{Ad}_{u_0}(\theta\oplus\psi),\psi]$$

So, $x = [\mathrm{Ad}_{u_0}(\theta\oplus\psi),\psi]$, and if we set $\phi = \mathrm{Ad}_{u_0}(\theta\oplus\psi)$, then ϕ is a weakly nuclear representation and since ψ is nuclearly absorbing representation, it follows that ϕ is nucleraly absorbing , as required. $\square$

Proposition 2.3.7. *Let A be a separable C^*-algebras and E a separable, unital, $\mathscr{Q}$-stable C^*-algebra. Also, let $\phi,\psi\colon A\to E$ be two nuclear *-homomorphisms, $\lambda\colon E\to M(I)$ the canonical *-homorphism and note that $(\lambda\phi,\lambda\psi) \in \mathscr{E}_{nuc}(A,I)$.*

If $\lambda\phi,\lambda\psi$ are nuclearly absorbing representations and $[\lambda\phi,\lambda\psi] = 0$ in $KK_{nuc}(A,I)$, then there exists $(u_n)_n \subseteq E$ sequence of unitaries such that

$$\|\phi(a) - u_n\psi(a)u_n^*\| \xrightarrow{n\to\infty} 0$$

for all $a\in A$.

Proof. Since $[\lambda\phi,\lambda\psi] = 0$ and $\lambda\phi$, $\lambda\psi$ nuclearly absorbing representations, by Theorem 2.3.5 we get that $\lambda\phi\oplus\lambda\psi \approxeq \lambda\psi\oplus\lambda\psi$ and $\lambda\phi\oplus\lambda\phi \approxeq \lambda\psi\oplus\lambda\phi$. In particular, since there is a unitary $u\in M_2(I+\mathbb{C}1_{M(I)})$ such that $\mathrm{Ad}_u\circ(\lambda\phi\oplus\lambda\psi) = \lambda\psi\oplus\lambda\phi$, then there is a sequence of unitaries $(u_n)_n \subseteq M_2(I+\mathbb{C}1_{M(I)})$ satisfying

$$\|\lambda\phi\oplus\lambda\phi(a) - u_n(\lambda\psi\oplus\lambda\psi(a))u_n^*\| \xrightarrow{n\to\infty} 0, \quad \forall\, a\in A$$

Now, since λ is unital, there are $u'_n \in M_2(I+\mathbb{C}1_E)$ unitaries, such that $\lambda_2(u'_n) = u_n$, for each

$n \in \mathbb{N}$. Thus,

$$\|\lambda\phi \oplus \lambda\phi(a) - u_n(\lambda\psi \oplus \lambda\psi(a))u_n^*\| = \|\lambda_2[\phi \oplus \phi(a) - u_n'(\psi \oplus \psi(a))u_n'^*]\|$$

and since the restriction of λ_2 to $M_2(I + \mathbb{C}1_E)$ is injective we obtain that

$$\|\phi \oplus \phi(a) - u_n'(\psi \oplus \psi(a))u_n'^*\| = \|\lambda\phi \oplus \lambda\phi(a) - u_n(\lambda\psi \oplus \lambda\psi(a))u_n^*\| \xrightarrow{n\to\infty} 0$$

i.e $\phi \oplus \phi$ and $\psi \oplus \psi$ are apprroximately unitarily equivalent as *-homomorphisms $A \to M_2(E)$.

Now, consider a unital embedding $M_2(\mathbb{C}) \hookrightarrow \mathscr{Q}$, which induces a unital embedding $E \otimes M_2(\mathbb{C}) \to E \otimes \mathscr{Q}$, and if we apply this embedding to the approximate unitary equivalence above we get that $\phi \otimes 1_{\mathscr{Q}}$ is approximately unitarily equivalent to $\psi \otimes 1_{\mathscr{Q}}$ as *-homorphisms $A \to E \otimes \mathscr{Q}$. So, let $\epsilon > 0$ and employ the sequence of *-homomorphisms $\theta_n \colon E \otimes \mathscr{Q} \to E$ constructed in the proof of Theorem 1.1.11, which satisfy that

$$\|\theta_n(x \otimes 1_{\mathscr{Q}}) - x\| \xrightarrow{n\to\infty} 0, \quad \forall\, x \in E$$

Then, there exist unitaries $(w_n)_n \subseteq E \otimes \mathscr{Q}$ and $N \in \mathbb{N}$ such that

$$\|\phi(a) - \theta_N(\phi(a) \otimes 1_{\mathscr{Q}})\| < \epsilon/3$$
$$\|\phi(a) \otimes 1_{\mathscr{Q}} - w_N(\psi(a) \otimes 1_{\mathscr{Q}})w_N^*\| < \epsilon/3$$
$$\|\psi(a) - \theta_N(\psi(a) \otimes 1_{\mathscr{Q}})\| < \epsilon/3$$

from which relations it follows that,

$$\|\phi(a) - \theta_N(w_N)\psi(a)\theta_N(w_N^*)\| < \epsilon$$

where $\theta_N(w_N))$ unitary in E, as desired. $\square$

2.4 Trace-kernel extensions

In this section we introduce the notion of a *trace-kernel extension* and the class of *admissible kernel* C^*-algebras. Again, as in the previous section, we will emphasize on specific results that will be rather important in the proofs of the following chapter. But before entering the main core of this section, we digress momentarily to define some notions that emerge in the sequel of the project.

Definition 2.4.1. *Let B, I be two C^*-algebras. An extension of B by I is a short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} E \xrightarrow{q} B \longrightarrow 0$$

of C^*-algebras. When A and B are fixed we refer to the above extension by the triple (ι, E, q). Moreover, E is called the extension algebra.

Note 2.4.2. An interesting fact about extensions is that there is a way to transform them into *-homomorphisms without losing any essential information. The *-homorphism that does this work, is called *Busby invariant*. For details see chapter 3 in [18].

There are plenty of types of extensions. Below we give the definition for a portion of them. Recall, that an element in a C^*-algebra is called full, if it is not contained to any proper two-sided ideal of the C^*-algebra. Moreover, a *-homomorphism $\phi \colon A \to B$ is called full, if $\phi(a)$ is a full element, for any $a \in A$, while if B is unital then ϕ is called unitizably full if the unitization $\tilde{\phi} \colon \tilde{A} \to B$ is full. It is a fact that if both A, B are unital, then ϕ is unitizably full if, and only if, ϕ is full and $1_B - \phi(1_A)$ is full.

Definition 2.4.3. *Let B, I be two C^*-algebras and let*

$$0 \longrightarrow I \xrightarrow{\iota} E \xrightarrow{q} B \longrightarrow 0$$

be the extension of B by I. Then,

i) we say that the extension is trivial if the short exact sequence above splits,

ii) we say that the extension is full if the corresponding Busby invariant is full $^* -$ homomorphism,

iii) we say that is weakly nuclear, if the splitting map is weakly nuclear and that is nuclearly absorbing, if it is equivalent to its own sum with any trivial weakly nuclear extension,

iv) we say that the extension algebra E has the purely large property if: for every $e \in E^+$ that is not contained in I, the hereditary subalgebra that it generates, $\overline{eIe}$, contains a stable subalgebra that is not contained to any proper two-sided ideal in I, i.e it is full in I

One more notion that we will need is the one of *corona factorization property.* Below, we collect equivalent conditions for a C^* algebra having the *corona factorization property.*

Definition 2.4.4 ([19], Definition 2.1)**.** *Let B be a separable C^*-algebra. We say that B has the corona factorization property if it satisfies one of the following equivalent conditions:*

i) every full extension of B is nuclearly absorbing,

ii) every full trivial extension of B is nuclearly absorbing,

iii) for every projection p which is full in M(B) there is an element x in M(B) such that $xpx^* = 1_{M(B)}$

iv) for every projection q which is full in M(B)/B there is an element y in M(B)/B such that $yqy^* = 1_{M(B)/B}$

We end this brief discussion about extensions and the corona factorization property with a result that incorporates these new notions and which will be employed in the last chapter. The proof is ommited and it can be found in [30].

Theorem 2.4.5. *If A is a separable C^*-algebra and B is a σ-unital C^*-algebra with the corona factorization property, then every unitizably full representation $A \to M(B \otimes \mathbb{K})$ is nuclearly absorbing.*

Now, we get back to the main scope of this section and in the following lines we set up the environment that we are going to work with.

Let B be a simple, unital C^*-algebra, with unique tracial state τ_B and define the 2-norm on B by $\|b\|_2 = \tau(b^*b)^{1/2}$, for all $b \in B$. Moreover, denote by $l^\infty(B)$ the C^*-algebra of bounded sequences in B and for a free ultrafilter ω on the natural numbers, we define

$$B_\omega = l^\infty(B)/\{b = (b_n)_n \in l^\infty(B) \colon \lim_{n \to \omega} \|b\| = 0\}$$
$$B^\omega = l^\infty(B)/\{b = (b_n)_n \in l^\infty(B) \colon \lim_{n \to \omega} \|b\|_2 = 0\}$$

Since τ_B is contractive, $\|b\| \leq \|b\|_2$ and therefore we can define the following extension

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0$$

where q_B is the quotient map, $J_B = Ker(q_B)$ and j_B the inclusion map. The C^*-algebra J_B is referred to as the *trace-kernel ideal* associated to B, while the extension defined above is called the *trace-kernel extension* associated to B.

Definition 2.4.6. *A C^*-algebra I is called admissible kernel if it has real rank zero and stable rank one, $K_0(I)$ is divisible, $K_1(I) = 0$, the von Neumann - Murray semigroup $D(I) = P_\infty(I)/_{\sim_0}$ is almost* *unpreforated, and every projection in $I \otimes \mathbb{K}$ is von Neumann - Murray equivalent to a projection in I.*

Our main goal is to prove the following result, which collects important characteristics of *trace-kernel extensions* and it will be used extensively in the sequel.

Proposition 2.4.7 ([30], Proposition 3.2)**.** *Let B be a simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique tracial state τ_B, such that every quasi-trace on B is a trace and $K_1(B) = 0$, then*

i) B^ω is a II_1-factor

ii) B_ω has real rank zero and stable rank one, has unique tracial state τ_{B_ω}, has strict comparison of positive elements with respect to its trace, is separably $\mathscr{Q}$-stable and has trivial K_1-group

iii) J_B is an admissible kernel

Proof. i) Firstly we claim that if $\pi_{\tau_B} : B \to B(H_{\tau_B})$ is the GNS-representation of B with respect to its unique tracial state τ_B, then $(\pi_{\tau_B}(B)'')^\omega \cong B_\omega / J_B$. To this end, set $N = \pi_{\tau_B}(B)''$, and consider the *-homomorphism

$$\Phi \colon B_\omega \to N^\omega$$

given by

$$\Phi([(b_1, b_2, b_3, ...)]) = [(\pi_{\tau_B}(b_1), \pi_{\tau_B}(b_2), \pi_{\tau_B}(b_3), ...)]$$

Φ is well defined: let $b \in B_\omega$ such that $b = 0$, since B is simple, π_{τ_B} is a faithful representation and therefore $\lim_{n\to\omega} \|b_n\| = 0$ implies that $\lim_{n\to\omega} \|\pi_{\tau_B}(b_n)\| = 0$ and since τ_B is contractive it follows that $\|\pi_{\tau_B}(b_n)\|_2 = \|b_n\|_2 \leq \|b_n\| = \|\pi_{\tau_B}(b_n)\|$, for all $n \in \mathbb{N}$, hence $\lim_{n\to\omega} \|\pi_{\tau_B}(b_n)\|_2 = 0$, which shows that $\Phi(b) = 0$.

Now, we aim to show that Φ is surjective. So, let $x \in \mathbb{N}$, self-adjoint, $\|x\| \leq 1$, then by Kaplansky's Density Theorem, there are $b_n \in B$, $n \in \mathbb{N}$, such that $\pi_{\tau_B}(b_n) \in (\pi_{\tau_B}(B))_1$, self-adjoint and $\pi_{\tau_B}(b_n) \xrightarrow{sot} x$. Furhermore, we want to restrict the norm of b_n, such that $\|b_n\| \leq \|x\|$, for all $n \in \mathbb{N}$. To do so, define a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} t, & |t| \leq \|x\| \\ \|x\|^2 / t, & |t| \geq \|x\| \end{cases}$$

since $\pi_{\tau_B}(b_n) \xrightarrow{sot} x$, Proposition 19.2 in [34] implies that $f(\pi_{\tau_B}(b_n)) \xrightarrow{sot} f(x) = x$ and moreover we observe that $\|f(\pi_{\tau_B}(b_n))\| = r(f(\pi_{\tau_B}(b_n))) = f(r(\pi_{\tau_B}(b_n))) =$

$$\begin{cases} \|\pi_{\tau_B}(b_n)\|, & \|\pi_{\tau_B}(b_n)\| \leq \|x\| \\ \|x\|^2 / \|\pi_{\tau_B}(b_n)\|, & \|\pi_{\tau_B}(b_n)\| \geq \|x\| \end{cases}$$

which shows that $\|f(\pi_{\tau_B}(b_n))\| \leq \|x\|$ for all $n \in \mathbb{N}$. Hence, if we set $c_n = f(b_n)$, we have that $c_n \xrightarrow{sot} x$ and $\|c_n\| = \|\pi_{\tau_B}(f(b_n))\| = \|f(\pi_{\tau_B}(b_n))\| \leq \|x\|$, for all $n \in \mathbb{N}$, as desired. Since any element in N is a linear combination of self-adjoint elements, we get that for any $x \in \mathbb{N}$ we can find such a sequence c_n in B satisfying these conditions.

Before continuing, consider the quotient maps

$$\rho_B \colon l^\infty(B) \to B_\omega$$

and

$$\rho_N \colon l^\infty(N) \to N^\omega$$

and let $x = \rho_N(x_1, x_2, x_3, ...)$, $x_n \in \mathbb{N}$, $n = 1, 2, 3, ...$ Then, for each $k \in \mathbb{N}$ we find by the previous observation an element $b_k \in B$ such that $\|b_k\| \leq \|x_k\|$ and moreover such that $\|\pi_\tau(b_k) - x_k\|_2 < 1/k$. Then, $\Phi(\rho_B(b_1, b_2, b_3, ...)) = \rho_N((\pi_{\tau_B}(b_1), \pi_{\tau_B}(b_2), \pi_{\tau_B}(b_3), ...))$, and since $\lim_k \|\pi_\tau(b_k) - x_k\|_2 = 0$, it follows that $\rho_N((\pi_{\tau_B}(b_1), \pi_{\tau_B}(b_2), \pi_{\tau_B}(b_3), ...)) = x$. Hence $\Phi(b) = x$, where $b = \rho_B((b_1, b_2, b_3, ...)) \in B_\omega$, showing that Φ is surjective.

At this point, in order to establish the desired isomorphism $(\pi_{\tau_B}(B)'')^\omega \cong B_\omega/J_B$, it remains to show that $J_B = Ker\Phi$. So, let $b = \rho_B(b_1, b_2, b_3, ...) \in J_B$, then $\lim_{n\to\omega} \|b_n\|_2 = 0$, and if ξ_{τ_B} is a cyclic vector for the GNS-representation $(\pi_{\tau_B}, H_{\tau_B})$, we obtain that

$$\langle \pi_{\tau_B}(b_n^* b_n)\xi_{\tau_B}, \xi_{\tau_B}\rangle = \tau_B(b_n^* b_n) \xrightarrow{n\to\omega} 0$$

Hence, $\lim_{n\to\omega} \|\pi_{\tau_B}(b_n\|_2 = 0$, or equivalently $\Phi(b) = \rho_N(\pi_{\tau_B}(b_1), \pi_{\tau_B}(b_2), \pi_{\tau_B}(b_3), ...) = 0$. On the other hand, if $b = \rho_B(b_1, b_2, b_3, ...) \in Ker\Phi$, then

$$\lim_{n\to\omega} \|\pi_{\tau_B}(b_n)\|_2 = 0 \implies \lim_{n\to\omega} \langle \pi_{\tau_B}(b_n^* b_n)\xi_{\tau_B}, \xi_{\tau_B}\rangle = 0 \implies \|b_n\|_2 = 0$$

hence $b \in J_B$, concluding that $J_B = Ker\Phi$. Thus, since Φ is also surjective, we obtain our first claim, namely that $(\pi_{\tau_B}(B)'')^\omega \cong B_\omega/J_B$. Moreover, since by consrtuction $B_\omega/J_B \cong B^\omega$, in order to show that B^ω is a II_1 - factor, it is enough to show that $(\pi_{\tau_B}(B)'')^\omega$ is a II_1 - factor.

To this end, let us first observe that since $\langle \pi_{\tau_B}(b_1)\pi_{\tau_B}(b_2)\xi_{\tau_B}, \xi_{\tau_B}\rangle = \langle \pi_{\tau_B}(b_2)\pi_{\tau_B}(b_1)\xi_{\tau_B}, \xi_{\tau_B}\rangle$, for any $b_1, b_2 \in B$, then weak density of $\pi_{\tau_B}(B)$ in N, implies that the following map

$$\tau_N : N \to \mathbb{C}$$

given by

$$\tau_N(x) = \langle x(\xi_{\tau_B}), \xi_{\tau_B}\rangle$$

is a normal tracial state on N, $\tau_N|_{\pi_{\tau_B}(B)} = \tau_B$, and we claim that it is faithful. To show this, it suffices for ξ_{τ_B} to be a separating vector for N. So, let $T \in \mathbb{N}$ such that $T\xi_{\tau_B} = 0$, then for any $b \in B$,

$$\|T\pi_{\tau_B}(b)\xi_{\tau_B}\|^2 = \tau_N(\pi_{\tau_B}(b^*)T^*T\pi_{\tau_B}(b)) = \tau_N(\pi_{\tau_B}(b)\pi_{\tau_B}(b^*)T^*T) = 0$$

Thus, T is zero in $\pi_{\tau_B}(B)\xi_{\tau_B}$, and since ξ_{τ_B} is cyclic, we obtain that $T = 0$, and in turn that ξ_{τ_B} is a separating vector of N, as claimed. Now, let $p \in \pi_{\tau_B}(B)' \cap N$ and define the function

$$\theta : N \to \mathbb{C}$$

by

$$\theta(x) = \tau_N(px) \quad x \in \mathbb{N}$$

Then, θ is a weakly continuous, positive linear functional, and so if we restrict it to $\pi_{\tau_B}(B)$ it is constant t times the unique tracial state on $\pi_{\tau_B}(B)$. Thus, by weak continuity of θ and weak density of $\pi_{\tau_B}(B)$ in N we get that $\theta(x) = t\tau_N(x)$, for any $x \in \mathbb{N}$. Therefore, $\tau_N(p) = t$, and $0 = \theta(1-p) = t\tau_N(1-p) = \tau_N(p)\tau_N(1-p)$. But, as p, $1-p$ are positive and τ_N is faithful, it follows that $p = 0$ or $1 - p = 0$, thus in any case p is a trivial projection. Since p was arbitrary and $\pi_{\tau_B}(B)' \cap N$ as a von Neumann algebra is the closed linear span of its projections, we conclude that $\pi_{\tau_B}(B)' \cap N = \mathbb{C}$, which shows that N is a factor. Now, the existence of a faithful, tracial state on N immediately implies that N is a finite factor, and that N is infinite dimensional, implies that N is a II_1 - factor (See [29], Corollary 12). Then, the ultrapower of N, N^ω is again a II_1 - factor by Theorem 17 in [29]. Hence, we conclude that B^ω is a II_1 - factor, as desired.

ii) That B has strict comparison of positive elements with respect to its trace follows from Theorem 1.2.22 and Theorem 1.2.25, and the assumption that any quasi-trace on B is a trace. Furthermore, since τ_B is faithful tracial state, then the induced trace on $M_n(B)$ is again faithful, for any $n \in \mathbb{N}$, and so B is stably finite. Hence, by Corollary 1.3.13, B has stable rank one, while B has real rank zero by Theorem 7.2 in [28]. Now, it is a fact that all three properties are preserved by ultraproducts; for strict comparison see Lemma 1.23 in [3] and for real rank zero see Proposition 3.2 in [31]. To see that B_ω has stable rank one, let $\epsilon > 0$ and $b = (b_n)_n \in l^\infty(B)$, then density of the invertible group in B, implies that for each b_n we can find $c_n \in GL(B)$ to be ϵ-close to b_n. But, since we have to be able to control the norm of these invertible elements, we employ the fact that we can select these $c_n \in B$ such that

$$\|c_n - b_n\| < \epsilon$$

and

$$\left\|c_n^{-1}\right\| \leq 2\epsilon^{-1}$$

Then, $c = (c_n) \in l^\infty(B)$ invertible and

$$\|b - c\| = \sup\{\|b_n - c_n\| : n \in \mathbb{N}\} < \epsilon$$

Hence, $l^\infty(B)$ has stable rank one. Now, for $\epsilon > 0$, let the quotient map $\pi_B \colon l^\infty(B) \to B_\omega$, and take $x = \pi_B(b)$, $b \in l^\infty(B)$. Then, there exists $c \in GL(l^\infty(B))$ such that $\|b - c\| < \epsilon$, and since $\pi_B(c) \in GL(B_\omega)$ and

$$\|\pi_B(c) - x\| = \|\pi_B(c) - \pi_B(b)\| \leq \|c - b\| < \epsilon$$

we deduce that B_ω has stable rank one as well. Now, since B is $\mathscr{Q}$-stable, that B_ω is separably $\mathscr{Q}$-stable follows from Proposition 1.4.16. Hence it remains to show that B_ω admits unique tracial state and that $K_1(B_\omega) = 0$. For the uniquensess of the tracial state on B_ω, firstly let $T(B_\omega)$ to be the set of all traces on B_ω, and

$$T_\omega(B_\omega) = \{\lim_{n\to\omega} \tau_n \colon \tau_n \text{ tracial state on } B_n = B\}$$

If τ_B is the unique tracial state on B, then the map τ defined by

$$\tau(b) = \lim_{n\to\omega} \tau_B(b_n), \quad b \in B_\omega$$

belongs in $T_\omega(B_\omega)$ and in fact uniqueness of τ_B shows that

$$T_\omega(B_\omega) = \{\tau\}$$

Now, it is a fact that $T(B_\omega) = \overline{T_\omega(B_\omega)}^{w^*}$ (see [23], Theorem 8), hence $T(B_\omega) = \{\tau\}$, as required.

Finally, we embark to show that $K_1(B_\omega) = 0$. Since B_ω has stable rank one Theorem 2.10 in [25], asserts that $K_1(B_\omega) \cong GL(B_\omega)/GL_0(B_\omega)$, where $GL_0(B_\omega)$ is the connected component of the identity in $GL(B_\omega)$. So, by using the well known map $GL(B_\omega) \to U(B_\omega)$, given by $x \mapsto x|x|^{-1}(= u)$, and assuming that $U(B_\omega)$ is path connected, we get that for any $x \in GL(B_\omega)$, x is homotopic to u(i.e $x \sim_h u$), and since $u \sim_h 1$, it follows that $x \sim_h 1$ in $GL(B_\omega)$. Thus, if we show that $U(B_\omega)$ is path connected then $K_1(B_\omega) \cong GL(B_\omega)/GL_0(B_\omega) = 0$. To this end, let $u \in U(B_\omega)$ and let $(u_n)_n \in l^\infty(B)$ representing u. Since B has stable rank one and $K_1(B) = 0$, by employing again Theorem 2.10 in [25], we get that $u_n \sim_h 1$, for all $n \in \mathbb{N}$. Moreover, since B has real rank zero it is a fact that B has the weak (FU) property, i.e the set of unitaries in B_ω with finite spectrum is dense in $U(B_\omega)$ (see main result in [21]). Hence, for any u_n we find $w_n \in U(B)$ with finite spectrum such that $\|u_n - w_n\| < 1/n$, and moreover we can find $h_n \in B$ self-adjoint, $\|h_n\| \leq \pi$ such that $w_n = \exp ih_n$ (See Lemma 2.1.3 and Proposition 2.1.6 in [20]). Set $h = \pi_B(h_1, h_2, h_3, ...) \in B_\omega$ and $w = \exp ih$, then since

$$\|u_n - w_n\| \xrightarrow{n\to\omega} 0$$

it follows that $u = w = \exp ih$ in B_ω, thus again by Proposition 2.1.6 in [20], $u \in U_0(B_\omega)$. Since u was an arbitrary element in $U(B_\omega)$, we get that $U(B_\omega)$ is path connected, hence $K_1(B_\omega) = 0$

iii) Let us show first that any projection in $J_B \otimes \mathbb{K}$ is von Neumann-Murray equivalent to a projection in J_B. Since considering projections in $J_B \otimes \mathbb{K}$ is the same as considering projections in $M_\infty(J_B)$, let $d \geq 1$ and $p \in M_d(J_B)$ a projection. Then $p = (p_{ij})_{ij}$, where $p_{ij} = (p_{ij}^n)_n \in J_B$, which implies that

$$0 = \lim_{n\to\omega} \left\|p_{ij}^n\right\|_2 = \lim_{n\to\omega} \tau_B((p_{ij}^n)^* p_{ij}^n) = \tau_{B_\omega}(p_{ij}^* p_{ij}), \quad \forall\, i, j = 1, 2, ..., d$$

In particular, $\tau_{B_\omega}(p_{ii}) = 0$, for all $i = 1, 2, ..., d$, hence

$$(\tau_{B_\omega} \otimes Tr_d)(p) = 0$$

where Tr_d is the usual normalised trace on $M_d(\mathbb{C})$. Now, observe that

$$(\tau_{B_\omega} \otimes Tr_d)(p) = 0 < 1 = (\tau_{B_\omega} \otimes Tr_d)(1_{B_\omega \oplus 0_{d-1}})$$

and since B_ω has strict comparison, there is a partial isometry $w \in M_d(B_\omega)$ such that $w^*w = p$ and $ww^* \leq 1_{B_\omega} \oplus 0_{d-1}$. As $w^*w \in M_d(J_B)$, then $ww^* = ww^*ww^* \in M_d(J_B)$, and therefore $ww^* = q \oplus 0_{d-1}$, for some projection $q \in J_B$. Since $q \oplus 0_{d-1}$ is von Neumann-Murray equivalent to q, we obtain that p is von Neumann-Murray equivalent to q. Thus, every projection in $J_B \otimes \mathbb{K}$ is von Neumann-Murray equivalent to a projection in J_B.

Now, we argue that $K_1(J_B) = 0$. Firstly, the trace-kernel extension induce the following exact sequence in K-groups (See Proposition 10.2.4 in [20])

$$K_0(B_\omega) \xrightarrow{K_0(q_B)} K_0(B^\omega) \xrightarrow{\delta_0} K_1(J_B) \xrightarrow{K_1(j_B)} K_1(B_\omega)$$

and since $K_1(B_\omega) = 0$, to prove that $K_1(J_B) = 0$, it suffices to show that $Ker(\delta_0) = Im(K_0(q_B)) = K_0(B^\omega)$, or equivalently that $K_0(q_B)$ is a surjective group homomorphism. So, let $t \in [0, 1]$ and find $t_n \in \mathbb{Q} \cap [0, 1]$ such that $\lim_{n\to\omega} t_n = t$. Since B is $\mathscr{Q}$-stable we find a unital embedding $\mathscr{Q} \to \mathscr{Q} \otimes B \cong B$, given by the map $x \mapsto x \otimes 1_B$. Then, for each $n \in \mathbb{N}$, we can find a projection $p_n \in B$ such that $\tau_B(p_n) = t_n$. Let p be the projection in B_ω defined by the sequence $(p_n)_n$, and we see that

$$\tau_{B_\omega}(p) = \lim_{n\to\omega} \tau_B(p_n) = t$$

hence, for any $t \in [0, 1]$ and $n \in \mathbb{N}$ we can select a projection $p \in M_n(B_\omega)$ such that $\tau_{B_\omega}(p_{ii}) = t$, $i = 1, 2, ..., n$. Thus, the induced trace $\hat{\tau}_{B_\omega} = \tau_{B_\omega} \otimes Tr_n \colon K_0(B_\omega) \to \mathbb{R}$, is surjective. Moreover, since B^ω is a II_1-factor, τ_{B^ω} is faithful, and so $\hat{\tau}_{B^\omega}$ is faithful, and by Proposition E in [29], we get that

$$\{\tau_{B^\omega}(p) \colon p \in Proj(B^\omega)\} = [0, 1]$$

so arguing as above we obtain that $\hat{\tau}_{B^\omega}$ is also surjective. Now, since by construction $\tau_{B_\omega} = \tau_{B^\omega} \circ q_B$, it follows that $\hat{\tau}_{B_\omega} = \hat{\tau}_{B^\omega} \circ K_0(q_B)$, hence $K_0(q_B)$ is a surjective group homomorphism, as desired.

Let us now show that $K_0(J_B)$ is a divisible group. Since B_ω is separably $\mathscr{Q}$-stable, then also J_B is separably $\mathscr{Q}$-stable by Proposition 1.4.16. Thus, we can find an increasing sequence $(J_i)_i$ of $\mathscr{Q}$-stable, separable C^*-subalgebras of J_B, such that $\overline{\cup_i J_i} = J_B$. Now, since each J_i is $\mathscr{Q}$-stable, we can view it as the inductive limit of the inductive sequence $(M_{k_j}(J_i), \phi_j)$, where $\phi_j \colon M_{k_j}(J_i) \to M_{k_{j+1}}(J_i)$, are injective, unital *-homomorphisms with multiplicity k_{j+1}/k_j, $j \in \mathbb{N}$. By continuity of K_0, we see $K_0(J_i)$, as the inductive limit of $(K_0(M_{k_j(J_i)}), K_0(\phi_j))$, and let $g \in K_0(J_i)$, $n \in \mathbb{N}$. Suppose that $g \in K_0(M_{k_j}(J_i))$, for some $j \in \mathbb{N}$, and if we select $m \in \mathbb{N}$ such that $k_m/k_j = n$, it follows that $K_0(\phi_{k_m,k_j})(g) = ng$. But, $K_0(\phi_{k_m,k_j})(g)$ and g represent the same element in $K_0(J_i)$, hence $ng = g$ in $K_0(J_i)$, and since g, n were arbitrary, we conclude that $K_0(J_i)$ is a divisible group, for any $i \in \mathbb{N}$. Now, since $\overline{\cup_i J_i} = J_B$, we can view J_B as the inductive limit of (J_i, ψ_i), where ψ_i are the inclusions, and using continuity of K_0 along the fact that inductive limits preserve divisibility, we obtain that $K_0(J_B)$ is a divisible group.

For proving that $D(J_B)$ is almost unperforated, we find again the increasing sequence $(J_i)_i$ and we see each J_i as an inductive limit exactly in the same fashion as above. Now, by continuity of $D(-)$, we can see $D(J_i)$ as the inductive limit of $(D(M_{k_j}(J_i), D(\phi_j))$ and let $m, m' \in \mathbb{N}$, $x_j \in D(M_{k_j}(J_i))$, $j = 1, 2$, such that $m' < m$ and $mx_1 \leq m'x_2$. Then, there exists $y \in D(J_i)$ such that $mx_1 = y$ and $y \leq m'x_2$, and observe that since $mx_1 = x_1 \oplus x_1 \oplus \cdots \oplus x_1$, then if $j_1, j_2 \in \mathbb{N}$, such that $k_{j_1}/k_1 = m$ and $k_{j_2}/k_2 = m'$, we get that

$$mx_1 = D(\phi(\phi_{k_{j_1},k_1}))(x_1)$$

and

$$m'x_2 = D(\phi(\phi_{k_{j_2},k_2}))(x_2)$$

But, $D(\phi(\phi_{k_{j_1},k_1}))(x_1)$ represents the same element as x_1 in $D(J_i)$ and similarly for x_2. Hence, $x_1 = y \leq m'x_2 = x_2$, showing that $D(J_i)$ is an almost unperforated semigroup for all $i \in \mathbb{N}$. So, by regarding again J_B as an inductive limit, using continuity of $D(-)$ and the fact that the property "almost unperforated" is preserved by inductive limits, we conclude that $D(J_B)$ is an almost unperforated semigroup.

Finally, that J_B has stable rank one and real rank zero follows from Corollary 2.8 in [5] and Theorem 4.3 in [26], respectively.

□

We end this chapter by introducing the UCT class of C^*-algebras without delving into details, and by stating one more result about admissible kernels that will be critical in the sequel. The proof is ommited and can be found in [30].

Definition 2.4.8 ([32], Definition 1.7)**.** *A separable C^*-algebra is said to satisfy the universal coefficient theorem (UCT) if*

$$0 \longrightarrow Ext_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \longrightarrow KK(A,B) \longrightarrow Hom_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \longrightarrow 0$$

is an exact sequence, for any σ-unital C^*-algebra B.

Proposition 2.4.9.

i) The property of being admissible kernel is separably inheritable.

ii) If I is an admissible kernel, then $M_n(I)$ is an admissible kernel for all $n \in \mathbb{N}$.

iii) If I is a separable admissible kernel, then it is stable and has the corona factorization property.

iv) If A is a separable $C^* - algebra$ satisfying the UCT and I is a separable admissible kernel, then the canonical homomorphism $KK_{nuc}(A,I) \to Hom_Z(K_0(A), K_0(I))$ is an isomorphism.

3 Main results

3.1 An existence and a classification result

In this chapter we present the main outcomes of this project. We will start with a preliminary result, an then we proceed to show one existence and one classification result. As a last step we will embark in improving these results, finally reaching to a rather interesting theorem about unital, simple AF-algebras with unique trace and divisible K_0-group. This will be the ending point of this project. All the material of this chapter is from [30].

Lemma 3.1.1. *Let the following commuting diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{j_1} & P & \xrightarrow{\alpha_1} & B_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} & & \\ 0 & \longrightarrow & I & \xrightarrow{j_2} & B_2 & \xrightarrow{\beta_2} & D & \longrightarrow & 0 \end{array}$$

of C^*-algebras with exact rows. If A is a C^*-algebra and $\phi_i \colon A \to B_i$, $i = 1, 2$ are *-homomorphisms satisfying that $\beta_1 \circ \phi_1 = \beta_2 \circ \phi_2$, then there exists a unique *-homomorphism $\phi \colon A \to P$ such that $a_i \circ \phi = \phi_i$, $i = 1, 2$. Moreover, ϕ is nuclear if, and only if, ϕ_1 and ϕ_2 are nuclear.

Proof. Let Q be the pullback C^*-algebra of

$$\begin{array}{ccc} & & B_1 \\ & & \downarrow{\scriptstyle \beta_1} \\ B_2 & \xrightarrow{\beta_2} & D \end{array}$$

i.e. $Q = \{(b_1, b_2) \in B_1 \oplus B_2 \colon \beta_1(b_1) = \beta_2(b_2)\}$ and define a *-homomorphism $\pi \colon P \to Q$ by $\pi(p) = (a_1(p), a_2(p))$. We claim that π is an *-isomorphism. For injectivity, let $p \in P$ such that $a_1(p) = a_2(p) = 0$, then $\exists\, x \in I$ such that $j_1(x) = p$, but since $0 = a_2(p) = a_2(j_1(x)) = j_2(x)$ and j_2 injective, we get that $x = 0$ and in turn that $p = 0$. Now, let $q = (b_1, b_2) \in Q$, and see that $\exists\, p \in P$, such that $a_1(p) = b_1$, which implies that $\beta_2(a_2(p) - b_2) = 0$. Hence, $j_2(x) = a_2(p) - b_2$ for some $x \in I$ and therefore $a_2(p - j_1(x)) = b_2$, but since $a_1(p - j_1(x)) = a_1(p) = b_1$, we conclude that π is surjective.

So, if we set $\phi(-) = \pi^{-1}(\phi_1(-), \phi_2(-)) \colon A \to P$, then ϕ is a well defined $*$-homomorphism, since $b_1\phi_1 = b_2\phi_2$, and it is trivially checked that $a_i\phi = \phi_i$, $i = 1, 2$. This ϕ is unique, since if there exists $\phi' \colon A \to P$ such that $a_i\phi' = \phi_i$, $i = 1, 2$, then $\phi(a) - \phi'(a) \in Ker(a_1) \cap Ker(a_2) = Ker(\pi) = 0$, for all $a \in A$.

Now, if ϕ is nuclear then it is obvious that ϕ_1 and ϕ_2 are nuclear. On the other hand, suppose that ϕ_1, ϕ_2 are nuclear *-homomorphisms and we claim that ϕ is nuclear as well. To this end, fix a C^*-algebra C and consider the canonical map

$$\rho \colon A \otimes_{\max} C \to A \otimes_{\min} C$$

Moreover, using the fact that maximal tensor product respects exact sequences we get the following commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes_{max} C & \xrightarrow{j_1 \otimes id} & P \otimes_{max} C & \xrightarrow{\alpha_1 \otimes id} & B_1 \otimes_{max} C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha_2 \otimes id} & & \downarrow{\scriptstyle \beta_1 \otimes id} & & \\ 0 & \longrightarrow & I \otimes_{max} C & \xrightarrow{j_2 \otimes id} & B_2 \otimes_{max} C & \xrightarrow{\beta_2 \otimes id} & D \otimes_{max} C & \longrightarrow & 0 \end{array}$$

of C^*-algebras with exact rows. Since ϕ_i is nuclear, there is a *-homomorphism $\psi_i \colon A \otimes_{\min} C \to$

$B_i \otimes_{\max} C$ such that $\psi_i \circ \rho = \phi_i \otimes_{\max} id_C$, for each $i = 1, 2$ by Corollary 3.8.8 in [6]. Then,

$$(\beta_1 \otimes_{\max} id_C) \circ \psi_1 = (\beta_2 \otimes_{\max} id_C) \circ \psi_2$$

and so by the first part of the proof, there exists unique *-homomorphism $\psi\colon A \otimes_{\min} C \to P \otimes_{\max} C$ satisfying that $(a_i \otimes id_C) \circ \psi = \psi_i$, for each $i = 1, 2$. But then $a_i \otimes id_C(\phi \otimes_{\max} id_C(x) - \psi \circ \rho(x)) = 0$, for all $x \in A \otimes_{\max} C$, $i = 1, 2$, and since $Ker(a_1 \otimes id_C) \cap Ker(a_2 \otimes id_C) = 0$, we get that

$$\phi \otimes_{\max} id_C = \psi \circ \rho$$

Hence, $\phi \otimes_{\max} id_C$ factors through $A \otimes_{\min} C$ and since C was arbitrary, ϕ is nuclear by Corollary 3.8.8 in [6]. $\square$

Before proceeding to the proof of the first major result of this section, let us collect some *-homomorphisms that are going to be employed without further explanation. First, recall from the previous chapter that for B simple, unital C^*-algebra we have the trace-kernel extension

$$0 \longrightarrow J_B \xrightarrow{j_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0$$

Moreover, for any C^*-algebra C, $\iota_2\colon C \to M_2(C)$ denotes the inclusion into the $(1,1)$-corner and for any *-homomorphism $f\colon A \to B$ we denote the induced map $M_n(A) \to M_n(B)$, again by f.

Proposition 3.1.2. *Suppose A is a separabe, unital, exact C^*-algebra satisfying the UCT and B simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique trace τ_B such that every quasi trace on B is a trace and $K_1(B) = 0$.*

If τ_A is a faithful, amenable trace on A and $\sigma\colon K_0(A) \to K_0(B_\omega)$ is a group homomorphism such that $\sigma([1_A]_0) = [1_{B_\omega}]_0$ and $\hat{\tau}_{B_\omega}\sigma = \hat{\tau}_A$, then there is a full, unital, nuclear *-homomorphism $\phi\colon A \to B_\omega$ such that $K_0(\phi) = \sigma$ and $\tau_{B_\omega}\phi = \tau_A$.

Proof. Since B is $\mathscr{Q}$-stable there is a unital embedding $\mathscr{Q} \overset{\iota}{\hookrightarrow} B$, which is also trace-preserving due to the uniqueness of the trace in $\mathscr{Q}$ and B. Moreover consider the induced map $\iota_\omega\colon \mathscr{Q}_\omega \to B_\omega$ which makes the following diagram commutative

$$\begin{tikzcd}
\mathscr{Q} \arrow[r, hook, "\iota"] \arrow[d] & B \arrow[d] \\
l^\infty(\mathscr{Q}) \arrow[r, hook, "\iota_\infty"] \arrow[d, "\pi_{\mathscr{Q}}"] & l^\infty(B) \arrow[d, "\pi_B"] \\
\mathscr{Q}_\omega \arrow[r, "\iota_\omega"] & B_\omega
\end{tikzcd}$$

Hence, ι_ω is a unital embedding *-homomorphism, and we observe that it is also trace-preserving, since if $x = \pi_{\mathscr{Q}}((x_n)_n) \in \mathscr{Q}_\omega$, then $\iota_\omega(x) = \pi_B((x_n)_n)$ and therefore

$$\tau_{B_\omega}\iota_\omega(x) = \lim_{n\to\omega} \tau_B(x_n) = \lim_{n\to\omega} \tau_{\mathscr{Q}}(x_n) = \tau_{\mathscr{Q}_\omega}(x)$$

Then, we compose ι_ω with the unital, full, nuclear, trace-preserving *-homomorphism $A \to \mathscr{Q}_\omega$ given in Theorem A.0.6, we get a unital, full, nuclear, *-homomorphism $\psi\colon A \to B_\omega$, satisfying $\tau_{B_\omega}\psi = \tau_A$. From these conditions on ψ the only non-trivial is that ψ is full, but this follows in the exact same fashion as in the proof of Theorem A.0.6.

Note that since $\hat{\tau}_{B_\omega} = \hat{\tau}_{B^\omega} K_0(q_B)$, and $\hat{\tau}_{B_\omega}\sigma = \hat{\tau}_A$, it follows that

$$\hat{\tau}_{B^\omega} K_0(q_B\psi) = \hat{\tau}_{B^\omega} K_0(q_B)\sigma$$

but, by Proposition 2.4.7, B^ω is a II_1-factor, which entails that $\hat{\tau}_{B^\omega}$ is an isomorphism, hence

$$K_0(q_B\psi) = K_0(q_B)\sigma$$

So the image of $\sigma - K_0(\psi)$ lies inside $Ker(K_0(q_B)) = ImK_0(j_B)$. Now, by the long exact sequence of K-groups induced by the trace-kernel extension and the fact that $K_1(B^\omega) = 0$ (see Proposition A.0.12) it follows that $K_0(j_B)$ is injective. Using injectivity of $K_0(j_B)$, we can define a group homomorphism $\kappa\colon K_0(A) \to K_0(J_B)$ by

$$\kappa([a]_0) = [y]_0$$

where $K_0(j_B)([y]_0) = \sigma - K_0(\psi)([a]_0)$, and note that $K_0(j_B)\kappa = \sigma - K_0(\psi)$.

Furthermore, ψ satisfies the conditions of Proposition A.0.9 and therefore there is $E_0 \subseteq B_\omega$ separable C^*-subalgebra such that the corestriction of ψ to E_0 is again unital, full and nuclear, while by Proposition A.0.11 we find $I_0 \subseteq J_B$ separable C^*-subalgebra and $k_0\colon K_0(A) \to K_0(I_0)$ group homomorphism such that, $k = K(\iota_0)k_0$, where $\iota_0\colon I_0 \hookrightarrow J_B$ is the inclusion map. As J_B is an admissible kernel and being an admissible kernel is a separably inheritable property by Proposition 2.4.7 and Proposition 2.4.9, respectively, we can then find $I \subseteq J_B$ separable admissible kernel containing I_0, $E \subseteq B_\omega$ separable C^*-algebra containing E_0 and $D \subseteq B^\omega$ separable C^*-algebra by Proposition 2.4.7, such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\hat{j}"] \arrow[d, "\iota_I"] & E \arrow[r, "\hat{q}"] \arrow[d, "\iota_E"] & D \arrow[r] \arrow[d, "\iota_D"] & 0 \\
0 \arrow[r] & J_B \arrow[r, "j_B"] & B_\omega \arrow[r, "q_B"] & B^\omega \arrow[r] & 0
\end{tikzcd} \tag{1}$$

commutes, and the vertical maps are the inclusions. Let $\hat{\psi}\colon A \to E$ be the corestriction of ψ to E and $\hat{\kappa}_0\colon K_0(A) \to K_0(I)$ the group homomorphism which factors through $K_0(I_0)$. As the corestriction of ψ to $E_0 \subseteq E$ is unital, full and nuclear then $\hat{\psi}$ is unital, full and nuclear and we now claim that $\iota_2\hat{\psi}\colon A \to M_2(E)$ is unitizably full and nuclear.

Nuclearity of $\iota_2\hat{\psi}$ is immediate, and for being unitizably full it suffices to show that $\iota_2\hat{\psi}$ is full and the element $1_{M_2(E)} - \iota_2\hat{\psi}(1_A)$ is full in $M_2(E)$. Let us observe first that if we denote by J the ideal in $M_2(E)$ containing $\left(\begin{smallmatrix} 1_E & 0 \\ 0 & 0 \end{smallmatrix}\right)$ then, if $v = \left(\begin{smallmatrix} 0 & 0 \\ 1_E & 0 \end{smallmatrix}\right)$ it follows that $v^*v = \left(\begin{smallmatrix} 1_E & 0 \\ 0 & 0 \end{smallmatrix}\right) \in J$ and so $v^* = v^*vv^* \in J$, which in turn implies that $vv^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_E \end{smallmatrix}\right) \in J$. Hence, J contains the identity $1_{M_2(E)}$, and therefore $J = M_2(E)$. Now, let $a \in A$ then that $\hat{\psi}(a)$ is full implies that there is $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in E$ such that

$$\sum_{i=1}^{n} x_i\hat{\psi}(a)x_i^* = 1_E$$

thus,

$$\begin{pmatrix} 1_E & 0 \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} x_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\psi}(a) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_i^* & 0 \\ 0 & 0 \end{pmatrix} \in \langle i_2\hat{\psi}(a) \rangle$$

and by the previous argument we get that $\langle i_2\hat{\psi}(a) \rangle = M_2(E)$, hence $i_2\hat{\psi}$ is a full *-homomorphism. Moreover, since $1_{M_2(E)} - i_2\hat{\psi}(1_A) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_E \end{smallmatrix}\right)$, using again the argument above we know that this element is full in $M_2(E)$, concluding that $i_2\hat{\psi}$ is a unitizably full *-homomorphism.

If $\lambda\colon M_2(E) \to M(M_2(I))$ is the canonical *-homomorphism, then $\lambda\iota_2\hat{\psi}$ is unitizably full, since λ is unital and $\iota_2\hat{\psi}$ is unitizably full, and note that since I is a separable admissible kernel then $M_2(I)$ is a separable admissible kernel and in turn is stable and has the corona factorization property by Proposition 2.4.9. Hence, $\lambda\iota_2\hat{\psi}$ is nuclearly absorbing by Theorem 2.4.5. Now, since A satisfies the UCT and $M_2(I)$ is a separable admissible kernel we obtain by Proposition 2.4.9 the following group isomorphism

$$KK_{nuc}(A, M_2(I)) \to Hom_{\mathbb{Z}}(K_0(A), K_0(M_2(I)))$$

and since $K_0(\iota_2)\hat{\kappa}_0 \in Hom_{\mathbb{Z}}(K_0(A), K_0(I))$, there exist a lifting $x \in KK_{nuc}(A, M_2(I))$. Now, $\lambda\iota_2\hat{\psi}$ is weakly nuclear, since it is nuclear, and nuclearly absorbing, thus by Proposition 2.3.6

there exists a weakly nuclear, nuclearly absorbing representation $\theta\colon A \to M(M_2(I))$, such that $x = [\theta, \lambda\iota_2\hat{\psi}]$. As A is exact and θ is weakly nuclear Proposition A.0.7 asserts that θ is nuclear.

Consider the following commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & M_2(I) \arrow[r, "\hat{j}"] \arrow[d, equal] & M_2(E) \arrow[r, "\hat{q}"] \arrow[d, "\lambda"] & M_2(D) \arrow[r] \arrow[d, "\rho_1"] & 0 \\ 0 \arrow[r] & M_2(I) \arrow[r] & M(M_2(I)) \arrow[r, "\rho_2"] & M(M_2(I))/M_2(I) \arrow[r] & 0 \end{tikzcd} \tag{2}$$

where ρ_1 is defined in a way that makes the RHS square commutative. Now, observe that $\rho_1\hat{q}\iota_2\hat{\psi} = \rho_2\lambda\iota_2\hat{\psi}$ and since $(\theta, \lambda\iota_2\hat{\psi})$ is a Cuntz pair, it follows that $\lambda\iota_2\hat{\psi}(a) - \theta(a) \in M_2(I)$ for all $a \in A$. Hence, $\rho_2(\lambda\iota_2\hat{\psi}(a) - \theta(a)) = 0$, for all $a \in A$, and therefore $\rho_2\theta = \rho_1\hat{q}\iota_2\hat{\psi}$. So, if we apply the previous lemma for θ and $\hat{q}\iota_2\hat{\psi}$, there exists $\hat{\phi}_2\colon A \to M_2(E)$ *-homomorphism satisfying that

$$\lambda\hat{\phi}_2 = \theta \quad \text{and} \quad \hat{q}\hat{\phi}_2 = \hat{q}\iota_2\hat{\psi}$$

while, since both θ and $\hat{q}\iota_2\hat{\psi}$ are nuclear *-homomorphisms, we get by the same lemma that $\hat{\phi}_2$ is nuclear as well.

Now, consider the group homomorphism induced by $\hat{j}$

$$\hat{j}_*\colon KK_{nuc}(A, M_2(I)) \to KK_{nuc}(A, M_2(E))$$

and note that, since $\lambda(\hat{\phi}_2(a) - \iota_2\hat{\psi}(a)) \in M_2(I)$, for all $a \in A$, then $(\hat{\phi}_2 - \iota_2\hat{\psi})(a) \in M_2(I)$, for all $a \in A$, while $\hat{\phi}_2$ and $\iota_2\hat{\psi}$ are nuclear, hence it follows from Proposition 2.2.21 that

$$j_*[\lambda\hat{\phi}_2, \lambda\iota_2\hat{\psi}] = [\hat{\phi}_2] - [\iota_2\hat{\psi}]$$

Since x lifts $K_0(\iota_2)\hat{\kappa}_0$, then $\hat{j}_*(x)$ lifts $K_0(\hat{j}\iota_2)\hat{\kappa}_0 = K_0(\iota_2\hat{j})\hat{\kappa}_0$, and in particular

$$K_0(\iota_2\hat{j})\hat{\kappa}_0 = K_0(\hat{\phi}_2) - K_0(\iota_2\hat{\psi})$$

Hence, if we set $\phi_2 = \iota_E\hat{\phi}_2\colon A \to M_2(B_\omega)$, then by how κ is constructed, commutativity of (1) and the relation above, we get

$$\begin{aligned} K_0(\phi_2) - K_0(\iota_2\psi) &= K_0(\iota_E\iota_2\hat{j})\hat{\kappa} + K_0(\iota_E\iota_2\hat{\psi}) - K_0(\iota_2\psi) \\ &= K_0(\iota_2 j_B)\kappa = K_0(\iota_2)\sigma - K_0(\iota_2\psi) \end{aligned}$$

So, $K_0(\phi_2) = K_0(\iota_2)\sigma$, from which it follows that

$$K_0(\phi_2)([1_A]) = K_0(\iota_2)\sigma([1_A]) = [1_{B_\omega}]$$

Now, as B_ω has stable rank one by Proposition 2.4.7, then B_ω has cancellation of projections by Proposition A.0.11, which implies that $\phi_2(1_A) \sim_0 1_{B_\omega}$ and in turn that there exists unitary $u \in M_2(B_\omega)$ satisfying $u\phi_2(1_A)u^* = 1_{B_\omega} \oplus 0_{B_\omega}$. Thus, for any $a \in A$,

$$\begin{aligned} \phi_2(a) = \phi_2(a)u^*(1_{B_\omega} \oplus 0_{B_\omega})u &\implies u\phi_2(a)u^* = u\phi_2(a)u^*(1_{B_\omega} \oplus 0_{B_\omega}) \\ &\implies \mathrm{Ad}_u\phi_2(a) = \phi(a) \oplus 0_{B_\omega} = \iota_2\phi(a) \end{aligned}$$

for some $\phi\colon A \to B_\omega$ unital, *-homomorphism.

We claim that ϕ is the desired *-homomorphism. Firstly, since $\mathrm{Ad}_u\phi_2(p) \sim_0 \phi(p)$ for any $p \in P_\infty(A)$, it follows that

$$K_0(\iota_2\phi) = K_0(\phi_2) = K_0(\iota_2)\sigma$$

and now, by stability of K_0 (See Proposition 4.3.8 in [20]), $K_0(\iota_2)$ is a group isomorphism, hence $K_0(\phi) = \sigma$. Moreover, by construction we have the following

$$q_B\phi_2 = q_B\iota_E\hat{\phi}_2 = \iota_D\hat{q}\hat{\phi}_2 = \iota_D\hat{q}\iota_2\hat{\psi} = q_B\iota_2\psi$$

thus,

$$(\tau_{B_\omega} \otimes Tr_{M_2(\mathbb{C})})\phi_2 = (\tau_{B_\omega} \otimes Tr_{M_2(\mathbb{C})})\iota_2\psi$$

But, as ψ is a trace-preserving *-homomorphism, it follows that

$$\begin{aligned}\tau_A = \tau_{B_\omega}\psi = 2(\tau_{B_\omega} \otimes Tr_{M_2(\mathbb{C})})\iota_2\psi = 2(\tau_{B_\omega} \otimes Tr_{M_2(\mathbb{C})})\phi_2 \\ = 2(\tau_{B_\omega} \otimes Tr_{M_2(\mathbb{C})})\iota_2\phi = \tau_{B_\omega}\phi\end{aligned}$$

showing that ϕ is trace-preserving. For nuclearity of ϕ, we observe that, since ϕ_2 is nuclear *-homomorphism and ϕ is a compression of ϕ_2, then ϕ is also a nuclear *-homomorphism. It remains to show that ϕ is full. To this end, note that faithfulness of τ_A, implies that for any $a \in A^+$, $a \neq 0$, $\tau_{B_\omega}\phi(a) = \tau_A(a) > 0$, and considering that B_ω has strict comparison of positive elements, Lemma 1.2.23 asserts that $\phi(a)$ is a full element in B_ω. Now, as any element in A is a linear combination of positive elements, we conclude that for any non-zero $a \in A$, $\phi(a)$ is full in B_ω. Hence ϕ is a full *-homomorphism and the proof is complete. □

Keeping the same conditions on A, B, apart from the existence result that we just proved, there is also a classification result for unital, full, nuclear *-homomorphisms from A to B_ω, satisfying some mild conditions. The machinery and the proof techniques used in Proposition 3.1.2, are the driving forces in proving this classification result as well, and therefore we prefer to omit the proof and instead move towards the direction of refining these two results. The statement of this classification result follows.

Proposition 3.1.3. *Suppose A is a separabe, unital, exact C^*-algebra satisfying the UCT and B simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique trace τ_B such that every quasi trace on B is a trace and $K_1(B) = 0$.*

If $\phi, \psi\colon A \to B_\omega$ are unital, full, nuclear *-homomorphisms such that $K_0(\phi) = K_0(\psi)$ and $\tau_{B_\omega}\phi = \tau_{B_\omega}\psi$, then there exists unitary $u \in B_\omega$ such that $\phi = \mathrm{Ad}_u\psi$.

It is the case that with a bit more work we can upgrade the existence result of the Proposition 3.1.2, by finding a $*$-homomorphism from A to B, instead of B_ω, satifying the same properties, where we keep the same conditions on A and B. The main problem to tackle in this direction, is that only approximately multiplicative maps from A to B can be produced directly from a *-homomorphism from A to B_ω. In order to address this issue, we introduce the notions of $(\mathscr{G}, \delta)$-multiplicative maps and K_0-triples. The same idea for refining Proposition 3.1.2 will be applicable for refining Proposition 3.1.3 too, as it will be evident in the following.

Let A, B be two C^*-algebras, $\mathscr{G} \subseteq A$ finite set and $\delta > 0$. Then we say that a linear, self-adjoint map $\phi\colon A \to B$ is $(\mathscr{G}, \delta)$-multiplicative if

$$\|\phi(aa') - \phi(a)\phi(a')\| < \delta, \quad \text{for all } a, a' \in \mathscr{G}$$

Now, a K_0-triple for a unital C^*-algebra A is a triple $(\mathscr{G}, \delta, P)$, where $\mathscr{G} \subseteq A$ finite set, $\delta > 0$, and $P \subseteq P_\infty(A)$ finite set of projections, such that whenever $\phi\colon A \to B$ is a $(\mathscr{G}, \delta)$-multiplicative map, then

$$\left\|\phi(p^2) - \phi(p)^2\right\| < 1/4, \quad \text{for all } p \in P$$

It must be evident that for any $P \subseteq P_\infty(A)$ finite set, we can find sufficiently large $\mathscr{G}$ and sufficiently small δ such that $(\mathscr{G}, \delta, P)$ becomes a K_0-triple for A. Also, note that, if $(\mathscr{G}, \delta, P)$ is a K_0-triple for A and $\phi\colon A \to B$ is a $(\mathscr{G}, \delta)$-multiplicative map then $1/2$ is not contained in the spectrum of $\phi(p)$. Thus, if χ is the characteristic function on $[1/2, \infty)$ defined on the real numbers, we can then define a map $\phi_\#\colon P \to K_0(B)$, by $\phi_\#(p) = [\chi(\phi(p))]_0$. In this way, every linear, self-adjoint, $(\mathscr{G}, \delta)$-multiplicative map $\phi\colon A \to$ corresponds to a function $\phi_\#\colon P \to K_0(B)$. Now we present, the "counterpart" of Proposition 3.1.2 in the setting of linear, self-adjoint,

$(\mathscr{G}, \delta)$-multiplicative maps and K_0-triples.

Lemma 3.1.4. *Suppose A is a separabe, unital, exact C^*-algebra satisfying the UCT and B simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique trace τ_B such that every quasi trace on B is a trace and $K_1(B) = 0$.*

If τ_A is a faithful, amenable trace on A and $\sigma\colon K_0(A) \to K_0(B)$ is a group homomorphism such that $\sigma([1_A]_0) = [1_B]_0$ and $\hat{\tau}_B\sigma = \hat{\tau}_A$, then for any K_0-triple $(\mathscr{G}, \delta, P)$ for A, there is a unital, completely positive, nuclear, $(\mathscr{G}, \delta)$-multiplicative map $\phi\colon A \to B$ such that $\sigma([p]_0) = \phi_\#(p)$ for all $p \in P$ and $|\tau_B\phi(a) - \tau_A(a)| < \delta$ for all $a \in \mathscr{G}$.

Proof. Let $(\mathscr{G}, \delta, P)$ be a K_0-triple for A and let $\iota_B\colon B \to B_\omega$ the diagonal embedding. Then, $K_{\iota_B}\sigma([1_A]_0) = [1_{B_\omega}]$ and $\hat{\tau}_{B_\omega}K_0(\iota_B)\sigma = \hat{\tau}_A$. Therefore, by Proposition 3.1.2 there is a unital, nuclear *-homomorphism $\phi_\omega\colon A \to B_\omega$, satisfying that $K_0(\phi_\omega) = K_0(\iota_B)\sigma$ and $\tau_{B_\omega}\phi_\omega = \tau_A$. Then, by Choi-Effros lifting theorem there exist $\phi_n\colon A \to B$ u.c.p, nuclear maps such that $\pi_\omega((\phi_n(a))_n) = \phi_\omega(a)$, for all $a \in A$, where $\pi_\omega\colon l^\infty(B) \to B_\omega$ the quotient map. Now, set

$$S_1 = \cap_{a,a' \in \mathscr{G}}\{n \geq 1\colon\ \|\phi_n(aa') - \phi_n(a)\phi_n(a')\| < \delta\}$$

and since $\lim_{n\to\omega}\|\phi_n(aa') - \phi_n(a)\phi_n(a')\| = 0$, for any $a, a' \in A$, we get that for any $a, a' \in \mathscr{G}$ there is $I_\delta \in \omega$, such that $I_\delta \subseteq \{n \geq 1\colon\ \|\phi_n(aa') - \phi_n(a)\phi_n(a')\| < \delta\}$. Hence, $\{n \geq 1\colon\ \|\phi_n(aa') - \phi_n(a)\phi_n(a')\| < \delta\} \in \omega$ and since $\mathscr{G}$ is finite, it follows that $S_1 \in \omega$. Also, note that for each $n \in S_1$, ϕ_n is $(\mathscr{G}, \delta)$-multiplicative.

Take $p \in P$, and $d, k \in \mathbb{N}$, such that $p \in M_d(A)$ and $\sigma([p]_0) = [e] - [f]$, where e, f projections in $M_k(B)$. Then, $K_0(\phi_\omega)([p]_0) = [\iota_B(e)]_0 - [\iota_B(f)]_0$ which implies that $\phi_\omega(p) \oplus \iota_B(f)$ is stably equivalent to $\iota_B(e)$. Thus, there exists $l \in \mathbb{N}$ and $u \in M_{d+k+l}(B_\omega)$ partial isometry such that

$$u^*u = \phi_\omega(p) \oplus \iota_B(f) \oplus 1_{B_\omega}^{\oplus l} \quad \text{and} \quad uu^* = 0_{B_\omega}^{\oplus d} \oplus \iota_B(e) \oplus 1_{B_\omega}^{\oplus l}$$

Since $\chi(\phi_\omega(p)) = \phi_\omega(p)$, under the identification $M_{d+k+l}(l^\infty(B)) \cong l^\infty(M_{d+k+l}(B))$ we find a bounded sequence $(u_n)_n \subseteq M_{d+k+l}(B)$ satisfying that

$$\lim_{n\to\omega}\left\|u_n^*u_n - \chi(\phi_n(p)) \oplus f \oplus 1_B^{\oplus l}\right\| = \lim_{n\to\omega}\left\|u_nu_n^* - 0_B^{\oplus d} \oplus e \oplus 1_B^{\oplus l}\right\| = 0$$

and observe that since P is finite and p was arbitrary

$$S = \cap_{p\in P}\{n \in S_1\colon (\phi_n)_\#(p) = \sigma([p]_0)\} \in \omega$$

Now, let

$$T = \cap_{a\in\mathscr{G}}\{n \geq 1\colon |\tau_B\phi_n(a) - \tau_A(a)| < \delta\}$$

As, $\lim_{n\to\omega}\tau_B(\phi_n(a)) = \tau_{B_\omega}\phi_\omega(a) = \tau_A(a)$, for all $a \in A$, we obtain that $T \in \omega$. Hence, $S \cap T \neq \emptyset$, and if we fix any $n \in S \cap T$, and set $\phi = \phi_n$, we get the desired unital, completely positive, nuclear, $(\mathscr{G}, \delta)$-multiplicative map. □

There is also a "counterpart" of the classification result (Proposition 3.1.3) and it has the following form.

Lemma 3.1.5. *Suppose A is a separabe, unital, exact C^*-algebra satisfying the UCT and B simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique trace τ_B such that every quasi trace on B is a trace and $K_1(B) = 0$.*

For any τ_A faithful trace on A, $F \subseteq A$ finite set, $\epsilon > 0$, there exists a K_0-triple $(\mathscr{G}, \delta, P)$ for A, such that if $\phi, \psi\colon A \to B$ unital, completely positive, nuclear $(\mathscr{G}, \delta)$-multiplicative maps with $\phi_\#(p) = \psi_\#(p)$ for all $p \in P$, and $|\tau_B\phi(a) - \tau_A(a)| < \delta$, $|\tau_B\psi(a) - \tau_A(a)| < \delta$, for all $a \in G$ then there exists unitary $u \in B$ such that

$$\|\phi(a) - u\psi(a)u^*\| < \epsilon, \quad \textit{for all } a \in F.$$

Proof. We assume that the result does not hold for some τ_A faithful trace on A, $F \subseteq A$ finite set and $\epsilon > 0$. Then, we note that since A is separable then $A \otimes \mathbb{K}$ is separable and since any subspace of a separable metric space is separable, we get that $P_\infty(A) = P(M_\infty(A))$ is separable. So, we can find an increasing sequence of finite sets $(P_n)_n$ with dense union in $P_\infty(A)$. Now, for each $n \in \mathbb{N}$, there is $N(n) \in \mathbb{N}$ such that $P_n \subseteq P(\cup_{j=1}^{N(n)} M_j(A))$, since P_n is finite. But, separability of A also allow us to choose $\mathscr{G}_n$ finite sets in A, such that $\cup_n \mathscr{G}_n$ dense in A, $\mathscr{G}_n \subseteq \mathscr{G}_{n+1}$ and $P_n \subseteq P(\cup_{j=1}^{N(n)} M_j(\mathscr{G}_n))$, for all $n \in \mathbb{N}$. Finally, we select δ_n to be a decreasing sequence to zero, such that whenever $\phi\colon A \to B$ is a $(\mathscr{G}_n, \delta_n)$-multiplicative map then its inflation $\phi\colon M_j(A) \to M_j(B)$ is $(M_j(\mathscr{G}_n), 1/4)$-multiplicative, for all $1 \leq j \leq N(n)$. This last assertion, guarantees that $(\mathscr{G}_n, \delta_n, P_n)$ are K_0-triples for A, for each $n \in \mathbb{N}$.

By our assumption on τ_A, F and ϵ, we can find for each K_0-triple $(\mathscr{G}_n, \delta_n, P_n)$, a pair of unital, nuclear, completely positive, $(\mathscr{G}_n, \delta_n)$-multiplicative maps $\phi_n, \psi_n\colon A \to B$ such that $(\phi_n)_\#(p) = (\psi_n)_\#(p)$, $|\tau_B\phi_n(a) - \tau_A(a)| < \delta_n$, $|\tau_B\psi_n(a) - \tau_A(a)| < \delta_n$, for all $a \in G_n$, $p \in P_n$, but such that for each unitary $u_n \in B$, there is $a \in F$ satisfying

$$\|\phi_n(a) - u_n\psi_n(a)u_n^*\| \geq \epsilon$$

Since ϕ_n, ψ_n are $(\mathscr{G}_n, \delta_n)$-multiplicative we have that

$$\|\phi_n(aa') - \phi_n(a)\phi_n(a')\| < \delta_n \text{ and } \|\psi_n(aa') - \psi_n(a)\psi_n(a')\| < \delta_n$$

for all $a, a' \in \mathscr{G}_n$ and $n \in \mathbb{N}$. Moreover, as $\delta_n \to 0$, $\mathscr{G}_n \subseteq \mathscr{G}_{n+1}$, for all $n \in \mathbb{N}$, and $\overline{\cup_n \mathscr{G}_n} = A$, it follows that

$$\lim_{n\to\omega} \|\phi_n(aa') - \phi_n(a)\phi_n(a')\| = \lim_{n\to\omega} \|\psi_n(aa') - \psi_n(a)\psi_n(a')\| = 0$$

for $a, a' \in A$, where ω is a free ultrafilter on the natural numbers. So, the maps $\phi_\omega, \psi_\omega\colon A \to B_\omega$, induced by $(\phi_n)_n, (\psi_n)_n$, are unital *-homomorphisms and employing the same arguments we see that

$$\tau_{B_\omega}\phi_\omega(a) = \lim_{n\to\omega} \tau_B\phi_n(a) = \tau_A(a) = \lim_{n\to\omega} \tau_B\psi_n(a) = \tau_{B_\omega}\psi_\omega$$

for all $a \in A$. Furhermore, as ϕ_n, ψ_n are nuclear, u.c.p maps and A is exact we obtain from Proposition A.0.3 that ϕ_ω, ψ_ω are nuclear *-homomorphisms, and since τ_A is a faithful trace and B_ω has strict comparison of positive elements with respect to its trace (see Proposition 2.4.7), we get by Lemma 2.2.3 that ϕ_ω, ψ_ω are also full *-homomorphisms.

Now, take $p \in P_n$ such that $p \in M_d(A)$ for some $d \in \mathbb{N}$ then, as $(\phi_k)_\#(p) = (\psi_k)_\#(p)$ for all $k \geq n$, or equivalently $[\chi(\phi_k(p))]_0 = [\chi(\psi_k(p))]_0$, for all $k \geq n$, and since B has cancellation of projections by the proof of Proposition 2.4.7, there are partial isometries $u_k \in M_d(B)$, satisfying

$$u_k^* u_k = \chi(\phi_k(p)) \text{ and } u_k u_k^* = \chi(\psi_k(p))$$

for all $k \geq n$. Let, $u = \pi_\omega((u_k)_k)$ be the corresponding element in B_ω, where $\pi_\omega\colon l^\infty(B) \to B_\omega$ the quotient map. Then,

$$u^*u = \pi_\omega((\chi(\phi_k(p)))_{k\geq n}) = \chi(\pi_\omega((\phi_k(p))_{k\geq n})) = \chi(\phi_\omega(p)) = \phi_\omega(p)$$

and similarly

$$uu^* = \chi(\psi_\omega(p)) = \psi_\omega(p)$$

So, $[\phi_\omega(p)]_0 = [\psi_\omega(p)]_0$ and since p was arbitrary, it follows that $K_0(\phi_\omega) = K_0(\psi_\omega)$.

Now, if we apply Proposition 3.1.3 to ϕ_ω and ψ_ω, we find a unitary $w \in B_\omega$ satisfying that $\phi_\omega = \mathrm{Ad}_w\psi_\omega$. From the proof of Proposition 2.4.7, B_ω has path connected unitary group, hence $w \sim_h 1_{B_\omega} = \pi_\omega(1_{l^\infty(B)})$, which implies that $w \in \pi_\omega(U(l^\infty(B)))$. So, if $(w_n)_n \subseteq B$ is the sequence of unitaries lifting w, then

$$\lim_{n\to\omega} \|\phi_n(a) - w_n\psi_n(a)w_n^*\| = 0$$

for all $a \in A$, and in fact there is $n_0 \in \mathbb{N}$ such that

$$\left\|\phi_{n_0}(a) - w_{n_0}\psi_{n_0}(a)w_{n_0}^*\right\| < \epsilon$$

for all $a \in F$, which is a contradiction. $\square$

These last two lemmas are the cornerstones in refining Proposition 3.1.2 and Proposition 3.1.3. The key factor lies in the idea that we can now globalise the local essence results of the last two lemmas in a way that all the K-theoretical information will remain intact after this transition. The rigorous statement of this refinement follows.

Theorem 3.1.6. *Suppose A is a separable, unital, exact C^*-algebra satisfying the UCT and B simple, unital, $\mathscr{Q}$-stable C^*-algebra with unique trace τ_B such that every quasi trace on B is a trace and $K_1(B) = 0$.*

1) *If τ_A is a faithful, amenable trace on A and $\sigma\colon K_0(A) \to K_0(B)$ is a group homomorphism such that $\sigma([1_A]_0) = [1_B]_0$ and $\hat{\tau}_B\sigma = \hat{\tau}_A$, then there exists a unital, faithful, nuclear *-homomorphism $\phi\colon A \to B$ satisfying $K_0(\phi) = \sigma$ and $\tau_B\phi = \tau_A$.*
2) *If $\phi, \psi\colon A \to B$ are unital, faithful, nuclear *-homomorphisms such that $K_0(\phi) = K_0(\psi)$ and $\tau_B\phi = \tau_B\psi$, then there exists a sequence of unitaries $u_n \in B$ such that*

$$\lim_{n\to\infty} \|\phi(a) - u_n\psi(a)u_n^*\| = 0$$

for all $a \in A$.

Proof. 1) Let $F_n \subseteq A$ be an increasing sequence of finite sets with dense union in A and $\epsilon_n > 0$ a sequence such that $\sum_n \epsilon_n < \infty$. Then, by the previous lemma we find $(\mathscr{G}_n, \delta_n, P_n)$ K_0-triples corresponding to the pairs (F_n, ϵ_n). Following the same ideas as in the proof of the last lemma we arrange $\mathscr{G}_n$, P_n and δ_n, such that $\mathscr{G}_n$, P_n are increasing sequences of finite subsets with dense union in A and $P_\infty(A)$, respectively, and δ_n is a sequence decreasing to zero.

Now, by Lemma 3.1.4 we find for each K_0-triple $(\mathscr{G}_n, \delta_n, P_n)$ a unital, completely positive, nuclear, $(\mathscr{G}_n, \delta_n)$-multiplicative map $\psi_n\colon A \to B$ satisfying that $\sigma([p]_0) = (\psi_n)_\#(p)$ and $|\tau_B\psi_n(a) - \tau_A(a)| < \delta_n$, for all $p \in P_n$ and $a \in \mathscr{G}_n$. Since by contstruction, for any $n \in \mathbb{N}$, ψ_{n+1} is $(\mathscr{G}_n, \delta_n)$-multiplicative and $(\mathscr{G}_{n+1}, \delta_{n+1})$-multiplicative map, by Lemma 3.1.5 there exists unitary $u_{n+1} \in B$ such that

$$\left\|\psi_n(a) - u_{n+1}\psi_{n+1}(a)u_{n+1}^*\right\| < \epsilon_n, \quad \text{for all } a \in F_n$$

Define $\phi_1 = \psi_1$ and $\phi_n = \mathrm{Ad}(u_1u_2\cdots u_n)\psi_n$ which are clearly unital, completely positive, nuclear, $(\mathscr{G}_n, \delta_n)$-multiplicative maps, and observe that for any $n \in \mathbb{N}$ and $a \in F_n$ we have that

$$\begin{aligned}\|\phi_n(a) - \phi_{n+1}(a)\| &= \|\mathrm{Ad}(u_1u_2\cdots u_n)\psi_n(a) - \mathrm{Ad}(u_1u_2\cdots u_{n+1})\psi_{n+1}(a)\| \\ &= \|\mathrm{Ad}(u_1u_2\cdots u_n)(\psi_n(a) - \mathrm{Ad}(u_{n+1})\psi_{n+1}(a))\| < \epsilon_n\end{aligned}$$

Hence, $(\phi_n(a))$ is a Cauchy sequence in B for any $a \in \cup_n F_n$. So, for $\epsilon > 0$ and $a \in A$ we can find $n \in \mathbb{N}$ and $a' \in F_n$ such that $\|a - a'\| < \epsilon/3$ and $\|\phi_n(a') - \phi_{n+1}(a')\| < \epsilon/3$, which implies that

$$\|\phi_n(a) - \phi_{n+1}(a)\| = \|\phi_n(a) - \phi_n(a') + \phi_n(a') - \phi_{n+1}(a') + \phi_{n+1}(a') - \phi_{n+1}(a)\| < \epsilon$$

From this, we obtain that $(\phi_n(a))_n$ is a Cauchy sequence in B for any $a \in A$ and therefore we are allowed to define a map $\phi\colon A \to B$ by

$$\phi(a) = \lim_{n\to\infty} \phi_n(a)$$

We claim that ϕ is the desired *-homomorphism. Firstly, we see that since for any $n \in \mathbb{N}$ and $a \in G_n$, we have that

$$\|\phi_k(aa') - \phi_k(a)\phi_k(a')\| < \delta_k, \quad \text{for all } k \geq n$$

it follows that $\phi(aa') = \phi(a)\phi(a')$, for any $a, a' \in \cup_n \mathscr{G}_n$. Thus, using that $\cup_n \mathscr{G}_n$ is dense in A, an $\epsilon/3$ argument as above, shows that $\phi(aa') = \phi(a)\phi(a')$, for all $a, a' \in A$, and as ϕ is already a u.c.p map, we get that ϕ is a *-homomorphism. Moreover, note that $\tau_B\phi_n = \tau_B\psi_n$, for any $n \in \mathbb{N}$. Hence, in a similar manner as above, we use that for any $n \in \mathbb{N}$ and $a \in \mathscr{G}_n$

$$|\tau_B\phi_k(a) - \tau_A(a)| < \delta_k, \quad \text{for all } k \geq n$$

to obtain first that $\tau_B\phi(a) = \tau_A(a)$, for all $a \in \cup_n \mathscr{G}_n$, and by an $\epsilon/3$ argument that $\tau_B\phi(a) = \tau_A(a)$, for all $a \in A$. That ϕ is trace-preserving, implies that ϕ is faithful, since τ_A is faithful. Now, It remains to show that $\sigma = K_0(\phi)$. To this end, firstly note that as unitary equivalence implies von Neumann-Murray equivalence of projections, it follows that for any $n \in \mathbb{N}$ and $p \in P_n$, $\sigma([p]_0) = (\psi_k)_\#(p) = (\phi_k)_\#(p)$, for any $k \geq n$. Thus, for any $n \in \mathbb{N}$, $p \in P_n$ and $k \geq n$

$$K_0(\phi)([p]_0) = [\phi(p)]_0 = [\chi(\phi(p))]_0 = [\chi(\phi_k(p))]_0 = \sigma([p]_0)$$

As (P_n) is a dense sequence of finite sets in $P_\infty(A)$, for any $n \in \mathbb{N}$ and $p \in P_n(A)$, there exists $q \in \cup_n P_n$ such that $\|p - q\| < 1$. But then $p \sim_0 q$ (See Proposition 2.2.4 in [20]) which shows that

$$\sigma([p]_0) = \sigma([q]_0) = K_0(\phi)([q]_0) = K_0(\phi)([p]_0)$$

concluding that $\sigma = K_0(\phi)$.

2) Since ϕ is a faithful *-homomorphism, then $\tau_A = \tau_B\phi$ defines a faithful trace on A. Now, let $\epsilon > 0$ and $(F_n)_n \subseteq A$ be an increasing sequence of finite sets with dense union in A. As, ϕ, ψ trivially satisfy the conditions of the previous lemma for any K_0-triple for A, there exists unitary $u_n \in B$ satisfying that

$$\|\phi(a) - u_n\psi(a)u_n^*\| < \epsilon/3 \quad \text{for all } a \in F_n$$

Since, for any $a \in A$ we can find $n \in \mathbb{N}$ and $b \in F_n$ such that $\|a - b\| < \epsilon/3$, we obtain that

$$\|\phi(a) - u_n\psi(a)u_n^*\| \leq \|\phi(a) - \phi(b)\| + \|\phi(b) - u_n\psi(b)u_n^*\| + \|\psi(a) - \psi(b)\| < \epsilon$$

as desired. □

As an ending point of this project, we restrict our attention to simple, unital AF-algebras with unique trace and divisible K_0-group and we aim to show that the conclusion of Theorem 3.1.6 still holds when the C^*-algebra B is replaced by an AF-algebra with the aforementioned conditions.

Corollary 3.1.7. *Suppose A is a separabe, unital, exact C^*-algebra satisfying the UCT and B simple, unital, AF-algebra with unique trace τ_B and divisible K_0-group.*

1) *If τ_A is a faithful, amenable trace on A and $\sigma\colon K_0(A) \to K_0(B)$ is a group homomorphism such that $\sigma([1_A]_0) = [1_B]_0$ and $\hat{\tau}_B\sigma = \hat{\tau}_A$, then there exists a unital, faithful *-homomorphism $\phi\colon A \to B$ satisfying $K_0(\phi) = \sigma$ and $\tau_B\phi = \tau_A$.*

2) *If $\phi, \psi\colon A \to B$ are unital, faithful *-homomorphisms such that $K_0(\phi) = K_0(\psi)$ and $\tau_B\phi = \tau_B\psi$, then there exists a sequence of unitaries $u_n \in B$ such that*

$$\lim_{n\to\infty} \|\phi(a) - u_n\psi(a)u_n^*\| = 0$$

for all $a \in A$.

Proof. As B is a unital AF-algebra, it is isomorphic to the inductive limit of an inductive sequence (B_n, ϕ_n), where B_n are finite dimensional C^*-algebras and $\phi_n\colon B_n \to B_{n+1}$ are unital

*-homomorphisms. Since B_n are finite dimensional C^*-algebras it is a well known fact that

$$B_n = M_{j_1}(\mathbb{C}) \oplus M_{j_2}(\mathbb{C}) \oplus \cdots \oplus M_{j_{k_n}}(\mathbb{C})$$

for some $j_i \in \mathbb{N}$, $i = 1, 2, ..., k_n$. Hence, by continuity of K_1-functor we obtain that $K_1(B) = 0$, since $K_1(B_n) = 0$ for any $n \in \mathbb{N}$. Moreover, by Exercise 2.9 and Paragraph 3.1.1 in [20], $K_0(M_{j_i}(\mathbb{C})) \cong \mathbb{Z}$, for all $i = 1, 2, ..., k_n$, and since K_0-functor preserves direct sums and is continuous we get that

$$K_0(B) \cong \varinjlim(\mathbb{Z}^{j_{k_n}}, K_0(\phi_n))$$

But, as $\mathbb{Z}^{j_{k_n}}$ are torsion free abelian groups for each $n \in \mathbb{N}$, it is easily verified that $K_0(B)$ is also a torsion free abelian group.

Now, using this observation, we claim that the group homomorphism

$$K_0(B) \xrightarrow{\Lambda} K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}$$

given by

$$[p]_0 \mapsto [p]_0 \otimes 1_{\mathbb{Q}}$$

where p is a projection in $M_\infty(B)$, is an isomorphism. Let us show first that it is surjective. Let $[p]_0 \otimes m/n \in K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $m, n \in \mathbb{Z}$. Then, $[p]_0 \otimes m/n = m([p]_0 \otimes 1/n)$, and since $K_0(B)$ is divisible, we find $[q]_0 \in K_0(B)$ such that $n[q]_0 = [p]_0$. Hence,

$$m([p]_0 \otimes 1/n) = m(n[q]_0 \otimes 1/n) = (m[q]_0 \otimes 1_{\mathbb{Q}}) = \Lambda(m[q]_0)$$

which proves that Λ is surjective. For injectivity, we use a standard fact from commutative algebra (see for example [4]) that for any $\mathbb{Z}$-module M, if $S = \mathbb{Z} \setminus \{0\}$, and $S^{-1}\mathbb{Z}$, $S^{-1}M$ are the localizations of $\mathbb{Z}$ and M with respect to the multiplicatively closed set S, then there exists an isomorphism

$$M \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} \to S^{-1}M$$

given by

$$m \otimes z/s \mapsto mz/s$$

As $S^{-1}\mathbb{Z} = \mathbb{Q}$, if we set $M = K_0(B)$, then it suffices to show that $[p]_0 = 0$ when $[p]_0/1_{\mathbb{Q}} = 0$ in $S^{-1}K_0(B)$. By the definition of localizations, $[p]_0/1_{\mathbb{Q}} = 0$ if there exists $s \in S = \mathbb{Z} \setminus \{0\}$, such that $s[p]_0 = 0$. But, as $K_0(B)$ is torsion free, this not possible, except $[p]_0 = 0$, which shows the injectivity of Λ.

Now, we view $B \otimes \mathscr{Q}$ as the inductive limit of the the inductive sequence $(M_{k_i}(B), \psi_i)$, where $\psi_i \colon M_{k_i}(B) \to M_{k_{i+1}}(B)$ are unital *-homomorphisms with multiplicity k_{i+1}/k_i, $i \geq 1$, and we define the following group homomorphisms

$$\rho_i \colon K_0(M_{k_i}(B)) \to K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}$$

by

$$\rho_i([p]_0) = [p]_0 \otimes 1/k_i$$

Then, by continuity of K_0, $K_0(B \otimes \mathscr{Q})$ is isomorphic to the inductive limit of $(K_0(M_{k_i}(B)), K_0(\psi_i))$, and observe that the following diagram

$$\begin{tikzcd}
K_0(M_{k_{i+1}}(B)) \arrow[dr, "\rho_{i+1}"] & \\
K_0(M_{k_i}(B)) \arrow[u, "K_0(\psi_i))"] \arrow[r, "\rho_i"] & K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{tikzcd}$$

commutes for all $i \geq 1$. Hence, by the universal property of inductive limits there exists a group homomorphism $\lambda \colon K_0(B \otimes \mathscr{Q}) \to K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ making the diagram

$$\begin{tikzcd} K_0(M_{k_i}(B)) \arrow[d, "\iota_i"] \arrow[dr, "\rho_i"] & \\ K_0(B \otimes \mathscr{Q}) \arrow[r, "\lambda"] & K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

commutative for any $i \geq 1$, where the vertical map amounts to the inclusion. Now, we claim that λ is a group isomorphism.

To this end, let $[p]_0 \in K_0(B \otimes \mathscr{Q})$ and suppose that $\lambda([p]_0) = 0$. Since, $[p]_0 \in K_0(M_{k_i}(B))$ for some $i \geq 1$, then by the commutative diagram above we have that $\rho_i([p]_0) = [p]_0 \otimes 1/k_i = 0$. So, $k_i[p]_0 \otimes 1_{\mathbb{Q}} = 0$ and as previously, it follows that $[p]_0 = 0$, since $K_0(B)$ is torsion free.

For surjectivity, it suffices to show that for any $[p]_0 \otimes 1/n \in K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, there exists $i \geq 1$ and $[p']_0 \in K_0(M_{k_i}(B))$ such that $\rho_i([p']_0) = [p]_0 \otimes 1/n$. But since we can find $i, j \in \mathbb{N}$ such that $k_j/k_i = n$, divisibility of $K_0(B)$, provides us with $[p']_0 \in K_0(B)$ such that $k_i[p]_0 = k_j[p']_0 \in K_0(M_{k_j}(B))$. Hence, it follows that

$$\rho_j(k_j[p']_0) = k_j[p']_0 \otimes 1/k_j = k_i[p]_0 \otimes 1/k_j = [p]_0 \otimes 1/n$$

thus λ is surjective.

Summarizing, we have shown that $K_0(B \otimes \mathscr{Q}) \cong K_0(B) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_0(B)$, and it is obvious by construction that this group isomorphism preserve distinguished order units and maps $K_0(B)^+$ onto $K_0(B \otimes \mathscr{Q})^+$. Thus by Elliot's classification of unital AF-algebras, we obtain that

$$B \cong B \otimes \mathscr{Q}$$

i.e B is $\mathscr{Q}$-stable. Moreover, as any AF-algebra is nuclear, B is nuclear, hence exact, and since any quasi-trace on exact C^*-algebra is a trace (see [17]), and any *-homomorphism $A \to B$ is nuclear, as long as B is nuclear, we can apply the previous theorem, from which the desired conclusion follows immediately.

$\square$

A Appendix

The main reason for creating this appendix it to collect results that are used heavily in the proofs of Chapter 3. As a starting point, we show that for any unital, separable, exact C^*-algebra in the UCT class which admits a faithful, amenable trace there is a unital, full, nuclear and trace-preserving *-homomorphism into an ultrapower of the universal UHF algebra, $\mathscr{Q}$, with respect to a fixed free ultrafilter defined on the natural numbers. The proof consists of a critical middle step concerning quasidiagonal traces and their relation to faithful, amenable traces. Therefore we start by giving the definitions of an amenable and quasidiagonal trace.

Definition A.0.1. *Let A be a C^*-algebra. We say that a trace τ_A on A is amenable if there exists a net of c.c.p maps $\phi_i\colon A \to M_{k_i}(\mathbb{C})$ such that*

$$i)\ \|\phi_i(aa') - \phi_i(a)\phi_i(a')\|_{2,Tr} \longrightarrow 0, \quad \text{for all } a, a' \in A$$

$$ii)\ Tr_{k(i)}\phi_i(a) \longrightarrow \tau_A(a), \quad \text{for all } a \in A$$

Moreover, we say that τ_A is quasidiagonal, if there exists a net of c.c.p maps $\phi_i\colon A \to M_{k_i}(\mathbb{C})$ satisfying condition (ii) and the following property

$$iii)\ \|\phi_i(aa') - \phi_i(a)\phi_i(a')\| \longrightarrow 0, \quad \text{for all } a, a' \in A$$

The first preliminary result that we are after is the following. We postpone its proof for a moment for reasons that we explain right after the statement

Proposition A.0.2 ([13], Proposition 1.4)**.** *Let A be a separable, unital, exact C^*-algebra and τ_A a trace on A. The following statements are equivalent:*

i) τ_A is quasidiagonal,

ii) there exists a unital, nuclear, trace-preserving *-homomorphism $\theta\colon A \to \mathscr{Q}_\omega$

In order to establish the nuclearity of the *-homomorphism θ above, we will need an argument showing that the c.c.p maps given in the definition of quasidiagonal trace induce a nuclear map into the product $l^\infty(\mathscr{Q})$. The following proposition does this important work for us.

Proposition A.0.3 ([8], Proposition 3.3)**.** *Let A be a separable C^*-algebra and let $\phi_n\colon A \to B_n$ be a sequence of nuclear c.c.p maps into C^*- algebras B_n. Then, if A is exact, the map $\phi = (\phi_n)_n\colon A \to \prod_n B_n$ is a nuclear c.c.p map.*

Proof. Since A is exact, it is a fact that there is a nuclear embedding $\psi\colon A \to C$, into some C^*-algebra, C. Let $\epsilon > 0$ and $F \subset A$ finite, then it suffices to find a c.c.p map $\lambda\colon C \to \prod_n B_n$ such that $\|\phi(a) - \lambda \circ \psi(a)\| < \epsilon$, for all $a \in F$.

Since ϕ_n are nuclear, we can find $k_n \in \mathbb{N}$ and c.c.p maps $\lambda_n\colon A \to M_{k(n)}$ and $\mu_n\colon M_{k(n)} \to B_n$ such that $\|\mu_n \circ \lambda_n(a) - \phi_n(a)\| < \epsilon$ for all $a \in F$. Now, by Arverson's Extention Theorem find c.c.p maps $\tilde{\lambda}_n\colon C \to M_{k(n)}$ satisfying $\tilde{\lambda}_n \circ \psi = \lambda_n$ and set $\lambda = (\mu_n \circ \tilde{\lambda}_n)_n\colon C \to \prod_n B_n$. Then λ is a c.c.p map and

$$\|\phi(a) - \lambda \circ \psi(a)\| = \sup_n \left\|\phi_n(a) - \mu_n \circ \tilde{\lambda}_n \circ \psi(a)\right\| = \sup_n \|\phi_n(a) - \mu_n \circ \lambda_n(a)\| < \epsilon$$

for all $a \in F$. Now by separability of A the desired conclusion follows. ☐

Note A.0.4. If we restrict our attention to the universal UHF-algebra $\mathscr{Q}$, since $\mathscr{Q}$ is nuclear, any c.c.p map into $\mathscr{Q}$ has to be nuclear. So if we additionally suppose that we have a c.c.p map from a separable, exact C^*-algebra A into $l^\infty(\mathscr{Q})$, then by the proposition above we obtain that it must be nuclear.

Now, we are ready to prove Proposition A.0.2.

Proof. (Proposition A.0.2) Suppose first that there exists $\theta\colon A \to \mathscr{Q}_\omega$ unital, nuclear, *-homomorphism. Then, by Choi-Efrros Lifting Theorem there exist a sequence $\theta_n\colon A \to \mathscr{Q}$ of c.c.p maps which satisfy the following conditions

$$\|\theta_n(aa') - \theta_n(a)\theta_n(a')\| \xrightarrow{n\to\omega} 0, \quad \text{for all } a, a' \in A$$

and

$$\lim_{n\to\omega} \tau_{\mathscr{Q}}\theta_n(a) = \tau_A(a), \quad \text{for all } a \in A$$

Now, as $\mathscr{Q} = \overline{\cup_n M_{k_n}(\mathbb{C})}$, where $(k_n) \subseteq \mathbb{N}$ and $k_{n+1}|k_n$ for all $\in \mathbb{N}$, we can find for sufficiently large n, surjective, contractive linear maps $\psi_n\colon \mathscr{Q} \to M_{k_n}(\mathbb{C})$, such that $\psi_n^2 = \psi_n$. Then, by Tomiyama's Theorem, we know that $\psi_n(ab) = \psi_n(a)\psi_n(b)$ and $\psi_n(ba) = \psi_n(b)\psi_n(a)$ for all $a \in M_{k_n}(\mathbb{C}), b \in \mathscr{Q}$. Now, this condition combined with the density of $\cup_n M_{k_n}(\mathbb{C})$ in $\mathscr{Q}$, show that

$$\|\psi_n\theta_n(aa') - \psi_n\theta_n(a)\psi_n\theta_n(a')\| \xrightarrow{n\to\omega} 0, \quad \text{for all } a, a' \in A$$

and

$$\lim_{n\to\omega} Tr_{k_n}\psi_n\theta_n(a) = \tau_A(a), \quad \text{for all } a \in A$$

Since, $\psi_n\theta_n\colon A \to M_{k_n}(\mathbb{C})$ is a sequence of c.c.p maps, it follows that τ_A is a quasidiagonal on A.

On the other hand, suppose that τ_A is a quasidiagonal trace. Then, as A is separable we can find a sequence $\theta_n\colon A \to M_{k_n}(\mathbb{C}) \subset \mathscr{Q}$ of c.c.p maps satisfying that

$$\|\theta_n(aa') - \theta_n(a)\theta_n(a')\| \xrightarrow{n\to\omega} 0, \quad \text{for all } a, a' \in A \quad (1)$$

and

$$\lim_{n\to\infty} Tr_{k_n}\theta_n(a) = \tau_A(a), \quad \text{for all } a \in A \quad (2)$$

As θ_n are nuclear c.c.p maps, Proposition A.0.3 implies that the induced map $\theta\colon A \to \mathscr{Q}_\omega$ is nuclear. Moreover, conditions (1), (2) imply that θ is also unital and trace-preserving *-homomorphism, hence θ has all the desired properties and the proof is done. □

The last fact needed in order to establish the result that we promised at the beginning of the appendix is the following. The proof is ommited.

Theorem A.0.5 ([13], Theorem 3.7)**.** *Any faithful, amenable trace on a separable, unital, exact C^*-algebra satisfying the UCT is quasidiagonal.*

Theorem A.0.6 ([30], Theorem 1.2)**.** *Let A be a separable, unital, exact C^*-algebra satisfying the UCT and τ_A faithful, amenable trace on A, then there is a unital, full, nuclear, trace-preserving *-homomorphism $\theta\colon A \to \mathscr{Q}_\omega$.*

Proof. Employing Proposition A.0.2 and Theorem A.0.5 we immediately get a unital, nuclear, trace-preserving *- homomorphism $\theta\colon A \to \mathscr{Q}_\omega$. Moreover, as $\mathscr{Q}_\omega$ has strict comparison of positive elements with respect to its trace (see Proposition 2.4.7), θ is trace-preserving and τ_A is a faithful trace, Lemma 1.2.23 implies that θ is a full *-homomorphism, as required. □

In the previous results we already used the fact that exact C^*-algebras are nuclearly embeddable. To this end, we recall that in Proposition 3.1.2 an another characterization of exact C^*-algebras in terms of c.p weakly nuclear maps was used critically. The proof of this characterization follows.

Proposition A.0.7 ([14], Proposition 3.2)**.** *If A is a C^*-algebra and B is a σ-unital C^*-algebra then A is exact iff any weakly nuclear c.p map $\phi\colon A \to M(B)$ is nuclear.*

Proof. First of all, note that if A is non unital, then we can extend ϕ to a u.c.p weakly nuclear map from the unitization $\tilde{A}$ of A into $M(B)$. Therefore, we may assume that ϕ and A are unital. Moreover, using that nuclearity is a local property, it is sufficient to show that for any unital, separable C^*-subalgebra A_0 of A, $\phi|_{A_0}$ is nuclear.

Let $A_0 \subseteq B(H)$ be a non-degenerate representation over some separable Hilbert space H. Then, as $\mathbb{K}$ is a closed two-sided ideal inside $B(H)$, $B(H) = M(\mathbb{K})$ (see Note 1.1.3). Let $\iota\colon A_0 \to M(\mathbb{K})$ be the unital inclusion. Since A is exact, then any separable C^*-subalgebra of A is exact and in particular A_0 is exact. Thus, A_0 is nuclearly embeddable and by Arverson's extention Theorem we get that ι must be nuclear. Now, consider the composition

$$\Phi\colon A_0 \overset{\iota}{\hookrightarrow} M(\mathbb{K}) \xhookrightarrow{1_{M(B)}\otimes id_{M(\mathbb{K})}} M(B)\otimes M(\mathbb{K}) \hookrightarrow M(B\otimes\mathbb{K})$$

which is again nuclear, and it is a fact that Φ absorbs any unital, weakly nuclear c.p map (see [9]).

So, if ϕ_0 is the map given by the following composition

$$A_0 \xrightarrow{\phi|_{A_0}} M(B) \xhookrightarrow{id_{M(B)}\otimes e_{11}} M(B)\otimes M(K) \overset{\iota_1}{\hookrightarrow} M(B\otimes K)$$

then ϕ_0 is a c.p map since ϕ is, and we claim that is weakly nuclear.

Let $b = b_1 \otimes k_1 \in B\otimes K$, consider the map $A_0 \to B\otimes K$ given by $a \mapsto b^*\phi_0(a)b$ and let $\rho_n\colon A \to M_{k(n)}(\mathbb{C})$, $\theta_n\colon M_{k(n)}(\mathbb{C}) \to B$ be the c.c.p maps exhibiting the weakly nuclearity of $\phi|_{A_0}$ for b_1 i.e, the nuclearity of the map $a \mapsto b_1^*\phi|_{A_0}(a)b_1$. Define $\rho_n' = \rho_n \otimes e_{11}\colon A \to M_{k_n}(\mathbb{C})$ and $\theta_n' = \theta_n \otimes k_1^*(-)k_1\colon M_{k_n}(\mathbb{C}) \to B\otimes\mathbb{K}$, and note that they are both c.c.p maps and they exhibit the nuclearity of $a \mapsto b^*\phi_0(a)b$ since

$$\|\theta_n' \circ \rho_n'(a) - b^*\phi_0(a)b\| = \|\theta_n \circ \rho_n \otimes k_1^* e_{11} k_1 - b_1^*\phi|_{A_0}(a)b_1 \otimes k_1^* e_{11} k_1\| =$$
$$\|(\theta_n \circ \rho_n(a) - b_1^*\phi|_{A_0}(a)b_1)\otimes k_1^* e_{11}k_1\| = \|(\theta_n \circ \rho_n(a) - b_1^*\phi|_{A_0}(a)b_1)\|\,\|k_1^* e_{11} k_1\| \to 0$$

As, $b \in B\otimes K$ was an arbitrary simple tensor, it follows that ϕ_0 is a weakly nuclear c.p map.

Now, since Φ absorbs ϕ_0, there is a sequence of unitaries $(u_n)_n \subseteq M(B\otimes K)$ such that

$$\|u_n\Phi(a)u_n^* - \Phi \oplus \phi_0(a)\| \to 0, \quad \text{for all } a \in A_0$$

So, if $s_1, s_2 \in M(B\otimes\mathbb{K})$ are isometries satisfying that $s_1s_1^* + s_2s_2^* = 1_{M(B\otimes\mathbb{K})}$, $s_1s_2^* = 0$ and $s_1^*s_2 = 0$, then we have that

$$\|\phi(a) - s_2^*u_n\Phi(a)u_n^*s_2\| = \|s_2^*(s_1\phi(a)s_1^* + s_2\phi(a)s_2^* - u_n\Phi(a)u_n^*)s_2\|$$
$$\leq \|u_n\Phi(a)u_n^* - \Phi\oplus\phi_0(a)\| \to 0$$

Set, $w_n = s_2^*u_n$, isometries in $M(B\otimes\mathbb{K})$, and use nuclearity of Φ to find $\lambda_n\colon A_0 \to M_{k(n)}(\mathbb{C})$ and $\mu_n\colon M_{k(n)}(\mathbb{C}) \to M(B\otimes K)$ c.c.p maps such that for $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ so that if $n \geq N_0$ then

$$\|\mathrm{Ad}_{w_n} \circ \mu_n \circ \lambda_n(a) - \phi_0(a)\| < \epsilon, \quad \text{for all} \quad a \in A_0$$

This shows that ϕ_0 is nuclear.

Finally, let

$$\Psi\colon M(B\otimes K) \xrightarrow{1_{M(B)}\otimes e_{11}(-)1_{M(B)}\otimes e_{11}} M(B)\otimes e_{11} \cong M(B)$$

and it is easy to see that Ψ is a surjective, projection, which is also contractive. Moreover, $\Psi \circ \phi_0 = \phi|_{A_0}$, which implies that $\phi|_{A_0}$ is a nuclear c.p map, as desired.

For the other direction, let $A_0 \subseteq A$ be a separable C^*-subalgebra, and let $\pi\colon A_0 \to M(\mathbb{K})$ be a faithful representation. By Arverson's extention theorem, we may extend π to a c.c.p map $\tilde{\pi}\colon A \to M(\mathbb{K})$, which is weakly nuclear since $\mathbb{K}$ is a nuclear C^*-algebra. Thus, by assumption $\tilde{\pi}$ is nuclear, and so π is nuclear. This shows that A_0 is nuclearly embeddable, hence by a well known fact, A_0 is exact. Now, using that A is exact if, and only if, any $A_0 \subseteq A$ separable C^*-subalgebra is exact, the desired result follows. $\square$

The second main goal of this appendix is to present two results used in tackling non-separability issues. We first show that under some rather mild assumptions, the properties of a *-homomorphism to be nuclear and full can be preserved after passing to a suitable corestriction. Before proving this claim, we need a preliminary result which has its own particular interest.

Lemma A.0.8. *Let A be a separable, unital C^*-algebra. Then, there is a sequence $(a_n)_n \subset A$ such that $(a_n)_n \cap I \subset_{dense} I$, for any closed two sided ideal $I \lhd A$.*

Proof. Firstly, we claim that if $a \in A^+$, $b \in I^+$ and $\delta = \|a-b\|$ then $(a-\epsilon)_+ \in I$ whenever $\|a-b\| < \epsilon$. To do so, define $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ continuous function such that $f(x) = 0$, when $x \leq \delta$, and $f(x) = 1$, when $x \geq \epsilon$. Then, as $a - b \in A_{\text{s.a}}$, it follows that $a - b \leq \|a-b\| = \delta$, so we have

$$(a-\epsilon)_+ = f(a)(a - \epsilon 1_A)f(a) \leq f(a)(a - \delta 1_A)f(a) \leq f(a)bf(a)$$

But, I as an ideal, is also a hereditary subalgebra of A, which implies that $(a-\epsilon)_+ \in I$, since $f(a)bf(a) \in I$.

Now, let $(b_n)_n \subseteq A^+$ be a dense sequence and take $c \in I^+$. Then, for any $k \in \mathbb{N}$, there is $n \in \mathbb{N}$, such that $\|b_n - c\| < \frac{1}{k}$. Hence, from the previous argument, $(b_n - \frac{1}{k})_+ \in I$ and moreover, $\left\|b_n - (b_n - \frac{1}{k})_+\right\| \leq \frac{1}{k}$, which shows that $\left\|(b_n - \frac{1}{k})_+ - c\right\| < \frac{2}{k}$. So, if we set

$$T = \{(b_n - \frac{1}{k})_+ \ : n, k \in \mathbb{N}\}$$

then T is a countable set and $T \cap I$ dense in I^+, for any $I \lhd A$. But, as any element in I can be expressed as a linear combination of elements in I^+, if we set

$$T' = \{a_1 - a_2 + ib_1 - ib_2\colon a_1, a_2, b_1, b_2 \in T\}$$

then T' is a countable set in A satisfying that $T' \cap I \subset_{dense} I$, for any $I \lhd A$, as desired. $\square$

Proposition A.0.9 ([30], Proposition 1.9)**.** *Let A be a separable C^*-algebra and B a unital C^*-algebra. If $\phi\colon A \to B$ is a full, nuclear *- homomorphism, then there is a separable, unital C^*-subalgebra $B_0 \subseteq B$ such that the corestriction of ϕ to B_0 is full and nuclear.*

Proof. Firstly, let $\phi_n\colon A \to M_{k(n)}(\mathbb{C})$, $\phi'_n\colon M_{k(n)}(\mathbb{C}) \to B$ be c.c.p maps such that

$$\|\phi'_n \circ \phi_n(a) - \phi(a)\| \xrightarrow{n\to\infty} 0$$

for all $a \in A$.

As A is separable, by Lemma A.0.8 we can find a sequence $(\alpha_n)_n \subseteq A$ so that for any $I \subseteq A$ closed two-sided ideal, $\exists\ n \in \mathbb{N}$ such that $\alpha_n \in I$. Now, for each $n \in \mathbb{N}$, $\phi(\alpha_n)$ is a full element in B, and threfore we can find $k(n) \in \mathbb{N}$ and $\beta_{n,i}, \beta'_{n,i} \in B$ satisfying

$$\sum_{i=1}^{k(n)} \beta_{n,i}\phi(\alpha_n)\beta'_{n,i} = 1_B$$

.

Let B_0 be the C^*-algebra generated by $\phi(A)$, $\phi'_n(M_{k(n)}(\mathbb{C}))$ and $\beta_{n,i}, \beta'_{n,i}$, for all $i = 1, 2, ..., k(n)$

and $n \geq 1$. Note that, as A and $M_{k_n}(\mathbb{C})$ are separable, it follows that $\phi(A)$ and $\phi'_n(M_{k(n)}(\mathbb{C})$ are separable, for all $n \geq 1$. Hence, B_0 is a separable C^*-subalgebra of B. Now, if ϕ_0 is the corestriction of ϕ in B_0, then ϕ_0 is nuclear, since $\phi'_n(M_{k(n)}(\mathbb{C})) \subset B_0$, for all $n \geq 1$

Let us now argue that ϕ_0 is a full *-homomorphism. If $a \in A \setminus \{0\}$ and $I = (\phi(a))$ is the ideal generated by $\phi(a)$ then, $\phi^{-1}(I)$ is a non-zero ideal in A and so there exists $n \in \mathbb{N}$ such that $\alpha_n \in I$. Hence, $\phi(a_n) \in I$ and therefore $1_B = \sum_{i=1}^{k(n)} \beta_{n,i}\phi(\alpha_n)\beta'_{n,i} \in I$, showing that $I = B_0$. But this clearly implies that ϕ_0 is full, as desired. ☐

Another result employed in the proofs of Chapter 3 for addressing non-separability issues is the following.

Proposition A.0.10 ([30], Proposition 1.10)**.** *Suppose that G is a countable abelian group and A is a C^*-algebra. For $i = 0, 1$ the natural group homomorphisms*

$$\varinjlim Hom_Z(G, K_i(A_0)) \to Hom_Z(G, K_i(A))$$

and

$$\varinjlim Ext^1_Z(G, K_i(A_0)) \to Ext^1_Z(G, K_i(A))$$

are isomorphisms where the limit is taken over all separable C^*-subalgebras A_0 of A.

Proof. Since G is a $\mathbb{Z}$-module there exist X, Y countable sets such that the sequence

$$0 \to \mathbb{Z}X \to \mathbb{Z}Y \to G \to 0$$

is exact. Now, using the long exact sequence induced by the $Ext_{\mathbb{Z}}$ functor, that $\mathbb{Z}$ is a projective Z-module and that $Hom_{\mathbb{Z}}(-, K_i(B))$ is a contravariant right exact functor for any C^*-algebra B, we get the following exact sequence

$$0 \to Hom_{\mathbb{Z}}(G, K_i(B)) \to Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_i(B)) \to Hom_{\mathbb{Z}}(\mathbb{Z}X, K_i(B)) \to Ext^1_{\mathbb{Z}}(G, K_i(B)) \to 0$$

Moreover, is a standard fact from Homological algebra that in the category of abelian groups, inductive limits commute with exact sequences, and therefore we have the following exact sequence

$$0 \to \varinjlim Hom_{\mathbb{Z}}(G, K_i(A_0)) \to \varinjlim Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_i(A_0)) \to \varinjlim Hom_{\mathbb{Z}}(\mathbb{Z}X, K_i(A_0)) \to$$
$$\varinjlim Ext^1_{\mathbb{Z}}(G, K_i(A_0)) \to 0$$

where the limit is taken over all separable C^*-subalgebras A_0 of A. Hence, by a diagram chasing, it suffices to show that

$$\varinjlim Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_i(A_0)) \to Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_i(A))$$

is a group isomorphism.

We may assume, without loss of generality, that A is unital. Let us start with case of $i = 0$.

For surjectivity, let $f \colon \mathbb{Z}Y \to K_0(A)$, then for each $y \in Y$ there exist $n(y) \geq 1$ and projections $p_y, q_y \in M_{n(y)}(A)$ such that $f(y) = [p_y]_0 - [q_y]_0$. Let A_0 be the separable C^*-subalgebra of A generated by the entries of p_y and q_y, for each $y \in Y$, and define $g \colon \mathbb{Z}Y \to K_0(A_0)$ by $g(y) = [p_y]_0 - [q_y]_0$. Then g is a well defined group homomorphism and if $\iota_0 \colon A_0 \hookrightarrow A$ is the inclusion, we get that $K_0(\iota_0) \circ g = f$.

For injectivity, take $f, g \in \varinjlim Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_0(A_0))$ group homomorphisms, and take A_0 unital, separable C^*-subalgebra of A such that $f, g \in Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_0(A_0))$. Now, suppose that $K_0(\iota_0)f = K_0(\iota_0)g$. As previously, for any $y \in Y$ there are $n(y) \geq 1$ and p_y, q_y, p'_y, q'_y projections in $M_{n(y)}(A)$ such that $f(y) = [p_y]_0 - [q_y]_0$ and $g(y) = [p'_y]_0 - [q'_y]_0$. Then, $[p_y \oplus q'_y]_0 = [p'_y \oplus q_y]_0$ in $K_0(A)$, i.e they are stably equivalent, and so we can find $s(y) \in \mathbb{N}$ such that $p_y \oplus q'_y \oplus 1_A^{\oplus s(y)} \sim_0 p'_y \oplus q_y \oplus 1_A^{\oplus s(y)}$. Thus, there is $u_y \in M_{2n(y)+s(y)}(A)$ partial isometry satisfying, $u_y^* u_y = p_y \oplus q'_y \oplus 1_A^{\oplus s}$ and $u_y u_y^* = p'_y \oplus q_y \oplus 1_A^{\oplus s}$, for each $y \in Y$. Now, if we set A_1 to be the C^*-subalgebra of A generated by

A_0 and the entries of u_y, for each $y \in Y$, then A_1 is separable, and by the above considerations $f(y) = g(y)$ in $K_0(A_1)$, for each $y \in Y$. Hence, if $\iota_{0,1} \colon A_0 \to A_1$ is the inclusion, we obtain that $K_0(\iota_{0,1})f = K_0(\iota_{0,1})g$, which shows that $f = g$ as elements in $\varinjlim Hom_{\mathbb{Z}}(\mathbb{Z}Y, K_0(A_0))$.

Finally, for $i = 1$, it is straightforward to see that if we employ Bott periodicity, the desired result follows. $\square$

We end this exposition with two propositions concerning the K-theoretical behaviour of stable rank one C^*-algebras and of von Neumann algebras. Both results are omnipresent in K-theory literature, but their versatility and usefulness in the current project lead us to include them in this appendix.

Proposition A.0.11. *Let A be a unital C^*-algebra. If A has stable rank one, then the von Neumann - Murray semigroup, $(D(A), +)$, has the cancellation property.*

Proof. First, we show that, if p, q projections in A and $p \sim_0 q$, then $1 - p \sim_0 1 - q$. To see this, initially let $v \in A$, such that $v^*v = p$ and $vv^* = q$. Then, for $\epsilon < \frac{1}{2}$, find $z \in GL(A)$, satisfying $\|z - v\| < \epsilon/3$, and let $z = u|z|$ be the (unitary) polar decomposition of z. Since, u, v are both contractions, we get that

$$\|z - v\| < \epsilon/3 \implies \||z| - u^*v\| < \epsilon/3 \implies \|(|z| - u^*v)^*(|z| - u^*v)\| < \epsilon/3$$
$$\implies \|z^*z - z^*v - v^*z + p\| < \epsilon/3 \qquad (1)$$

and,

$$\|z^*v - p\| < \epsilon/3 \qquad (2)$$
$$\|v^*z - p\| < \epsilon/3 \qquad (3)$$

Hence, by $(1), (2), (3)$

$$\|z^*z - p\| = \|z^*z - z^*v + z^*v - p - p - v^*z + v^*z + p\|$$
$$\leq \|z^*z - z^*v - v^*z + p\| + \|z^*v - p\| + \|v^*z - p\| < \epsilon$$

so,

$$\|uz^*zu^* - upu^*\| = \|zz^* - upu^*\| < \epsilon \qquad (4)$$

Similarly, we see that $\|zz^* - q\| < \epsilon$, and combining it with (4), we obtain

$$\|upu^* - q\| < 2\epsilon < 1$$

which implies that $p \sim_u upu^* \sim_u q$, and in turn that $1 - p \sim_0 1 - q$.

Now, let $r \in A$ projection, such that $p \perp r$, $q \perp r$ and $p+r \sim_0 q+r$. Then, $1-(p+r) \sim 1-(q+r)$, and let $w \in A$, be the partial isometry exhibiting this equivalene. If we set $v = w + r$, we get that $v^*v = 1 - (p + r) + r$ and $vv^* = 1 - (q + r) + r$, i.e $1 - (p + r) + r \sim_0 1 - (q + r) + r$, thus, by the first part again, it follows that $p \sim q$, and the proof is complete. $\square$

Proposition A.0.12. *Let M be a (unital) von Neumann algebra. Then $K_1(M) = 0$*

Proof. Let $u \in M$ be a unitary and define the Borel measurable function $\sigma \colon \mathbb{T} \to [0, 2\pi)$, by $\sigma(\exp(it)) = t$. Using Borel functional calculus, set $h = \sigma(u)$, which is a self-adjoint in M. Since, $\exp(i\sigma(x)) = x$ for every $x \in \mathbb{T}$, we have that $\exp(ih) = u$, which shows that $u \in U_0(M)$ i.e u belongs to the connected component of the identity. Thus $[u]_1 = 0$, and since u was arbitrary, we conclude that $K_1(M) = 0$. $\square$

References

[1] W. Arveson, Notes on extensions of C^*-algebras, *Duke Math. J. (2)* **44** (1977), 329–355.

[2] B. Blackadar, Operator Algebras: Theory of C^*-algebras and von Neumann algebras, Encyclopedia of Mathematical Sciences, Vol 122, Springer-Verlag, Berlin Heidelberg New York, 2006.

[3] J. Bosa, N. Brown, Y. Sato, A. Tikuisis, S. White, W. Winter, Covering dimension of C^*-algebras and 2-coloured classification, *Mem. Amer. Math. Soc. (1233)* **257** (2019). (arXiv:1506.03974 [math.OA])

[4] S. Bosch, Algebraic Geometry and Commutative Algebra, Universitext, Springer-Verlag, London, 2013

[5] L. Brown, G. Pedersen, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.

[6] N. Brown, N. Ozawa, C^*-Algebras and Finite-Dimensional Approximations, Graduate Studies in Mathematics, **88**, Amer. Math. Soc., Providence, RI, 2008

[7] J. Cuntz, N. Higson, Kuiper's theorem for Hilbert modules, Operator algebras and mathematical physics (Iowa City, Iowa, 1985), *Contemp. Math., Amer. Math. Soc.* **62** (1987), 429–435.

[8] M. Dadarlat, Quasidiagonal morphisms and homotopy, *J. Funct. Anal. (1)* **151** (1997), 213–233.

[9] M. Dadarlat, S. Eilers, On the classification of nuclear C^*-algebras, *Proc. Lond. Math. Soc. (3)* **85** (2002), 168–210.

[10] M. Dadarlat, S. Eilers, Asymptotic unitary equivalence in KK-theory, *K-Theory* **23** (2001), 305–322.

[11] E. Effros, E. G. Rosenberg, J.: C*-algebras with approximately inner flip, *Pac. J. Math.* **77** (1978), 417-443.

[12] J. Gabe, E. Ruiz, Absorbing representations with respect to closed operator convex cones, preprint (2015). (arXiv:1503.07799 [math.OA])

[13] J. Gabe, Quasidiagonal traces on exact C^*-algebras, *J. Funct. Anal.* **272** (2017), 1104–1120.

[14] J. Gabe, Lifting theorems for completely positive maps, preprint (2016). (arXiv:1508.00389v3 [math.OA])

[15] K.R. Gixdearland, D. Handelman, Rank functions and K_0 of regular rings, *J. Pure Appl. Algebra* **7** (1976), 195-216.

[16] A. Guichardet, Tensor Products of C*-algebras, *Part I, Aarhus University Lecture Notes Series No. 12, Aarhus,* 1969.

[17] U.Haagerup, Quasitraces on exact C^*-algebras are traces, 2014. (arXiv:1403.7653v1 [math.OA])

[18] K. K. Jensen, K. Thomsen, Elements of KK-Theory, Mathematics: Theory and Applications, Birkhäuser, Boston, MA, 1991.

[19] D. Kucerovsky, P. Ng, The corona factorization property and approximate unitary equivalence, *Houston J. Math.* **32** (2006), 531–550.

[20] F. Larsen, N.J Laustsen, M. Rørdam, An introduction to K-theory for C^*-algebras (London Mathematical Society student texts 49), Cambridge: Cambridge University Press, 2000.

[21] H. Lin, Exponential length of C^*-algebras with real rank zero and the Brown-Pedersen conjectures, *J. Funct. Anal.* **114** (1993), 1–11.

[22] G. J. Murphy, C^*-algebras and operator theory, Academic Press, London, 1990.

[23] N. Ozawa, Dixmier approximation and symmetric amenability for C^*-algebras, *Math. Sci. Univ. Tokyo* **20** (2013), 349–374.

[24] G. K. Pedersen, C^*-algebras and their automorphism groups, London: Academic Press, 1979.

[25] M. Rieffel, The homotopy groups of the unitary groups of non-commutative tori, *J. Operator Theory* **17** (1987), 237–254.

[26] M. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. Lond. Math. Soc. (3)* **46** (1983), 301–333.

[27] M. Rørdam, On the structure of simple C*-algebras tensored with a UHF-algebra, *J. Funct. Anal.* **100** (1991), 1-17.

[28] M. Rørdam, On the structure of simple C*-algebras tensored with a UHF-algebra, *II, J. Funct. Anal.* **107** (1992), 255-269.

[29] M. Rørdam, Topics in Operators Algebras, *University of Copenhagen, Lecture Notes I, II, Fall Semester*, 2021.

[30] C. Schafhauser, Subalgebras of simple AF-algebras, 2019. (arXiv:1807.07381v2 [math.OA])

[31] C. Schafhauser, A new proof of the Tikuisis-White-Winter theorem, to appear in *J. Reine Angew. Math.* (arXiv:1701.02180 [math.OA])

[32] A. Tikuisis, S. White, W. Winter, Quasidiagonality of nuclear C^*-algebras, *Ann. of Math. (2)* **185** (2017), 229–284.

[33] A. Toms, W. Winter, Strongly self-absorbing C^*-algebras, *Transactions of the American Mathematical Society* **359.8** (2007), 3999-4029.

[34] K. Zhu, An introduction to operator algebras (1.st ed., Studies in Advanced Mathematics), Boca Raton, USA: CRC Press, 1993.