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Quantum correlations, C*-algebras and Tsirelson's conjecture

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1 Introduction

Consider the following setup: Two scientists Alice and Bob are situated in two spatially separated spaces with no means of communication. They both receive information from a source emitting a quantum system from which each can measure one of n observables and respond with one of k measurements. Say Alice measures observable x and Bob receives observable y, what is the probability, denoted $p(a, b; x, y)$, of Alice responding with measurement a and Bob simultaneously responding with measurement b?

This project presents and works through three models; the classical model, quantum commuting model and quantum spatial model. These models concerns so-called *correlation matrices*; matrices of dimension $(nk)^2$ whose entries belong to $[0, 1]$. Though these models try to describe a physical situation, this project is purely a mathematical study of the sets of correlation matrices. We first consider the classical model which only concerns probability measures. Physical experiments in the 1980s done by physicist Alain Aspect show that to describe the physical world completely, quantum mechanics is necessary. This is the motivation for introducing quantum correlation matrices. First, we describe the quantum commuting correlation matrices, $C_{qc}(n, k)$, using states on the maximal tensor products of the universal C^*-algebra of the k-fold free product of the cyclic group of order n. In doing so, one concludes that the set of quantum commuting correlation matrices is closed. Here, the assumption that we require certain projections to commute reflects that Alice and Bob can measure simultaneously.

Next, we introduce the quantum spatial correlation matrices, $C_{qs}(n, k)$. It is known and highly non-trivial to show that $C_{qs}(n, k)$ is not closed for $n, k \geq 2$ and will not be shown in this project. Instead, we aim to describe the closure of $C_{qs}(n, k)$ in a similar way as $C_{qc}(n, k)$ using the minimal tensor product. These sets are all related as will be shown in the project. Lastly, we touch upon Tsirelson's conjecture, asking whether $\overline{C_{qs}}(n, k) = C_{qc}(n, k)$. Here, we prove that this is the case when you take finite-dimensional Hilbert spaces and show that Tsirelson's conjecture is equivalent to several other interesting problems, including Connes Embedding Problem. In the following, the cases of either n, k equal 1 are uninteresting and degenerate, hence we will always consider $n, k \geq 2$.

2 Classical correlation matrices

We first describe the classical model of correlation matrices.

Let $M_m([0,1])$ be the set of $m \times m$ matrices with entries in $[0,1]$. For $n, k \geq 1$ one can identify $M_{nk}([0,1])$ with $M_n(M_k([0,1]))$. For $T \in M_n(M_k([0,1]))$ denote its entries by $T^{x,y}_{a,b}$ for $1 \leq a, b \leq k$ and $1 \leq x, y \leq n$. Fix x, y and let $T^{x,y}$ denote the $k \times k$ matrix with entries $T^{x,y}_{a,b}$ for $1 \leq a, b \leq k$. Then T becomes the block matrix

$$T = \begin{pmatrix} T^{1,1} & T^{1,2} & \dots & T^{1,n} \\ T^{2,1} & T^{2,2} & \dots & T^{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ T^{n,1} & T^{n,2} & \dots & T^{n,n} \end{pmatrix}, \qquad T^{x,y} \in M_k([0,1]), 1 \leq x, y \leq n.$$

Definition 2.1. Let $n, k \geq 2$. Denote $C_c(n,k) \subset M_{nk}([0,1])$ the set of *classical correlation matrices*,

$$T = \left[\mu(A^x_a \cap B^y_b)\right]_{a,b;x,y}, \quad 1 \leq a, b \leq k, 1 \leq x, y \leq n$$

where Ω is a compact Hausdorff space, $(\Omega, \mathcal{A}, \mu)$ is a probability space and $\{A^x_a\}_{1 \leq a \leq k}$, $\left\{B^y_b\right\}_{1 \leq b \leq k}$ are measurable partitions of Ω for each $1 \leq x, y \leq n$.

Remark 2.2. One can also define $C_c(n,k)$ without the assumption of Ω being compact Hausdorff. In fact, any $T \in C_c(n,k)$ can be realized with a finite probability space hence also one that is compact Hausdorff.

Proposition 2.3. *$C_c(n,k)$ is convex.*

Proof. Let $t \in (0,1)$ and consider $T, S \in C_c(n,k)$ given by

$$T = \left[\mu(A^x_a \cap B^y_b)\right]_{a,b;x,y}, S = \left[\nu(C^x_a \cap D^y_b)\right]_{a,b;x,y}$$

where $(X, \mathcal{A}, \mu), (Y, \mathcal{A}', \nu)$ are probability spaces and $\{A^x_a\}_{1 \leq a \leq k}$, $\left\{B^y_b\right\}_{1 \leq b \leq k}$ and $\{C^x_a\}_{1 \leq a \leq k}$, $\left\{D^y_b\right\}_{1 \leq b \leq k}$ are measurable partitions of X, respectively, Y. Define $\tilde{X} = X \times \{0\}$, $\tilde{Y} = Y \times \{1\}$. The disjoint union of X and Y is defined by $X \sqcup Y = \tilde{X} \cup \tilde{Y}$. It is straightforward to see that

$$\mathcal{C} := \left\{ C \subset X \sqcup Y \mid C \cap \tilde{X} \in \mathcal{A}, C \cap \tilde{Y} \in \mathcal{A}' \right\}$$

is a σ-algebra. For $0 \leq t \leq 1$, define $\rho : \mathcal{C} \to [0,1]$ by $\rho(C) = t\mu(C \cap \tilde{X}) + (1-t)\nu(C \cap \tilde{Y})$. That ρ is a probability measure is a consequence of μ and ν being probability measures and is shown by simple computations. Lastly, note that $\{A^x_a \sqcup C^x_a\}_{1 \leq a \leq k}$, $\left\{B^y_b \sqcup D^y_b\right\}_{1 \leq b \leq k}$ are partitions of $X \sqcup Y$ and letting $K^{x,y}_{a,b} = (A^x_a \sqcup C^x_a) \cap (B^y_b \sqcup D^y_b)$ it follows that

$$\rho(K^{x,y}_{a,b}) = t\mu(K^{x,y}_{a,b} \cap \tilde{X}) + (1-t)\nu(K^{x,y}_{a,b} \cap \tilde{Y}) = t\mu(A^x_a \cap B^y_b) + (1-t)\nu(C^x_a \cap D^y_b)$$

showing $tT + (1-t)S = \left[\rho((A^x_a \sqcup C^x_a) \cap (B^y_b \sqcup D^y_b))\right]_{a,b;x,y} \in C_c(n,k)$ as wanted. $\square$

Definition 2.4. Let $n, k \geq 2$. Define $\mathcal{E}(n,k)$ to be the set of extreme points of $M_{nk}([0,1])$. Let $\mathcal{E}^\dagger(n,k) = C_c(n,k) \cap \mathcal{E}(n,k)$ be the set of *deterministic correlation matrices.*

Remark 2.5. $\mathcal{E}(n,k)$ are exactly the matrices in $M_{nk}([0,1])$ with entries belonging to $\{0,1\}$. Moreover, both $\mathcal{E}(n,k)$ and $\mathcal{E}^\dagger(n,k)$ are finite.

For a non-empty subset A of a vector space X, we denote the *convex hull of* A by $\operatorname{conv}(A)$. For $K \subset X$ non-empty and convex we denote by $\partial_e K$ the set of extreme points of K.

Lemma 2.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $Prob(\Omega)$ be the probability measures on Ω. The extreme points of $Prob(\Omega)$ is the set of $\nu \in Prob(\Omega)$ for which $\nu(\mathcal{A}) = \{0,1\}$.*

Proof. Let $\nu \in \operatorname{Prob}(\Omega)$ such that $\nu(\mathcal{A}) = \{0,1\}$. Assume there is $0 < t < 1$ and $\nu_1, \nu_2 \in \operatorname{Prob}(\Omega)$ such that $\nu = t\nu_1 + (1-t)\nu_2$. If $A \in \mathcal{A}$ with $\nu(A) = 1$, then the convex combination forces $\nu_1(A) = \nu_2(A) = 1$. Similarly, for $A \in \mathcal{A}$ with $\nu(A) = 0$ we get $\nu_1(A) = \nu_2(A) = 0$ showing $\nu_1 = \nu_2 = \nu$, so $\nu \in \partial_e \operatorname{Prob}(\Omega)$.

For the converse, let $\nu \in \partial_e \operatorname{Prob}(\Omega)$. Assume for contradiction that there exists $A \in \mathcal{A}$ with $0 < \nu(A) < 1$. Then, as ν is a probability measure, $0 < \nu(A^c) < 1$. Let $\varepsilon, \tilde{\varepsilon} > 0$ be such that

$$\nu = \frac{1}{2}\left((1+\varepsilon)\nu_{|A} + (1-\tilde{\varepsilon})\nu_{|A^c}\right) + \frac{1}{2}\left((1-\varepsilon)\nu_{|A} + (1+\tilde{\varepsilon})\nu_{|A^c}\right),$$

where $(1+\varepsilon)\nu_{|A} + (1-\tilde{\varepsilon})\nu_{|A^c}$ and $(1-\varepsilon)|_A + (1+\tilde{\varepsilon})|_{A^c}$ are probability measures by choice of ε and $\tilde{\varepsilon}$. Since ν is a probability measure, we get $\nu(\mathcal{A}) = \{0,1\}$, as wanted. $\square$

Proposition 2.7. *$C_c(n,k) = conv\ \mathcal{E}^\dagger(n,k)$. In particular $C_c(n,k)$ is closed.*

Proof. By definition, $\mathcal{E}^\dagger(n,k) \subset C_c(n,k)$ so the inclusion $\operatorname{conv}(\mathcal{E}^\dagger(n,k)) \subset C_c(n,k)$ is a consequence of $C_c(n,k)$ convex.

For the converse inclusion, let $T \in C_c(n,k)$. Then T is of the form $T = \left[\mu(A_a^x \cap B_b^y)\right]_{a,b;x,y}$ for some probability space $(\Omega, \mathcal{A}, \mu)$ and partitions $\{A_a^x\}_{1\leq a\leq k}, \{B_b^y\}_{1\leq b\leq k}$ of Ω. Define

$$\Delta = \left\{ \left[\nu(A_a^x \cap B_b^y)\right]_{a,b;x,y} \mid \nu \in \operatorname{Prob}(\Omega) \right\}.$$

Let $\operatorname{Prob}(\Omega)$ be equipped with the weak*-topology. As Ω is compact Hausdorff, $\operatorname{Prob}(\Omega)$ is compact. Moreover, Δ is the image of the weak*-continuous mapping

$$\operatorname{Prob}(\Omega) \ni \nu \mapsto \left[\nu(A_a^x \cap B_b^y)\right]_{a,b;x,y} \in \Delta,$$

implying Δ is compact. Moreover, $\operatorname{Prob}(\Omega)$ is convex hence Δ is convex and by definition, $T \in \Delta$. It follows from Lemma 2.6 that $\partial_e \Delta$ are the matrices $\left[\nu(A_a^x \cap B_b^y)\right]_{a,b;x,y}$ for which $\nu \in \partial_e \operatorname{Prob}(\Omega)$. By Krein-Milman we get $\Delta = \overline{\operatorname{conv}\ \partial_e \Delta}$. It follows $T \in \overline{\operatorname{conv}\ \mathcal{E}^\dagger(n,k)}$ implying $T \in \operatorname{conv}\ \mathcal{E}^\dagger(n,k)$, as $\mathcal{E}^\dagger(n,k)$ is a finite set and the convex hull of a finite set is closed. Thus, $C_c(n,k)$ is closed. $\square$

Remark 2.8. As $\mathcal{E}^\dagger(n,k) \subset C_c(n,k) \subset M_{nk}([0,1])$ and the elements of $\mathcal{E}^\dagger(n,k)$ are extreme in $M_{nk}([0,1])$ it follows that $\mathcal{E}^\dagger(n,k) \subset \partial_e C_c(n,k)$. The converse follows from the above proposition. This shows that $\partial_e C_c(n,k) = \mathcal{E}^\dagger(n,k)$. Combined with the above, we conclude that $C_c(n,k)$ is a closed convex finite polytope spanned by its extreme points.

3 Quantum commuting correlation matrices

We now move on the quantum models. We will introduce two models for quantum correlation matrices, beginning with the *quantum commuting correlation matrices*.

Definition 3.1. An n-tuple of operators $\{P_j\}_{j=1}^n \subset B(H)$ for a Hilbert space H and $n \geq 1$ is called a *Positive Valued Measure (PVM)* if each P_j is a projection and $\sum_{j=1}^n P_j = 1$.

Definition 3.2. Let $n, k \geq 2$. Denote $C_{qc}(n,k) \subset M_{nk}([0,1])$ the set of *quantum commuting correlation matrices*,

$$T = \left[\left\langle P_a^x Q_b^y \xi, \xi \right\rangle\right]_{a,b;x,y}, \quad 1 \leq a, b \leq k, 1 \leq x, y \leq n$$

where H is a Hilbert space and $\xi \in H$ is a unit vector and for each $1 \leq x, y \leq n$, $\{P_a^x\}_{1 \leq a \leq k}$, $\left\{Q_b^y\right\}_{1 \leq b \leq k}$ are commuting PVMs on H in the sense that

$$P_a^x Q_b^y = Q_b^y P_a^x, \quad 1 \leq a, b \leq k, \ 1 \leq x, y \leq n.$$

Proposition 3.3. *$C_{qc}(n,k)$ is convex.*

Proof. Let $T, S \in C_{qc}(n,k)$. Then for each $1 \leq x, y \leq n$ there exist commuting PVMs $\{P_a^x\}_{1 \leq a \leq k}, \left\{Q_b^y\right\}_{1 \leq b \leq k}$ and $\{K_a^x\}_{1 \leq a \leq k}, \left\{L_b^y\right\}_{1 \leq b \leq k}$ on Hilbert spaces H_T, respectively, H_S, and unit vectors $\xi_T \in H_T$, $\xi_S \in H_S$ such that

$$T = \left[\left\langle P_a^x Q_b^y \xi_T, \xi_T \right\rangle\right]_{a,b;x,y}, \quad S = \left[\left\langle K_a^x L_b^y \xi_S, \xi_S \right\rangle\right]_{a,b;x,y}.$$

Let $H = H_T \oplus H_S$ and consider $\{P_a^x \oplus K_a^x\}_{1 \leq a \leq k}, \left\{Q_b^y \oplus L_b^y\right\}_{1 \leq b \leq k}$. We claim these are commuting PVM's on H. Firstly, note that by the properties of the direct sum, for each $1 \leq x, y \leq n$, $1 \leq a, b \leq k$ it follows that $P_a^x \oplus K_a^x$ and $Q_b^y \oplus K_b^y$ are projections. Moreover,

$$\sum_{a=1}^k (P_a^x \oplus K_a^x) = \sum_{a=1}^k P_a^x \oplus \sum_{a=1}^k K_a^x = 1_{H_T} \oplus 1_{H_S} = 1_H$$

so $\{P_a^x \oplus K_a^x\}_{1 \leq a \leq k}$ is a PVM on H. Likewise, $\left\{Q_b^y \oplus L_b^y\right\}_{1 \leq b \leq k}$ is a PVM on H. Note that for $\xi = \xi_1 \oplus \xi_2 \in H$,

$$(P_a^x \oplus K_a^x)(Q_b^y \oplus L_b^y)(\xi_1 \oplus \xi_2) = P_a^x Q_b^y(\xi_1) \oplus K_a^x L_b^y(\xi_2) = (Q_b^y \oplus L_b^y)(P_a^x \oplus K_a^x)(\xi_1 \oplus \xi_2)$$

since $P_a^x Q_b^y = Q_b^y P_a^x$ and $K_a^x L_b^y = L_b^y K_a^x$. Hence, $\{P_a^x \oplus K_a^x\}_{1 \leq a \leq k}, \left\{Q_b^y \oplus L_b^y\right\}_{1 \leq b \leq k}$ are commuting PVMs on H. For $t \in (0,1)$, let $\xi = \sqrt{t}\xi_T \oplus \sqrt{1-t}\xi_S \in H_T \oplus H_S$. Then

$$\begin{aligned}\left\langle (P_a^x \oplus K_a^x)(Q_b^y \oplus L_b^y)\xi, \xi \right\rangle &= \left\langle P_a^x Q_b^y(\sqrt{t}\xi_T) \oplus K_a^x L_b^y(\sqrt{1-t}\xi_S), \sqrt{t}\xi_T \oplus \sqrt{1-t}\xi_S \right\rangle \\ &= \sqrt{t}^2 \left\langle P_a^x Q_b^y \xi_T, \xi_T \right\rangle + \sqrt{1-t}^2 \left\langle K_a^x L_b^y \xi_S, \xi_S \right\rangle = tT + (1-t)S,\end{aligned}$$

showing that $tT + (1-t)S \in C_{qc}(n,k)$ as wanted. $\square$

Proposition 3.4. *There exists a Hilbert space H and commuting PVMs $\{P_a^x\}_{1\le a\le k}$, $\{Q_b^y\}_{1\le b\le k}$ on H such that for each $S \in C_{qc}(n,k)$,*

$$S = \left[\left\langle P_a^x Q_b^y \xi_S, \xi_S \right\rangle\right]_{a,b;x,y}$$

for a unit vector $\xi_S \in H$.

Proof. For each $T \in C_{qc}(n,k)$ there exists a Hilbert space H_T, a unit vector $\psi_T \in H_T$ and commuting PVMs $\{(P_T)_a^x\}_{1\le a\le k}, \{(Q_T)_b^y\}_{1\le a\le k}$ on H_T such that

$$T = \left[\left\langle (P_T)_a^x (Q_T)_b^y \psi_T, \psi_T \right\rangle\right]_{a,b;x,y}$$

Let

$$H = \bigoplus_{T\in C_{qc}(n,k)} H_T = \left\{ \xi = (\xi_T)_{T\in C_{qc}(n,k)} \mid \|\xi\|^2 = \sum_T \|\xi_T\|^2 < \infty \right\}.$$

For each $1 \le a,b \le k$, $1 \le x,y \le n$, define

$$P_a^x = \bigoplus_{T\in C_{qc}(n,k)} (P_T)_a^x, \quad Q_b^y = \bigoplus_{T\in C_{qc}(n,k)} (Q_T)_b^y.$$

For each $S \in C_{qc}(n,k)$, define $\xi(S) \in H$ by

$$\xi(S)_T = \begin{cases} 0 & \text{if } T \ne S \\ \psi_S & \text{if } T = S. \end{cases}$$

Then $\|\xi(S)\| = 1$, and

$$\left\langle P_a^x Q_b^y \xi(S), \xi(S) \right\rangle = \sum_T \left\langle (P_T)_a^x (Q_T)_b^y \xi(S)_T, \xi(S)_T \right\rangle = \left\langle (P_S)_a^x (Q_S)_b^y \psi_S, \psi_S \right\rangle = S_{a,b}^{x,y},$$

as wanted. □

The proposition shows that we can describe $C_{qc}(n,k)$ without varying over Hilbert spaces and PVMs as in the definition. The rest of this section aims towards describing $C_{qc}(n,k)$ using states on a C^*-algebra. A unitary representation of a countable discrete group G on a Hilbert space H is a homomorphism $\pi : G \to \mathcal{U}(H)$.

Definition 3.5. Let G be a discrete group. Denote by $\mathbb{C}[G]$ the *group $*$-algebra of G*, the vector space of all complex-valued functions $f : G \to \mathbb{C}$ with finite support. For $f \in \mathbb{C}[G]$, define the *universal norm*

$$\|f\|_u = \sup\left\{ \|\pi(f)\| \mid \pi : \mathbb{C}[G] \to B(H) \text{ is a unitary representation} \right\}.$$

The *full group C^*-algebra of G* is defined as $C^*(G) := \overline{\mathbb{C}[G]}^{\|\cdot\|_u}$.

Remark 3.6. By construction $C^*(G)$ is the C^*-algebra together with a unitary representation $\eta_G : G \to C^*(G)$ that satisfies the following universal property: Given a unital C^*-algebra A, for any unitary representation $\pi : G \to \mathcal{U}(A)$ there exist a unique $*$-homomorphism $\tilde{\pi} : C^*(G) \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\eta_G"'] & \mathcal{U}(A) \arrow[d, hook, "\iota"] \\ C^*(G) \arrow[r, dashed, "\exists!\tilde{\pi}"'] & A \end{tikzcd}$$

The full group C^*-algebra is also called *the universal C^*-algebra.*

Lemma 3.7. *Let $\mathbb{Z}_n$ denote the cyclic group of order n and let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. Then $C^*(\mathbb{Z}_n) \cong \mathbb{C}^n$ and moreover given a PVM $\{P_j\}_{j=1}^n$ on a Hilbert space H there exists a unique $*$-homomorphism $\pi : C^*(\mathbb{Z}_n) \to B(H)$ satisfying $\pi(e_j) = P_j$.*

Proof. For the first part, note that as $\mathbb{Z}_n$ is abelian, $\mathbb{C}[\mathbb{Z}_n]$ is commutative hence $C^*(\mathbb{Z}_n)$ is commutative. Since $\mathbb{Z}_n$ is a finite group with n elements and $\mathbb{C}[\mathbb{Z}_n]$ has basis $\{\delta_g \mid g \in \mathbb{Z}_n\}$, $\dim \mathbb{C}[\mathbb{Z}_n] = n$ so $\dim C^*(\mathbb{Z}_n) = n$. This implies $C^*(\mathbb{Z}_n)$ is isomorphic to $\mathbb{C}^n$ as C^*-algebras since $\mathbb{C}^n$ is the unique commutative C^*-algebra of dimension n. Thus, we can describe

$$C^*(\mathbb{Z}_n) = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \ldots \oplus \mathbb{C}e_n,$$

where e_i is identified with the i'th basis vector in $\mathbb{C}^n$ by the isomorphism above. Let $\{P_j\}_{j=1}^n$ be a PVM on H. Define $\pi : C^*(\mathbb{Z}_n) \to B(H)$ by $\pi(e_j) = P_j$. Expanding linearly to $\mathbb{C}^n$, we obtain that π is a unique $*$-homomorphism with the wanted property. □

The reader is assumed to be familiar with the free product of groups.

Definition 3.8. The free product $A * B$ is the unital C^*-algebra satisfying the following universal property: Given any unital C^*-algebra D and unit-preserving $*$-homomorphisms $\pi_A : A \to D, \pi_B : B \to D$, there exists a unique $*$-homomorphism $\pi : A * B \to D$ such that the following diagram is commutative:

$$\begin{tikzcd} & A \arrow[dl, "\iota_A"'] \arrow[dr, "\pi_A"] & \\ A * B \arrow[rr, dashed, "\exists!\pi"] & & D \\ & B \arrow[ul, "\iota_B"] \arrow[ur, "\pi_B"'] & \end{tikzcd}$$

Here, we view $A, B \subset A * B$ by the inclusions $\iota_A : A \to A * B, \iota_B : B \to A * B$ given by

$$\iota_A(a) = a1, \quad \iota_B(b) = 1b.$$

Proposition 3.9. *Let G_1, G_2 be discrete groups. Then $C^*(G_1 * G_2) \cong C^*(G_1) * C^*(G_2)$.*

Proof. By the universal property of the free product of groups, for any unital C^*-algebra D and $\pi_1 : G_1 \to \mathcal{U}(D), \pi_2 : G_2 \to \mathcal{U}(D)$ unitary representations there exists a unique $\pi : G_1 * G_2 \to \mathcal{U}(D)$ such that the following diagram is commutative:

$$\begin{tikzcd} & G_1 \arrow[dl, "\iota_1"'] \arrow[dr, "\pi_1"] & \\ G_1 * G_2 \arrow[rr, dashed, "\exists! \pi"] & & \mathcal{U}(D) \\ & G_2 \arrow[ul, "\iota_2"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

As G_1, G_2 are subgroups of $G_1 * G_2$ we have canonical inclusions $\iota_1 : C^*(G_1) \to C^*(G_1 * G_2)$ and $\iota_2 : C^*(G_2) \to C^*(G_1 * G_2)$. By the universal property of the full group C^*-algebra there exist unique $*$-homomorphisms $\tilde{\pi}_i : C^*(G_i) \to D$ for $i = 1, 2$ and $\tilde{\pi} : C^*(G_1 * G_2) \to D$ that, combined with the commutative diagram above, makes the following diagram commute:

$$\begin{tikzcd} & C^*(G_1) \arrow[dl, "\tilde{\iota}_1"'] \arrow[dr, "\tilde{\pi}_1"] & \\ C^*(G_1 * G_2) \arrow[rr, dashed, "\exists! \tilde{\pi}"] & & D \\ & C^*(G_2) \arrow[ul, "\tilde{\iota}_2"] \arrow[ur, "\tilde{\pi}_2"'] & \end{tikzcd}$$

Hence $C^*(G_1 * G_2)$ satisfies the universal property of $C^*(G_1) * C^*(G_2)$, so we obtain

$$C^*(G_1 * G_2) \cong C^*(G_1) * C^*(G_2).$$

□

From now on, let $\Gamma = (\mathbb{Z}_n)^{*k}$ be the k-fold free product of $\mathbb{Z}_n$. Using the above proposition we get $C^*(\Gamma) \cong C^*(\mathbb{Z}_n)^{*k}$. Recalling $C^*(\mathbb{Z}_n) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, we have the following

$$C^*(\Gamma) = \bigoplus_{i=1}^{n} \mathbb{C}e_i^1 * \bigoplus_{i=1}^{n} \mathbb{C}e_i^2 * \ldots * \bigoplus_{i=1}^{n} \mathbb{C}e_i^k,$$

where e_i^j denotes the i'th basis vector for the $j'th$ copy of $C^*(\mathbb{Z}_n)$. This C^*-algebra will be an important component in describing $C_{qc}(n, k)$ later on.

Lemma 3.10. *For each $1 \leq j \leq n$, let $\left\{P_i^j\right\}_{1\leq i\leq k}$ be a PVM on a Hilbert space H. There exists a unique $*$-homomorphism $\pi : C^*(\Gamma) \to B(H)$ satisfying $\pi(e_i^j) = P_i^j$.*

Proof. By Lemma 3.7, for all $1 \leq i \leq n$ and $1 \leq m \leq k$, there exist $*$-homomorphisms

$$\pi_m : \bigoplus_{i=1}^{n} \mathbb{C}e_i^m \to B(H), \quad e_i^m \mapsto P_i^m.$$

By the universal property of the free product, there exists a unique $*$-homomorphism $\pi_{1,2} : C^*(\mathbb{Z}_n) * C^*(\mathbb{Z}_n) \to B(H)$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & C^*(\mathbb{Z}_n) \arrow[dl, "\iota"'] \arrow[dr, "\pi_1"] & \\
C^*(\mathbb{Z}_n) * C^*(\mathbb{Z}_n) \arrow[rr, dashed, "\exists!\pi_{1,2}"] & & B(H) \\
 & C^*(\mathbb{Z}_n) \arrow[ul, "\iota"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Then $P_i^1 = \pi_1(e_i^1) = \pi_{1,2} \circ \iota(e_i^1)$ and $P_i^2 = \pi_2(e_i^2) = \pi_{1,2} \circ \iota(e_i^2)$. Using the universal property for the free product with $\pi_{1,2}$ and π_3 and so on yields the wanted $*$-homomorphism $\pi : \mathbb{C}^*(\Gamma) \to B(H)$ satisfying $P_i^j = \pi_\Gamma(e_i^j)$. □

Let $A \otimes_{\text{alg}} B$ denote the algebraic tensor product of two unital C^*-algebras A, B. Define the *maximal norm* on $A \otimes_{\text{alg}} B$ by

$$||x||_{\max} = \sup \left\{ ||\pi(x)|| \ \mid \pi : A_1 \otimes_{\text{alg}} A_2 \to B(H) \text{ is a } *\text{-homomorphism} \right\}.$$

The completion of $A \otimes_{\text{alg}} B$ with respect to the maximal norm is called *the maximal tensor product*, denoted by $A \otimes_{\max} B$. The maximal tensor product satisfies the following universal property: Given a unital C^*-algebra D and unit-preserving $*$-homomorphisms with commuting images $\pi_A : A \to D, \pi_B : B \to D$, there exists a unique $*$-homomorphism $\pi : A \otimes_{\max} B \to D$ making the following diagram commute;

$$\begin{tikzcd}
 & A \arrow[dl, "\iota_A"'] \arrow[dr, "\pi_A"] & \\
A \otimes_{\max} B \arrow[rr, dashed, "\exists!\pi"] & & D \\
 & B \arrow[ul, "\iota_B"] \arrow[ur, "\pi_B"'] &
\end{tikzcd}$$

where $\iota_A : A \to A \otimes_{\max} B$ and $\iota_B : B \to A \otimes_{\max} B$ are the inclusion maps. For notational purposes e_a^x denotes the a'th basis vector for the x'th copy of $C^*(\mathbb{Z}_n)$ and e_b^y denotes the b'th basis vector for the y'th copy of $C^*(\mathbb{Z}_n)$. From now on, we always consider $1 \leq x, y \leq n$ and $1 \leq a, b \leq k$ unless otherwise specified.

Lemma 3.11. *Given commuting PVMs $\{P_a^x\}_{1\le a\le k}$, $\{Q_b^y\}_{1\le b\le k}$ on a Hilbert space H there exists a unique $*$-homomorphism $\pi : C^*(\Gamma)\otimes_{max} C^*(\Gamma)\to B(H)$ satisfying $\pi(e_a^x\otimes 1)=P_a^x$ and $\pi(1\otimes e_b^y)=Q_b^y$.*

Proof. Let $\pi_\Gamma : C^*(\Gamma)\to B(H)$ be the unique $*$-homomorphism obtained in Lemma 3.10. By the universal property of the max tensor product, there exists a unique $*$-homomorphism $\pi : C^*(\Gamma)\otimes_{\max} C^*(\Gamma)\to B(H)$ such that $P_a^x=\pi_\Gamma(e_a^x)=\pi\circ\iota(e_a^x)=\pi(e_a^x\otimes 1)$ and likewise $Q_b^y=\pi_\Gamma(e_b^y)=\pi\circ\iota(e_b^y)=\pi(1\otimes e_b^y)$. $\square$

This leads to the following description of the quantum commuting correlation matrices. Given a unital C^*-algebra A, we denote by $S(A)$ the set of states on A.

Theorem 3.12. *Let $n,k\ge 2$. Then*

$$C_{qc}(n,k)=\left\{\left[\rho(e_a^x\otimes e_b^y)\right]_{a,b;x,y}\ \mid\ \rho\in S(C^*(\Gamma)\otimes_{max} C^*(\Gamma))\right\}.$$

Proof. Let $T\in C_{qc}(n,k)$ be given and write

$$T=\left[\left\langle P_a^x Q_b^y\xi_T,\xi_T\right\rangle\right]_{a,b;x,y}$$

for PVMs $\{P_a^x\}_{1\le a\le k}$, $\{Q_b^y\}_{1\le b\le k}$ on H and unit vector $\xi_T\in H$ as in Proposition 3.4. Let $\pi : C^*(G)\otimes_{\max} C^*(G)\to B(H)$ be the unique $*$-homomorphism obtained in Lemma 3.11. Let $\rho : C^*(\Gamma)\otimes_{\max} C^*(\Gamma)\to\mathbb{C}$ be given by $\rho(x)=\langle\pi(x)\xi,\xi\rangle$. We claim ρ is a state:

Firstly, ρ is clearly linear and $\rho(1)=||\xi||^2=1$. Let $x\in C^*(\Gamma)\otimes_{\max} C^*(\Gamma)$ be positive and $y\in C^*(\Gamma)\otimes_{\max} C^*(\Gamma)$ be such that $x=y^*y$. Then

$$\rho(x)=\langle\pi(x)\xi,\xi\rangle=\langle\pi(y)^*\pi(y)\xi,\xi\rangle=||\pi(y)\xi||^2\ge 0,$$

showing ρ is a state. Moreover,

$$T=\left[\rho(e_a^x\otimes e_b^y)\right]_{a,b;x,y}=\left[\left\langle\pi(e_a^x\otimes e_b^y)\xi,\xi\right\rangle\right]_{a,b;x,y}=\left[\left\langle P_a^x Q_b^y\xi,\xi\right\rangle\right]_{a,b;x,y}\in C_{qc}(n,k),$$

as wanted.

For the converse inclusion, let $\rho\in S(C^*(\Gamma)\otimes_{\max} C^*(\Gamma))$ be given. By the GNS construction, there exists a Hilbert space H_ρ, a $*$-homomorphism $\pi_\rho : C^*(\Gamma)\otimes_{\max} C^*(\Gamma)\to B(H_\rho)$ and a unit vector $\xi_\rho\in H_\rho$ such that $\rho(\omega)=\langle\pi_\rho(\omega)\xi_\rho,\xi_\rho\rangle$ for $\omega\in C^*(\Gamma)\otimes_{\max} C^*(\Gamma)$. Consider the family $\{P_a^x\}_{1\le a\le k}$ of projections on H_ρ given by $P_a^x=\pi_\rho(e_a^x\otimes 1)$. Likewise, let $\{Q_b^y\}_{1\le b\le k}$ be the family of projections on H_ρ defined by $Q_b^y=\pi_\rho(1\otimes e_b^y)$. Moreover

$$\sum_{a=1}^k P_a^x=\sum_{a=1}^k \pi_\rho(e_a^x\otimes 1)=\pi_\rho\left(\sum_{a=1}^k e_a^x\otimes 1\right)=\pi_\rho(1)=1$$

and similarly, $\sum_{b=1}^{k} Q_b^y = 1$, Lastly,

$$P_a^x Q_b^y = \pi_\rho(e_a^x \otimes e_b^y) = Q_b^y P_a^x$$

showing $\{P_a^x\}_{1\leq a\leq k}$ and $\{Q_b^y\}_{1\leq b\leq k}$ are commuting PVMs on H_ρ. Thus,

$$\left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y} = \left[\langle \pi_\rho(e_a^x \otimes e_b^y)\xi_\rho, \xi_\rho\rangle\right]_{a,b;x,y} = \left[\langle P_a^x Q_b^y \xi_\rho, \xi_\rho\rangle\right]_{a,b;x,y} \in C_{qc}(n,k),$$

which proves the inclusion. □

From this theorem, we can immediately conclude the following;

Corollary 3.13. *$C_{qc}(n,k)$ is closed.*

Proof. It is well-known that the set of states on a C^*-algebra is a compact and convex subset of the unit ball with respect to the weak*-topology. The surjective mapping

$$\varphi : S(C^*(\Gamma) \otimes_{\max} C^*(\Gamma)) \to C_{qc}(n,k), \quad \varphi(\rho) = \left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y}$$

is weak$*$-continuous and affine hence $\varphi(S(C^*(\Gamma)) \otimes_{\max} C^*(\Gamma))) = C_{qc}(n,k)$ is closed. □

Remark 3.14. One can also show $C_{qc}(n,k)$ is closed in the following way: Using the Hilbert space H and PVMs given by Proposition 3.4, define $\mathcal{A}(n,k) \subset B(H)$ to be the C^*-algebra generated by $\{P_a^x\}_{1\leq a\leq k}, \{Q_b^y\}_{1\leq b\leq k}$. Then, using techniques similar to the ones used in the proof of Theorem 3.12, it can be shown that

$$C_{qc}(n,k) = \left\{ \left[\rho(P_a^x Q_b^y)\right]_{a,b;x,y} \mid \rho \in S(\mathcal{A}(n,k)) \right\}.$$

It follows that $C_{qc}(n,k)$ is closed by the same argument as above.

4 Quantum spatial correlation matrices

The second quantum model is that of quantum spatial correlation matrices.

Let A_1, A_2 be unital C^*-algebras and consider the *spatial norm* on $A_1 \otimes_{\text{alg}} A_2$

$$||x||_{min} = \sup \{||(\pi_1 \otimes \pi_2)(x)|| \mid \pi_i : A_i \to B(H_i) \text{ is a } *\text{-homomorphism}, \ i = 1, 2\}.$$

where $\pi_1 \otimes \pi_2 : A_1 \otimes_{\text{alg}} A_2 \to B(H_1 \otimes H_2)$ is defined by $(\pi_1 \otimes \pi_2)(v \otimes w) = \pi_1(v) \otimes \pi_2(w)$. *The spatial tensor product*, $A_1 \otimes_{\min} A_2$, is the completion of $A_1 \otimes_{\text{alg}} A_2$ with respect to the minimal norm. We denote $A_1 \otimes_{\min} A_2$ by $A_1 \otimes A_2$.

Definition 4.1. Let $n, k \geq 2$. Denote $C_{qs}(n, k) \subset M_{nk}([0, 1])$ the set of *quantum spatial correlation matrices*

$$T = \left[\left\langle P_a^x \otimes Q_b^y \xi, \xi \right\rangle\right]_{a,b;x,y}$$

where $\xi \in H_A \otimes H_B$ is a unit vector for Hilbert spaces H_A, H_B and for each $1 \leq x, y \leq n$, $\{P_a^x\}_{1 \leq a \leq k}$ is a PVM on a H_A, and $\left\{Q_b^y\right\}_{1 \leq b \leq k}$ is a PVM on H_B.

Proposition 4.2. *$C_{qs}(n, k)$ is convex.*

Proof. Let $T, S \in C_{qs}(n, k)$ and $0 < t < 1$. Then there exist Hilbert spaces $H_{A_1}, H_{B_1}, H_{A_2}, H_{B_2}$, $\{P_a^x\}_{1 \leq a \leq k}, \left\{Q_b^y\right\}_{1 \leq b \leq k}$ PVMs on H_{A_1}, respectively H_{B_1}, and $\{K_a^x\}_{1 \leq a \leq k}, \left\{L_b^y\right\}_{1 \leq b \leq k}$ PVMs on H_{A_2}, respectively H_{B_2}, and unit vectors $\psi_1 \in H_{A_1} \otimes H_{B_1}, \psi_2 \in H_{A_2} \otimes H_{B_2}$ such that

$$T = \left[\left\langle P_a^x \otimes Q_b^y \psi_1, \psi_1 \right\rangle\right]_{a,b;x,y}, \quad S = \left[\left\langle K_a^x \otimes L_b^y \psi_2, \psi_2 \right\rangle\right]_{a,b;x,y}.$$

It is clear that $\{P_a^x \oplus K_a^x\}_{1 \leq a \leq k}$ are projections on $H_{A_1} \oplus H_{A_2}$ and that $\{Q_b^y \oplus L_b^y\}_{1 \leq b \leq k}$ are projections on $H_{B_1} \oplus H_{B_2}$. Moreover,

$$\sum_{a=1}^{k} (P_a^x \oplus K_a^x) = \sum_{a=1}^{k} P_a^x \oplus \sum_{a=1}^{k} K_a^x = 1_{H_{A_1}} \oplus 1_{H_{A_2}},$$

and $\sum_{b=1}^{k} (Q_b^y \oplus L_b^y) = 1_{H_{B_1}} \oplus 1_{H_{B_2}}$. Thus, $\{P_a^x \oplus K_a^x\}_{1 \leq a \leq k}$ and $\{Q_b^y \oplus L_b^y\}_{1 \leq b \leq k}$ are PVMs on $H_{A_1} \oplus H_{A_2}$, respectively $H_{B_1} \oplus H_{B_2}$. It is true that

$$(H_{A_1} \oplus H_{A_2}) \otimes (H_{B_1} \oplus H_{B_2}) \cong (H_{A_1} \otimes H_{B_1}) \oplus (H_{A_1} \otimes H_{B_2}) \oplus (H_{A_2} \otimes H_{B_1}) \oplus (H_{A_2} \otimes H_{B_2}).$$

Define $\varphi \in (H_{A_1} \oplus H_{A_2}) \otimes (H_{B_1} \oplus H_{B_2})$ by $\varphi = \sqrt{t}\psi_1 \oplus 0 \oplus 0 \oplus \sqrt{1-t}\psi_2$. Then $||\varphi|| = 1$, and moreover

$$\left\langle (P_a^x \oplus K_a^x) \otimes (Q_b^y \oplus L_b^y)\varphi, \varphi \right\rangle = t \left\langle (P_a^x \otimes Q_b^y \psi_1, \psi_1 \right\rangle + (1-t) \left\langle (K_a^x \otimes L_b^y \psi_1, \psi_1 \right\rangle.$$

Hence,

$$tT + (1-t)S = \left[\left\langle (P_a^x \oplus K_a^x) \otimes (Q_b^y \oplus L_b^y)\varphi, \varphi \right\rangle\right]_{a,b;x,y} \in C_{qs}(n, k),$$

as wanted. $\square$

It is highly non-trivial to show that $C_{qs}(n,k)$ is not closed for $n,k \geq 2$. The main theorem of this section is the following characterization of the closure of $C_{qs}(n,k)$;

Theorem 4.3. *Let $n,k \geq 2$. Then*

$$\overline{C_{qs}}(n,k) = \left\{ \left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y} \mid \rho \in S(C^*(\Gamma) \otimes_{min} C^*(\Gamma)) \right\}.$$

In order to prove this, we need to further describe the set on the right-hand side.

A subspace $\mathcal{M} \subset B(H)$ is called an *operator space* if $\mathcal{M}$ contains the identity I and is self-adjoint, i.e., $T \in \mathcal{M}$ implies $T^* \in \mathcal{M}$. A *state on* $\mathcal{M}$ is a linear functional $\rho : \mathcal{M} \to \mathbb{C}$ satisfying $\rho(I) = ||\rho|| = 1$. Denote by $S(\mathcal{M})$ the set of states on $\mathcal{M}$. It follows from Hahn-Banach that each $\rho \in S(\mathcal{M})$ extends to a linear functional $\overline{\rho} \in B(H)$ satisfying $1 = ||\overline{\rho}|| = \overline{\rho}(I)$, i.e., $\overline{\rho} \in S(B(H))$. In particular, $\overline{\rho}$ is positive.

Proposition 4.4. *Let $\mathcal{M}$ be a finite dimensional operator space on H and for each unit vector $\xi \in H$ let $\omega_\xi : B(H) \to \mathbb{C}$ denote the vector state given by $\omega_\xi(T) = \langle T\xi, \xi\rangle$. Then*

$$S(\mathcal{M}) = \overline{conv\{\omega_\xi|_{\mathcal{M}} \mid \xi \in H, \ \ ||\xi|| = 1\}}.$$

For a proof see Corollary 4.3.10 in [Kadison and Ringrose, 1983].

In the following let $\{e_a^x\}_{1 \leq a \leq k}, \{e_b^y\}_{1 \leq b \leq k}$ be as in Lemma 3.11. A representation of a unital $C*$-algebra A on a Hilbert space H is a $*$-homomorphism $\pi : A \to B(H)$.

Lemma 4.5. *Let $\pi : C^*(\Gamma) \to B(H)$ be a faithful representation. For $1 \leq x,y \leq n$, $1 \leq a,b \leq k$ set $\pi(e_a^x) = P_a^x$ and $\pi(e_b^y) = Q_b^y$. Let $\varphi = \pi \otimes \pi : C^*(\Gamma) \otimes C^*(\Gamma) \to B(H \otimes H)$. Then φ is a faithful representation with $P_a^x \otimes Q_b^y = \varphi(e_a^x \otimes e_b^y)$ and*

$$\mathcal{M} = span\left\{P_a^x \otimes Q_b^y \mid 1 \leq a,b \leq k, 1 \leq x,y \leq n\right\}$$

is a finite-dimensional operator space on $B(H \otimes H)$. Moreover

$$\left\{ \left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y} \mid \rho \in S(C^*(\Gamma) \otimes C^*(\Gamma)) \right\} = \left\{ \left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \mid \sigma \in S(\mathcal{M}) \right\}.$$

Proof. That φ is a faithful representation follows from π being faithful, the property of the minimal tensor product and

$$\varphi(e_a^x \otimes e_b^y) = \pi(e_a^x) \otimes \pi(e_b^y) = P_a^x \otimes Q_b^y.$$

By definition, $\mathcal{M}$ is finite-dimensional with $\dim \mathcal{M} \leq (nk)^2$. Moreover, as $\{P_a^x\}_{1 \leq a \leq k}$ and $\{Q_b^y\}_{1 \leq b \leq k}$ are PVMs on H, $\sum_b \sum_a P_a^x \otimes Q_b^y = 1 \otimes 1 \in \mathcal{M}$ and $\mathcal{M}$ is self-adjoint. We conclude that $\mathcal{M}$ is an operator space. It remains to show the equality of sets;

Let $\rho \in S(C^*(\Gamma) \otimes C^*(\Gamma))$. Note that as φ is faithful, $\rho \circ \varphi^{-1} : \varphi(C^*(\Gamma) \otimes C^*(\Gamma)) \to \mathbb{C}$ is well-defined and moreover is a state, as ρ is a state. By definition, $\mathcal{M} \subset \varphi(C^*(\Gamma) \otimes C^*(\Gamma))$. Thus, $\sigma = (\rho \circ \varphi^{-1})|_{\mathcal{M}}$ is a state on $\mathcal{M}$ for which

$$\sigma(P_a^x \otimes Q_b^y) = \sigma(\varphi(e_a^x \otimes e_b^y)) = (\rho \circ \varphi^{-1})\varphi(e_a^x \otimes e_b^y) = \rho(e_a^x \otimes e_b^y),$$

showing the wanted inclusion.

For the converse, take $\sigma \in S(\mathcal{M})$. Extend to $\overline{\sigma}$ on $B(H \otimes H)$ and define $\rho = \overline{\sigma} \circ \varphi$. Then $\rho \in S(C^*(\Gamma) \otimes C^*(\Gamma))$ with

$$\rho(e_a^x \otimes e_b^y) = \overline{\sigma}(\varphi(e_a^x \otimes e_b^y)) = \overline{\sigma}(P_a^x \otimes Q_b^y) = \sigma(P_a^x \otimes Q_b^y),$$

which finishes the proof. □

We are now ready to prove the main theorem of this section. For notational purposes, set

$$\mathcal{K} := \left\{ [\rho(e_a^x \otimes e_b^y)] \mid \rho \in S(C^*(\Gamma) \otimes_{\min} C^*(\Gamma)) \right\}.$$

Proof of Theorem 4.3. For the inclusion $\overline{C_{qs}}(n,k) \subset \mathcal{K}$ note that $\mathcal{K}$ is the image under the weak*-continuous and affine map $\rho \mapsto \left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y}$ for $\rho \in S(C^*(\Gamma) \otimes_{\min} C^*(\Gamma))$. As $S(C^*(\Gamma) \otimes_{\min} C^*(\Gamma))$ is compact, $\mathcal{K}$ is closed. Hence it suffices to show $C_{qs}(n,k) \subset \mathcal{K}$. Let $T \in C_{qs}(n,k)$ be given and write $T = \left[\left\langle P_a^x \otimes Q_b^y \xi, \xi \right\rangle\right]_{a,b;x,y}$ where $\{P_a^x\}_{1 \le a \le k}$ and $\left\{Q_b^y\right\}_{1 \le b \le k}$ are PVMs on H_A, respectively H_B and $\xi \in H_A \otimes H_B$ is a unit vector. Let $\pi_A : C^*(\Gamma) \to B(H_A)$ and $\pi_B : C^*(\Gamma) \to B(H_B)$ be the $*$-homomorphisms given by Lemma 3.10. Consider the representation $\pi_A \otimes \pi_B : C^*(\Gamma) \otimes C^*(\Gamma) \to B(H_A \otimes H_B)$. Define $\rho : C^*(\Gamma) \otimes C^*(\Gamma) \to \mathbb{C}$ by

$$\rho(x) = \langle (\pi_A \otimes \pi_B)(x)\xi, \xi \rangle,$$

for $x \in C^*(\Gamma) \otimes C^*(\Gamma)$. That ρ is linear, $\rho(1) = 1$ and ρ is positive is straightforward to see. Hence ρ is a state. By construction,

$$\left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y} = \left[\left\langle (P_a^x \otimes Q_b^y)\xi, \xi \right\rangle\right]_{a,b;x,y} \in \mathcal{K},$$

as wanted.

For the converse inclusion, let $\pi : C^*(\Gamma) \to B(H)$ be a faithful representation and define $\mathcal{M}$ as in Lemma 4.5. It suffices to show that $\left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \in \overline{C_{qs}}(n,k)$ for $\sigma \in S(\mathcal{M})$. See first for a unit vector $\xi \in H \otimes H$ that

$$\left[\omega_\xi|_{\mathcal{M}}(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} = \left[\left\langle (P_a^x \otimes Q_b^y)\xi, \xi \right\rangle\right]_{a,b;x,y} \in C_{qs}(n,k).$$

As $C_{qs}(n,k)$ is convex it follows that $\operatorname{conv}\{\omega_\xi|_{\mathcal{M}} \mid \xi \in H, \ ||\xi|| = 1\} \subset C_{qs}(n,k)$. Take $\sigma \in S(\mathcal{M})$ and note that by Proposition 4.4, $\sigma \in \overline{\operatorname{conv}\{\omega_\xi|_{\mathcal{M}} \mid \xi \in H, \ ||\xi|| = 1\}}$, implying $\left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \in \overline{C_{qs}}(n,k)$, as wanted. □

We move on to a further description of $C_{qs}(n,k)$. For this, we restrict to the correlation matrices with finite-dimensional Hilbert spaces;

Definition 4.6. Let $n,k \geq 2$. Denote $C_{qs}^{\mathrm{fin}}(n,k) \subset C_{qs}(n,k)$ the set of quantum spatial correlation matrices with finite-dimensional Hilbert spaces H_1 and H_2.

Theorem 4.7. *$C_{qs}^{fin}(n,k)$ is dense in $C_{qs}(n,k)$.*

In order to prove this we will take a detour into the world of RFD $C*$-algebras.

Definition 4.8. Let A be a unital C^*-algebra and H a Hilbert space. Denote by $\mathrm{Rep}(A,H)$ the set of all representations of A on H equipped with the coarsest topology for which the maps $\pi \mapsto \pi(a)\xi \in H$ are continuous for all $a \in A$.

Definition 4.9. A representation $\pi \in \mathrm{Rep}(A,H)$ is *finite-dimensional* if $\pi(A) \subset H$ is finite-dimensional. A representation $\pi \in \mathrm{Rep}(A,H)$ is *residually finite-dimensional* (RFD) if π can be approximated by finite-dimensional representations.

Definition 4.10. A state $\rho \in S(A)$ in A is said to be finite-dimensional if the GNS-representation π_ρ is finite-dimensional. Let $S^{\mathrm{fin}}(A)$ denote the set of finite-dimensional states.

Definition 4.11. A family of representations $(\pi_\alpha)_{\alpha\in I}$ is called *separating* if for all $x \in A$ $\pi_\alpha(x) = 0$ for all $\alpha \in I$ if and only if $x = 0$. A C^*-algebra A is *residually finite-dimensional (RFD)* if A has a separating family of finite-dimensional representations.

Theorem 4.12. *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) $S^{fin}(A)$ is dense in $S(A)$;

(ii) Every cyclic representation of A is RFD;

(iii) Every representation of A is RFD;

(iv) A admits a faithful RFD representation

(v) A is RFD.

Lemma 4.13. *Let A be a unital C^*-algebra. Let $\pi \in Rep(A,H)$ non-degenerate and suppose there exists a net $(\pi_\alpha)_\alpha \subset Rep(A,H)$ that converges to π. If $(\rho_\alpha)_\alpha \subset Rep(A,H)$ is another net on the same directed set such that for each $a \in A$ the restriction of $\rho_\alpha(a)$ to the essential space $H_\alpha \subset H$ of π_α coincides with $\pi_\alpha(a)$, then ρ_α also converges to π.*

We refer the reader to [Exel and Loring, 2012] for a proof of Thm. 4.12 and Lemma 4.13.

Theorem 4.14. *Let A,B be unital C^*-algebras. Then $A * B$ is RFD if and only if A and B are RFD.*

Proof. Assume $A * B$ is RFD. The RFD property passes to subalgebras, implying A and B both are RFD.

For the other implication, we use the equivalence of (iv) and (v) in Theorem 4.12. Thus, let π be a faithful non-degenerate representation of $A_1 * A_2$ on H. For each $i = 1, 2$, define $\pi_i = \pi|_{A_i}$ to be the restriction of π to A_i. Since A_i is RFD, the equivalence of (v) and (iii) in Theorem 4.12 gives that π_i is RFD, hence there exists a net $(\pi^i_\alpha)_\alpha$ of finite-dimensional representations converging to π_i. Note that we can use a common directed set by taking the product of the directed sets.

Consider now $\pi^i_\alpha : A_i \to B(H^i_\alpha)$. Now, for each α, choose a finite-dimensional subspace $K_\alpha \subset H$ such that $H^1_\alpha, H^2_\alpha \subset K_\alpha$ with $\dim(K_\alpha)$ a common multiple of $\dim(H^1_\alpha)$ and $\dim(H^2_\alpha)$. Let ρ^i_α be a representation of A_i on H^i_α such that $\rho^i_\alpha = A_i \to B(K_\alpha)$ and $\rho^i_\alpha|_{B(H^i_\alpha)} = \pi^i_\alpha$. Note that π_1, π_2 are unital hence non-degenerate, so using Lemma 4.13, it follows ρ^i_α converges to π_i.

For each α, define then $\rho_\alpha = \rho^1_\alpha * \rho^2_\alpha$ which is a well-defined finite-dimensional representation of $A_1 * A_2$ on H. Moreover, ρ_α converges to π by definition, so the equivalence from Theorem 4.12 gives $A * B$ is RFD. $\square$

It follows by definition that finite-dimensional C^*-algebras are RFD. As $\dim(C^*(\mathbb{Z}_n)) = n$ we get $C^*(\mathbb{Z}_n)$ is RFD. Using Theorem 4.14 we conclude that

$$C^*(\Gamma) \cong C^*(\mathbb{Z}_n) * ... * C^*(\mathbb{Z}_n)$$

is RFD. We are now ready to prove the density of $C^{\text{fin}}_{qs}(n, k)$ in $C_{qs}(n, k)$.

Proof of Theorem 4.7. Let $\pi_n : C^*(\Gamma) \to B(H_n)$ such that $\{\pi_n\}_{n \geq 1}$ is a separating sequence of finite dimensional representations. We know such a sequence exists as $C^*(\Gamma)$ is both RFD and separable. Define

$$H = \bigoplus_{n \geq 1} H_n,$$

and let

$$\pi = \bigoplus_{n \geq 1} \pi_n : C^*(\Gamma) \to B(H).$$

For $\xi = (\xi_n)_{n \geq 1} \in H$ we have $\pi(x)\xi = (\pi_n(x)\xi_n)_{n \geq 1}$ for $x \in C^*(\Gamma)$. By definition, π is a representation and moreover it is faithful as the sequence $\{\pi_n\}_{n \geq 1}$ is separating. Let

$$H^{(m)} = \bigoplus_{n=1}^{m} H_n = \{\xi \in H \mid \xi_n = 0 \text{ for all } n > m\} \subset H.$$

Here we consider $H^{(m)}$ as a finite-dimensional subspace of dimension m. Define faithful representation $\varphi = \pi \otimes \pi : C^*(\Gamma) \otimes C^*(\Gamma) \to B(H \otimes H)$ and $\mathcal{M}$ as in Lemma 4.5. Note that by definition of $H^{(m)}$, $P^x_a \otimes Q^y_b$ is invariant under $H^{(m)} \otimes H^{(m)}$ so we can consider the restriction of $P_a^x \otimes Q_b^y$ to said subspace. Consider $\sigma = \omega_\xi|_{\mathcal{M}} \in S(\mathcal{M})$ for some unit vector $\xi \in H^{(m)} \otimes H^{(m)} \subset H \otimes H$ and any $m \geq 1$. Then

$$\left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \in C_{qs}^{\mathrm{fin}}(n,k).$$

See that $\bigcup_{m\geq 1} H^{(m)} \otimes H^{(m)}$ is dense in $H \otimes H$. Take $\xi \in H \otimes H$ and let $(\xi_n)_{n\geq 1}$ be a sequence in $\bigcup_{m\geq 1} H^{(m)} \otimes H^{(m)}$ such that $\xi_n \to \xi$. Then $\omega_{\xi_n}|_{\mathcal{M}} \to \omega_\xi|_{\mathcal{M}}$ and it follows from the above that

$$\left[\omega_\xi|_{\mathcal{M}}(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \in \overline{C_{qs}^{\mathrm{fin}}}(n,k).$$

Note that it is a consequence of Lemma 4.5 and the proof of Theorem 4.3 that

$$C_{qs}(n,k) \subset \left\{ \left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} \mid \sigma \in S(\mathcal{M}) \right\}.$$

Thus, for $T \in C_{qs}(n,k)$ let $\sigma \in S(\mathcal{M})$ such that $T = \left[\sigma(P_a^x \otimes Q_b^y)\right]_{a,b;x,y}$. It follows from Proposition 4.4 that $T \in \overline{C_{qs}^{\mathrm{fin}}}(n,k)$, as wanted. $\square$

5 Tsirelson's conjecture

Having introduced three models of correlation matrices, it is natural to ask how they relate to one another. In these studies, one naturally reaches Tsirelson's conjecture, which will be stated later on. First, we show some results concerning the classical correlation matrices.

Definition 5.1. Let $n, k \geq 1$. Let $\Delta(k)$ denote the set of matrices $T \in M_k([0,1])$ with entries $T_{a,b}$ for $1 \leq a, b \leq k$ such that $\sum_{a,b=1}^{k} T_{a,b} = 1$. Furthermore, let $\Delta(n,k)$ denote the set of matrices $T \in M_n(M_k([0,1]))$ for which $T^{x,y} \in \Delta(k)$ for all $1 \leq x, y \leq n$.

Definition 5.2. Let $n, k \geq 2$. Define $\mathcal{E}^*(n,k) = \Delta(n,k) \cap \mathcal{E}(n,k)$.

A *matrix unit* in $M_n([0,1])$ is a matrix E for which one entry is equal to 1 and all other entries are equal to 0. Recall the definition of $\mathcal{E}(n,k)$ and note that a matrix $E \in \mathcal{E}^*(n,k)$ if and only if each $E^{x,y}$ is a matrix unit for all $1 \leq x, y \leq n$. Thus, for each $E \in \mathcal{E}^*(n,k)$ there exists a unique map $f_E : [n] \times [n] \to [k] \times [k]$, where $[n] = \{1, \ldots, n\}$, such that

$$E_{a,b}^{x,y} = \begin{cases} 1 & (a,b) = f_E(x,y), \\ 0 & \text{else.} \end{cases}$$

A function $f : [n] \times [n] \to [k] \times [k]$ is *independent* if there exist $f_1, f_2 : [n] \to [k]$ such that $f(x,y) = (f_1(x), f_2(y))$. A matrix $E \in \mathcal{E}^*(n,k)$ for which f_E is independent is called an *independent deterministic correlation matrix*.

Definition 5.3. Let $n, k \geq 2$. Let $\mathcal{E}^*_{\text{indep}}(n,k) \subset \mathcal{E}^*(n,k)$ denote the set of independent correlation matrices.

Recall $\mathcal{E}^\dagger(n,k) = C_c(n,k) \cap \mathcal{E}(n,k)$.

Proposition 5.4. *$\mathcal{E}^\dagger(n,k) = \mathcal{E}^*_{indep}(n,k)$.*

Proof. Note that $\mathcal{E}^*_{\text{indep}}(n,k) \subset \mathcal{E}(n,k)$ so it suffices to show $\mathcal{E}^*_{\text{indep}}(n,k) \subset C_c(n,k)$. Take $E \in \mathcal{E}^*_{\text{indep}}(n,k)$ and let $f_1, f_2 : [n] \to [k]$ be such that $f_E(x,y) = (f_1(x), f_2(y))$. Let $(\Omega, \mathcal{A}, \mu)$ be a one-point probability space. For $1 \leq x, y \leq n$ and $1 \leq a, b \leq k$ define

$$A_a^x = \begin{cases} \Omega, & \text{if } a = f_1(x) \\ \emptyset, & \text{else} \end{cases}, \quad B_b^y = \begin{cases} \Omega, & \text{if } b = f_2(y) \\ \emptyset, & \text{else} \end{cases}.$$

Then $\{A_a^x\}_{1 \leq a \leq k}, \{B_b^y\}_{1 \leq b \leq k}$ are partitions of Ω and it follows that $\mu(A_a^x \cap B_b^y) = E_{a,b}^{x,y}$.

For the converse inclusion, let $E \in C_c(n,k) \cap \mathcal{E}(n,k)$ and write

$$E = \left[E_{a,b}^{x,y}\right]_{a,b;x,y} = \left[\mu(A_a^x \cap B_b^y)\right]_{a,b;x,y}$$

for some probability space $(\Omega, \mathcal{A}, \mu)$ and partitions $\{A_a^x\}_{1\le a\le k}, \{B_b^y\}_{1\le b\le k}$. Fix $1 \le x, y \le n$ and let a_x, b_y be such that $E_{a_x,b_y}^{x,y} = 1$. That such a_x, b_y exist follows as $\{A_a^x\}_{1\le a\le k}$ and $\{B_b^y\}_{1\le b\le k}$ are partitions of Ω. Clearly $\mu(A_{a_x}^x \cap B_{b_y}^y) = 1$ forces $\mu(A_{a_x}^x) = \mu(B_{b_y}^y) = 1$. Since

$$1 = \mu(\Omega) = \sum_{a=1}^k \mu(A_a^x), \quad 1 = \mu(\Omega) = \sum_{b=1}^k \mu(B_b^y)$$

we get $\mu(A_a^x) = \mu(B_b^y) = 0$ for all $(a, b) \neq (a_x, b_y)$. Define now $f_1, f_2 : [n] \to [k]$ by $f_1(x) = a_x, f_2(y) = b_y$. Then

$$E_{a,b}^{x,y} = \mu(A_a^x \cap B_b^y) = \begin{cases} 1, & \text{if } (a,b) = (f_1(x), f_2(y)) \\ 0, & \text{else} \end{cases}$$

showing $E \in \mathcal{E}_{\text{indep}}^*(n, k)$. $\square$

Proposition 5.4 combined with Proposition 2.7 shows that $C_c(n, k)$ is spanned by the independent deterministic correlation matrices. In particular, we can describe $C_c(n, k)$ without reference to the specific probability spaces and partitions as in Definition 2.1.

Recall that $C_{qs}(n, k)$ is not closed.

Proposition 5.5. $C_c(n, k) \subsetneq C_{qs}(n, k)$.

Proof. That $C_c(n, k) \neq C_{qs}(n, k)$ is immediate since $C_c(n, k)$ is closed but $C_{qs}(n, k)$ is not.

For the inclusion, since $C_{qs}(n, k)$ is convex, it suffices to show $\partial_e C_c(n, k) \subset C_{qs}(n, k)$, cf. Proposition 2.7. By Proposition 5.4, $\partial_e C_c(n, k) = \mathcal{E}_{\text{indep}}^*(n, k)$. Let $E \in \mathcal{E}_{\text{indep}}^*(n, k)$ be given. Then there exist $f_1, f_2 : [k] \to [n]$ such that

$$E_{a,b}^{x,y} = \begin{cases} 1, & \text{if } (a,b) = (f_1(x), f_2(y)) \\ 0, & \text{else} \end{cases}$$

Let $H = \mathbb{C} \otimes \mathbb{C}$. For each $1 \le x, y \le n$, let $a_x = f_1(x), b_y = f_2(y)$ and define

$$P_a^x = \begin{cases} 1, & \text{if } a = a_x \\ 0, & \text{else} \end{cases}, \quad Q_b^y = \begin{cases} 1, & \text{if } b = b_y \\ 0, & \text{else} \end{cases}$$

Then $\{P_a^x\}_{1\le a\le k}, \{Q_b^y\}_{1\le b\le k}$ are a PVMs on $\mathbb{C}$ and given a unit vector $\xi \in H$,

$$\langle P_a^x \otimes Q_b^y \xi, \xi \rangle = \begin{cases} 1, & \text{if } (a,b) = (a_x, b_y) \\ 0, & \text{else} \end{cases}$$

hence $E = [\langle P_a^x \otimes Q_b \xi, \xi \rangle]_{a,b;x,y} \in C_{qs}(n, k)$. $\square$

Proposition 5.6. $C_{qs}(n, k) \subsetneq \overline{C_{qs}}(n, k) \subset C_{qc}(n, k)$.

Proof. Let $T \in C_{qs}(n,k)$, write $T = \left[\left\langle P_a^x \otimes Q_b^y \xi, \xi \right\rangle\right]_{a,b;x,y}$ for $\{P_a^x\}_{1\leq a\leq k}$ and $\left\{Q_b^y\right\}_{1\leq b\leq k}$ PVMs on Hilbert spaces H_A, respectively H_B, and $\xi \in H_A \otimes H_B$ a unit vector. Then $\{P_a^x \otimes 1\}_{1\leq a\leq k}, \left\{1 \otimes Q_b^y\right\}_{1\leq b\leq k}$ are commuting PVMs on the Hilbert space $H_A \otimes H_B$, implying $T \in C_{qc}(n,k)$. The inclusion $\overline{C_{qs}}(n,k) \subset C_{qc}(n,k)$ now follows from the fact that $C_{qc}(n,k)$ is closed. □

It is now clear that $C_c(n,k) \subset C_{qc}(n,k)$. One can show that $\mathcal{E}^*_{\text{indep}}(n,k) = C_{qc}(n,k) \cap \mathcal{E}(n,k)$ meaning all deterministic quantum correlation matrices are classical.

Whether or not the latter inclusion in Proposition 5.6 is indeed an equality of sets is a famous conjecture of Tsirelson;

Conjecture 5.7 (Tsirelson)**.** *$\overline{C_{qs}}(n,k) = C_{qc}(n,k)$ for all $n,k \geq 2$.*

We are able to prove a weaker version concerning $C_{qs}^{\text{fin}}(n,k)$ and $C_{qc}^{\text{fin}}(n,k)$ called *Tsirelson's Theorem.* We define $C_{qc}^{\text{fin}}(n,k)$ in a similiar way as $C_{qs}^{\text{fin}}(n,k)$ in Definition 4.6 with a finite dimensional Hilbert space H. In order to prove Tsirelson's Theorem, we need the following lemma.

Lemma 5.8. *Let $A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C})$ be a finite-dimensional C^*-algebra acting on the Hilbert space $H = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \oplus \cdots \oplus \mathbb{C}^{n_m}$. Then*

$$S(A) = conv\left\{\omega_\xi|_A \mid \xi \in H, ||\xi|| = 1\right\}.$$

Proof. A state ρ on $\mathcal{A}$ extends to a state on $B(H)$. It is well-known that when H is finite-dimensional, any state on $B(H)$ is a convex combination of vector states, as wanted. □

Theorem 5.9 (Tsirelson)**.** *$C_{qs}^{fin}(n,k) = C_{qc}^{fin}(n,k)$.*

Proof. It is straightforward that $C_{qs}^{\text{fin}}(n,k) \subset C_{qc}^{\text{fin}}(n,k)$. For the reverse, let $T \in C_{qc}^{\text{fin}}(n,k)$ be given by

$$T = \left[\left\langle P_a^x Q_b^y \xi, \xi \right\rangle\right]_{a,b;x,y}$$

for commuting PVM's $\{P_a^x\}_{1\leq a\leq k}, \left\{Q_b^y\right\}_{1\leq b\leq k}$ on a finite-dimensional Hilbert space H and a unit vector $\xi \in H$. Define

$$\mathcal{P} = \{P_a^x \mid 1 \leq a \leq k, 1 \leq x \leq n\}, \quad \mathcal{Q} = \left\{Q_b^y \mid 1 \leq b \leq k, 1 \leq y \leq n\right\}$$

and let A_1 and A_2 be the sub-C^*-algebras of $B(H)$ generated by $\mathcal{P}$ and $\mathcal{Q}$ respectively and A be the sub-C^*-algebra of $B(H)$ generated by $\mathcal{P} \cup \mathcal{Q}$. Note that each element in $\mathcal{P}$ commutes with all elements in $\mathcal{Q}$ as they are defined by sets of commuting PVMs. In particular, A_1 and A_2 commute and moreover there are canonical inclusion of A_i into A, $i = 1,2$. By the universal property of the maximal tensor product there is a unital $*$-homomorphism

$\psi : A_1 \otimes_{\max} A_2 \to A$ such that $\psi(a \otimes b) = ab$ for all $a \in A_1$, $b \in A_2$. Moreover, ψ is surjective by definition of A. Since A_1, A_2 are finite they are in particular finite-dimensional so the maximal and minimal tensor product agree, hence we can consider $\psi : A_1 \otimes A_2 \to A$. Let $\rho : A \to \mathbb{C}$ be the state given by $\rho(S) = \langle S\xi, \xi \rangle$. By definition, $T = \left[\rho(P_a^x Q_b^y)\right]_{a,b;x,y}$. Consider now $\tilde{\rho} = \rho \circ \psi : A_1 \otimes A_2 \to \mathbb{C}$. Since ρ is a state and ψ is a $*$-homomorphism, $\tilde{\rho}$ is a state and moreover

$$\tilde{\rho}(P_a^x \otimes Q_b^y) = \rho(\psi(P_a^x \otimes Q_b^y)) = \rho(P_a^x Q_b^y),$$

hence $T = \left[\tilde{\rho}(P_a^x \otimes Q_b^y)\right]_{a,b;x,y}$. Now, since A_1, A_2 are finite dimensional, we have

$$A_i = M_{n(1,i)}(\mathbb{C}) \oplus M_{n(2,i)} \oplus \cdots \oplus M_{n(k,i)}, \quad i = 1, 2$$

and that A_i acts on the finite dimensional Hilbert space

$$H_i = \mathbb{C}^{n(1,i)} \oplus \mathbb{C}^{n(2,i)} \oplus \cdots \mathbb{C}^{n(k,i)}, \quad i = 1, 2.$$

Since $A_1 \otimes A_2$ has a representation on $H_1 \otimes H_2$ and $A_1 \otimes A_2$ is finite-dimensional, Lemma 5.8 gives

$$S(A_1 \otimes A_2) = \operatorname{conv}\left\{\omega_\xi|_{A_1 \otimes A_2} \mid \xi \in H_1 \otimes H_2, ||\xi|| = 1\right\}.$$

Thus, we can write $\tilde{\rho} = \sum_{j=1}^r \lambda_j \omega_{\xi_j}$ where $\lambda_j > 0, \sum_{j=1}^r \lambda_j = 1$ and $\xi_j \in H_1 \otimes H_2$. Then

$$T = \left[\tilde{\rho}(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} = \left[\sum_{j=1}^r \lambda_j \omega_{\xi_j}(P_a^x \otimes Q_b^y)\right]_{a,b;x,y} = \sum_{j=1}^r \lambda_j \left[\left\langle (P_a^x \otimes Q_b^y)\xi_j, \xi_j \right\rangle\right]_{a,b;x,y}$$

and as $\left[\left\langle (P_a^x \otimes Q_b^y)\xi_j, \xi_j \right\rangle\right]_{a,b;x,y} \in C_{qs}^{\text{fin}}(n,k)$, convexity of $C_{qs}^{\text{fin}}(n,k)$ gives $T \in C_{qs}^{\text{fin}}(n,k)$. $\square$

We now have the following picture of inclusions:

$$\begin{array}{ccccccc}
 & & C_{qs}^{\text{fin}}(n,k) & & = & & C_{qc}^{\text{fin}}(n,k) \\
 & & \cap & & & & \cap \\
C_c(n,k) & \subset & C_{qs}(n,k) & \subsetneq & \overline{C_{qs}}(n,k) & \subset & C_{qc}(n,k)
\end{array}$$

A motivation for studying Tsirelson's conjecture is the relation between Tsirelson's conjecture and the *Connes embedding problem*. To understand the statement of the Connes embedding problem, we need a few definitions. Recall that a von Neumann algebra M is a *factor* if the center of M is trivial, i.e., $Z(M) = \mathbb{C}1$.

Definition 5.10. A factor M is of type II_1 (called a II_1-factor) if it is infinite-dimensional and admits a faithful tracial state $\tau : M \to \mathbb{C}$.

We say M is *hyperfinite* if it contains a dense $*$-subalgebra which is an increasing union of finite-dimensional $*$-algebras. Let $\mathcal{R}$ be the unique hyperfinite II_1-factor up to isomorphism. For any free ultrafilter ω, define the ultrapower of $\mathcal{R}$ by

$$\mathcal{R}^\omega = \prod_{n=1}^{\infty} \mathcal{R}/I^\omega, \quad I^\omega = \left\{ (x_n) \in \prod_{n=1}^{\infty} \mathcal{R} \mid \lim_{\omega} ||x_n||_2 = 0 \right\}.$$

It can be shown that $\mathcal{R}^\omega$ is also a II_1-factor.

Conjecture 5.11 (The Connes Embedding Problem)**.** *Every separable II_1-factor embeds into $\mathcal{R}^\omega$.*

Theorem 5.12. *The following are equivalent:*

(i) Tsirelson's conjecture holds,

(ii) $\overline{C_{qc}^{fin}}(n,k) = C_{qc}(n,k)$ for all $n, k \geq 2$,

(iii) $C^*(\Gamma) \otimes_{max} C^*(\Gamma) = C^*(\Gamma) \otimes C^*(\Gamma)$ for all $n, k \geq 2$,

(iv) $C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$,

(v) The Connes Embedding Problem is true.

Proof. $(i) \Rightarrow (ii)$. Suppose $\overline{C_{qs}}(n,k) = C_{qc}(n,k)$. Since $C_{qs}^{\mathrm{fin}}(n,k)$ is dense in $C_{qs}(n,k)$, Tsirelson's theorem gives

$$C_{qc}(n,k) = \overline{C_{qs}}(n,k) = \overline{C_{qs}^{\mathrm{fin}}}(n,k) = \overline{C_{qc}^{\mathrm{fin}}}(n,k),$$

as wanted.

$(ii) \Rightarrow (i)$. Let $n, k \geq 2$. Suppose $\overline{C_{qc}^{\mathrm{fin}}}(n,k) = C_{qc}(n,k)$. Using again that $C_{qs}^{\mathrm{fin}}(n,k)$ is dense in $C_{qs}(n,k)$ and Tsirelson's Theorem, we get

$$C_{qc}(n,k) = \overline{C_{qc}^{\mathrm{fin}}}(n,k) = \overline{C_{qs}^{\mathrm{fin}}}(n,k) = \overline{C_{qs}}(n,k),$$

as wanted.

$(iii) \Rightarrow (i)$. This follows directly by

$$\begin{aligned} C_{qc}(n,k) &= \left\{ \left[\rho(e_a^x \otimes e_b^y)\right]_{a,b;x,y} \mid \rho \in S(C^*(\Gamma) \otimes_{\max} C^*(\Gamma)) \right\} \\ &= \left\{ \left[\rho(e_a^x \otimes e_b^y\right]_{a,b;x,y} \mid \rho \in S(C^*(\Gamma) \otimes C^*(\Gamma)) \right\} = \overline{C_{qs}}(n,k). \end{aligned}$$

$(i) \Rightarrow (iii)$. This was proved by Ozawa.

The equivalences $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ were proved in [Kirchberg, 1993]. Moreover, to prove $(iii) \Leftrightarrow (iv)$ one can use techniques of Kirchberg. $\square$

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