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Crossed Products and Cantor Systems

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Introduction and prerequisites

This is a thesis written during the period June,2000-February,2001. The work has partly been done at Copenhagen University (where I have had my education), partly at Mathematical Sciences Research Institute, Berkeley, California (where I had the pleasure to spend three months during the autumn 2000).

The reader is assumed to have a basic knowledge of C*-algebras. Thus I frequently use results from Murphy [15] and Rørdam et al. [22]. I was tempted to do the same about Davidson [4]. Yet that book is quite hard to read, and for that reason I have included proofs of some of the theorems in Davidson [4].

Finally, I would like to thank the large number of persons, who have contributed to my education at Copenhagen University. Special thanks are due to my supervisor Mikael Rørdam for many fruitful conversations while working with this thesis. I also thank Christian Skau (for answering many of my preliminary questions concerning Cantor systems) and Ian Putnam (for suggesting Theorem 5.6.6 below).

Søren Essendrop Boel, Copenhagen.

Summary

Here is a brief summary of the contents of this thesis.

In Chapter 1 we introduce the basic topological concepts, including a treatment of dynamical systems and the Cantor set. In particular, Putnam systems are introduced as the dynamical systems arising from actions on a totally disconnected compact metric space. An important subclass of Putnam systems is the Cantor systems arising from minimal actions on the Cantor set.

In Chapter 2 we turn to the C*-algebraic crossed products arising from Cantor systems (or more general dynamical systems). The chapter includes a representation of the crossed product as operators on a Hilbert space, a (partial) proof of simplicity, and a (partial) computation of the K-theory.

In Chapter 3 we give a proof of the fact that the crossed products arising from Cantor systems are classified by their K-theory. We prove that crossed products arising from Cantor systems are local $A\mathbb{T}$-algebras, and the classification result then follows from a famous work by Elliot [8].

In Chapter 4 we investigate the connection between Cantor systems and the Bratteli-Vershik systems arising from a certain class of Bratteli diagrams. We prove a famous result due to Giordano et al. [9], stating a dynamical criterion for the crossed products arising from Cantor systems to be isomorphic.

In Chapter 5 we give an account of the question of stable rank of the crossed products arising from Putnam systems. We generalize a result by Putnam [19] and end up giving a dynamical criterion for the crossed products to have stable rank one.

Contents

Chapter 1

Cantor Systems and Putnam Systems

This chapter deals with the fundamental building blocks of this thesis, namely totally disconnected compact spaces and dynamical systems. In the first section below we discuss totally disconnected spaces, a certain subclass of which is referred to as Cantor sets. In the second section we introduce dynamical systems, and in the third section we introduce some relations between dynamical systems.

1.1 Totally disconnected spaces

Definition 1.1.1. *Let X be a topological space, and, for each point $x \in X$, let $\Gamma(x)$ denote the connected component containing x (cf. Rudin [23], Definition 10.1). Then X is said to be totally disconnected if*

$$\forall x \in X : \Gamma(x) = \{x\}.$$

We will give a criterion for a compact Hausdorff space to be totally disconnected. First a lemma:

Lemma 1.1.2. *Let X be a compact Hausdorff space, let $x \in X$, and let $(V_i)_{i \in I}$ be the family of the clopen sets in X which contain x. Then $\cap_{i \in I} V_i = \Gamma(x)$, where $\Gamma(x)$ denotes the connected component containing x.*

Proof. Put $D = \cap_{i \in I} V_i$. We must prove that

$$D = \Gamma(x).$$

"$\supseteq$": Suppose that V_i is clopen and $x \in V_i$. Then $V_i \cap \Gamma(x)$ is nonempty and clopen with respect to the topology on $\Gamma(x)$. This implies that $V_i \cap \Gamma(x) = \Gamma(x)$, that is, $\Gamma(x) \subseteq V_i$.
"$\subseteq$": It suffices to prove that D is connected. Assume to reach a contradiction that $D = D_1 \sqcup D_2$ ("$\sqcup$" means "disjoint union"), where $x \in D_1$ and D_1 and D_2 are nonempty and closed relatively to D.

Since D is closed, D_1 and D_2 are also closed relatively to X. Now choose (using the fact that X has the T_4-property) disjoint open sets W_1 and W_2 such that $D_j \subseteq W_j, j = 1, 2$. In particular $D \subseteq W_1 \cup W_2$.

The set $V_i \cap \complement(W_1 \cup W_2)$ is closed for every $i \in I$ and

$$\bigcap_{i \in I}(V_i \cap \complement(W_1 \cup W_2)) = D \cap \complement(W_1 \cup W_2) = \emptyset.$$

We can therefore choose $i_1, \dots i_n \in I$, such that

$$\bigcap_{j=1}^{n} V_{i_j} \cap \complement(W_1 \cup W_2) = \emptyset.$$

Define a clopen set $V = \cap_{j=1}^{n} V_{i_j}$ and note that $V \subseteq W_1 \cup W_2$. Now $V \cap W_1$ and $V \cap W_2$ are open. Hence $V \cap W_1 = V \setminus (V \cap W_2)$ is clopen, and since $x \in V \cap W_1$, we get that $D \subseteq V \cap W_1 \subseteq W_1$. We therefore conclude that $D_2 = \emptyset$. □

Theorem 1.1.3. *Suppose that X is a compact Hausdorff space. Then the following are equivalent:*

1. *X is totally disconnected;*

2. $\dim(X) = 0$ *(that is, X has a basis consisting of clopen sets).*

Proof. "$1 \Rightarrow 2$": Suppose that U is open and $x \in U$. Since X is totally disconnected, $\Gamma(x) = \{x\}$. By Lemma 1.1.2 we may therefore to every $y \in \complement U$ pick a clopen set W_y such that $y \in W_y$ and $x \notin W_y$. Now use compactness of $\complement U$ to pick $y_1, \dots, y_n$ such that $\complement U \subseteq W_{y_1} \cup \dots \cup W_{y_n}$. Then

$$V = \complement(W_{y_1} \cup \dots \cup W_{y_n})$$

is clopen, and $x \in V \subseteq U$.
"$2 \Rightarrow 1$": Follows from Lemma 1.1.2. □

We will need the notion of a partition $\mathcal{P}$ of a (totally disconnected) compact Hausdorff space X:

Definition 1.1.4. *Let X be a compact Hausdorff space, and let $\mathcal{P} = \{X_1, \dots, X_m\}$ be a finite family of nonempty disjoint clopen sets satisfying $X = \cup_{i=1}^{m} X_i$. Then $\mathcal{P}$ is said to be a partition of X, and $C(\mathcal{P})$ will denote the finite dimensional C^*-algebra* $\operatorname{span}\{1_E | E \in \mathcal{P}\} \subseteq C(X)$.

Let $k \in \mathbb{N}$ and put $C_k = \{0, 1, \dots, k-1\}^{\mathbb{N}_0}$. Given $x, y \in C_k$ put $d_k(x, y) = 2^{-n}$, where n is the smallest integer such that $x_n \neq y_n$ (with $n = \infty$ if $x = y$). Then it is well-known (or easily seen), that (C_k, d_k) is a totally disconnected compact metric space without isolated points. We will now prove that spaces satisfying these properties are mutually homeomorphic. First a lemma:

Lemma 1.1.5. *Suppose that X is a totally disconnected compact Hausdorff space without isolated points, and $W \subseteq X$ is clopen and nonempty. Then we can write $W = W_1 \sqcup W_2$, where W_1 and W_2 both are clopen and nonempty.*

Proof. Pick an $x \in W$. Since x is not an isolated point, we can find $y \in W$ such that $y \neq x$. Now pick a clopen set W_1 such that $x \in W_1$ and $y \notin W_1$ (the Hausdorff property), and put $W_2 = W \setminus W_1$. □

Theorem 1.1.6. *Suppose (X, d) is a metric space satisfying*

1. *X is compact,*

2. *X is totally disconnected, and*

3. *X does not contain any isolated points.*

Then X is homeomorphic to $C_2 = \{0,1\}^{\mathbb{N}_0}$, where C_2 is equipped with the metric d_2 from above.

Proof. Suppose (X, d) satisfies condition 1, 2 and 3. Because of condition 1 and 2, X has a basis consisting of clopen sets. Since X is compact, we can find a finite covering $W_1 \cup \cdots \cup W_n$ of X such that, for each i in $\{1, \ldots, n\}$, $diam(W_i) \leq 1$ and W_i is clopen. Since clopen sets respects difference between sets (A, B clopen $\Rightarrow A \setminus B$ clopen), we may assume that $W_1, \ldots, W_n$ are disjoint.

By splitting into smaller sets we may assume that n has the form 2^{n_1}, cf. Lemma 1.1.5. We show that each W_i satisfies 1, 2 and 3:
1: W_i is a closed subset of a compact metric space.
2: W_i inherits a basis of clopen sets from X. Hence $\dim(W_i) = 0$, and by Theorem 1.1.3 and condition 1 we get that W_i is totally disconnected.
3: Since W_i is open, every isolated point in W_i is also isolated in X.

We can therefore again split every W_i into nonempty, clopen sets:

$$W_i = W_{i,1} \sqcup \cdots \sqcup W_{i,2^{n_2}} \text{ and } \operatorname{diam}(W_{i,j}) \leq \frac{1}{2}$$

and so on. Continuing this process, we get a sequence of partitions

$$\mathcal{P}_n = \{W_{\mathbf{h}} \mid \mathbf{h} \in \prod_{j=1}^{n} F_j\},$$

where $F_j = \{1, \ldots, 2^{n_j}\}$ for some integer n_j. Note that $\cup_{n=1}^{\infty} \mathcal{P}_n$ forms a basis for the topology on X. Put

$$\sigma(\alpha) = \bigcap_{n=1}^{\infty} W_{\pi_n(\alpha)}, \ \alpha \in \prod_{j=1}^{\infty} F_j$$

where π_n is the projection onto the first n coordinates. For any $\alpha \in \prod_{j=1}^{\infty} F_j$ we have that $\operatorname{diam}(W_{\pi_n(\alpha)}) \to 0$. Hence $\sigma(\alpha)$ is a singleton. Identifying $\sigma(\alpha)$ with the element it contains, we will consider σ to be a function from $\prod_{j=1}^{\infty} F_j$ to X. It is clear that σ is bijective. Let us endow $\prod_{j=1}^{\infty} F_j$ with the product topology induced by the discrete topology on F_j, and prove that σ is continuous:

Suppose $\alpha_k \to \alpha$ as $k \to \infty$; we must prove that $\sigma(\alpha_k) \to \sigma(\alpha)$. Let $W_{\mathbf{h}}$ be a neighborhood of $\sigma(\alpha)$ for some $\mathbf{h} \in \prod_{j=1}^{n} F_j$, and choose $K \in \mathbb{N}$ such that $k \geq K$ implies $\pi_n(\alpha_k) = \pi_n(\alpha) = \mathbf{h}$. Then $k \geq K$ implies $\sigma(\alpha_k) \in W_{\mathbf{h}}$.

Being a continuous bijective map between two compact metric spaces, σ is a homeomorphism.

Finally note that C_2 is homeomorphic to $\prod_{j=1}^{\infty} F_j$. □

Theorem 1.1.6 allows us to define *the Cantor set* as any set satisfying the conditions from Theorem 1.1.6. The standard way of introducing the Cantor set is the *middlethird construction* (cf. Pedersen [16], Exercise 1.6.8), and it follows from Theorem 1.1.6 that the middlethird construction indeed gives us a Cantor set. We will soon construct a couple of different pictures of the Cantor set.

We will need the following theorem:

Theorem 1.1.7. *Suppose X is a totally disconnected compact metric space. Then we may choose partitions $\mathcal{P}_n$ of X, such that $\mathcal{P}_n$ is coarser than $\mathcal{P}_{n+1}$, each set in $\mathcal{P}_n$ has diameter at most $1/n$, and*

$$C(X) = \overline{\bigcup_{n=1}^{\infty} C(\mathcal{P}_n)}.$$

In particular, $C(X)$ is an AF-algebra.

Proof. The construction of $\mathcal{P}_n$ is similar to the one in the proof of Theorem 1.1.6. The only difference is that we cannot be sure that we are able to split every clopen set arbitrarily many times. This situation will occur, if the clopen set contains only a single element, but in that case the diameter of the set is zero.

To see that $C(X) = \overline{\cup_{n=1}^{\infty} C(\mathcal{P}_n)}$, observe that $\cup_{n=1}^{\infty} C(\mathcal{P}_n)$ is a point-separating *-subalgebra of $C(X)$ which contains the constants, and apply the Stone-Weierstrass Theorem. □

1.2 Dynamical systems

Let us now define the notion of a dynamical system:

Definition 1.2.1. *A dynamical system is a pair (X, σ), where X is a compact Hausdorff space and $\sigma : X \to X$ is a homeomorphism.*

There is a certain ambiguity in the literature concerning the term *σ-invariant*. We therefore introduce two notions of σ-invariance in the following definition:

Definition 1.2.2. *Let (X, σ) be a dynamical system, and $F \subseteq X$. Then F is said to be σ-invariant, if $\sigma(F) \subseteq F$. If $\sigma(F) = F$ we say that F is strongly σ-invariant.*

Let F be a nonempty, closed, (strongly) σ-invariant subset of X. Then F is said to be a (strongly) minimal set, if F is minimal among the nonempty, closed, (strongly) σ-invariant subsets, that is, whenever F' is a nonempty, closed, (strongly) σ-invariant subset satisfying $F' \cap F \neq \emptyset$, then $F \subseteq F'$.

Let us immediately observe, that the two notions of *minimal* and *strongly minimal* coincide:

Theorem 1.2.3. *Let (X, σ) be a dynamical system, and $F \subseteq X$. Then*

$$F \text{ is minimal} \iff F \text{ is strongly minimal.}$$

Proof. "$\Rightarrow$": The only thing to remark is, that F is automatically strongly σ-invariant, since otherwise $\cap_{n\in\mathbb{Z}}\sigma^n(F)$ would be a nonempty, closed, σ-invariant, proper subset of F.
"$\Leftarrow$": Let F' be a nonempty, closed, σ-invariant subset satisfying $F' \subset F$, and put $H = \cap_{n\in\mathbb{Z}}\sigma^n(F')$. Then we reach a contradiction by observing that H is a nonempty, closed, strongly σ-invariant, proper subset of F. □

Thus from now on we will only use the word *minimal*. Every dynamical system has at least one minimal set:

Theorem 1.2.4. *Let (X, σ) be a dynamical system. Then X contains a minimal set.*

Proof. Let $(F_i)_{i\in I}$ be a totally ordered family of closed, nonempty, strongly σ-invariant subsets. Then $F = \cap_{i\in I}F_i \neq \emptyset$ (by compactness of X), and F is clearly closed and strongly σ-invariant. Thus F is a minorant, and hence Zorn's Lemma implies the existence of minimal sets. □

If (X, σ) is a dynamical system and $x \in X$, we have the notion of the orbit of x:

$$\mathrm{orbit}_\sigma(x) = \{\sigma^n(x) \mid n \in \mathbb{Z}\}.$$

We have the following theorem (which is a slight modification of Theorem 1.1 in Herman et al. [11]):

Theorem 1.2.5. *Let (X, σ) be a dynamical system. Then the following conditions are equivalent:*

1. *For every point $x \in X$, $\mathrm{orbit}_\sigma(x)$ is dense in X.*
2. *X is a minimal set.*
3. *For every nonempty open subset U of X, $\cup_{n\in\mathbb{Z}}\sigma^n(U) = X$.*
4. *For every nonempty open subset U of X, there exists an $N \in \mathbb{N}$, such that $\cup_{n=-N}^{N}\sigma^n(U) = X$.*

Proof. $3 \iff 4$: Follows from the fact that X is compact.
$1 \Rightarrow 2$: Assume that $F \neq \emptyset$ is a closed strongly σ-invariant subset of X. Pick a point $x \in F$, and use the assumption of strong σ-invariance of F to deduce that $X = \overline{\{\sigma^n(x) \mid n \in \mathbb{Z}\}} \subseteq F$.
$2 \Rightarrow 3$: Let U be a nonempty open subset of X, and note that

$$F = X \setminus \bigcup_{n\in\mathbb{Z}} \sigma^n(U)$$

is a closed strongly σ-invariant proper subset of X. Hence $F = \emptyset$.
$3 \Rightarrow 1$: Put $F = \overline{\{\sigma^n(x) \mid n \in \mathbb{Z}\}}$. If $U = X \setminus F \neq \emptyset$, then we get a contradiction, since $\sigma(F) = F$. Hence we conclude that $F = X$. □

Definition 1.2.6. *If (X, σ) satisfies one of the equivalent conditions from Theorem 1.2.5, then σ is called a minimal homeomorphism, and (X, σ) is called a minimal dynamical system.*

We are now able to introduce the two classes of dynamical systems, which will be the subject of this thesis:

Definition 1.2.7. *If X is a totally disconnected compact metric space and $\sigma : X \to X$ is a homeomorphism, then (X, σ) is called a Putnam system.*

If X is the Cantor set and $\sigma : X \to X$ is a minimal homeomorphism, then (X, σ) is called a Cantor system.

The reader should notice, that while the term *Cantor system* was used in for instance Giordano et al. [9], the term *Putnam system* is non-standard. The term is justified by the many contributions by Ian Putnam to the understanding of these systems. Note that every Cantor system is a Putnam system.

A couple of examples:

Example 1.2.8. *Let p be a prime number, and recall the set $\mathbb{Q}_p$ of p-adic rationals endowed with the* field norm $|\cdot|_p$ *(See Koblitz [13] for details). Consider the set $\mathbb{Z}_p$ of p-adic integers:*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

It is shown in Koblitz [13], that there is a bijective correspondence between elements in $C_p = \{0, \ldots, p-1\}^{\mathbb{N}_0}$ and elements in $\mathbb{Z}_p$, given by the map

$$\varphi(x) = \sum_{i=0}^{\infty} x_i p^i, \ x = (x_0, x_1, \ldots) \in C_p,$$

where the infinite sum is norm convergent in $\mathbb{Z}_p$. Let us prove that φ is continuous, when C_p is equipped with the metric d_p from above: Let $\epsilon > 0$, and choose $J \in \mathbb{N}$ such that $p^{-J} < \epsilon$. Then

$$d_p(x, y) \leq \frac{1}{2^J} \Rightarrow |\varphi(x) - \varphi(y)|_p = |\sum_{i=J}^{\infty} (x_i - y_i) p^i|_p \leq \frac{1}{p^J} < \epsilon$$

as desired. We conclude that $\mathbb{Z}_p$ is a Cantor set.

Let us consider the map $\sigma_p(x) = x + 1, x \in \mathbb{Z}_p$. σ_p is a homeomorphism, since σ_p is continuous (in fact an isometry), and $x \mapsto x - 1$ is an inverse. Moreover, let x_0 be any element in $\mathbb{Z}_p$, and consider the orbit of x_0 under σ_p. Since the map sending an element x to $x + x_0$ is also a homeomorphism and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, it follows that the orbit of x_0 under σ_p is dense in $\mathbb{Z}_p$. Thus σ_p is minimal, that is, $(\mathbb{Z}_p, \sigma_p)$ is a Cantor system. In Remark 4.4.3 below we will have more to say about $(\mathbb{Z}_p, \sigma_p)$.

We will later construct a much wider class of Cantor systems, namely the ones arising from *properly ordered Bratteli diagrams* (see Chapter 4).

Example 1.2.9. *Let us now consider the full n-shift (See Lind and Marcus [14]): Let $n \geq 2$ be an integer, and let $X = \prod_{j=-\infty}^{\infty} \{0, \ldots, n-1\}$ with the metric $d(x, y) = 2^{-N}$, where N is the largest integer, such that $x_{[-N;N]} = y_{[-N;N]}$ (with $N = \infty$ if $x = y$, and $N = -1$ if $x_0 \neq y_0$). It follows from Theorem 1.1.6 that X becomes a Cantor set.*

Now define $\sigma : X \to X$ to be the left shift map: $\sigma(x)_k = x_{k+1}, k \in \mathbb{Z}$. It is easy to see that σ is a homeomorphism. Thus (X, σ) is a Putnam system. Note that σ has a lot of periodic points, and that every finite orbit is a minimal set. In particular (X, σ) is not a Cantor system.

Let us for later use observe, that X has no nontrivial, clopen, strongly σ-invariant subsets. Assume V is a nontrivial, clopen, strongly σ-invariant subset of X, and consider balls $D(x, 2^{1-N}) \subseteq V$ and $D(y, 2^{1-N}) \subseteq \complement V$. Now construct a biinfinite sequence z by

$$z = \ldots x_{[-N;N]} y_{[-N;N]} . x_{[-N;N]} y_{[-N;N]} \cdots,$$

(where "." is placed to the left of the 0'th coordinate), and note that $\sigma^N(z) \in D(x, 2^{1-N})$ and $\sigma^{-N-1}(z) \in D(y, 2^{1-N})$, a contradiction.

We will need the following remark in Chapter 5:

Remark 1.2.10. *Let (X, σ) be a Putnam system, and assume σ is minimal. Then there are two possibilities: Either X is finite, or X is homeomorphic to the Cantor set.*

Indeed, assume that x is an isolated point in X. Then $\{x\}$ is an open set, and therefore minimality of σ implies that there is a positive integer N such that

$$X = \bigcup_{j=-N}^{N} \sigma^j(\{x\}),$$

proving that X is finite.

It is a consequence of Remark 1.2.10, that not all compact sets allow a minimal homeomorphism. Thus the set $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ does not allow a minimal homeomorphism.

We end this chapter by stating a number of important relations between dynamical systems:

1.3 Relations between dynamical systems

We have the following definition:

Definition 1.3.1. *Let (X_1, σ_1) and (X_2, σ_2) be dynamical systems.*

1. *(X_1, σ_1) and (X_2, σ_2) are called* orbit equivalent *if there exists a homeomorphism $F : X_1 \to X_2$, such that $F(\mathrm{orbit}_{\sigma_1}(x)) = \mathrm{orbit}_{\sigma_2}(F(x)), x \in X_1$ (F is called an* orbit map*).*

2. *If (X_1, σ_1) and (X_2, σ_2) are orbit equivalent and F is an orbit map, then we may define functions $m, n : X_1 \to \mathbb{Z}$ by*

$$F(\sigma_1^{m(x)}(x)) = \sigma_2(F(x)), x \in X_1,$$

and

$$F(\sigma_1(x)) = \sigma_2^{n(x)}(F(x)), x \in X_1$$

(m and n are called the orbit cocycles associated to F).

(X_1, σ_1) and (X_2, σ_2) are called strong orbit equivalent*, if there exists an orbit map from (X_1, σ_1) to (X_2, σ_2), such that the associated orbit cocycles each have at most one point of discontinuity.*

3. *(X_1, σ_1) and (X_2, σ_2) are called* conjugate*, if there exists a homeomorphism $F : X_1 \to X_2$ such that $F \circ \sigma_1 = \sigma_2 \circ F$ (F is called a* conjugacy*).*

4. *Let $x_1 \in X_1$ and $x_2 \in X_2$. Then $(X_i, \sigma_i, x_i), i = 1, 2$, are called pointed dynamical systems.*

(X_1, σ_1, x_1) and (X_2, σ_2, x_2) are called pointed topologically conjugate*, if there exists a conjugacy $F : (X_1, \sigma_1) \to (X_2, \sigma_2)$ such that $F(x_1) = x_2$.*

Note that

$$\text{conjugacy} \;\Rightarrow\; \text{strong orbit equivalence} \;\Rightarrow\; \text{orbit equivalence}.$$

Note also that orbit equivalence, conjugacy and pointed topological conjugacy are equivalence relations. That strong orbit equivalence between Cantor systems is an equivalence relation will follow from Theorem 4.4.1 below.

We will need the following theorem before we prove Theorem 4.4.1:

Theorem 1.3.2. *Let (X_1, σ_1), (X_2, σ_2), (Y_1, ν_1) and (Y_2, ν_2) be dynamical systems, and suppose (X_i, σ_i) is conjugate to (Y_i, ν_i), $i = 1, 2$. Let us also assume that (Y_1, ν_1) and (Y_2, ν_2) are strong orbit equivalent.*

Then (X_1, σ_1) and (X_2, σ_2) are strong orbit equivalent.

Proof. Let $G_i : X_i \to Y_i$ be a homeomorphism such that $G_i \circ \sigma_i = \nu_i \circ G_i$, $i = 1, 2$, and choose a homeomorphism $H : Y_1 \to Y_2$ such that

$$H(\nu_1^{\bar{m}(y)}(y)) = \nu_2(H(y)) \text{ and } H(\nu_1(y)) = \nu_2^{\bar{n}(y)}(H(y)), \; y \in Y_1,$$

where $\bar{m}$ and $\bar{n}$ are continuous in $Y_1 \setminus \{y_m\}$ and $Y_1 \setminus \{y_n\}$, respectively.

Put $F = G_2^{-1} \circ H \circ G_1$, $m = \bar{m} \circ G_1$ and $n = \bar{n} \circ G_1$. Then

$$\begin{aligned} F(\sigma_1^{m(x)}(x)) &= G_2^{-1} \circ H \circ G_1(\sigma_1^{m(x)}(x)) \\ &= G_2^{-1}(H(\nu_1^{m(x)}(G_1(x)))) \\ &= G_2^{-1}(H(\nu_1^{\bar{m}(G_1(x))}(G_1(x)))) \\ &= G_2^{-1}(\nu_2(H(G_1(x)))) \\ &= \sigma_2(F(x)), \; x \in X_1, \end{aligned}$$

and similarly $F(\sigma_1(x)) = \sigma_2^{n(x)}(F(x)), x \in X_1$. Moreover m and n are continuous in every point, possibly except for $x_m = G_1^{-1}(y_m)$ and $x_n = G_1^{-1}(y_n)$ respectively. We therefore conclude that (X_1, σ_1) and (X_2, σ_2) are strong orbit equivalent. ☐

Chapter 2

Dynamical Systems and Crossed Products

In this chapter we study the crossed product $C(X) \rtimes_\sigma \mathbb{Z}$ associated to a dynamical system (cf. Raeburn [20] or Davidson [4]). We start by recalling some basic properties about crossed products. Then we prove a lemma known as Rohlin's Lemma, and with Rohlin's Lemma we are able to obtain a representation of $C(X) \rtimes_\sigma \mathbb{Z}$, and simplicity of $C(X) \rtimes_\sigma \mathbb{Z}$ in the case where (X, σ) is minimal. We end the chapter by examining the K-theory of $C(X) \rtimes_\sigma \mathbb{Z}$.

2.1 Crossed products

Let $\mathcal{A}$ be a unital C*-algebra, and let $\alpha \in \mathrm{Aut}(\mathcal{A})$. The crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is a unital C*-algebra with the following properties:

1. There is a unital injective *-homomorphism $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \rtimes_\alpha \mathbb{Z}$ (we will from now on suppress $\iota_{\mathcal{A}}$ and simply consider $\mathcal{A}$ to be a subalgebra of $\mathcal{A} \rtimes_\alpha \mathbb{Z}$).

2. There is a unitary $u \in \mathcal{A}$ such that

$$\forall a \in \mathcal{A} : uau^* = \alpha(a).$$

3. $\mathcal{A} \rtimes_\alpha \mathbb{Z} = \overline{\mathrm{span}}\{au^n \mid a \in \mathcal{A}, n \in \mathbb{Z}\}$.

4. Whenever $\mathcal{B}$ is a unital C*-algebra, v is a unitary in B, and $\psi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism satisfying $v\psi(a)v^* = \psi(\alpha(a)), a \in \mathcal{A}$, then there exists a *-homomorphism

$$\Psi : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{B}$$

satisfying $\Psi(au^n) = \psi(a)v^n, a \in \mathcal{A}, n \in \mathbb{Z}$.

Property 4 above is called the universal property of crossed products. Note that the crossed product is uniquely determined by property 1-4.

We may describe the elements in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ as norm limits of finite sums of the form

$$\sum_{n=-N}^{N} a_n u^n$$

(where some of the a_n's might be zero). We will refer to these finite sums as *standard elements*.

The following theorem is often used tacitly:

Theorem 2.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\alpha \in \mathrm{Aut}(\mathcal{A})$ and $\beta \in \mathrm{Aut}(\mathcal{B})$, and let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism which satisfies that $\psi \circ \alpha = \beta \circ \psi$. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ and $\mathcal{B} \rtimes_\beta \mathbb{Z}$ are isomorphic.*

Proof. Note that $u_\mathcal{B}\psi(a)u_\mathcal{B}^* = \psi(\alpha(a)),\ a \in \mathcal{A}$. Therefore the universal property implies the existence of a *-homomorphism $\Psi : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{B} \rtimes_\beta \mathbb{Z}$, which satisfies $\Psi(au_\mathcal{A}^n) = \psi(a)u_\mathcal{B}^n, a \in \mathcal{A}, n \in \mathbb{Z}$.

Similarly we may construct a *-homomorphism $\Gamma : \mathcal{B} \rtimes_\beta \mathbb{Z} \to \mathcal{A} \rtimes_\alpha \mathbb{Z}$, which satisfies $\Gamma(bu_\mathcal{B}^n) = \psi^{-1}(b)u_\mathcal{A}^n, b \in \mathcal{B}, n \in \mathbb{Z}$.

Finally, one sees that $\Psi \circ \Gamma = \Gamma \circ \Psi = \mathrm{Id}$ by checking the equalities on standard elements and extending by continuity. $\square$

Suppose $\mathcal{A}$ is a unital C*-algebra. For any $\lambda \in \mathbb{T}$, λu is a unitary in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ and $\lambda u a(\lambda u)^* = \alpha(a),\ a \in \mathcal{A}$. Hence there is a *-homomorphism $\rho_\lambda : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{A} \rtimes_\alpha \mathbb{Z}$, such that $\rho_\lambda(a) = a$ and $\rho_\lambda(u) = \lambda u$.

Checking the equality on standard elements and extending by continuity we get that $\rho_\lambda \circ \rho_{\bar\lambda} = \rho_{\bar\lambda} \circ \rho_\lambda = Id$: Thus ρ_λ is an automorphism. It can be proved that $t \mapsto \rho_{e^{2\pi it}}(x),\ t \in [0;1]$, is norm-continuous for fixed $x \in \mathcal{A} \rtimes_\alpha \mathbb{Z}$. Hence the following definition makes sense:

Definition 2.1.2. *Let $\mathcal{A}$ be a unital C^*-algebra, and $\alpha \in \mathrm{Aut}(\mathcal{A})$. Put*

$$\Phi(x) = \int_\mathbb{T} \rho_{e^{2\pi it}}(x)dt,\ x \in \mathcal{A} \rtimes_\alpha \mathbb{Z}.$$

Φ (or, if it is not clear which crossed product Φ refers to, $\Phi_\mathcal{A}$ or $\Phi_{\mathcal{A},\alpha}$) is called the (canonical) conditional expectation corresponding to $\mathcal{A} \rtimes_\alpha \mathbb{Z}$.

Throughout the thesis the letter Φ will refer to a conditional expectation. The following theorem is proved in Davidson [4] (Theorem VIII.2.1) and justifies the terminology in Definition 2.1.2:

Theorem 2.1.3. *Using notation from Definition 2.1.2 we have that $\Phi : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{A}$ is a linear norm-decreasing map. Moreover Φ is faithful, positive, and idempotent.*

Although true in the case where x is a standard element, in general it is false that every element x in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ can be expressed as the norm limit $x = \sum \Phi(xu^{-n})u^n$. However, we have the following theorem (see Davidson [4], Theorem VIII.2.2):

Theorem 2.1.4. *Let $\mathcal{A}$ be a unital C*-algebra and let $x \in \mathcal{A} \rtimes_\alpha \mathbb{Z}$. Then*

$$\sum_{n=-N}^{N} (1 - \frac{|n|}{N+1})\Phi(xu^{-n})u^n \to x \text{ as } N \to \infty.$$

Let us now consider the non-unital case. Let $\mathcal{A}$ be a non-unital C*-algebra and α an automorphism on $\mathcal{A}$. Extend α to an automorphism $\tilde{\alpha}$ on $\tilde{\mathcal{A}}$. We now define $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ to be the following ideal in $\tilde{\mathcal{A}} \rtimes_{\tilde{\alpha}} \mathbb{Z}$:

$$\mathcal{A} \rtimes_\alpha \mathbb{Z} = \overline{\text{span}}\{au^n \mid a \in \mathcal{A}, n \in \mathbb{Z}\}.$$

Theorem 2.1.5. *The non-unital crossed product has the following universal property: Suppose $\mathcal{B}$ is a unital C*-algebra, v is a unitary in $\mathcal{B}$, and $\psi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism satisfying*

$$\psi(\alpha(a)) = v\psi(a)v^*, \ a \in \mathcal{A}.$$

*Then there exists a *-homomorphism $\Psi : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{B}$ such that*

$$\Psi(au^n) = \psi(a)v^n, n \in \mathbb{Z}, \ a \in \mathcal{A}.$$

Proof. Put $\bar{\psi}(a, \lambda) = \psi(a) + \lambda 1_{\mathcal{B}}, (a, \lambda) \in \tilde{\mathcal{A}}$ ($1_{\mathcal{B}}$ is the unit in $\mathcal{B}$). Then it is easy to see that $\bar{\psi}$ is a *-homomorphism. Moreover

$$\begin{aligned} \bar{\psi}(\tilde{\alpha}(a, \lambda)) &= \psi(\alpha(a)) + \lambda 1_{\mathcal{B}} \\ &= v\psi(a)v^* + \lambda 1_{\mathcal{B}} \\ &= v\bar{\psi}(a, \lambda)v^*, \ (a, \lambda) \in \tilde{A}. \end{aligned}$$

Hence it is possible to find a *-homomorphism $\bar{\Psi} : \tilde{\mathcal{A}} \rtimes_{\tilde{\alpha}} \mathbb{Z} \to \mathcal{B}$ such that

$$\bar{\Psi}((a, \lambda)u^n) = \bar{\psi}(a, \lambda)v^n, \ n \in \mathbb{Z}, \ (a, \lambda) \in \tilde{A}.$$

Put $\Psi = \bar{\Psi}|_{\mathcal{A} \rtimes_\alpha \mathbb{Z}}$, and observe that Ψ has the desired property. □

Note that we have an analogue of Theorem 2.1.1 in the non-unital case. We also have a conditional expectation obtained by restricting the conditional expectation on $\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}$ to $\mathcal{A} \rtimes_\alpha \mathbb{Z}$.

If (X, σ) is a dynamical system, the map $f \mapsto f \circ \sigma^{-1}, f \in C(X)$, is clearly an automorphism on $C(X)$. With an abuse of notation we also denote this map by σ, and thus we have a crossed product $C(X) \rtimes_\sigma \mathbb{Z}$.

In Chapter 3 we will need some basic properties of $C(X) \rtimes_\sigma \mathbb{Z}$, including a representation of $C(X) \rtimes_\sigma \mathbb{Z}$ as operators on $L^2(X)$ (equipped with a certain measure) and simplicity of $C(X) \rtimes_\sigma \mathbb{Z}$ in the case where (X, σ) is a Cantor system. We will also need the K-theory of $C(X) \rtimes_\sigma \mathbb{Z}$. The goal of the rest of this chapter is to obtain these properties.

2.2 Rohlin's Lemma

In this section we will prove a lemma from ergodic theory, known as Rohlin's Lemma. First we introduce a certain type of measures.

Definition 2.2.1. *Let (X, σ) be a dynamical system. A probability measure μ is said to be σ-invariant, if $\mu(A) = \mu(\sigma(A))$ whenever A is a Borel subset of X.*

μ is called ergodic with respect to (X, σ), if μ is σ-invariant, and $\mu(A) \in \{0, 1\}$ whenever A is a σ-invariant Borel subset of X.

If μ is an ergodic probability measure, then the triple (X, σ, μ) is said to be an ergodic dynamical system. A minimal ergodic dynamical system (X, σ, μ) is an ergodic dynamical system with the additional property that σ is minimal.

We have the following theorem, stating that it is always possible to equip a dynamical system with an ergodic measure:

Theorem 2.2.2. *Let (X, σ) be a dynamical system. Then there exists a measure μ on X, which is ergodic with respect to (X, σ).*

Proof. See Davidson [4], Proposition VIII.3.2. □

We state the following remark from measure theory saying, when two sets are equal in a measure-theoretic sense:

Remark 2.2.3. *Let (X, μ) be a measure space. If A and B are Borel subsets of X, it is not so hard to see that the relation $\sim$, given by*

$$A \sim B \Leftrightarrow \mu(A \setminus B) = \mu(B \setminus A) = 0,$$

is an equivalence relation, and that the relation $\leq$, given by

$$[A] \leq [B] \Leftrightarrow \mu(A \setminus B) = 0,$$

is a well-defined order-relation on the equivalence classes. We will say that A is essentially smaller *than B, if $[A] \leq [B]$. The term* essentially larger *is defined similarly. If $[A] \leq [B]$ and C is another Borel set, then $\mu(B \setminus A) = \mu(B) - \mu(A)$, $[B] = [A \cup B]$, and $\mu(A \cap C) \leq \mu(B \cap C)$; if $\sigma : X \to X$ is a homeomorphism and μ is σ-invariant, then $[A] \leq [B]$ if and only if $[\sigma(A)] \leq [\sigma(B)]$ (these properties are easy to verify and are often used tacitly).*

The following lemma shows that an ergodic measure automatically has some additional properties:

Lemma 2.2.4. *Let (X, σ, μ) be an ergodic dynamical system, and suppose that $A \subseteq X$ is essentially σ-invariant, i.e. $\mu(\sigma(A) \setminus A) = 0$. Then $\mu(A) \in \{0, 1\}$.*

Proof. The set $\bigcup_{j=0}^{\infty} \sigma^j(A)$ is σ-invariant. Therefore either $\mu(\bigcup_{j=0}^{\infty} \sigma^j(A)) = 0$ (in which case $\mu(A) = 0$), or $\mu(\bigcup_{j=0}^{\infty} \sigma^j(A)) = 1$. Suppose the last condition is satisfied, and pick a null set N such that $\sigma(A) \subseteq A \cup N$. By induction one gets that

$$\bigcup_{j=0}^{K} \sigma^j(A) \subseteq A \cup \bigcup_{j=0}^{K} \sigma^j(N), \ K \in \mathbb{N}.$$

Hence

$$\mu(\bigcup_{j=0}^{\infty}\sigma^j(A)) = \sup_{K\in\mathbb{N}}\mu(\bigcup_{j=0}^{K}\sigma^j(A)) \leq \sup_{K\in\mathbb{N}}\mu(A\cup\bigcup_{j=0}^{K}\sigma^j(N)) = \mu(A).$$

Therefore, $1 \leq \mu(A) \leq 1$, so that $\mu(A) = 1$. □

The following lemma is known as Rohlin's Lemma. Note that an ergodic measure μ on a minimal dynamical system (X, σ), where X is infinite, automatically satisfies that the support of μ is infinite.

Lemma 2.2.5. *Let (X, σ, μ) be an ergodic dynamical system, such that the support of μ is infinite. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$ there is a Borel subset F of X, such that*

1. *the sets $\sigma^i(F), 0 \leq i \leq n-1$, are disjoint, and*
2. *$\mu(\cup_{i=0}^{n-1}\sigma^i(F)) > 1-\epsilon$.*

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$. Start by choosing an integer k, such that $k > \epsilon^{-1}$.

Let us for short in the following say that a Borel subset E of X satisfying

1. $\mu(E) > 0$, and
2. $E, \ldots, \sigma^{kn-1}(E)$ are essentially disjoint, i.e.

$$\mu(\sigma^i(E)\cap\sigma^j(E)) = 0, 0 \leq i < j \leq kn-1,$$

has *property (α)*.

Pick a Borel subset E_0 of X such that $0 < \mu(E_0) < \frac{1}{kn}$. Indeed this is seen to be possible by picking disjoint open neighborhoods around $kn+1$ points in $\mathrm{supp}(\mu)$, and use that any such neighborhood must have positive measure.

We claim that it is possible to pick a subset E of E_0 such that E satisfies property (α). Let us prove the claim by induction after m, $0 < m \leq kn-1$ (the first step in the induction argument is obvious).

Suppose we have picked a subset E of E_0, such that $E, \sigma(E), \ldots, \sigma^{m-1}(E)$ are essentially disjoint and $\mu(E) > 0$. Now if $\mu(E \setminus \sigma^m(E)) > 0$ we are done. If not, we have that

$$\mu(E\cap\sigma^m(E)) = \mu(E).$$

Continuing the argument replacing E by $E \cap \sigma^m(E)$, we either find a subset of E having the desired properties, or, by induction and σ-invariance of μ, we get

$$\mu(\bigcap_{l=-\infty}^{\infty}\sigma^{lm}(E)) = \lim_{L\to\infty}\mu(\bigcap_{l=-L}^{L}\sigma^{lm}(E)) = \mu(E).$$

Put $E' = \cap_{l=-\infty}^{\infty}\sigma^{lm}(E)$ and

$$A = E' \cup \cdots \cup \sigma^{m-1}(E').$$

A is σ-invariant since $E' = \sigma^m(E')$, and $\mu(A) \geq \mu(E') = \mu(E) > 0$. Therefore ergodicity of σ implies that $\mu(A) = 1$. In particular we see that $1 \leq m\mu(E) \leq kn\mu(E)$. This means that $\mu(E) \geq \frac{1}{kn}$, contradicting the way we picked E_0.

We now apply Zorn's Lemma to obtain a maximal set up to a null set with property (α): Suppose $([E_\lambda])_{\lambda\in\Lambda}$ is a totally ordered family of equivalence classes of sets with property (α) (see Remark 2.2.3 above). Pick an increasing subsequence $([E_{\lambda_n}])_{n\in\mathbb{N}}$ such that

$$\sup_{\lambda\in\Lambda} \mu(E_\lambda) = \sup_{n\in\mathbb{N}} \mu(E_{\lambda_n}),$$

and put $E' = \cup_{n=1}^{\infty} E_{\lambda_n}$. Note that E' is measurable. Note also that $[E']$ indeed majorize every $[E_\lambda]$, since either $[E_\lambda] \leq [E_{\lambda_n}]$ for some n (in which case $[E_\lambda] \leq [E_{\lambda_n}] \leq [E']$), or $[E_\lambda] \geq [E_{\lambda_n}]$ for every n (in which case

$$\mu(E_\lambda \setminus E') \leq \inf_{n\in\mathbb{N}} \mu(E_\lambda \setminus E_{\lambda_n}) = \inf_{n\in\mathbb{N}} (\mu(E_\lambda) - \mu(E_{\lambda_n})) \leq 0,$$

whence $[E_\lambda] \leq [E']$). Moreover E' satisfies property (α), since

$$\begin{aligned}
\mu(\sigma^i(E') \cap \sigma^j(E')) &= \mu(\bigcup_{m,m'=1}^{\infty} \sigma^i(E_{\lambda_m}) \cap \sigma^j(E_{\lambda_{m'}})) \\
&\leq \sum_{m,m'=1}^{\infty} \mu(\sigma^i(E_{\lambda_{\max(m,m')}}) \cap \sigma^j(E_{\lambda_{\max(m,m')}})) \\
&= 0, \ 0 \leq i, j \leq kn-1.
\end{aligned}$$

Therefore Zorn's Lemma ensures the existence of maximal classes, and let us for simplicity assume that our set E represents such a maximal class. Since by assumption $[E] = [E \setminus (\cup_{l=1}^{kn-1} \sigma^l(E))]$ we may assume, that $E, \ldots, \sigma^{kn-1}(E)$ are disjoint.

Let $G = \sigma^{kn-1}(E)$. For each x in G, let $N(x)$ denote the smallest positive integer such that $\sigma^{N(x)}(x) \in E$, and set it equal to ∞ if no such integer exists. Let

$$G_j = \{x \in G \mid N(x) = j\}$$

and

$$G' = \{x \in G \mid N(x) > kn\}.$$

By the very construction of G_j and G' we have the following disjoint union:

$$G = \bigcup_{j=1}^{kn} G_j \cup G'.$$

Note that G' cannot have positive measure, since otherwise $E \cup \sigma(G')$ would be an essentially larger set than E with property (α).

We claim that we have a disjoint union

$$M = \bigcup_{j=0}^{kn-1} \sigma^j(E) \cup \bigcup_{j=1}^{kn} \bigcup_{i=1}^{j-1} \sigma^i(G_j).$$

By construction of E, the sets involving $\sigma^j(E)$ are disjoint. This fact implies that $\bigcup_{j=1}^{kn} \bigcup_{i=1}^{j-1} \sigma^i(G_j)$ is also a disjoint union. Now suppose, that $\sigma^l(x) = \sigma^i(y)$ for some $x \in E$ and $y \in G_j$, $1 \le i \le j-1, 0 \le l \le kn-1$. If $i \le l$ we immediately get a contradiction, and if $i > l$ then $\sigma^{i-l}(y) \in E$ and $0 < i - l \le j - 1 < j = N(y)$, which is also a contradiction.

The set M is clearly essentially translation invariant (modulo the null set G'). Ergodicity of μ therefore implies that M has measure 1 (since E was assumed to have positive measure). We may now define the set F as

$$F = \bigcup_{s=0}^{k-1} \sigma^{sn}(E) \cup \bigcup_{j=1}^{kn} \bigcup_{0 \le s < \frac{j}{n} - 1} \sigma^{sn+1}(G_j)$$

That F has the desired property follows from a calculation but is also immediate from the diagram below (in the case $k = n = 3$):

$$\begin{tikzcd}
\mathbf{E} \arrow[r] & \sigma(E) \arrow[r] & \sigma^2(E) \arrow[r] & \sigma^{\mathbf{3}}(\mathbf{E}) \arrow[r] & \sigma^4(E) \arrow[r] & \sigma^5(E) \arrow[r] & \sigma^{\mathbf{6}}(\mathbf{E}) \arrow[r] & \sigma^7(E) \arrow[r] & \sigma^8(E) \\
& \sigma(G_2) & \sigma(G_3) \arrow[d] & \sigma(\mathbf{G_4}) \arrow[d] & \sigma(\mathbf{G_5}) \arrow[d] & \sigma(\mathbf{G_6}) \arrow[d] & \sigma(\mathbf{G_7}) \arrow[d] & \sigma(\mathbf{G_8}) \arrow[d] & \sigma(\mathbf{G_9}) \arrow[d] \\
& & \sigma^2(G_3) & \sigma^2(G_4) \arrow[d] & \sigma^2(G_5) \arrow[d] & \sigma^2(G_6) \arrow[d] & \sigma^2(G_7) \arrow[d] & \sigma^2(G_8) \arrow[d] & \sigma^2(G_9) \arrow[d] \\
& & & \sigma^3(G_4) & \sigma^3(G_5) \arrow[d] & \sigma^3(G_6) \arrow[d] & \sigma^3(G_7) \arrow[d] & \sigma^3(G_8) \arrow[d] & \sigma^3(G_9) \arrow[d] \\
& & & & \sigma^4(G_5) & \sigma^4(G_6) \arrow[d] & \sigma^{\mathbf{4}}(\mathbf{G_7}) \arrow[d] & \sigma^{\mathbf{4}}(\mathbf{G_8}) \arrow[d] & \sigma^{\mathbf{4}}(\mathbf{G_9}) \arrow[d] \\
& & & & & \sigma^5(G_6) & \sigma^5(G_7) \arrow[d] & \sigma^5(G_8) \arrow[d] & \sigma^5(G_9) \arrow[d] \\
& & & & & & \sigma^6(G_7) & \sigma^6(G_8) \arrow[d] & \sigma^6(G_9) \arrow[d] \\
& & & & & & & \sigma^7(G_8) & \sigma^7(G_9) \arrow[d] \\
& & & & & & & & \sigma^8(G_9)
\end{tikzcd}$$

In the diagram, the sets included in F are typed in boldface. Evaluating σ on F n times exhausts M except for k triangles, each containing $\frac{n(n-1)}{2}$ sets. Because of disjointness of the sets in the diagram above, the total measure of the k triangles must be less than or equal to $(n-1)\mu(G) = (n-1)\mu(E) \le n\mu(E) \le n\frac{1}{kn} < \epsilon$. Therefore F has the desired property. $\square$

2.3 Representing the crossed product on a Hilbert space

Let (X, σ, μ) be a minimal ergodic dynamical system, and let us consider the Hilbert space $L^2(X, \mu) = L^2(X)$. Note that if Y is a measurable subset of X, then there is a natural decomposition $L^2(X) = L^2(Y) \oplus L^2(X \setminus Y)$.

$C(X)$ is represented faithfully on $L^2(X)$ as multiplication operators. Indeed, let $f \in C(X)$ be a non-zero function, and pick an open set V such that $f(x) \neq 0, x \in V$. By minimality of σ there exists an $N \in \mathbb{N}$ such that $\cup_{n=-N}^{N} \sigma^n(V) = X$, and σ-invariance of μ now gives that $\mu(V) \geq \frac{1}{2N+1} > 0$. Hence $M_f \neq 0$.

Moreover we may define a unitary U on $L^2(X)$ by

$$U(\xi) = \xi \circ \sigma^{-1}, \ \xi \in L^2(X).$$

It is easy to see that $UM_fU^* = M_{\sigma(f)}, f \in C(X)$. We will now prove that if X is infinite, then $C(X) \rtimes_\sigma \mathbb{Z} \cong \mathcal{A}_\sigma^\mu$, where

$$\mathcal{A}_\sigma^\mu = \mathrm{C}^*(\{M_f \mid f \in C(X)\}, U) = \overline{\mathrm{span}}\{\sum_{j=-n}^{n} M_{f_j} U^j \mid n \in \mathbb{N}_0\}.$$

We will need the following lemma:

Lemma 2.3.1. *Let (X, σ, μ) be a minimal ergodic dynamical system, where X is infinite. On the algebra A_σ^μ the map*

$$\Psi(\sum_{j=-n}^{n} M_{f_j} U^j) = M_{f_0}, \ \sum_{j=-n}^{n} M_{f_j} U^j \in \mathcal{A}_\sigma^\mu,$$

extends to a well-defined linear contraction from A_σ^μ to $\{M_f \mid f \in C(X)\}$.

Proof. We must prove that

$$||\sum_{j=-n}^{n} M_{f_j} U^j|| \geq ||M_{f_0}||.$$

Let $\epsilon > 0$, put $E = \{x \in X \mid |f_0(x)| \geq ||f_0||_\infty - \epsilon\}$, and put $\gamma = \mu(E) > 0$. Apply Lemma 2.2.5 and σ-invariance of μ to obtain a Borel set F, such that $(\sigma^i(F))_{i=-n}^{n}$ are disjoint, and $\mu(\cup_{i=-n}^{n} \sigma^i(F)) > 1 - \gamma$. Since $\mu(\cup_{i=-n}^{n} \sigma^i(F)) + \mu(E) > 1$, we have

$$\sum_{i=-n}^{n} \mu(\sigma^i(F) \cap E) = \mu((\cup_{i=-n}^{n} \sigma^i(F)) \cap E) > 0.$$

Choose m, such that $\mu(\sigma^m(F) \cap E) > 0$. Replacing F with $\sigma^m(F)$ we may assume that $\mu(F \cap E) > 0$. Let ξ be a unit vector in $L^2(F \cap E)$ ($\subseteq L^2(X)$) and note that $U(L^2(Y)) = L^2(\sigma(Y))$, whenever Y is a measurable subset of X. Thus the vectors $M_{f_j} U^j \xi$ are mutually orthogonal, and by Pythagoras' theorem we get

$$\begin{aligned} ||\sum_{j=-n}^{n} M_{f_j} U^j||^2 &\geq ||\sum_{j=-n}^{n} M_{f_j} U^j \xi||^2 \geq ||M_{f_0} \xi||^2 \\ &\geq (||f_0||_\infty - \epsilon)^2 ||\xi||^2 = (||M_{f_0}|| - \epsilon)^2. \end{aligned}$$

□

Theorem 2.3.2. *Let (X, σ, μ) be a minimal ergodic dynamical system, where X is infinite. Then A_σ^μ is *-isomorphic to $C(X) \rtimes_\sigma \mathbb{Z}$.*

Proof. By the universal property of the crossed product, we have a *-homomorphism $F : C(X) \rtimes_\sigma \mathbb{Z} \to A_\sigma^\mu$, which is onto. The only problem is to prove that F is injective. To prove that, note that we have the following commuting diagram (which actually does commute, since it commutes on the standard elements in $C(X) \rtimes_\sigma \mathbb{Z}$, and the involved maps are continuous):

$$\begin{tikzcd} C(X) \rtimes_\sigma \mathbb{Z} \arrow[r, "F"] \arrow[d, "\Phi"] & A_\sigma^\mu \arrow[d, "\Psi"] \\ C(X) & \{M_f | f \in C(X)\} \arrow[l, "\simeq"'] \end{tikzcd}$$

Here Φ is the conditional expectation from $C(X) \rtimes_\sigma \mathbb{Z}$ onto $C(X)$. Suppose that $F(a) = 0$. Replacing a with a^*a we may assume that a is positive. Now

$$F(a) = 0 \Rightarrow \Psi(F(a)) = 0 \iff \Phi(a) = 0 \Rightarrow a = 0$$

by Theorem 2.1.3. □

Using Theorem 2.3.2 and Lemma 2.2.5, one can prove the following lemma:

Lemma 2.3.3. *Let (X, σ) is a minimal dynamical system, where X is infinite. Then for each a in $C(X) \rtimes_\sigma \mathbb{Z}$ and $\epsilon > 0$, there are unimodular functions $\theta_1, \ldots, \theta_m$ in $C(X)$ such that*

$$||\Phi(a) - \frac{1}{m} \sum_{s=1}^{m} \theta_s a \bar{\theta}_s|| < \epsilon.$$

Proof. See Davidson [4], Lemma VIII.3.7. □

We can now prove that $C(X) \rtimes_\sigma \mathbb{Z}$ is simple, if X is infinite and σ is minimal:

Theorem 2.3.4. *Let (X, σ) be a minimal dynamical system, where X is infinite. Then $C(X) \rtimes_\sigma \mathbb{Z}$ is simple.*

Proof. Let $\mathcal{I}$ be a non-zero ideal and pick a positive element $a \in \mathcal{I} \setminus \{0\}$. Lemma 2.3.3 now implies that $\Phi(a) \in \mathcal{I}$, and by Theorem 2.1.3 $f = \Phi(a)$ is a positive non-zero function. Hence we may choose an open set V such that $f(x) > 0, x \in V$, and minimality of σ now ensures the existence of an $N \in \mathbb{N}$, such that

$$\bigcup_{n=-N}^{N} \sigma^n(V) = X.$$

Now $g = \sum_{n=-N}^{N} u^n f(u^*)^n = \sum_{n=-N}^{N} f \circ \sigma^{-n}$ is an invertible element of $\mathcal{I}$. □

We also have the following corollary to Lemma 2.3.3:

Corollary 2.3.5. *Let (X,σ) be a minimal dynamical system where X is infinite, and let τ be a trace state on $C(X)\rtimes_\sigma \mathbb{Z}$. Then $\tau = \tau\circ\Phi$ (where Φ is the conditional expectation).*

Proof. Let $a \in C(X)\rtimes_\sigma \mathbb{Z}$, and let $\epsilon > 0$. Use Lemma 2.3.3 to pick unimodular functions $\theta_1,\ldots,\theta_m \in C(X)$, such that $||\Phi(a) - \frac{1}{m}\sum_{s=1}^m \theta_s a\bar{\theta}_s|| < \epsilon$. Now

$$\begin{aligned} ||\tau(\Phi(a)) - \tau(a)|| &= ||\tau(\Phi(a)) - \tau(\frac{1}{m}\sum_{s=1}^m \theta_s a\bar{\theta}_s)|| \\ &\leq ||\Phi(a) - \frac{1}{m}\sum_{s=1}^m \theta_s a\bar{\theta}_s|| \\ &< \epsilon. \end{aligned}$$

□

2.4 The K-theory of the crossed product

This section deals with the K-theory of $C(X)\rtimes_\sigma \mathbb{Z}$, in the case where (X,σ) is a Putnam system. Let us begin by introducing the ordered group $K_0(X,\sigma)$:

Definition 2.4.1. *Let (X,σ) be a Cantor system, and consider the group homomorphism $\delta_\sigma : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$, given by*

$$\delta_\sigma(f) = f - f\circ\sigma^{-1},\ f \in C(X,\mathbb{Z}).$$

We now define

$$K_0(X,\sigma) = C(X,\mathbb{Z})/\delta_\sigma(C(X,\mathbb{Z})) = \{[f] \mid f \in C(X,\mathbb{Z})\}$$

and $K_0^+(X,\sigma) = \{[f] \mid f \in C(X,\mathbb{Z}), f \geq 0\}$.

Lemma 2.4.2. *$(K_0(X,\sigma), K_0^+(X,\sigma))$ is an ordered group.*

Proof. Indeed $K_0^+(X,\sigma)$ is closed under addition and satisfies $K_0(X,\sigma) = K_0^+(X,\sigma) - K_0^+(X,\sigma)$. Let us prove that $K_0^+(X,\sigma)\cap(-K_0^+(X,\sigma)) = \{0\}$. Take a translation invariant measure μ on X (which is strictly positive on open sets by minimality of σ), and put

$$\tau(f) = \int_X f d\mu,\ f \in C(X,\mathbb{Z}).$$

It is clear that τ is a positive linear functional on $C(X,\mathbb{Z})$. Moreover translation invariance of μ implies that $\delta_\sigma(C(X,\mathbb{Z})) \subseteq \mathrm{Ker}\tau$. Hence τ induces a homomorphism $\tilde{\tau}$ on $K_0(X,\sigma)$.

Now assume $[f] = -[g]$ where $f,g \geq 0$. Then

$$\tau(f+g) = \tilde{\tau}([f+g]) = \tilde{\tau}([f]+[g]) = 0,$$

whence $f = g = 0$. □

Note that $[1_X]$ is a (canonical) order unit for $(K_0(X,\sigma), K_0^+(X,\sigma))$. We will need the following theorem combining Putnam [19] (Theorem 1.1), and Giordano et al. [9] (Corollary 1, p. 61):

Theorem 2.4.3. *Let (X,σ) be a Putnam system. Then we have the following:*

1. *If X contains no non-trivial, strongly σ-invariant, clopen subsets, then $K_1(C(X) \rtimes_\sigma \mathbb{Z}) \cong \mathbb{Z}$, and $K_1(C(X) \rtimes_\sigma \mathbb{Z})$ is generated by $[u]_1$.*

2. *If (X,σ) is a Cantor system, then $K_0(C(X) \rtimes_\sigma \mathbb{Z}) \cong K_0(X,\sigma)$ by an order isomorphism preserving the canonical distinguished order units.*

A substantial part of the information in Theorem 2.4.3 follows from Pimsner and Voiculescu's exact sequence (cf. Blackadar [1], Theorem 10.2.1), which in our situation states that the sequence

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\mathrm{Id}-K_0(\sigma)} & K_0(C(X)) & \xrightarrow{K_0(\iota)} & K_0(C(X) \rtimes_\sigma \mathbb{Z}) \\ \Delta_1 \big\uparrow & & & & \big\downarrow \Delta_0 \\ K_1(C(X) \rtimes_\sigma \mathbb{Z}) & \xleftarrow[K_1(\iota)]{} & K_1(C(X)) & \xleftarrow[\mathrm{Id}-K_1(\sigma)]{} & K_1(C(X)) \end{array}$$

is exact (ι denotes the inclusion map $\iota : C(X) \to C(X) \rtimes_\sigma \mathbb{Z}$, and Δ_0 and Δ_1 are some group homomorphisms). Let us for brevity denote the sequence above by the PV-sequence. We will now prove part of Theorem 2.4.3 using the PV-sequence. For that purpose, we will need a few lemmas:

Lemma 2.4.4. *Let X be a totally disconnected compact metric space. Then $K_0(C(X)) \cong C(X,\mathbb{Z})$, and $K_1(C(X)) = 0$.*

Proof. In Rørdam et al. [22], Exercise 3.4, it is stated, that the map

$$\dim([p]_0 - [q]_0) = \mathrm{Tr} \circ p - \mathrm{Tr} \circ q, \ [p]_0 - [q]_0 \in K_0(C(X)),$$

is a group isomorphism. Hence $K_0(C(X)) \cong C(X,\mathbb{Z})$.

That $K_1(C(X)) = 0$ is a consequence of Theorem 1.1.7. □

Lemma 2.4.5. *If (X,σ) is a Putnam system, and X contains no non-trivial, strongly σ-invariant, clopen subsets, then*

$$\begin{aligned} \mathrm{Ker}(\mathrm{Id} - K_0(\sigma)) = \{[p]_0 - [q]_0 \mid p, q \in P_n(C(X)), \\ \mathrm{Tr} \circ p - \mathrm{Tr} \circ q \textit{ is constant}\} \cong \mathbb{Z}. \end{aligned}$$

Proof. Suppose $p, q \in P_n(C(X))$. Then (using the dim-isomorphism from Lemma 2.4.4)

$$\begin{array}{rcl} [p]_0 - [q]_0 & \in & \mathrm{Ker}(\mathrm{Id} - K_0(\sigma)) \iff \\ [p]_0 - [q]_0 & = & [\sigma(p)]_0 - [\sigma(q)]_0 \iff \\ \mathrm{Tr} \circ p - \mathrm{Tr} \circ q & = & \mathrm{Tr} \circ p \circ \sigma^{-1} - \mathrm{Tr} \circ q \circ \sigma^{-1} \iff \\ \mathrm{Tr} \circ p - \mathrm{Tr} \circ q & = & \mathrm{const.}, \end{array}$$

since $\{x \in X \mid \mathrm{Tr}(p(x)) - \mathrm{Tr}(q(x)) = k\}$ is clopen and strongly σ-invariant if $\mathrm{Tr} \circ p - \mathrm{Tr} \circ q = \mathrm{Tr} \circ p \circ \sigma^{-1} - \mathrm{Tr} \circ q \circ \sigma^{-1}$.

To prove that the group is isomorphic to $\mathbb{Z}$, observe that the map $\psi([p]_0 - [q]_0) = \mathrm{Tr}(p(x_0)) - \mathrm{Tr}(q(x_0))$ is a group isomorphism for any $x_0 \in X$ (note that ψ is the composition of the dim-isomorphism and the evaluation homomorphism). □

We will now use the PV-sequence to prove part of Theorem 2.4.3:

Theorem 2.4.6. *Let (X, σ) be a Putnam system. Then $K_0(C(X) \rtimes_\sigma \mathbb{Z}) \cong K_0(X, \sigma)$ (as non-ordered groups). If we also assume that X contains no non-trivial, strongly σ-invariant, clopen subsets, then $K_1(C(X) \rtimes_\sigma \mathbb{Z}) \cong \mathbb{Z}$ (with a non-specified generator).*

Proof. Looking at the PV-sequence one sees, that Δ_1 is injective, and $K_0(\iota)$ is surjective (Δ_0, $K_1(\iota)$ and $\mathrm{Id} - K_1(\sigma)$ are just the zero-mappings).

By Lemma 2.4.4 we have

$$\begin{aligned} K_0(C(X) \rtimes_\sigma \mathbb{Z}) &\cong K_0(C(X))/\mathrm{Im}(\mathrm{Id} - K_0(\sigma)) \\ &\cong \dim(K_0(C(X)))/\mathrm{Im}(\dim \circ (\mathrm{Id} - K_0(\sigma)) \circ \dim^{-1}) \\ &= C(X, \mathbb{Z})/\delta_\sigma(C(X, \mathbb{Z})) = K_0(X, \sigma) \end{aligned}$$

(as non-ordered groups).

Moreover, if X has no non-trivial, clopen, strongly σ-invariant subsets, Lemma 2.4.5 implies

$$K_1(C(X) \rtimes_\sigma \mathbb{Z}) \cong \mathrm{Im}\Delta_1 = \mathrm{Ker}(\mathrm{Id} - K_0(\sigma)) \cong \mathbb{Z}$$

(with a non-specified generator). □

Chapter 3

Classification of Crossed Products Arising from Cantor Systems

In the rest of this thesis we will deal with Putnam systems and Cantor systems. This chapter is devoted to proving that crossed products arising from Cantor systems are classified by their ordered K_0-group with distinguished order unit. Along the way we will obtain some other results, which will be important in Chapter 5.

3.1 Kakutani-Rohlin partitions

The following is a lemma which we will need several times. The lemma was first obtained by Putnam in [18] and is analogous to Lemma 2.2.5.

Lemma 3.1.1. *Let (X, σ) be a Putnam system and let $\mathcal{P}$ be a partition of X. Suppose Y is a clopen subset of X satisfying $\cup_{n\in\mathbb{Z}}\sigma^n(Y) = X$.*

Then we can find $K, J_1, \ldots, J_K \in \mathbb{N}$ and a partition

$$\mathcal{Q} = \{Y(k, j) \mid 1 \leq k \leq K, 1 \leq j \leq J_k\}$$

of X, such that we have

1. $\forall 1 \leq j < J_k : \ \sigma(Y(k, j)) = Y(k, j + 1)$,
2. $\bigcup_{k=1}^{K} Y(k, J_k) = Y$,
3. $\bigcup_{k=1}^{K} Y(k, 1) = \sigma(Y)$, *and*
4. *$\mathcal{Q}$ is finer than $\mathcal{P}$.*

Proof. Define a function $\lambda : Y \to \mathbb{R}$ by

$$\lambda(y) = \inf\{n \in \mathbb{N} \mid \sigma^n(y) \in Y\}, \ y \in Y.$$

Compactness of X implies the existence of an $N \in \mathbb{N}$, such that $\cup_{n=-N}^{N}\sigma^n(Y) = X$. Since σ is bijective, we have $\cup_{n=-2N-1}^{-1}\sigma^n(Y) = X$: In particular λ is well-defined. *λ is upper semi-continuous*: Let $t \in \mathbb{R}$, and note that

$$\lambda^{-1}(]-\infty; t[) = (\bigcup_{1\leq n<t} \sigma^{-n}(Y)) \cap Y$$

which is clopen, since Y is clopen.
λ *is lower semi-continuous*: If $t \in \mathbb{R} \setminus \mathbb{Z}$ then

$$\lambda^{-1}(]t;\infty[) = \complement(\lambda^{-1}(]-\infty;t[)),$$

and if $t \in \mathbb{Z}$ then

$$\lambda^{-1}(]t;\infty[) = \complement(\lambda^{-1}(]-\infty;t+1/2[)).$$

In both cases we see that $\lambda^{-1}(]t;\infty[)$ is clopen by the first argument.

Being lower and upper semi-continuous we conclude that λ is continuous (see Pedersen [16], 1.5.10). Thus $\lambda(Y) \subseteq \mathbb{Z}$ is compact and thereby finite:

$$\lambda(Y) = \{J_1, \ldots, J_K\}.$$

We now define

$$Y(k,j) = \sigma^j(\lambda^{-1}(\{J_k\})), 1 \leq k \leq K, 1 \leq j \leq J_k.$$

Let us comment upon the assertions 1, 2 and 3 one by one:
1: Obvious.
2: "$\subseteq$": Assume $y \in Y(k, J_k)$. Then $\lambda(\sigma^{-J_k}(y)) = J_k$, and therefore $y \in Y$.
"$\supseteq$": Suppose $y \in Y$, and pick the least positive j such that $\sigma^{-j}(y) \in Y$. Then we have that $j = J_k$ for some $1 \leq k \leq K$, and $y \in Y(k, J_k)$.
3: Since $Y = \cup_{k=1}^K \lambda^{-1}(\{J_k\})$ we have that

$$\sigma(Y) = \cup_{k=1}^K \sigma(\lambda^{-1}(\{J_k\})) = \cup_{k=1}^K Y(k,1).$$

Let us now prove that $\mathcal{Q} = \{Y(k,j) \mid 1 \leq k \leq K, 1 \leq j \leq J_k\}$ is a partition of X. It is clear that the sets $Y(k,j)$ are clopen.

Suppose $Y(k,j) \cap Y(k',j') \neq \emptyset$, that is, $\sigma^j(y) = \sigma^{j'}(y')$ for some y, y' satisfying $\lambda(y) = J_k$ and $\lambda(y') = J_{k'}$. Let us for example assume $j < j'$. Then $y = \sigma^{j'-j}(y')$, and it follows that $j' - j \geq J_{k'}$. On the other hand we also have that $J_{k'} - 1 \geq j' - j$, which is a contradiction. Hence $j = j'$, and now it is clear that also $k = k'$. Thus the sets in $\mathcal{Q}$ are disjoint.

Finally $\cup_{k=1}^K \cup_{j=1}^{J_k} Y(k,j) = X$, since it follows from property 1, 2 and 3 that $\cup_{k=1}^K \cup_{j=1}^{J_k} Y(k,j)$ is a σ-invariant subset of X containing Y. We therefore conclude that $\mathcal{Q}$ is a partition of X.

These considerations lead to the following picture of X:

$$\begin{tikzcd}
Y(1,J_1) & Y(2,J_2) & \cdots & Y(K,J_K) \\
\vdots \arrow[u, "\sigma"] & \vdots \arrow[u, "\sigma"] & \cdots & \vdots \arrow[u, "\sigma"] \\
Y(1,1) \arrow[u, "\sigma"] & Y(2,1) \arrow[u, "\sigma"] & \cdots & Y(K,1) \arrow[u, "\sigma"]
\end{tikzcd}$$

We will refer to the sets above as "floors" and "towers". Thus tower number k has exactly J_k floors. Note that Y is the union of the top floors (by property 2), and that σ maps each of the top floors into the union of the first floors (by property 3).

Let $\mathcal{P} = \{X_1, \ldots, X_m\}$ be another partition of X, and consider the first tower. Start by splitting the first floor of the first tower into sets, each lying inside a set in $\mathcal{P}$. Then split the rest of the tower by "going up with σ". Thereby we get a certain number of towers, each of height J_1. Now split the second floor in each of the new towers, and split the rest of each of the new towers by going up and down with σ. This does not change the fact, that each of the sets in the first floors lie inside a set in $\mathcal{P}$. Repeating this procedure floor by floor eventually gives us a splitting of the first tower into (a lot of) thinner towers, each of height J_1, where each of the floors lies inside a set in $\mathcal{P}$.

Then do the same for each of the remaining $K-1$ towers, and note that property 1, 2 and 3 are preserved. Finally we are left with a new partition finer than $\mathcal{P}$, which satisfies property 1, 2 and 3. □

We will refer to the partition above as a Kakutani-Rohlin partition (or just a KR-partition) of (X, σ) finer than $\mathcal{P}$. The set

$$Y = \cup_{k=1}^{K} Y(k, J_k)$$

will be referred to as the *top floor* of the KR-partition.

Remark 3.1.2. *If (X, σ) is in fact a Cantor system and Y is a clopen subset of X, the assumption about Y in Lemma 3.1.1 is automatically satisfied. Moreover, if $N \in \mathbb{N}$, by replacing Y with a smaller clopen set Y', we may arrange that each tower is taller than N, and that each of the disjoint sets $Y', \ldots, \sigma^N(Y')$ lies inside a set from $\mathcal{P}$.*

To see this, let $N \in \mathbb{N}$, choose $y \in Y$, and note that the points $y, \ldots, \sigma^N(y)$ are distinct. We may therefore pick a clopen neighborhood Y_n around each point $\sigma^n(y), 0 \leq n \leq N$, such that $Y_0, \ldots, Y_N$ are mutually disjoint and each set Y_n lies inside a set from $\mathcal{P}$. Now put

$$Y' = \bigcap_{n=0}^{N} \sigma^{-n}(Y_n)$$

and use Lemma 3.1.1 to pick a KR-partition of X with top floor Y'.

It is clear, that the sets $Y', \ldots, \sigma^N(Y')$ are disjoint, and that each set lies inside a set from $\mathcal{P}$: Hence it follows from the proof of Lemma 3.1.1 that $J_k > N$ for each $1 \leq k \leq K$. Finally note that these conditions are not affected by the splitting of the towers.

3.2 The local A$\mathbb{T}$-property

In this section we are going to prove a remarkable theorem due to Putnam [19]. First we have a couple of lemmas:

Lemma 3.2.1. *Let $K, J_1, \dots, J_K \in \mathbb{N}$, and let $e_{ij}^{(k)}, 1 \le k \le K,\ 1 \le i,j \le J_k$, be the matrix units in $M_{J_1}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C})$. Define q to be the matrix with 1 below the diagonal and 0 elsewhere:*

$$q = \sum_{k=1}^{K} \sum_{j=2}^{J_k} e_{j,j-1}^{(k)}.$$

Then $M_{J_1}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C}) \cong C^*(\{e_{ii}^{(k)}, 1 \le k \le K,\ 1 \le i \le J_k\}, q)$.

Proof. Let $1 \le k \le K$ and $1 \le i,j \le J_k$. Then $e_{ij}^{(k)} = e_{i1}^{(k)}(e_{j1}^{(k)})^*$, and by induction it is easy to verify that $e_{i1}^{(k)} = e_{ii}^{(k)} q^{i-1}$ if $1 < i \le J_k$. Thus we are done. ☐

Lemma 3.2.2. *Suppose $\mathcal{B}$ is a unital C^*-algebra, $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$ containing the unit of $\mathcal{B}$, and $(f_{ij})_{1 \le i,j \le n}$ is a system of matrix units in $\mathcal{B}$ such that*

$$\forall a \in \mathcal{A}\ \forall i,j : a f_{ij} = f_{ij} a.$$

Then $M_n(\mathcal{A}) \cong C^*(\mathcal{A}, \{f_{ij}\})$.

Proof. Define $\psi : M_n(\mathcal{A}) \to \mathcal{B}$ by

$$\psi\Big(\sum_{1 \le i,j \le n} a_{ij} \otimes e_{ij}\Big) = \sum_{1 \le i,j \le n} a_{ij} f_{ij}.$$

It is straightforward to check that ψ is a *-homomorphism. We will now prove that ψ maps $M_n(\mathcal{A})$ bijectively onto $C^*(\mathcal{A}, \{f_{ij}\})$.
ψ is injective: Assume $\sum a_{ij} f_{ij} = 0$, and let $i_0, j_0 \in \{1, \dots, n\}$. Then

$$\begin{aligned} a_{i_0 j_0} &= \sum_{k=1}^{n} a_{i_0 j_0} f_{kk} = \sum_{k=1}^{n} f_{k i_0} a_{i_0 j_0} f_{i_0 j_0} f_{j_0 k} \\ &= \sum_{k=1}^{n} f_{k i_0} \Big(\sum_{i,j} a_{ij} f_{ij}\Big) f_{j_0 k} = 0 \end{aligned}$$

showing, that ψ is injective.
$\psi(M_n(\mathcal{A})) = C^*(\mathcal{A}, f_{ij})$: "$\subseteq$" is clear. Conversely, let $a \in \mathcal{A}$ and note that $\psi(\sum_{i=1}^{n} a \otimes e_{ii}) = a$. Hence $\mathcal{A} \subseteq \psi(M_n(\mathcal{A}))$. Moreover for any $1 \le i,j \le n$

$$f_{ij} = \psi(1_{\mathcal{B}} \otimes e_{ij}) \in \psi(M_n(\mathcal{A})),$$

proving that $\psi(M_n(\mathcal{A})) \supseteq C^*(\mathcal{A}, f_{ij})$. ☐

Lemma 3.2.3. *Suppose $\mathcal{C}$ is a C^*-algebra, and $\mathcal{A}$ and $\mathcal{B}$ are *-invariant subsets of $\mathcal{C}$. Let us moreover assume that $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A} = 0$. Then*

$$C^*(\mathcal{A}, \mathcal{B}) = C^*(\mathcal{A}) + C^*(\mathcal{B}) \cong C^*(\mathcal{A}) \oplus C^*(\mathcal{B}).$$

Proof. If we prove that $C^*(\mathcal{A})+C^*(\mathcal{B}) \cong C^*(\mathcal{A})\oplus C^*(\mathcal{B})$, the validity of the equality sign will follow.

Define a map $\Psi : C^*(\mathcal{A}) \oplus C^*(\mathcal{B}) \to \mathcal{C}$ by $\Psi(x, y) = x + y$. The assumption implies that

$$C^*(\mathcal{A})C^*(\mathcal{B}) = C^*(\mathcal{B})C^*(\mathcal{A}) = 0.$$

Hence it is easy to see, that Ψ is a *-homomorphism.

Suppose $x+y = 0$ for some $(x, y) \in C^*(\mathcal{A})\oplus C^*(\mathcal{B})$, and choose an approximating unit (u_λ) for $C^*(\mathcal{A})$. Then

$$0 = u_\lambda(x + y) = u_\lambda x \to x.$$

Hence $x = y = 0$, and we conclude that Ψ is injective. Thus Ψ is a *-isomorphism onto its image. $\square$

The next two lemmas deal with Hilbert spaces:

Lemma 3.2.4. *Let $H = H_1 \oplus H_2$ be the direct sum of Hilbert spaces H_1 and H_2, and suppose $U \in B(H)$ is a unitary satisfying $U(H_1) = H_1$. Then U acts as a unitary on H_2.*

Proof. Let $\xi_1 \in H_1$ and $\xi_2 \in H_2$. Since

$$< U\xi_2|\xi_1 >=< \xi_2|U^*\xi_1 >=< \xi_2|U^{-1}\xi_1 >= 0$$

and

$$< U^{-1}\xi_2|\xi_1 >=< U^*\xi_2|\xi_1 >=< \xi_2|U\xi_1 >= 0$$

we get, that U maps H_2 bijectively onto H_2. Now the fact that U is an isometry implies that U is a unitary on H_2. $\square$

Lemma 3.2.5. *Let $H = \oplus_{i=1}^n H_i$, where each H_i is a Hilbert space. Let $T \in B(H)$, put $T_i = T|_{H_i}$, and suppose the subspaces $T_i(H_i)$ are mutually orthogonal. Then*

$$||T|| \leq \max_{1\leq i\leq n}||T_i||,$$

where $||T_i||$ is the norm of the bounded operator $T_i : H_i \to H$.

Proof. Let $x = \sum_{i=1}^n x_i \in H$. Then

$$\begin{aligned} ||Tx||^2 &= ||\sum_{i=1}^n T_i x_i||^2 \\ &= \sum_{i=1}^n ||T_i x_i||^2 \\ &\leq \max||T_i||^2||x||^2, \end{aligned}$$

as desired. $\square$

We are now ready for the main theorem of Chapter 3:

Theorem 3.2.6. *Let (X, σ) be a Cantor system, let u be the canonical unitary in $C(X) \rtimes_\sigma \mathbb{Z}$, let $\mathcal{P}$ be a partition of X, and let $\epsilon > 0$. Then there exists a C*-subalgebra $\mathcal{A}$ of $C(X) \rtimes_\sigma \mathbb{Z}$, such that*

1. $\mathcal{A} \cong M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C})$ *for a suitable choice of integers* $J_1, \ldots, J_K$*,*
2. $C(\mathcal{P}) \subseteq \mathcal{A}$*, and*
3. $\exists u' \in U(\mathcal{A}) : ||u - u'|| < \epsilon$*.*

Proof. Fixing an ergodic (or just σ-invariant) measure μ on X (see Theorem 2.2.2), we will use the picture of $C(X) \rtimes_\sigma \mathbb{Z}$ as a C*-subalgebra of $B(L^2(X))$ described in Theorem 2.3.2. Thus $C(X) \rtimes_\sigma \mathbb{Z}$ is the C*-subalgebra of $B(L^2(X))$ generated by u and $C(X)$, where u is the unitary defined by

$$u(\xi) = \xi \circ \sigma^{-1},\ \xi \in L^2(X),$$

and $C(X)$ is identified with the corresponding multiplication operators.

Let $\epsilon > 0$ and let $\mathcal{P}$ be a partition of X. Choose $N \in \mathbb{N}$ such that $\frac{\pi}{N} < \epsilon$, and choose a KR-partition $\mathcal{P}_0 = \{Y(k,j) \mid 1 \leq k \leq K, 1 \leq j \leq J_k\}$ with top floor Y such that $\mathcal{P}_0$ is finer than $\mathcal{P}$ (see Lemma 3.1.1). By Remark 3.1.2 we may assume that $J_k > N$, $1 \leq k \leq K$, that the sets $Y, \ldots, \sigma^N(Y)$ are disjoint, and that each of the sets $Y, \ldots, \sigma^N(Y)$ lies inside a set from $\mathcal{P}$. Put

$$e_{ij}^{(k)} = 1_{Y(k,i)} u^{i-j},\ 1 \leq k \leq K, 1 \leq i,j \leq J_k.$$

Let us check that $(e_{ij}^{(k)})$ is a set of matrix units (see Rørdam et al. [22], p. 110). Using the fact that $\sigma(1_A) = 1_{\sigma(A)}$ whenever A is a clopen set, we have

$$\begin{aligned} e_{ij}^{(k)} e_{mn}^{(l)} &= 1_{Y(k,i)} u^{i-j} 1_{Y(l,m)} u^{m-n} \\ &= 1_{Y(k,i) \cap \sigma^{i-j}(Y(l,m))} u^{i-j+m-n}. \end{aligned}$$

We claim

$$Y(k,i) \cap \sigma^{i-j}(Y(l,m)) = \begin{cases} Y(k,i) & \text{whenever } j = m, k = l, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $j = m$ and $k = l$ it is clear that $Y(k,i) \cap \sigma^{i-j}(Y(l,m)) = Y(k,i)$. Now suppose $Y(k,i) \cap \sigma^{i-j}(Y(l,m)) \neq \emptyset$. Then also

$$Y(k,j) \cap Y(l,m) = \sigma^{j-i}(Y(k,i) \cap \sigma^{i-j}(Y(l,m)) \neq \emptyset,$$

and therefore $k = l$ and $j = m$.

This proves that $e_{ij}^{(k)} e_{jl}^{(k)} = e_{il}^{(k)}$, and that $e_{ij}^{(k)} e_{mn}^{(l)} = 0$ if $k \neq l$ or $j \neq m$. We also have

$$(e_{ij}^{(k)})^* = u^{j-i} 1_{Y(k,i)} = 1_{\sigma^{j-i}(Y(k,i))} u^{j-i} = 1_{Y(k,j)} u^{j-i} = e_{ji}^{(k)},$$

proving that

$$\mathcal{A}_0 = \operatorname{span}\{e_{ij}^{(k)} \mid 1 \le k \le K, 1 \le i,j \le J_k\}$$

is isomorphic to $M_{J_1}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C})$. Observe that

$$C(\mathcal{P}_0) = \operatorname{span}\{e_{jj}^{(k)} \mid 1 \le k \le K, 1 \le j \le J_k\}$$

and

$$\sum_{k=1}^{K}\sum_{j=2}^{J_k} e_{j,j-1}^{(k)} = \sum_{k=1}^{K}\sum_{j=2}^{J_k} 1_{Y(k,j)}u = 1_{X\setminus\sigma(Y)}u = u1_{X\setminus Y}.$$

By Lemma 3.2.1 we therefore have that $\mathcal{A}_0 = C^*(C(\mathcal{P}_0), u1_{X\setminus Y})$. Let us define maps σ_0 and ψ_0 by

$$\psi_0(x) = \begin{cases} \sigma(x) & x \notin Y \\ \sigma^{1-J_k}(x) & x \in Y(k, J_k),\ 1 \le k \le K \end{cases}$$

and $\sigma_0 = \psi_0^{-1} \circ \sigma$. By construction ψ_0 cycles around in each of the K towers; in particular ψ_0 is bijective and thus a homeomorphism. Hence σ_0 is also a homeomorphism. Note that

$$\psi_0^{-1}(x) = \begin{cases} \sigma^{-1}(x) & x \notin \sigma(Y) \\ \sigma^{J_k-1}(x) & x \in Y(k,1). \end{cases}$$

We conclude that σ_0 is the identity outside Y.

Since μ is σ-invariant, it is also σ_0- and ψ_0-invariant. Therefore u_0 and v_0 defined by $u_0\xi = \xi \circ \sigma_0^{-1}$, $\xi \in L^2(X)$, and $v_0\xi = \xi \circ \psi_0^{-1}$, $\xi \in L^2(X)$, are unitaries. Moreover $\psi_0 \circ \sigma_0 = \sigma$ implies that $v_0u_0 = u$.

Note that

$$v_0\xi = \xi \circ \psi_0^{-1} = \xi \circ \sigma^{-1} = u\xi,\ \xi \in L^2(X \setminus Y),$$

and

$$v_0\xi = \xi \circ \psi_0^{-1} = \xi \circ \sigma^{J_k-1} = u^{1-J_k}\xi,\ \xi \in L^2(Y(k, J_k)).$$

Hence we get that

$$v_0 e_{ii}^{(k)} = \begin{cases} ue_{ii}^{(k)} = e_{i+1,i}^{(k)} & 1 \le i < J_k, \\ u^{1-J_k}e_{J_kJ_k}^{(k)} = e_{1,J_k}^{(k)} & i = J_k, \end{cases}$$

and we therefore have

$$v_0 = v_0(\sum_{k=1}^{K}\sum_{i=1}^{J_k} e_{ii}^{(k)}) = \sum_{k=1}^{K}(\sum_{i=1}^{J_k-1} e_{i+1,i}^{(k)} + e_{1,J_k}^{(k)}) \in \mathcal{A}_0.$$

It now follows that $u_0 = v_0^{-1}u \in C(X) \rtimes_\sigma \mathbb{Z}$. Also observe that

$$u_0\xi = \xi \circ \sigma_0^{-1} = \xi, \xi \in L^2(X \setminus Y),$$

since σ_0 is the identity on $X \setminus Y$. Hence $u_0(L^2(Y)) = L^2(Y)$.

Let us now construct a new KR-partition $\mathcal{P}_1$ with top floor $Z = Y(1, J_1)$ and

$$\mathcal{P}_1 = \{Z(k,j) \mid 1 \le k \le K',\ 1 \le j \le J'_k\}.$$

We demand that $\mathcal{P}_1$ is finer than $\mathcal{P}_0$. As before we obtain a finite dimensional C*-subalgebra $\mathcal{A}_1 \subseteq C(X) \rtimes_\sigma \mathbb{Z}$ (we will not need to write down the matrix units), and again we have that $\mathcal{A}_1 = C^*(C(\mathcal{P}_1), u1_{X\setminus Z})$. $C(\mathcal{P}_0) \subseteq C(\mathcal{P}_1)$ since $\mathcal{P}_0$ is coarser than $\mathcal{P}_1$, and $u1_{X\setminus Y} = u1_{X\setminus Z}1_{X\setminus Y} \in \mathcal{A}_1$ since $Z \subseteq Y$: Thus we have $\mathcal{A}_0 \subseteq \mathcal{A}_1$.

In an analogous way we define homeomorphisms σ_1 and ψ_1, and unitaries u_1 and v_1. Again we get that $v_1 u_1 = u$, $v_1 \in \mathcal{A}_1$, $u_1 \in C(X) \rtimes_\sigma \mathbb{Z}$, $v_1 = u$ and $u_1 = I$ on $L^2(X \setminus Z)$, and $u_1(L^2(Z)) = L^2(Z)$ (we denote the identity in $B(L^2(X))$ by I). Hence $u_1 e^{(1)}_{J_1 J_1} = e^{(1)}_{J_1 J_1} u_1$.

Put $\mathcal{B} = C^*(\mathcal{A}_0, u_1)$. We will now prove

$$\mathcal{B} \cong M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C}).$$

Put $p_1 = \sum_{j=1}^{J_1} e^{(1)}_{jj}$, and define

$$\hat{u} = \sum_{j=1}^{J_1} e^{(1)}_{jJ_1} u_1 e^{(1)}_{J_1 j}.$$

Note that $\hat{u}$ is a unitary in $C^*(\{e^{(1)}_{ij} \mid 1 \le i,j \le J_1\}, \hat{u})$, since $\hat{u}\hat{u}^* = \hat{u}^*\hat{u} = p_1$ and p_1 is the unit in $C^*(\{e^{(1)}_{ij} \mid 1 \le i,j \le J_1\}, \hat{u})$. Moreover $\hat{u} \in \mathcal{B}$. In fact $\mathcal{B} = C^*(\mathcal{A}_0, \hat{u})$, since

$$u_1 = e^{(1)}_{J_1 J_1} u_1 e^{(1)}_{J_1 J_1} + I - e^{(1)}_{J_1 J_1} = e^{(1)}_{J_1 J_1} \hat{u} e^{(1)}_{J_1 J_1} + I - e^{(1)}_{J_1 J_1}.$$

To prove $\mathcal{B} \cong M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C})$, by Lemma 3.2.2 and Lemma 3.2.3 we only have to prove that

1. $\hat{u}$ (and $\hat{u}^*$) commutes with all the matrix units $e^{(1)}_{ij}, 1 \le i,j \le J_1$, and
2. $\mathrm{sp}(\hat{u}) = \mathbb{T}$ in $C^*(\{e^{(1)}_{ij} \mid 1 \le i,j \le J_1\}, \hat{u})$.

1: Follows directly from the definition of $\hat{u}$.
2: $u = u_1 v_1$; therefore $[u]_1 = [u_1]_1 + [v_1]_1$ in $K_1(C(X) \rtimes_\sigma \mathbb{Z})$ (see Rørdam et al. [22], Prop. 8.1.4 and Prop. 8.1.6). Since $[u]_1 \ne 0$ by Theorem 2.4.3, and $v_1 \in \mathcal{A}_1$ has finite spectrum (as a unitary in a finite dimensional C*-algebra) and therefore satisfies $[v_1]_1 = 0$ (see Rørdam et al. [22], Lemma 2.1.3), we conclude that $[u_1]_1 \ne 0$. Therefore (again by [22], Lemma 2.1.3) $\mathrm{sp}(u_1) = \mathbb{T}$.

Now $\hat{u}$ is a unitary in the unital C*-algebra $C^*(p_1, \hat{u})$; thus $\mathrm{sp}(\hat{u}) \subseteq \mathbb{T} = \mathrm{sp}(u_1)$. Hence it is enough to show that $\mathrm{sp}(u_1) \subseteq \mathrm{sp}(\hat{u}) \cup \{1\}$ (by compactness of the spectrum). Therefore, suppose $\lambda \notin \mathrm{sp}(\hat{u}) \cup \{1\}$, and pick $b \in C^*(p_1, \hat{u})$ such that $b(\hat{u} - \lambda p_1) = (\hat{u} - \lambda p_1)b = p_1$. Put

$$a = e^{(1)}_{J_1 J_1} b e^{(1)}_{J_1 J_1} + \frac{1}{1-\lambda}(I - e^{(1)}_{J_1 J_1})$$

and note that

$$u_1 - \lambda I = e^{(1)}_{J_1J_1}(\hat{u} - \lambda p_1)e^{(1)}_{J_1J_1} + (1-\lambda)(I - e^{(1)}_{J_1J_1}).$$

It is now obvious that a is an inverse for $u_1 - \lambda I$ (recall that $e^{(1)}_{J_1J_1}$ commutes with $\hat{u}$ and p_1). Therefore $\lambda \notin \mathrm{sp}(u_1)$.

Now we have a C*-algebra $\mathcal{B}$ of the desired form. Since $\mathcal{A}_0 \subseteq \mathcal{B}$, we also have that $C(\mathcal{P}) \subseteq \mathcal{B}$. In order to get an approximating unitary we will however need to twist $\mathcal{B}$ by a certain unitary z. We will choose z in such a way that z commutes with $C(\mathcal{P})$ and u_1, and zv_0z^* approximates v_1. Then $C(\mathcal{P}) = zC(\mathcal{P})z^* \subseteq z\mathcal{B}z^*$ and $zv_0u_1z^* = zv_0z^*u_1$ approximates $v_1u_1 = u$, and thus $\mathcal{A} = z\mathcal{B}z^*$ has all the desired properties.

$v_1v_0^*$ is a unitary in the finite dimensional C*-algebra $\mathcal{A}_1$. Hence the spectrum is a finite subset of $\mathbb{T}$. Pick a $\delta > 0$, such that $\frac{\pi+\delta}{N} < \epsilon$ and $-e^{i\delta} \notin \mathrm{sp}(v_1v_0^*)$, and put

$$f(z) = e^{\frac{1}{N}\mathrm{Log}_\delta(z)} = e^{\frac{1}{N}i\mathrm{Arg}_\delta(z)},\ z \in \mathrm{sp}(v_1v_0^*).$$

Put $w = f(v_1v_0^*)$ and note that $w^N = v_1v_0^*$ and $||w - I|| = ||f-1||_u < \frac{\pi+\delta}{N} < \epsilon$.

We have that $v_1v_0^* = v_1v_0^{-1} = uu^{-1} = I$ on $L^2(X \setminus \sigma(Y))$, and hence $v_1v_0^*$ acts as a unitary on $L^2(\sigma(Y))$ (see Lemma 3.2.4). Thus every polynomial in $v_1v_0^*$ and $v_0v_1^*$ leaves $L^2(\sigma(Y))$ and $L^2(X \setminus \sigma(Y))$ invariant. Hence the same is true for w (being a norm limit of such polynomials), and since w is a unitary, we conclude that $w(L^2(\sigma(Y))) = L^2(\sigma(Y))$ (by an argument similar to the one in Lemma 3.2.4).

Define an operator z by

$$z = \begin{cases} u^{j-1}w^{N-j+1}u^{-j+1} & \text{on } L^2(\sigma^j(Y)), 1 \le j \le N \\ I & \text{on } L^2(X \setminus (\sigma(Y) \cup \cdots \cup \sigma^N(Y))). \end{cases}$$

Since $u(L^2(\sigma^j(Y))) = L^2(\sigma^{j+1}(Y))$, z is surjective. Because of Pythagoras' Theorem, z is an isometry. Thus z is a unitary. Moreover $z \in \mathcal{A}_1$, because $\mathcal{A}_1$ contains I, w, $u1_{X\setminus Y}$, and the projections on each of the subspaces $L^2(\sigma(Y)), \ldots, L^2(\sigma^N(Y))$, and $L^2(X \setminus (\sigma(Y) \cup \cdots \cup \sigma^N(Y)))$. In fact, it is not hard to see that

$$z = \sum_{j=1}^{N} 1_{\sigma^j(Y)}(u1_{X\setminus Y})^{j-1}w^{N-j+1}((u1_{X\setminus Y})^*)^{j-1}1_{\sigma^j(Y)} + 1_{X\setminus(\sigma(Y)\cup\cdots\cup\sigma^N(Y))}.$$

Put $\mathcal{A} = z\mathcal{B}z^*$. We now prove the following:

1. $\mathcal{A}$ contains an approximating unitary, and
2. z commutes with $C(\mathcal{P})$.

1: Let us consider the operator $zv_0 - v_1z$. We have that $z = I$ on $L^2(Y)$, and v_0 carries $L^2(Y)$ onto $L^2(\sigma(Y))$ where $z = w^N = v_1v_0^*$. Hence

$$zv_0 - v_1z = v_1v_0^*v_0 - v_1 = 0$$

on $L^2(Y)$.

Note that $v_0 = v_1 = u$ on $L^2(\sigma^j(Y)), 1 \leq j \leq N$. Thus for any $1 \leq j \leq N$ we have that

$$zv_0 - v_1 z = u^j w^{N-j} u^{-j+1} - u^j w^{N-j+1} u^{-j+1}$$

on $L^2(\sigma^j(Y))$, whence $(zv_0 - v_1 z)(L^2(\sigma^j(Y))) \subseteq L^2(\sigma^{j+1}(Y))$ and

$$||(zv_0 - v_1 z)|_{L^2(\sigma^j(Y))}|| = ||(I - w)|_{L^2(\sigma(Y))}|| \leq ||I - w|| < \epsilon.$$

Finally, $v_0 = v_1 = u$ on $L^2(X \setminus (Y \cup \cdots \cup \sigma^N(Y)))$, and $z = I$ on the subspaces $L^2(X \setminus (Y \cup \cdots \cup \sigma^N(Y)))$ and $L^2(X \setminus (\sigma(Y) \cup \cdots \cup \sigma^{N+1}(Y)))$. Thus $zv_0 - v_1 z = 0$ on $L^2(X \setminus (Y \cup \cdots \cup \sigma^N(Y)))$.

Since $Y, \ldots, \sigma^N(Y)$ are disjoint, we have that $L^2(\sigma(Y)), \ldots, L^2(\sigma^{N+1}(Y))$ are mutually orthogonal. Hence we may apply Lemma 3.2.5 to conclude that

$$||zv_0 - v_1 z|| \leq \max_{1 \leq j \leq N} ||(zv_0 - v_1 z)|_{L^2(\sigma^j(Y))}|| < \epsilon.$$

Note that $z = I$ on $L^2(Y)$, $u_1 = I$ on $L^2(X \setminus Y)$, $z(L^2(X \setminus Y)) = L^2(X \setminus Y)$, and $u_1(L^2(Y)) = L^2(Y)$. Hence $zu_1 = u_1 z$, whence $z^* u_1 = z^{-1} u_1 = u_1 z^{-1} = u_1 z^*$. Thus $u' = zv_0 z^* u_1 = zv_0 u_1 z^*$ is a unitary in $\mathcal{A}$, and

$$||u - u'|| = ||v_1 u_1 - zv_0 z^* u_1|| = ||v_1 z - zv_0|| < \epsilon.$$

2: Let us write $\mathcal{P}$ as $\mathcal{P} = \{X_1, \ldots, X_m\}$. To prove that z commutes with $C(\mathcal{P})$, it suffices to prove that $z1_{X_i} = 1_{X_i} z$ for $i \in \{1, \ldots, m\}$. Hence let $i \in \{1, \ldots, m\}$, and recall that each of the sets $\sigma(Y), \ldots, \sigma^N(Y)$ is contained in one of the sets in $\mathcal{P}$. Moreover $z(L^2(\sigma^j(Y))) = L^2(\sigma^j(Y))$ for $1 \leq j \leq N$, and $z = I$ on

$$X \setminus (\sigma(Y) \cup \cdots \cup \sigma^N(Y)).$$

These facts imply that $z(L^2(X_i)) = L^2(X_i)$ and $z(L^2(X \setminus X_i)) = L^2(X \setminus X_i)$, whence $1_{X_i} z = 1_{X_i} z 1_{X_i} = z 1_{X_i}$, as desired. □

Theorem 3.2.6 has several important applications.

In the next section we will see how it follows from Theorem 3.2.6, that crossed products arising from Cantor systems are completely classified by their K-theory.

3.3 Classification of the crossed products

Definition 3.3.1. *Let $\mathcal{B}$ be a C^*-algebra. Then we will say that $\mathcal{B}$ is a* local A$\mathbb{T}$-algebra, *if given a finite subset $\{a_1, \ldots, a_n\}$ of $\mathcal{B}$ and $\epsilon > 0$, there exists a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that*

1. *$\mathcal{A}$ is isomorphic to a direct sum of matrix algebras over $C(\mathbb{T})$ and $\mathbb{C}$, and*
2. *$\exists a'_1, \ldots, a'_n \in \mathcal{A}\ \forall 1 \leq j \leq n : ||a'_j - a_j|| < \epsilon$.*

If $\mathcal{B} = \underrightarrow{\lim} \mathcal{A}_k$, where each $\mathcal{A}_k$ is a direct sum of matrix algebras over $C(\mathbb{T})$, then $\mathcal{B}$ is said to be an A$\mathbb{T}$-algebra.

It is a consequence of Theorem 3.2.6 that the crossed product associated to a Cantor system is a local $A\mathbb{T}$-algebra:

Corollary 3.3.2. *Let (X, σ) be a Cantor system, let $\{a_1, \ldots, a_n\}$ be a finite subset of $C(X) \rtimes_\sigma \mathbb{Z}$, and let $\epsilon > 0$. Then there exists a C^*-subalgebra $\mathcal{A}$ of $C(X) \rtimes_\sigma \mathbb{Z}$, such that*

1. *$\mathcal{A}$ contains the unit of $C(X) \rtimes_\sigma \mathbb{Z}$,*
2. *$\mathcal{A} \cong M_{J_1}(C(\mathbb{T})) \oplus M_{J_2}(\mathbb{C}) \oplus \cdots \oplus M_{J_K}(\mathbb{C})$ for some integers $J_1, \ldots, J_K \in \mathbb{N}$, and*
3. *$\exists a'_1, \ldots, a'_n \in \mathcal{A} : ||a_j - a'_j|| < \epsilon$.*

In particular, $C(X) \rtimes_\sigma \mathbb{Z}$ is a local $A\mathbb{T}$-algebra.

Proof. Let $\epsilon > 0$ and let $a_1, \ldots, a_n \in C(X) \rtimes_\sigma \mathbb{Z}$. We may assume that $a_1, \ldots, a_n$ are standard elements. Choose $m \in \mathbb{N}$ and $f_{ij} \in C(X)$ such that

$$a_i = \sum_{j=-m}^{m} f_{ij} u^j, \ 1 \leq i \leq n.$$

Choose (by Theorem 1.1.7) a partition $\mathcal{P}$ and functions $p_{ij} \in C(\mathcal{P})$ such that

$$\forall i, j : ||f_{ij} - p_{ij}|| < \frac{\epsilon}{4m+2}.$$

Put $T = \max||p_{ij}||$, and choose a $\delta > 0$ such that

$$||b - u|| < \delta \Rightarrow \forall j \in \{-m, \ldots, m\} : ||b^j - u^j|| < \frac{\epsilon}{(4m+2)T}$$

(see Pedersen [16], Proposition 4.1.8). Now use Theorem 3.2.6 to pick an algebra $\mathcal{A}$ of the form in Theorem 3.2.6 and a unitary $u' \in \mathcal{A}$, such that $C(\mathcal{P}) \subseteq \mathcal{A}$ and $||u - u'|| < \delta$. Put

$$a'_i = \sum_{j=-m}^{m} p_{ij} u'^j, \ 1 \leq i \leq n,$$

and note that $a'_i \in \mathcal{A}$ and

$$\begin{aligned} ||a_i - a'_i|| &\leq \sum_{j=-m}^{m} ||f_{ij} - p_{ij}|| ||u^j|| + \sum_{j=-m}^{m} ||p_{ij}|| ||u^j - u'^j|| \\ &< \sum_{j=-m}^{m} \frac{\epsilon}{4m+2} + \sum_{j=-m}^{m} T \frac{\epsilon}{(4m+2)T} = \epsilon. \end{aligned}$$

□

We will need the following lemma:

Lemma 3.3.3. *Let (X, σ) be a Cantor system. Then $C(X) \rtimes_\sigma \mathbb{Z}$ is separable.*

Proof. It follows from Theorem 1.1.7 that $C(X)$ is separable. Take $\mathcal{F}$ to be a countable dense subset of $C(X)$. Now

$$\bigcup_{N=1}^{\infty}\{\sum_{n=-N}^{N} f_n u^n \mid f_n \in \mathcal{F}\}$$

is a countable dense subset of $C(X) \rtimes_\sigma \mathbb{Z}$. □

Recall that any separable, local AF-algebra is an AF-algebra (cf. Davidson [4], Theorem III.3.4). Elliot has proved a similar result for A$\mathbb{T}$-algebras:

Theorem 3.3.4. *Let $\mathcal{A}$ be a separable, local A$\mathbb{T}$-algebra. Then $\mathcal{A}$ is an A$\mathbb{T}$-algebra.*

Proof. See Elliot [8], p. 184. □

It follows from Corollary 3.3.2, Lemma 3.3.3 and Theorem 3.3.4 that $C(X) \rtimes_\sigma \mathbb{Z}$ is an A$\mathbb{T}$-algebra, when (X, σ) is a Cantor system. We will also need that $C(X) \rtimes_\sigma \mathbb{Z}$ has real rank zero:

Theorem 3.3.5. *Let (X, σ) be a Cantor system. Then $C(X) \rtimes_\sigma \mathbb{Z}$ has real rank zero (that is, the invertible self-adjoints are dense among the self-adjoints).*

Proof. By Theorem 1.3 in Blackadar et al. [2] it is enough to prove that the projections in $C(X) \rtimes_\sigma \mathbb{Z}$ separates the tracial states in $C(X) \rtimes_\sigma \mathbb{Z}$. Therefore assume τ and τ' are tracial states that take the same value on the projections. Hence $\tau = \tau'$ on $C(\mathcal{P})$ for any partition $\mathcal{P}$ of X, and by Theorem 1.1.7 we conclude that $\tau = \tau'$ on $C(X)$. Now apply Corollary 2.3.5 to get that $\tau = \tau \circ \Phi = \tau' \circ \Phi = \tau'$ on $C(X) \rtimes_\sigma \mathbb{Z}$ (Φ denotes the conditional expectation on $C(X) \rtimes_\sigma \mathbb{Z}$). □

We will now state a remarkable classification result due to Elliot:

Theorem 3.3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be simple, separable A$\mathbb{T}$-algebras with real rank zero, and assume that*

$$(K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_{\mathcal{A}}]_0) \cong (K_0(\mathcal{B}), K_0(\mathcal{B})^+, [1_{\mathcal{B}}]_0)$$

and $K_1(\mathcal{A}) \cong K_1(\mathcal{B})$. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Proof. Follows from Elliot [8], Theorem 7.1. □

We finally have the following corollary, which states that crossed products arising from Cantor systems are completely classified by their K-theory:

Corollary 3.3.7. *Let (X_1, σ_1) and (X_2, σ_2) be Cantor systems. Then $C(X_1) \rtimes_{\sigma_1} \mathbb{Z}$ and $C(X_2) \rtimes_{\sigma_2} \mathbb{Z}$ are isomorphic, if and only if*

$$K_0(C(X_1) \rtimes_{\sigma_1} \mathbb{Z}) \cong K_0(C(X_2) \rtimes_{\sigma_2} \mathbb{Z})$$

by an order isomorphism preserving the distinguished order units.

Proof. Let (X, σ) be a Cantor system. As already noted, it follows from Corollary 3.3.2, Lemma 3.3.3 and Theorem 3.3.4 that $C(X) \rtimes_\sigma \mathbb{Z}$ is a separable A$\mathbb{T}$-algebra. By Theorem 2.3.4 and Theorem 3.3.5 we get that $C(X) \rtimes_\sigma \mathbb{Z}$ is a simple C*-algebra with real rank zero, and Theorem 2.4.3 implies that $K_1(C(X) \rtimes_\sigma \mathbb{Z}) \cong \mathbb{Z}$. Thus Theorem 3.3.6 gives the desired result. □

Chapter 4

Cantor Systems, Bratteli Diagrams, and the Isomorphism Problem

This chapter is devoted to give a dynamical criterion for $C(X_1)\rtimes_{\sigma_1}\mathbb{Z}$ and $C(X_2)\rtimes_{\sigma_2}\mathbb{Z}$ to be isomorphic, in the case where (X_1,σ_1) and (X_2,σ_2) are Cantor systems.

An answer to that question was given in Giordano et al. [9], Theorem 2.1 (Giordano et al. also studied several other interesting relations between Cantor systems, which we will not discuss). The fruitful idea behind their work was to identify arbitrary Cantor systems with a certain type of Cantor systems called *Bratteli-Vershik systems*, where the underlying space is a Bratteli diagram. For that reason we will begin by introducing the notion of a Bratteli diagram.

4.1 Bratteli diagrams

In his study of AF-algebras, Bratteli was the first to encounter a certain type of infinite graphs, subsequently known as Bratteli diagrams (see Bratteli [3]). We will define Bratteli diagrams as follows [1]:

Definition 4.1.1. *A Bratteli diagram* (V,E,r,s) *(or with an abuse of notation just* (V,E)*) consists of a vertex set* V*, an edge set* E*, and maps* r *and* s *from* E *to* V *(called the range and source maps), such that the following properties are satisfied:*

1. $V_0=\{v_0\}$ *is a one-point set,*
2. $V=\sqcup_{i=0}^{\infty}V_i$*, where each* V_i *is finite,*
3. $E=\sqcup_{i=1}^{\infty}E_i$*, where each* E_i *is finite,*
4. $s(E_i)=V_{i-1}, i=1,2,\ldots,$
5. $r(E_i)=V_i, i=1,2,\ldots.$

We will refer to V_i *and* E_i *as the vertex set respectively the edge set at level* i*.*

[1] Our definition will be identical to the one given in Bratteli [3], p. 201, except for the condition that there is only one vertex at level 0.

Note that condition 4 and 5 in the definition above guarantees, that no edges and vertices are "stranded". Note also that we have a natural notion of isomorphism between Bratteli diagrams (V, E) and (V', E'): A bijective map between the diagrams, that intertwine the range and source maps. We will often identify isomorphic Bratteli diagrams.

Let (V, E) be a Bratteli diagram, let $m < n$, and consider two arbitrary vertices $v_m \in V_m$ and $v_n \in V_n$. By $P(v_m, v_n)$ we shall mean the set of all paths from v_m to v_n:

$$\begin{aligned} P(v_m, v_n) = \{(e_{m+1}, \ldots, e_n) \mid s(e_{m+1}) &= v_m,\ r(e_n) = v_n, \\ r(e_{i-1}) &= s(e_i), i = m+1, \ldots, n\}. \end{aligned}$$

We also define $P(v_m, V_n) = \cup_{v_n \in V_n} P(v_m, v_n)$ and $P(V_m, V_n) = \cup_{v_m \in V_m} P(v_m, V_n)$.

We will now introduce the notion of a telescoping of a Bratteli diagram:

Definition 4.1.2. *Given a Bratteli diagram (V, E) and a sequence*

$$m_0 = 0 < m_1 < m_2 < \ldots$$

in $\mathbb{N}_0$, we define the telescoping of (V, E) to $(m_n)_{n \in \mathbb{N}_0}$ as the diagram (V', E'), where $V'_n = V_{m_n}, n \in \mathbb{N}_0$, and E'_n is the set of paths from V'_{n-1} to V'_n:

$$E'_n = P(V_{m_{n-1}}, V_{m_n}),\ n \in \mathbb{N}.$$

The range and source maps are the obvious ones.

If B and B' are Bratteli diagrams satisfying that B is a telescoping of B' (to some sequence), then B and B' have several properties in common (cf. Theorem 4.1.5 below). Note however that telescoping is not an equivalence relation; it is neither transitive nor symmetric (it is not so hard to construct counterexamples). Yet any relation generates an equivalence relation, and thus we have the following definition:

Definition 4.1.3. *On the class of Bratteli diagrams, the symbol $\sim$ will denote the equivalence relation generated by telescoping and isomorphism.*

Let us now construct an ordered group from a Bratteli diagram:

Remark 4.1.4. *Given a Bratteli diagram $B = (V, E)$, we have an ordered abelian group $K_0(V, E)$ given by the inductive limit*

$$\mathbb{Z} = \mathbb{Z}^{|V_0|} \xrightarrow{A_1} \mathbb{Z}^{|V_1|} \xrightarrow{A_2} \cdots$$

where A_k is a $|V_k| \times |V_{k-1}|$-matrix, and $(A_k)_{wv} = |P(v, w)|, v \in V_{k-1}, w \in V_k, k \in \mathbb{N}$ (see Effros [6], Chapter 2 and 3).

If we denote the canonical map from $\mathbb{Z}^{|V_k|}$ to $K_0(V, E)$ by η_k, we have the distinguished order unit $\eta_0(1)$. Indeed $\eta_0(1)$ is an order unit, since the images of 1 under the incidence matrices A_k are strictly positive (by the way we defined Bratteli diagrams).

To any Bratteli diagram $B = (V, E)$ we also associate a unital AF-algebra $AF(V, E)$ in a canonical way similar to the way we have constructed $K_0(V, E)$ (cf. Bratteli [3], p. 201, for the details). We have the following remarkable theorem:

Theorem 4.1.5. *Let (V, E) and (V', E') be Bratteli diagrams. Then the following conditions are equivalent:*

1. $K_0(V, E) \cong K_0(V', E')$ *(as ordered groups with distinguished units),*
2. $\mathrm{AF}(V, E) \cong \mathrm{AF}(V', E')$,
3. $(V, E) \sim (V', E')$.

Proof. "1 $\iff$ 2": See Rørdam et al. [22], Theorem 7.3.4, or Elliot [7].
"2 $\iff$ 3": See Bratteli [3], Theorem 2.7. □

4.2 Properly ordered Bratteli diagrams

Let us now introduce the notion of a simple Bratteli diagram:

Definition 4.2.1. *Let $B = (V, E)$ be a Bratteli diagram. B is then said to be simple, if*

1. *For any $m \in \mathbb{N}_0$, there is an $n > m$, such that every vertex on level m is connected by a path to every vertex on level n.*
2. *There is an infinite sequence $(n_k)_{k\in\mathbb{N}}$, such that each E_{n_k} contains two or more edges.*

Condition 2 is not part of the "standard definition" of a simple Bratteli diagram (cf. Davidson [4], Corollary III.4.3). It will nevertheless turn out to be important, when we proceed to construct a Cantor set from a simple Bratteli diagram (see Theorem 4.3.1 below).

Note, that any telescoping of a simple Bratteli diagram gives a simple Bratteli diagram. We have the following remark:

Remark 4.2.2. *A Bratteli diagram $B = (V, E)$ is simple, if and only if B has a telescoping $B' = (V', E')$, such that for each n and each $v_n \in V'_n, v_{n+1} \in V'_{n+1}$ there are at least two edges connecting v_n and v_{n+1}.*

Indeed one way is obvious. Conversely, suppose that $B = (V, E)$ is simple, and pick a sequence $m_0 = 0 < m_1 < m_2 < \dots$, such that every vertex in $V_{m_{n-1}}$ is connected to every vertex in V_{m_n}, and there are at least two paths between $V_{m_{n-1}}$ and V_{m_n}. Now let B' be the telescoping of B to the sequence $0 < m_2 < m_4 < \dots$, and note that B' has the desired properties.

We now introduce a relation between simple Bratteli diagrams, which will turn out to be identical to $\sim$ (we include simplicity in the definition, but the reader may notice that there is an obvious analogue for arbitrary Bratteli diagrams).

Definition 4.2.3. *Let B, B' and B'' be simple Bratteli diagrams. We will say that B' is connected to B'' (by B), and we write $B' \bowtie B''$, if telescoping B to $0 < 1 < 3 < \dots$ gives (a Bratteli diagram isomorphic to) a telescoping of B', and telescoping B to $0 < 2 < 4 < \dots$ gives (a Bratteli diagram isomorphic to) a telescoping of B''.*

Lemma 4.2.4. *"$\bowtie$" is an equivalence relation.*

Proof. It is obvious that $\bowtie$ is reflexive, and it is easy to see that it is symmetric (assume B' is connected to B'' by B, and telescope B to the sequence $0 < 2 < 3 < 4 < \dots$).

The only nontrivial thing to prove is that $\bowtie$ is transitive. To that end, suppose that $B' = (V', E')$ is connected to $B'' = (V'', E'')$ by $B^1 = (V^1, E^1)$ and B'' is connected to $B''' = (V''', E''')$ by $B^2 = (V^2, E^2)$. We want to construct a simple Bratteli diagram B, such that B' is connected to B''' by B.

By performing a telescoping of B^1 and B^2 we may well assume that $V_0^1 < V_1^2 < V_2^1 < V_3^2 < \dots$ in B'', where we have ordered the vertex sets according to the level at which they occur in B''.

Note that the paths in B'' from V_0^1 to V_2^1 correspond bijectively to the paths in B^1 from V_0^1 to V_2^1, the paths in B'' from V_1^2 to V_3^2 correspond bijectively to the paths in B^2 from V_1^2 to V_3^2, and so on.

We will now define a Bratteli diagram $\tilde{B}$. Define the vertex sets in $\tilde{B}$ by

$$\tilde{V}_0 = V_0^1, \tilde{V}_1 = V_1^1, \tilde{V}_2 = V_2^1, \tilde{V}_3 = V_3^2, \tilde{V}_4 = V_4^2, \tilde{V}_5 = V_5^2, \tilde{V}_6 = V_6^1, \tilde{V}_7 = V_7^1, \dots,$$

and let us define the edge sets as follows: Connect $\tilde{V}_0$ to $\tilde{V}_1$ and $\tilde{V}_1$ to $\tilde{V}_2$ using the edges from B^1. Connect $\tilde{V}_2$ to $\tilde{V}_3$ using the edges from B''. Connect $\tilde{V}_3$ to $\tilde{V}_4$ and $\tilde{V}_4$ to $\tilde{V}_5$ using the edges from B^2. Connect $\tilde{V}_5$ to $\tilde{V}_6$ using the edges from B'' (and so on). We include the following diagram to illustrate our construction of $\tilde{B}$:

$$\begin{tikzcd}
B' & B'' & B''' \\
V_0^1 & V_0^1 = V_0^2 \arrow[dl] & V_0^2 \\
V_1^1 \arrow[dr] & V_1^2 & \\
 & V_2^1 \arrow[d] & V_2^2 \\
V_3^1 & V_3^2 \arrow[dr] & \\
 & V_4^1 & V_4^2 \arrow[dl] \\
V_5^1 & V_5^2 \arrow[d] & \\
 & V_6^1 \arrow[dl] & V_6^2 \\
V_7^1 \arrow[dr] & V_7^2 & \\
 & \vdots &
\end{tikzcd}$$

Let $B = (V, E)$ be the telescoping of $\tilde{B}$ to the sequence $0 < 1 < 4 < 7 < \dots$. Because of our assumptions about B'', telescoping B to the sequence $0 < 1 < 3 < \dots$ gives us a telescoping of the uneven numbers of B^1, and hence a telescoping of B'. Similarly we get a telescoping of B''' by telescoping B to the sequence $0 < 2 < 4 < \dots$. Finally, it is obvious that B is simple, since for instance B' is a simple diagram which is a telescoping of B. ☐

We have got the following theorem:

Theorem 4.2.5. *Let $B' = (V', E')$ and $B'' = (V'', E'')$ be simple Bratteli diagrams. Then*

$$B' \bowtie B'' \iff B' \sim B''.$$

Proof. We split the proof in two parts:
"$\Rightarrow$": Obvious.
"$\Leftarrow$": If B' is a telescoping of B'', it is clear that $B' \bowtie B''$. Now the fact that $\bowtie$ is an equivalence relation implies that $B' \sim B'' \Rightarrow B' \bowtie B''$. ☐

We will now describe the notion of an ordered Bratteli diagram:

Definition 4.2.6. *An ordered Bratteli diagram $B = (V, E, \geq)$ is a triple, satisfying the following conditions:*

1. *(V, E) is a Bratteli diagram, and*
2. *$\geq$ is a partial order on E, such that $e, e' \in E$ are comparable if and only if $r(e) = r(e')$.*

We have a natural notion of isomorphism between ordered Bratteli diagrams: A bijective order-preserving map intertwining the range and source maps.

Also note that we have a natural induced lexicographic order on the paths in $P(V_k, V_l)$, given by

$$(x_{k+1}, \dots, x_l) < (y_{k+1}, \dots, y_l)$$

if and only if for some i with $k + 1 \leq i \leq l$, $x_j = y_j$ for $i < j \leq l$ and $x_i < y_i$.

If $B = (V, E, \geq)$ is an ordered Bratteli diagram, we will introduce the two sets E_{min} and E_{max}:

$$E_{min} = \{e \in E \mid \text{ e is a minimal element}\},$$
$$E_{max} = \{e \in E \mid \text{ e is a maximal element}\}.$$

Since there corresponds a maximal and a minimal edge to each vertex, both sets will always be infinite.

A minimal (resp. maximal) infinite path in B is an infinite path $(e_1, e_2, \dots)$ with edges from E_{min} (resp. E_{max}).

Similarly a finite minimal (resp. maximal) path is a path $(e_1, \dots, e_n) \in P(V_0, V_n)$ with edges from E_{min} (resp. E_{max}). Equivalently, a finite path $(e_1, \dots, e_n) \in P(V_0, V_n)$ is minimal (resp. maximal), if $(e_1, \dots, e_n)$ is a minimal (resp. maximal) element in $P(V_0, V_n)$, where $P(V_0, V_n)$ is equipped with the lexicographic order from above.

We have the following theorem:

Theorem 4.2.7. *Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. Then there exists a minimal and a maximal infinite path in B.*

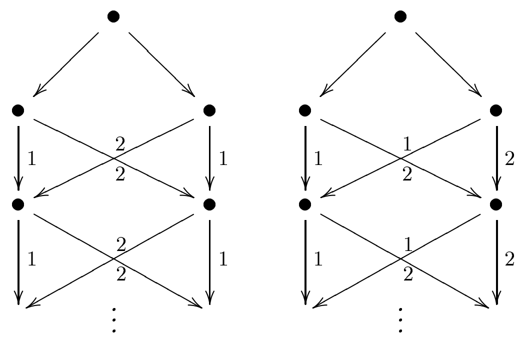
Proof. Choose at each level n a vertex v_n, and choose a minimal path $p_n = (e_1, \dots, e_n)$ from v_0 to v_n (start by picking a minimal edge e_n from v_n to level $n-1$, and continue by induction).

Now there must be a subsequence (p_{n_i}) of (p_n) and a vertex w_1 at level 1, such that each p_{n_i} goes through w_1 for all i. Similarly there must be a vertex w_2 and a subsequence $(p_{n_{i_j}})$, such that each path $p_{n_{i_j}}$ goes through w_2 for all j. Continuing by induction we get an infinite sequence of vertices w_i, each connected to the next by a minimal edge. Now the path consisting of these minimal edges constitutes a minimal infinite path in B.

The existence of a maximal infinite path is handled similarly. $\square$

Contrary to the question of existence, uniqueness is not automatic:

Example 4.2.8. *Consider the following diagrams:*



Both diagrams are examples of simple, ordered Bratteli diagrams. Yet the diagram at left has two infinite minimal and maximal paths.

The uniqueness question leads to the following definition of a properly ordered Bratteli diagram [2]:

Definition 4.2.9. *A properly ordered Bratteli diagram $B = (V, E, \geq)$ is an ordered Bratteli diagram $B = (V, E, \geq)$, such that (V, E) is simple, and there are unique minimal and maximal infinite paths $y_{min} = (a_1, a_2, \dots)$ and $y_{max} = (b_1, b_2, \dots)$.*

From now on, whenever $B^* = (V^*, E^*, \geq^*)$ is a properly ordered Bratteli diagram, it will be understood that $y^*_{min} = (a^*_1, a^*_2, \dots)$ and $y^*_{max} = (b^*_1, b^*_2, \dots)$ are the unique minimal and maximal paths ("$*$" denotes a symbol or, as in Definition 4.2.9, just empty space).

The following theorem states that up to a telescoping it is always possible to make a simple Bratteli diagram a properly ordered Bratteli diagram:

[2] Called *simple ordered Bratteli diagram* in Giordano et al. [9].

Theorem 4.2.10. *Let $B' = (V', E')$ be a simple Bratteli diagram. Then there exists a telescoping $B = (V, E)$ of B' and an order "$\geq$" on B, such that $(V, E, \geq)$ is a properly ordered Bratteli diagram.*

Proof. Let B be a telescoping of B', such that $|P(v, w)| \geq 1, v \in V_n, w \in V_{n+1}$. It is clear that B is simple. Let us write

$$V_n = \{v(1, n), \ldots, v(k_n, n)\}, n \in \mathbb{N}_0.$$

If $e, e' \in E_n$ satisfies that $r(e) = r(e')$, then $s(e) = v(i, n-1)$ and $s(e') = v(j, n-1)$ for some i and j in $\{1, \ldots, k_{n-1}\}$. Hence we may define

$$e \leq e' \iff i \leq j$$

and, by ordering e and e' arbitrarily if $s(e) = s(e')$, we obtain an order on B.

Let a_n be the minimal edge with $r(a_n) = v(1, n)$, and let b_n be the maximal edge with $r(b_n) = v(k_n, n)$. It follows that $r(a_n) = s(a_{n+1}) = v(1, n)$ and $r(b_n) = s(b_{n+1}) = v(k_n, n)$. Hence it is clear that $(a_1, a_2, \ldots)$ is a minimal path and $(b_1, b_2, \ldots)$ is a maximal path.

To examine uniqueness, let us for instance assume that $(c_1, c_2, \ldots)$ is an infinite minimal path in B, and let $n \in \mathbb{N}$. Then, by construction of the order, c_{n+1} must have $v(1, n)$ as its source. Hence c_n and a_n are comparable, and we conclude that $c_n = a_n$. □

We are now going to introduce an equivalence relation between properly ordered Bratteli diagrams, in a way similar to Definition 4.1.3. First we need to define telescoping of an ordered Bratteli diagram:

Definition 4.2.11. *Let $B = (V, E, \geq)$ be an ordered Bratteli diagram, and assume we have a sequence*

$$m_0 = 0 < m_1 < m_2 < \ldots$$

of integers. We then define the telescoping of B to $(m_n)_{n \in \mathbb{N}_0}$ as the diagram $B' = (V', E', \geq')$, where $V'_n = V_{m_n}, n \in \mathbb{N}_0$, $E'_n = P(V_{m_{n-1}}, V_{m_n}), n \in \mathbb{N}$, and the edges in E'_n are ordered by the lexicographic ordering.

Since a minimal path must consist of minimal edges, we have that any telescoping of a properly ordered Bratteli diagram is again a properly ordered Bratteli diagram. This leads to the following definition:

Definition 4.2.12. *On the class of properly ordered Bratteli diagrams, the symbol $\approx$ will denote the equivalence relation generated by telescoping and isomorphism.*

4.3 From properly ordered Bratteli diagrams to Cantor systems

We will now show how to construct a Cantor system from a properly ordered Bratteli diagram.

Let $B = (V, E)$ be a Bratteli diagram, and let X_B be the set of infinite paths in B:

$$X_B = \{(e_1, e_2, \dots) \in \prod_{i=1}^{\infty} E_i \mid r(e_i) = s(e_{i+1})\}.$$

Put $\tilde{X} = \prod_{i=1}^{\infty} E_i$, and equip E_i with the discrete topology and $\tilde{X}$ with the product topology. Then $\tilde{X}$ is compact because of Tychonoff's Theorem, and the topology is induced by the metric $\tilde{d}$, given by

$$\tilde{d}(x, z) = \begin{cases} 0 & \text{if } x = z \\ \frac{1}{2^n} & \text{if } n \text{ is the least integer such that } x_n \neq z_n \end{cases}$$

($x = (x_n)$ and $z = (z_n)$). We also have that $\tilde{X}$ is totally disconnected. In fact, the topology on $\tilde{X}$ has got a basis of clopen cylinder sets of the form

$$\tilde{U}(e_1, \dots, e_k) = \{x \in \tilde{X} \mid x_i = e_i, i = 1, \dots, k\}.$$

Equip X_B with the inherited topology from $\tilde{X}$. This means, that a basis for the topology on X_B is given by the cylinder sets

$$U(e_1, \dots, e_k) = \{x \in X_B \mid x_i = e_i, i = 1, \dots, k\},$$

where $r(e_i) = s(e_{i+1}), i = 1, \dots, k-1$ (throughout the chapter $U(p), p \in P(V_0, V_k)$, will denote a cylinder set as above).

Theorem 4.3.1. *Let $B = (V, E)$ be a simple Bratteli diagram. Then X_B is a Cantor set.*

Proof. The only thing that remains to be proved is that X_B is a closed subset of $\tilde{X}$ without isolated points.

But X_B is closed, since

$$X_B = \bigcap_{i=1}^{\infty} \{x \in \tilde{X} \mid r(x_i) = s(x_{i+1})\},$$

and each set $\{x \in \tilde{X} \mid r(x_i) = s(x_{i+1})\}$ is a finite union of closed sets of the form $\{x \in \tilde{X} \mid x_i = a, x_{i+1} = b\}$.

Moreover simplicity of B implies that every cylinder set contains an infinite number of points (recall Definition 4.2.1). Hence no point in X_B is isolated. $\square$

We will now construct a minimal homeomorphism λ_B on X_B, in the case where B is a properly ordered Bratteli diagram. First we need a couple of lemmas:

Lemma 4.3.2. *Let $(X_i, d_i), i = 1, 2$, be compact metric spaces, and let $x \in X_1$. Assume $\sigma : X_1 \to X_2$ is bijective and satisfies that $\sigma|_{X_1 \setminus \{x\}}$ is a homeomorphism onto its image. Then σ is a homeomorphism.*

Proof. Let $(x_n) \subseteq X_1 \setminus \{x\}$ be a sequence such that $x_n \to x$, and assume $\sigma(x_n) \not\to \sigma(x)$. Pick a subsequence $(\sigma(x_{n_k})) \subseteq X_2 \setminus \{\sigma(x)\}$ and a point $z \in X_2 \setminus \{\sigma(x)\}$, such that $\sigma(x_{n_k}) \to z$. Using the continuity of the inverse of σ restricted to $X_2 \setminus \{\sigma(x)\}$ and the uniqueness of limit points, one now gets that $x = \sigma^{-1}(z)$, a contradiction. We thus conclude that σ is continuous, and hence σ is a homeomorphism. ☐

Lemma 4.3.3. *Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram. Then*

$$\{U(p) \mid p \in P(v_0, V_n), p \textit{ not maximal }, n \in \mathbb{N}\}$$

is a basis for the topology on $X_B \setminus \{y_{max}\}$. We get a similar result if maximal is replaced by minimal everywhere.

Proof. Let $x = (x_1, x_2, \dots) \neq y_{max}$ and choose $N \in \mathbb{N}$ such that x_N is not maximal. Then

$$\{U(x_1, \dots, x_n) \mid n \geq N\}$$

is a neighborhood basis for x in $X_B \setminus \{y_{max}\}$. Now apply Pedersen [16], Proposition 1.2.12, to get the desired result. The case, where maximal is replaced by minimal everywhere, is handled similarly. ☐

Let us now construct the minimal homeomorphism. Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram, and define

$$\lambda_B(y_{max}) = y_{min}.$$

Then suppose $x = (x_n) \in X_B \setminus \{y_{max}\}$. Choose the least N such that $x_N \notin E_{max}$, and choose a successor e_N to x_N. Going backwards in the diagram, choose minimal edges $e_1, \dots, e_{N-1}$ such that $(e_1, \dots, e_{N-1}, e_N) \in P(V_0, V_N)$, and finally put

$$\lambda_B(x) = (e_1, \dots, e_N, x_{N+1}, \dots).$$

Thus we have a map $\lambda_B : X_B \to X_B$, and it is easy to see that λ_B is bijective (the inverse is defined in a similar way).

Moreover λ_B satisfies that $\lambda_B(U(p)) = U(q)$, when p is a non-maximal path and q is the successor of p. This gives a bijective correspondence between sets in

$$\{U(p) \mid p \text{ is not maximal}\}$$

and

$$\{U(q) \mid q \text{ is not minimal}\},$$

and it follows by Lemma 4.3.3 that the map $\lambda_B|_{X_B \setminus \{y_{max}\}}$ is a homeomorphism from $X_B \setminus \{y_{max}\}$ to $X_B \setminus \{y_{min}\}$. Thus Lemma 4.3.2 implies that λ_B is a homeomorphism. We have the following theorem:

Theorem 4.3.4. *(X_B, λ_B) is a Cantor system, and λ_B satisfies $\lambda_B(U(p)) = U(q)$, whenever p is a non-maximal path, and q is the successor of p in the lexicographic order.*

Proof. We only need to prove that λ_B is minimal. Hence consider a cylinder set $U(e_1, \dots, e_N)$ in X_B, and let $x \in X_B$. We must prove that the orbit of x intersects $U(e_1, \dots, e_N)$.

By simplicity of the diagram there exists an $M > N$, such that $r(e_N)$ is connected to every vertex at level M. In particular we may choose edges $e_{N+1}, \dots, e_M$, such that $(e_{N+1}, \dots, e_M)$ is a path from $r(e_N)$ to $s(x_{M+1})$. Now

$$z = (e_1, \dots, e_M, x_{M+1}, \dots)$$

is an infinite path in $U(e_1, \dots, e_N)$, and it follows that $\lambda_B^k(x) = z$ for some $k \in \mathbb{Z}$ (see also Remark 4.3.6 below). ☐

Systems of the form (X_B, λ_B), where B is a properly ordered Bratteli diagram, are called Bratteli-Vershik systems. We need to introduce some convenient notation:

Definition 4.3.5. *Let $B = (V, E)$ be a Bratteli diagram and let $x, z \in X_B$: Thus $x = (x_1, x_2, \dots)$ and $z = (z_1, z_2, \dots)$. We then define*

$$x_{[j;k]} = (x_j, \dots, x_k), 1 \leq j < k < \infty,$$

and $x_{[j;\infty[} = (x_j, x_{j+1}, \dots)$, $j \in \mathbb{N}$. Moreover we say that x and z are cofinal *(or x and z* have the same tail*), if $x_{[N;\infty[} = z_{[N;\infty[}$ for some $N \in \mathbb{N}$.*

We have the following important remark concerning the orbit of λ_B:

Remark 4.3.6. *Let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram and let $x = (x_1, x_2, \dots) \in X_B$. If x is not cofinal with y_{max} or y_{min}, then*

$$\mathrm{orbit}_{\lambda_B}(x) = \{z \in X_B \mid \exists N \in \mathbb{N} : x_{[N;\infty[} = z_{[N;\infty[}\}.$$

In other words, $\mathrm{orbit}_{\lambda_B}(x)$ *is the set of infinite paths in X_B which are cofinal with x (if x is cofinal with y_{max} or y_{min}, then* $\mathrm{orbit}_{\lambda_B}(x)$ *is the set of infinite paths in X_B which are cofinal with y_{max}* or *y_{min}). Indeed this follows directly from the definition of λ_B.*

We now state a fundamental theorem regarding the correspondence between properly ordered Bratteli diagrams and pointed Cantor systems. It will follow from the theorem that, up to pointed topological conjugacy, every Cantor system is a Bratteli-Vershik system (recall Definition 1.3.1).

Theorem 4.3.7. *There is a bijective correspondence between equivalence classes of properly ordered Bratteli diagrams and equivalence classes of pointed Cantor systems (with the equivalence relations "$\approx$" and "pointed topologically conjugate", respectively), induced by the map*

$$B = (V, E, \geq) \mapsto (X_B, \lambda_B, y_{max}).$$

Proof. For a proof of the theorem, we refer the reader to Herman et al. [11], Theorem 4.7[3]. For a sketch of the proof, see also Durand et al. [5], Theorem 5. We will only sketch how to obtain a properly ordered Bratteli diagram from a pointed Cantor system.

If (X, σ, y) is a pointed Cantor system, then construct a sequence $(\mathcal{P}_n)_{n \in \mathbb{N}_0}$ of KR-partitions with top floor Y_n, such that

[3] Herman et al. study a slightly more general class of systems in [11].

1. $\mathcal{P}_0 = \{X\}$,
2. $\forall n \in \mathbb{N}_0 : \mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$,
3. $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ and $\cap_{n=1}^{\infty} Y_n = \{y\}$, and
4. $\cup_{n=0}^{\infty} \mathcal{P}_n$ generates the topology on X.

(this is not so hard, using Lemma 3.1.1 and Remark 3.1.2). Now define a Bratteli diagram as follows: Begin by defining the vertices at level n to be the towers in the n'th KR-partition $\mathcal{P}_n$.

Then let T' be a tower at level $n+1$. Starting at the first floor and going up through the tower, T' traverses towers at level n (this follows from property 2 and 3 above, and the properties listed in Lemma 3.1.1). Draw an edge from T' to each traversed tower, and order the edges according to the order at which the corresponding towers are traversed.

Let us prove that the diagram is simple[4]. Let $m \in \mathbb{N}$, and pick $K \in \mathbb{N}$ such that, for every clopen set in $\mathcal{P}_m$, at least two of the points $y, \ldots, \sigma^{-K}(y)$ belong to the set (this is possible, since $\text{orbit}_\sigma(y)$ is dense and $\mathcal{P}_m$ contains a finite number of clopen sets). Then choose a clopen set U such that $y \in U$, $\sigma^{-j}(U)$ is a subset of a set from $\mathcal{P}_m$ for every $j \in \{0, 1, \ldots, K\}$, and $U, \ldots, \sigma^{-K}(U)$ are disjoint (this is similar to Remark 3.1.2), and finally choose $n > m$ such that $Y_n \subseteq U$. It now follows that every vertex at level m is connected to every vertex at level n by at least two paths. Thus the diagram is simple.

Let us finally note that it follows by a computation that the resulting Bratteli diagram is in fact properly ordered (see Herman et al. [11], p. 842). $\square$

The importance of Theorem 4.3.7 lies in the fact that it allows us to investigate abstract Cantor systems by investigating the very concrete Bratteli-Vershik systems. An example of that is given in Theorem 4.3.9 below. First we need a lemma:

Lemma 4.3.8. *Suppose (X_1, σ_1) and (X_2, σ_2) are Cantor systems, which are conjugate. Then $K_0(X_1, \sigma_1)$ is order isomorphic to $K_0(X_2, \sigma_2)$ by a map preserving the distinguished order units.*

Proof. Let $F : X_1 \to X_2$ be a homeomorphism satisfying $F \circ \sigma_1 = \sigma_2 \circ F$. Define $\Psi : K_0(X_1, \sigma_1) \to K_0(X_2, \sigma_2)$ by

$$\Psi([f]) = [f \circ F^{-1}], f \in C(X_1, \mathbb{Z}).$$

Then it is easy to check, that Ψ is a well-defined group isomorphism, $\Psi(K_0^+(X_1, \sigma_1)) = K_0^+(X_2, \sigma_2)$, and $\Psi([1_{X_1}]) = [1_{X_2}]$. $\square$

Theorem 4.3.9. *Let (X, σ) be a Cantor system, let $x \in X$ be arbitrary, and let $B = (V, E, \geq)$ be a properly ordered Bratteli diagram corresponding to (X, σ, x). Then $K_0(X, \sigma) \cong K_0(V, E)$ as ordered groups by a map preserving the distinguished order units.*

[4] This will supplement the proof in Herman et al. [11].

Proof. We will give a brief sketch of the detailed proof in Johansen [12], Theorem 4.10 (see also Herman et al. [11], Theorem 5.4). By Theorem 4.1.5, Theorem 4.3.7 and Lemma 4.3.8 we may assume $(X, \sigma, x) = (X_B, \lambda_B, y_{max})$.

The idea is now to identify elements in $\mathbb{Z}^{|V_n|}$ with vectors

$$\vec{f_n} = (f_v)_{v \in V_n}, f \in C(X_B, \mathbb{Z}), n \geq \mathrm{lev}(f),$$

where $\mathrm{lev}(f)$ is the least integer such that the continuous function f is constant on each cylinder set $U(p), p \in P(v_0, V_{\mathrm{lev}(f)})$, and

$$f_v = \sum_{p \in P(v_0, v)} f(U(p)).$$

The sum should be interpreted as a sum of integers: The constant value of f on each cylinder set $U(p)$.

We now define a map $\gamma_n : \mathbb{Z}^{|V_n|} \to K_0(X_B, \lambda_B)$ by $\gamma_n(\vec{f_n}) = [f]$, and it follows by a computation that γ_n is a positive well-defined homomorphism.

Finally one checks that $(K_0(X_B, \lambda_B), \{\gamma_n\}_{n=0}^{\infty})$ is an inductive limit of the inductive sequence from Remark 4.1.4.

Johansen does not comment upon order units, but if we use the notation from Remark 4.1.4, it is clear that the canonical isomorphism from $K_0(V, E)$ to $K_0(X_B, \lambda_B)$ takes $\eta_0(1)$ to

$$\gamma_0(1) = \gamma_0((\vec{1_{X_B}})_0) = [1_{X_B}],$$

as desired. □

4.4 The isomorphism problem

The main theorem in this chapter is Theorem 4.4.1 due to Giordano et al. [9]. The proof was subsequently shortened in Glasner et al. [10].

Theorem 4.4.1. *Suppose (X_1, σ_1) and (X_2, σ_2) are Cantor systems. Then the following three conditions are equivalent:*

1. *(X_1, σ_1) and (X_2, σ_2) are strong orbit equivalent,*
2. *$K_0(X_1, \sigma_1)$ is order isomorphic to $K_0(X_2, \sigma_2)$ by a map preserving the distinguished order units,*
3. *$C(X_1) \rtimes_{\sigma_1} \mathbb{Z}$ and $C(X_2) \rtimes_{\sigma_2} \mathbb{Z}$ are *-isomorphic.*

Proof. "2 $\iff$ 3": Follows from Theorem 2.4.3 and Corollary 3.3.7.
"1 $\Rightarrow$ 2": Assume (X_1, σ_1) and (X_2, σ_2) are strong orbit equivalent and let $F : X_1 \to X_2$ be an orbit map, such that the associated cocycles $m, n : X_1 \to \mathbb{Z}$ (defined as in Definition 1.3.1) has at most one point of discontinuity each. Put $\tilde{\sigma}_2 = F^{-1} \circ \sigma_2 \circ F$. Then $(X_1, \tilde{\sigma}_2)$ is a Cantor system, since

$$\overline{\mathrm{orbit}_{\tilde{\sigma}_2}(x)} = \overline{F^{-1}(\mathrm{orbit}_{\sigma_2}(F(x)))} = F^{-1}(\overline{\mathrm{orbit}_{\sigma_2}(F(x))}) = X_1, \ x \in X_1.$$

Moreover we have that $\sigma_1^{m(x)}(x) = \tilde{\sigma}_2(x)$ and $\sigma_1(x) = \tilde{\sigma}_2^{n(x)}(x)$, $x \in X_1$.

By Lemma 4.3.8 we may therefore assume $X = X_1 = X_2$, and thus we have that $\sigma_i : X \to X, i = 1, 2$, satisfies $\sigma_1^{m(x)}(x) = \sigma_2(x)$ and $\sigma_1(x) = \sigma_2^{n(x)}(x)$ for $x \in X$.

Let us denote the discontinuity point of m and n by x_m and x_n respectively. By Definition 2.4.1, it is now enough to prove that

$$\delta_{\sigma_1} C(X, \mathbb{Z}) = \delta_{\sigma_2} C(X, \mathbb{Z}).$$

"$\subseteq$": Let $f \in C(X, \mathbb{Z})$; we must prove that $f - f \circ \sigma_1^{-1} \in \delta_{\sigma_2} C(X, \mathbb{Z})$. Since f is a finite sum of characteristic functions and $\delta_{\sigma_2} C(X, \mathbb{Z})$ is a subgroup, we may assume that $f = 1_E$ where E is clopen. There are now two possibilities:

$x_n \notin E$: Continuity of n on the compact set E implies $n(E) = \{n_1, \ldots, n_r\}$ for some integers $n_1, \ldots, n_r$. Put

$$E_k = \{x \in E \mid n(x) = n_k\}, k = 1, \ldots, r,$$

and note that E is the disjoint union of the clopen sets $E_1, \ldots, E_r$. Hence

$$1_E = \sum_{k=1}^{r} 1_{E_k}$$

and

$$1_{\sigma_1(E)} = \sum_{k=1}^{r} 1_{\sigma_1(E_k)} = \sum_{k=1}^{r} 1_{\sigma_2{}^{n_k}(E_k)}.$$

Since

$$\begin{aligned} 1_{E_k} - 1_{\sigma_2^{n_k}(E_k)} &= (1_{E_k} - 1_{\sigma_2(E_k)}) + \cdots + (1_{\sigma_2^{n_k-1}(E_k)} - 1_{\sigma_2^{n_k}(E_k)}) \\ &= (1_{E_k} - 1_{E_k} \circ \sigma_2^{-1}) + \cdots + (1_{\sigma_2^{n_k-1}(E_k)} - 1_{\sigma_2^{n_k-1}(E_k)} \circ \sigma_2^{-1}) \end{aligned}$$

if n_k is positive, and

$$1_{E_k} - 1_{\sigma_2^{n_k}(E_k)} = -(1_{\sigma_2^{n_k}(E_k)} - 1_{\sigma_2^{-n_k}(\sigma_2^{n_k}(E_k))}),$$

it follows that $1_{E_k} - 1_{\sigma_2^{n_k}(E_k)} \in \delta_{\sigma_2} C(X, \mathbb{Z}), 1 \leq k \leq r$. Hence

$$1_E - 1_{\sigma_1(E)} \in \delta_{\sigma_2} C(X, \mathbb{Z}).$$

$x_n \in E$: Then the equality $1_E - 1_{\sigma_1(E)} = -(1_{\complement E} - 1_{\sigma_1(\complement E)})$ reduces the second case to the first case.

"$\supseteq$": A similar argument, this time using the continuity of m.

"$2 \Rightarrow 1$": Assume $K_0(X_1, \sigma_1)$ and $K_0(X_2, \sigma_2)$ are order isomorphic by a map preserving the distinguished order units, and choose arbitrary points $x_1 \in X_1$ and $x_2 \in X_2$.

Choose properly ordered Bratteli diagrams $B^i = (V^i, E^i, \geq^i)$ corresponding to (X_i, σ_i, x_i), $i = 1, 2$ (see Theorem 4.3.7), and note that the assumption and Theorem 4.3.9 implies that $K_0(V^1, E^1) \cong K_0(V^2, E^2)$ as ordered groups with distinguished order units.

This means by Theorem 4.1.5 that $(V^1, E^1) \sim (V^2, E^2)$. By Theorem 4.2.5 we may therefore pick a simple (not ordered) Bratteli diagram $B = (V, E)$, such that a telescoping of B to $0 < 1 < 3 < \ldots$ gives us a telescoping of (V^1, E^1), and a telescoping of B to $0 < 2 < 4 < \ldots$ gives us a telescoping of (V^2, E^2).

Simplicity of B ensures, that we may assume

$$|P(v_n, v_{n+1})| \geq 2, n \in \mathbb{N}_0, v_n \in V_n, v_{n+1} \in V_{n+1}$$

(see Remark 4.2.2). Moreover, since telescoping of properly ordered Bratteli diagrams makes us stay inside our equivalence class (see the comment just before Definition 4.2.12), we may as well assume that the telescopings of B in fact give us (V^1, E^1) and (V^2, E^2).

As usual, let X_B be the Cantor set constructed from B, X_{B^i} the Cantor sets constructed from B^i, and let the unique minimal and maximal elements in B^1 and B^2 be denoted by $y^i_{min} = (a^i_1, a^i_2, \ldots)$ and $y^i_{max} = (b^i_1, b^i_2, \ldots)$ for $i = 1, 2$.

We now construct an orbit map F from (X_{B^1}, λ_{B^1}) to (X_{B^2}, λ_{B^2}), such that the associated orbit cocycles have at most one point of discontinuity each: Then, by Theorem 1.3.2, (X_1, σ_1) and (X_2, σ_2) are strong orbit equivalent. We will define F as $F = H^{-1} \circ G$, where $G : X_{B^1} \to X_B$ and $H : X_{B^2} \to X_B$ are homeomorphisms. We now proceed to construct G and H.

Let us start by choosing a bijective map $G_1 : E^1_1 \to E_1 = P(V_0, V_1)$ and put $a_1 = G_1(a^1_1)$ and $b_1 = G_1(b^1_1)$.

Now use the assumptions about B to pick $a_2 \in E_2$, such that $r(a_2) = r(a^2_1)$ and $s(a_2) = r(a_1)$. Also find $b_2 \in E_2 \setminus \{a_2\}$, such that $r(b_2) = r(b^2_1)$ and $s(b_2) = r(b_1)$, and define $H_1(a^2_1) = (a_1, a_2)$ and $H_1(b^2_1) = (b_1, b_2)$. Extend this map to a bijective map

$$H_1 : E^2_1 \to P(V_0, V_2)$$

respecting the range and source maps (i.e. $s(e) = s(H_1(e))$ and $r(e) = r(H_1(e))$, $e \in E^2_1$).

Now choose $a_3 \in E_3$, such that $r(a_3) = r(a^1_2)$ and $s(a_3) = r(a_2)$. Also find $b_3 \in E_3 \setminus \{a_3\}$, such that $r(b_3) = r(b^1_2)$ and $s(b_3) = r(b_2)$, and define $G_2(a^1_2) = (a_2, a_3)$ and $G_2(b^1_2) = (b_2, b_3)$. Extend G_2 to a bijective map

$$G_2 : E^1_2 \to P(V_1, V_3)$$

respecting the range and source maps.

Proceeding by induction we get bijective maps

$$G_n : E^1_n \to P(V_{2n-3}, V_{2n-1})$$

and

$$H_n : E^2_n \to P(V_{2n-2}, V_{2n}),$$

and elements $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ in X_B, such that G_n and H_n respect the range and source maps, and $G_n(a_n^1) = (a_{2n-2}, a_{2n-1})$, $G_n(b_n^1) = (b_{2n-2}, b_{2n-1})$, $H_n(a_n^2) = (a_{2n-1}, a_{2n})$, and $H_n(b_n^2) = (b_{2n-1}, b_{2n})$.

Now define $G : X_{B^1} \to X_B$ by

$$G(x_1^1, x_2^1, \dots) = (G_1(x_1^1), G_2(x_2^1), \dots),\ (x_1^1, x_2^1, \dots) \in X_{B^1},$$

and similarly define $H : X_{B^2} \to X_B$ by

$$H(x_1^2, x_2^2, \dots) = (H_1(x_1^2), H_2(x_2^2), \dots)\ (x_1^2, x_2^2, \dots) \in X_{B^2}.$$

G and H are clearly bijective maps. Moreover

$$G(U(x_1^1, \dots, x_n^1)) = U(G_1(x_1^1), \dots, G_n(x_n^1))$$

and

$$H(U(x_1^2, \dots, x_n^2)) = U(H_1(x_1^2), \dots, H_n(x_n^2))$$

showing that G and H are homeomorphisms.

Put $F = H^{-1} \circ G$. It follows directly from the definition of G and H that F has the following nice properties (recall Definition 4.3.5):

1. $x_{[1;k]} = z_{[1;k]} \Rightarrow F(x)_{[1;k-1]} = F(z)_{[1;k-1]}, x, z \in X_{B^1}$,
2. $x_{[k;\infty[} = z_{[k;\infty[} \Rightarrow F(x)_{[k;\infty[} = F(z)_{[k;\infty[}, x, z \in X_{B^1}$,
3. $x_{[1;k]} = z_{[1;k]} \Rightarrow F^{-1}(x)_{[1;k]} = F^{-1}(z)_{[1;k]}, x, z \in X_{B^2}$,
4. $x_{[k;\infty[} = z_{[k;\infty[} \Rightarrow F^{-1}(x)_{[k+1;\infty[} = F^{-1}(z)_{[k+1;\infty[}, x, z \in X_{B^2}$,
5. $F(y_{min}^1) = y_{min}^2$ and $F(y_{max}^1) = y_{max}^2$.

If x and z are two arbitrary points in X_{B^1}, then, by property 2 and 4, x and z are cofinal if and only if $F(x)$ and $F(z)$ are cofinal. By Remark 4.3.6 and property 5 we therefore conclude

$$F(\mathrm{orbit}_{\lambda_{B^1}}(x)) = \mathrm{orbit}_{\lambda_{B^2}}(F(x)), x \in X_{B^1}.$$

Thus F is an orbit map from (X_{B^1}, λ_{B^1}) to (X_{B^2}, λ_{B^2}).

Let $m, n : X_{B^1} \to \mathbb{Z}$ be the associated orbit cocycles:

$$F(\lambda_{B^1}^{m(x)}(x)) = \lambda_{B^2}(F(x)), x \in X_{B^1},$$

and

$$F(\lambda_{B^1}(x)) = \lambda_{B^2}^{n(x)}(F(x)), x \in X_{B^1}.$$

Let us now show that n is continuous on $X_{B^1} \setminus \{y_{max}^1\}$. Hence let $x \neq y_{max}^1$, pick the least k such that $x_k \notin E_{max}^1$, and put

$$U = U(x_1, \dots, x_{k+1}).$$

We want to prove that n is constant on U, that is, $n(x) = n(z)$ for an arbitrary $z \in U$: Therefore let $z \in U$. We now make three observations:

For the first, $x_{[1;k+1]} = z_{[1;k+1]}$ and $z_k = x_k \notin E^1_{max}$ implies that $\lambda_{B^1}(x)_{[1;k+1]} = \lambda_{B^1}(z)_{[1;k+1]}$, whence $F(\lambda_{B^1}(x))_{[1;k]} = F(\lambda_{B^1}(z))_{[1;k]}$.

For the second, $x_{[1;k+1]} = z_{[1;k+1]}$ implies that $F(x)_{[1;k]} = F(z)_{[1;k]}$.

For the third, $z_k = x_k \notin E^1_{max}$ implies that $z_{[k+1;\infty[} = \lambda_{B^1}(z)_{[k+1,\infty[}$, whence $F(z)_{[k+1;\infty[} = F(\lambda_{B^1}(z))_{[k+1,\infty[}$. In particular $F(z)$ and $F(\lambda_{B^1}(z))$ go through the same vertex in V_k. Similarly $F(x)$ and $F(\lambda_{B^1}(x))$ go through the same vertex in V_k, and since $F(x)_{[1;k]} = F(z)_{[1;k]}$ we conclude that $F(x)$, $F(\lambda_{B^1}(x))$, $F(z)$ and $F(\lambda_{B^1}(z))$ all go through the same vertex in V_k.

Using these three observations it now follows from the definition of λ_{B^2}, that the (positive or negative) number of iterates it takes to go from $F(x)$ to $F(\lambda_{B^1}(x))$ is equal to the number of iterates it takes to go from $F(z)$ to $F(\lambda_{B^1}(z))$: Thus $n(x) = n(z)$ as desired.

Let us now consider the other orbit cocycle m. We have that

$$\lambda_{B^1}^{m\circ F^{-1}(x)}(F^{-1}(x)) = F^{-1}(\lambda_{B^2}(x)), x \in X_{B^2},$$

and hence an argument similar to the proof of the continuity of n on $X_{B^1} \setminus \{y^1_{max}\}$ gives that $m \circ F^{-1}$ is continuous on $X_{B^2} \setminus \{y^2_{max}\}$. We therefore get that m is continuous on $X_{B^1} \setminus \{y^1_{max}\}$, and thus the proof is complete. $\square$

Recall the notion of a simple dimension group (see Effros [6] for definitions and a characterization in terms of Riesz' interpolation property). We have the following well-known theorem:

Theorem 4.4.2. *Let (G, G^+, u) be an ordered group different from $\mathbb{Z}$ with order unit u. Then (G, G^+, u) is a simple dimension group, if and only if $(G, G^+, u) \cong K_0(V, E)$ for some simple Bratteli diagram (V, E).*

It follows from Theorem 4.2.10, Theorem 4.3.9, and Theorem 4.4.2, that every simple dimension group different from $\mathbb{Z}$ with distinguished order unit may be obtained as $K_0(C(X) \rtimes_\sigma \mathbb{Z})$ for some Cantor system (X, σ). In particular we have the following remark:

Remark 4.4.3. *Recall the definition of a UHF-algebra (see Rørdam et al. [22], p. 125). A supernatural number is a formal product $n = \prod_{j=1}^{\infty} p_j^{n_j}$, where $n_j \in \{1, 2, \ldots, \infty\}$ specifies the multiplicity of the j'th prime number p_j in the factorization of n. We may also write n as $n = \prod_{j=1}^{\infty} k_j$, where each k_j is an integer, and thus we may define a UHF-algebra $\mathcal{A}_n$ by the inductive limit of the sequence*

$$\mathbb{C} \xrightarrow{\varphi_1} M_{k_1}(\mathbb{C}) \xrightarrow{\varphi_2} M_{k_1 k_2}(\mathbb{C}) \longrightarrow \cdots$$

where φ_j is the canonical embedding with multiplicity k_j. It is proved in Rørdam et al. [22], that the definition does not depend on the choice of factors $k_1, k_2, \ldots$, and that we have a bijective correspondence between supernatural numbers n and (isomorphism classes of) UHF-algebras $\mathcal{A}_n$. The corresponding Bratteli diagram $B^n = (V^n, E^n)$ is given by

$$\begin{tikzcd}
\bullet \arrow[d, "k_1 \ edges"] \\
\bullet \arrow[d, "k_2 \ edges"] \\
\bullet \arrow[d] \\
\vdots
\end{tikzcd}$$

Choosing any order "$\geq^n$" on the sets $E_k^n, k \in \mathbb{N}$, it is clear that B^n becomes a properly ordered Bratteli diagram if n is not an integer (recall that an infinite number of the sets E_k^n should have at least 2 elements for the Bratteli diagram to be simple).

The associated minimal homeomorphism λ_{B^n} is often called an odometer map, since its action resembles the one of an odometer in a car. By Rørdam et al. [22], Theorem 7.4.5, and the theorems above, we have for supernatural numbers m and n:

$$\begin{aligned}
m = n \quad &\iff \quad \mathcal{A}_m \cong \mathcal{A}_n \\
&\iff \quad K_0(V^m, E^m) \cong K_0(V^n, E^n) \\
&\iff \quad K_0(X_{B^m}, \lambda_{B^m}) \cong K_0(X_{B^n}, \lambda_{B^n}) \\
&\iff \quad (X_{B^m}, \lambda_{B^m}) \text{ and } (X_{B^n}, \lambda_{B^n}) \text{ are strong orbit equivalent.}
\end{aligned}$$

(where the group isomorphisms should be unit and order preserving). In particular Rørdam et al. [22], Example 7.4.6, implies, that there is an uncountable family of pair-wise non-isomorphic equivalence classes with regard to any of the three relations from Theorem 4.4.1.

Let us finally mention the set $\mathbb{Z}_p$ of p-adic integers endowed with the homeomorphism $\sigma_p(x) = x + 1, x \in \mathbb{Z}_p$ (see Example 1.2.8). We have a canonical conjugacy $F : (X_{B^n}, \lambda_{B^n}) \to (\mathbb{Z}_p, \sigma_p)$, $n = p^\infty$, and it follows that the following three conditions are equivalent for primes p and q:

1. $(\mathbb{Z}_p, \sigma_p)$ and $(\mathbb{Z}_q, \sigma_q)$ are conjugate,

2. $(\mathbb{Z}_p, \sigma_p)$ and $(\mathbb{Z}_q, \sigma_q)$ are strong orbit equivalent,

3. $p = q$.

Chapter 5

The Stable Rank of the Crossed Product

Chapter 5 is devoted to a study of the stable rank of $C(X) \rtimes_\sigma \mathbb{Z}$, when (X, σ) is a Putnam system.

The concept of stable rank was introduced by Rieffel in [21], and it follows from Rieffel [21], Theorem 7.1, that the stable rank of $C(X) \rtimes_\sigma \mathbb{Z}$ is either one or two.

Putnam gave in [19] a necessary and sufficient condition for the stable rank to be one, under the assumption that X has no clopen strongly σ-invariant subsets (see Theorem 5.5.6 below). This is a mild restriction (see Lemma 5.1.3 and Theorem 5.3.7 below). Yet it is possible to find an example, which is not covered by these theorems (see Example 5.6.7). In Theorem 5.6.6 we generalize Theorem 5.5.6, proving a necessary and sufficient condition for the stable rank to be one[1].

5.1 Stable rank of a C*-algebra

As already mentioned the concept of stable rank (in short: sr) of a C*-algebra was introduced by Rieffel in [21]. We will only study the concept of stable rank one, and for that purpose the following definition will be sufficient:

Definition 5.1.1. *Let $\mathcal{A}$ be a unital C*-algebra. Then*

$$\mathrm{sr}(\mathcal{A}) = 1 \iff \overline{GL_1(\mathcal{A})} = \mathcal{A}.$$

We need a wide class of C*-algebras having stable rank one:

Lemma 5.1.2. $\mathrm{sr}(C(\mathbb{T})) = 1$ *and* $\mathrm{sr}(\mathbb{C}) = 1$.

Proof. It is obvious that $\mathrm{sr}(\mathbb{C}) = 1$.

Now, suppose that $f \in C(\mathbb{T})$. Invoking the Stone-Weierstrass theorem we may assume that f has a finite number of zeros $z_1, \ldots, z_n$ (approximate f by finite Laurent polynomials, and observe that a finite Laurent polynomial only has a finite number of zeros on $\mathbb{T}$).

[1] The theorem was conjectured to the author by Putnam during the author's stay at MSRI, Berkeley, in the autumn 2000.

Let $\epsilon > 0$, pick disjoint, connected neighborhoods $D_1, \ldots, D_n$ around $z_1, \ldots, z_n$ such that $f(D_i) \subseteq D(0, \epsilon)$, and on each of the connected pieces $D_1, \ldots, D_n$ of $\mathbb{T}$ substitute f with a continuous function on D_i with values in the "punctured" disc $D(0, \epsilon) \setminus \{0\}$. $\square$

Lemma 5.1.3. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be n unital C*-algebras. Then*

$$\operatorname{sr}(\mathcal{A}_1) = \cdots = \operatorname{sr}(\mathcal{A}_n) = 1 \iff \operatorname{sr}(\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n) = 1.$$

Proof. "$\Rightarrow$": Let $a = (a_1, \ldots, a_n)$ be an element in $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ and let $\epsilon > 0$. Pick invertible elements $b_i \in \mathcal{A}_i$ such that

$$||a_i - b_i|| < \epsilon, i = 1, \ldots, n,$$

and note that $b = (b_1, \ldots, b_n)$ is an invertible element satisfying $||a - b|| < \epsilon$.
"$\Leftarrow$": A similar argument. $\square$

If $\mathcal{A}$ is a unital C*-algebra then $\tilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$. Thus it follows from Lemma 5.1.3 that $\operatorname{sr}(\mathcal{A}) = 1$ if and only if $\operatorname{sr}(\tilde{\mathcal{A}}) = 1$.

Lemma 5.1.4. *For any unital C*-algebra $\mathcal{A}$ and any $n \in \mathbb{N}$, we have that*

$$\operatorname{sr}(\mathcal{A}) = 1 \iff \operatorname{sr}(M_n(\mathcal{A})) = 1.$$

Proof. The proof involves only standard facts about matrix algebras; we omit the proof and refer the reader to Davidson [4], Theorem VIII.8.7, or Rieffel [21], Theorem 3.3. $\square$

We will need the following neat description of the unitization of a C*-algebra:

Lemma 5.1.5. *Suppose $\mathcal{B}$ is a unital C*-algebra, and $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ such that $1_{\mathcal{B}} \notin \mathcal{A}$. Then $\mathcal{A} + \mathbb{C}1_{\mathcal{B}} \cong \tilde{\mathcal{A}}$. In particular*

$$\widetilde{C_0(Z)} \cong C_0(Z) + \mathbb{C}1_X \subseteq C(X)$$

whenever Z is a proper open subset of a compact Hausdorff space X.

Proof. Define $\psi : \tilde{\mathcal{A}} \to \mathcal{B}$ by $\psi(a, \lambda) = a + \lambda 1_{\mathcal{B}}, (a, \lambda) \in \tilde{\mathcal{A}}$. It is easy to check that ψ is an injective *-homomorphism; thus $\tilde{\mathcal{A}} \cong \operatorname{Im}\psi = \mathcal{A} + \mathbb{C}1_{\mathcal{B}}$. $\square$

We finally obtain two large classes of C*-algebras having stable rank one:

Theorem 5.1.6. *Let $\mathcal{A}$ be a C*-algebra.*

1. *If $\mathcal{A}$ is finite dimensional, then* $\operatorname{sr}(\mathcal{A}) = 1$.

2. *If $\mathcal{A}$ is an AF-algebra, then* $\operatorname{sr}(\tilde{\mathcal{A}}) = 1$.

Proof. 1: If $\mathcal{A}$ is finite dimensional, then $\mathcal{A}$ is isomorphic to a direct sum of matrix algebras. Hence the result follows from Lemma 5.1.2, Lemma 5.1.3, and Lemma 5.1.4.
2: Suppose $(\mathcal{A}_n)$ is an increasing sequence of finite dimensional C*-algebras and $\mathcal{A} = \overline{\cup_{n=1}^{\infty}\mathcal{A}_n}$.

Then it is easy to see that $\tilde{\mathcal{A}} = \overline{\cup_{n=1}^{\infty}(\mathcal{A}_n + \mathbb{C}1_{\tilde{\mathcal{A}}})}$, and it follows by Lemma 5.1.5 that $\mathcal{A}_n + \mathbb{C}1_{\tilde{\mathcal{A}}}$ is a unital C*-algebra and $\mathrm{sr}(\mathcal{A}_n + \mathbb{C}1_{\tilde{\mathcal{A}}}) = \mathrm{sr}(\tilde{\mathcal{A}}_n) = \mathrm{sr}(\mathcal{A}_n) = 1$. Also observe that $GL_1(\mathcal{A}_n + \mathbb{C}1_{\tilde{\mathcal{A}}}) \subseteq GL_1(\tilde{\mathcal{A}})$.

Now let $b \in \tilde{\mathcal{A}}$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ and $b' \in \mathcal{A}_n + \mathbb{C}1_{\tilde{\mathcal{A}}}$ such that $||b' - b|| < \epsilon/2$, choose $b'' \in GL_1(\mathcal{A}_n + 1_{\tilde{\mathcal{A}}})$ such that $||b'' - b'|| < \epsilon/2$, and note that $||b'' - b|| < \epsilon$. □

We now turn to investigate the stable rank of the crossed products $C(X) \rtimes_\sigma \mathbb{Z}$, where (X, σ) is a Putnam system. We begin with the case where (X, σ) is minimal.

5.2 The minimal case

If (X, σ) is a minimal Putnam system, by Remark 1.2.10 there are two possibilities: Either (X, σ) is a Cantor system or X is finite. We will now prove, that in each of the two cases $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$.

Theorem 5.2.1. *Let (X, σ) be a Cantor system. Then* $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$.

Proof. Let $a \in C(X) \rtimes_\sigma \mathbb{Z}$ and $\epsilon > 0$, and pick an algebra $\mathcal{A}$ of the form in Corollary 3.3.2 and $a' \in \mathcal{A}$, such that $||a - a'|| < \frac{\epsilon}{2}$. By Lemma 5.1.2, Lemma 5.1.3 and Lemma 5.1.4 we conclude that $\mathcal{A}$ has stable rank one. Hence it is possible to find an invertible element $a'' \in \mathcal{A}$ such that $||a' - a''|| < \frac{\epsilon}{2}$. Now observe that

$$||a - a''|| \leq ||a - a'|| + ||a' - a''|| < \epsilon.$$

□

Let us now turn to the case where X is finite. We will need the following lemma:

Lemma 5.2.2. *Let $p(z) = \sum_{j=-K}^{K} \lambda_j z^j$, $z \neq 0$ be a Laurent polynomial. Then, for any $j \in \{-K, \ldots, K\}$,*

$$\sup_{z \in \mathbb{T}} |p(z)| \geq |\lambda_j|.$$

Proof. The Residue Theorem (see Rudin [23]) implies that

$$\int_{|z|=1} p(z) \frac{1}{z^{j+1}} dz = 2\pi i \lambda_j.$$

Now, using a well-known norm estimate for complex integrals, we get that

$$2\pi|\lambda_j| \leq 2\pi \sup_{|z|=1} |\frac{p(z)}{z^{j+1}}| = 2\pi \sup_{|z|=1} |p(z)|,$$

as desired. □

Theorem 5.2.3. *Let X be a finite set endowed with the discrete topology, and let σ be a minimal homeomorphism on X. Then $C(X) \rtimes_\sigma \mathbb{Z} \cong M_n(C(\mathbb{T}))$, where n is the number of elements in X.*

Proof. Since σ is minimal, σ is forced to be a cyclic permutation of the elements in X. We may therefore assume that $X = \{x_1, \ldots, x_n\}$, and σ is the shift map:

$$\sigma(x_i) = \begin{cases} x_{i+1} & i < n \\ x_1 & i = n. \end{cases}$$

Let $(e_{ij})_{1 \leq i,j \leq n}$ be the matrix units in $M_n(\mathbb{C})$, and let us (with a slight abuse of notation) identify e_{ij} with $1 \otimes e_{ij} \in M_n(C(\mathbb{T})) = C(\mathbb{T}) \otimes M_n(\mathbb{C})$. Define a map $\pi : C(X) \to M_n(C(\mathbb{T}))$ by

$$\pi(f) = \sum_{i=1}^{n} f(x_i) e_{ii}, \;\; f \in C(X).$$

It is easy to see that this definition makes π a *-homomorphism. Put

$$v = e_{1n} + z \otimes e_{21} + e_{32} + \cdots + e_{n,n-1}.$$

By induction we get that

$$\begin{aligned} v^j = e_{1,n-j+1} + z \otimes e_{2,n-j+2} + \cdots + z \otimes e_{j,n} + \\ z \otimes e_{j+1,1} + e_{j+2,2} + \cdots + e_{n,n-j} \end{aligned} \tag{5.1}$$

whenever $2 \leq j \leq n-1$, and

$$v^n = z \otimes e_{11} + \cdots + z \otimes e_{nn} = z \otimes I. \tag{5.2}$$

Note in particular that if $m = Kn$ for some $K \in \mathbb{Z}$, then v^m has the function z^K in the diagonal; if m is not divided by n, then v^m has zeros in the diagonal.

It follows from a direct computation that v is a unitary in $M_n(C(\mathbb{T}))$ and $v\pi(f)v^* = \pi(\sigma(f)), f \in C(X)$. The universal property for crossed products therefore implies the existence of a *-homomorphism $\pi \times v : C(X) \rtimes_\sigma \mathbb{Z} \to M_n(C(\mathbb{T}))$ satisfying

$$\pi \times v(\sum_{k=-N}^{N} f_k u^k) = \sum_{k=-N}^{N} \pi(f_k) v^k, \;\; \sum_{k=-N}^{N} f_k u^k \in C(X) \rtimes_\sigma \mathbb{Z}.$$

We need to prove that $\pi \times v$ is bijective:
$\pi \times v$ is surjective: Since $\pi \times v(1_{\{x_i\}}) = e_{ii}, i = 1, \ldots, n$, and $\pi \times v(u) = v$, it follows that $\pi \times \; v(C(X) \rtimes_\sigma \mathbb{Z}) \supseteq C^*(e_{11}, \ldots, e_{nn}, v)$. Hence it is enough to prove that

$$M_n(C(\mathbb{T})) = C^*(e_{11}, \ldots, e_{nn}, v).$$

Since the Laurent polynomials are dense in $C(\mathbb{T})$, we only have to prove that $(e_{ij})_{1 \leq i,j \leq n}$ and (for instance) $z \otimes e_{11}$ belong to $C^*(e_{11}, \ldots, e_{nn}, v)$. But

1. $e_{i2} = e_{ii}ve_{i-1,i-1}v \cdots ve_{33}v,\ 3 \le i \le n,$

2. $e_{i1} = e_{i2}e_{21} = e_{i2}v^{-n}v,\ 2 \le i \le n,$

3. $e_{ij} = e_{i1}e_{1j} = e_{i1}e_{j1}^*,\ 1 \le i, j \le n$, and

4. $z \otimes e_{11} = (z \otimes I)e_{11} = v^n e_{11},$

which proves the inclusion.
$\pi \times v$ is injective: Our proof is similar to the proof of Theorem 2.3.2. Since the e_{ii}'s are linearly independent we have that π is injective. The problem is therefore to construct a continuous linear map Ψ (acting like the conditional expectation), such that the following diagram commutes:

$$\begin{tikzcd} C(X) \rtimes_\sigma \mathbb{Z} \arrow[r, "\pi \times v"] \arrow[d, "\Phi"] & M_n(C(\mathbb{T})) \arrow[d, "\Psi"] \\ C(X) \arrow[r, "\pi"'] & \mathrm{span}(e_{ii}) \end{tikzcd}$$

(where Φ is the conditional expectation). Being a surjective continuous map, $\pi \times v$ maps the dense subset of standard elements in $C(X) \rtimes_\sigma \mathbb{Z}$ to a dense subset in $M_n(C(\mathbb{T}))$. Hence we define

$$\Psi(\sum_{k=-N}^{N} \pi(f_k)v^k) = \pi(f_0),\ \sum_{k=-N}^{N} \pi(f_k)v^k \in M_n(C(\mathbb{T})).$$

To prove that Ψ is a well-defined linear contraction (and thus extends to $M_n(C(\mathbb{T}))$), it is enough to prove that

$$||\pi(f_0)|| \le ||\sum_{k=-nK}^{nK} \pi(f_k)v^k||$$

for any $f_{-nK}, \dots, f_{nK} \in C(X)$ (the peculiar limits will simplify the notation below!). Estimating the (i,i)'th entry using a well-known norm estimate for matrices (see Murphy [15], Remark 3.4.1), and applying Lemma 5.2.2 and equation 5.1 and 5.2 above, we have for any $i \in \{1, \dots, n\}$

$$||\sum_{k=-nK}^{nK} \pi(f_k)v^k|| \ge ||\sum_{j=-K}^{K} f_{nj}(x_i)z^j|| \ge |f_0(x_i)|$$

as desired, since π is an isometry. ☐

Corollary 5.2.4. *If X is finite and $\sigma : X \to X$ is minimal, then* $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$.

Proof. Follows directly from Theorem 5.2.3, Lemma 5.1.2, and Lemma 5.1.4. ☐

Theorem 5.2.1 and Corollary 5.2.4 solves the minimal case.

For later use, we state the following theorem, which Putnam derives from Theorem 3.2.6 and Theorem 5.2.3:

Theorem 5.2.5. *Let (X,σ) be a Putnam system, and suppose that σ is minimal. Then the map $\eta : GL_1(C(X) \rtimes_\sigma \mathbb{Z})/GL_1^0(C(X) \rtimes_\sigma \mathbb{Z}) \to K_1(C(X) \rtimes_\sigma \mathbb{Z})$ given by*

$$\eta([a]) = [a|a|^{-1}]_1, a \in GL_1(C(X) \rtimes_\sigma \mathbb{Z}),$$

is a group isomorphism.

Proof. See Putnam [19], Corollary 2.4. □

In Section 5.5 below we will continue our study of the stable rank of crossed products arising from Putnam systems. Before we can do that, we will have to establish some other results.

5.3 A short exact sequence for crossed products

This section is devoted to prove, that any strongly invariant subset of a dynamical system induces a short exact sequence of crossed products. We begin with the following theorem:

Theorem 5.3.1. *Let $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, let γ, α and β be automorphisms on $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ respectively, and suppose we have the following commuting diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r, "\psi"] \arrow[d, "\gamma"] & \mathcal{A} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathcal{B} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r, "\psi"] & \mathcal{A} \arrow[r, "\pi"] & \mathcal{B} \arrow[r] & 0.
\end{tikzcd}$$

Then we have an exact sequence

$$0 \longrightarrow \mathcal{I} \rtimes_\gamma \mathbb{Z} \xrightarrow{\Psi} \mathcal{A} \rtimes_\alpha \mathbb{Z} \xrightarrow{\Pi} \mathcal{B} \rtimes_\beta \mathbb{Z} \longrightarrow 0,$$

where Ψ and Π are given by

$$\begin{aligned}
\Psi(cu_{\mathcal{I}}^n) &= \psi(c)u_{\mathcal{A}}^n,\ c \in \mathcal{I},\ n \in \mathbb{Z}, \text{ and} \\
\Pi(au_{\mathcal{A}}^n) &= \pi(a)u_{\mathcal{B}}^n,\ a \in \mathcal{A},\ n \in \mathbb{Z}.
\end{aligned}$$

Proof. The existence of Ψ and Π follows from the universal property of crossed products (cf. Theorem 2.1.5), since $u_{\mathcal{A}}\psi(c)u_{\mathcal{A}}^* = \alpha(\psi(c)) = \psi(\gamma(c)), c \in \mathcal{I}$, and $u_{\mathcal{B}}\pi(a)u_{\mathcal{B}}^* = \beta(\pi(a)) = \pi(\alpha(a)), a \in \mathcal{A}$.
Π *is surjective*: Follows from the fact that Π hits every standard element in $\mathcal{B} \rtimes_\beta \mathbb{Z}$ and Im(Π) is a C*-algebra.
Ψ *is injective*: Suppose $\Psi(x) = 0$ for a positive element $x \in \mathcal{I} \rtimes_\gamma \mathbb{Z}$. We must prove that $x = 0$. Since $\Phi_{\mathcal{I}}$ is faithful, it is enough to prove that $\Phi_{\mathcal{I}}(x) = 0$ (as usual $\Phi_{\mathcal{I}}$, $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$ denotes the conditional expectation on $\mathcal{I} \rtimes_\gamma \mathbb{Z}$, $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ and

$\mathcal{B} \rtimes_\beta \mathbb{Z}$ respectively). Starting by checking agreement on standard elements and then extending by continuity, we get the following commuting diagram:

$$\begin{tikzcd} \mathcal{I} \rtimes_\gamma \mathbb{Z} \arrow[r, "\Phi_\mathcal{I}"] \arrow[d, "\Psi"] & \mathcal{I} \arrow[d, "\psi"] \\ \mathcal{A} \rtimes_\alpha \mathbb{Z} \arrow[r, "\Phi_\mathcal{A}"] & \mathcal{A} \end{tikzcd}$$

Hence $\Psi(x) = 0 \Rightarrow \Phi_\mathcal{A}(\Psi(x)) = 0 \Rightarrow \psi(\Phi_\mathcal{I}(x)) = 0 \Rightarrow \Phi_\mathcal{I}(x) = 0$, as desired.
$\mathrm{Ker}\Pi = \mathrm{Im}\Psi$:
"$\supseteq$": Observe that $\Pi \circ \Psi = 0$ on standard elements and extend by continuity.
"$\subseteq$": We will need a couple of lemmas:

Lemma 5.3.2. *Let $n \in \mathbb{N}$. Then*

$$\forall x \in \mathcal{A} \rtimes_\alpha \mathbb{Z} : \Phi_\mathcal{B}(\Pi(x)u_\mathcal{B}^{-n}) = \pi(\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})).$$

Proof. Let us start by verifying the equality on standard elements:

$$\begin{aligned} \Phi_\mathcal{B}(\Pi(\sum_{k=-N}^{N} a_k u_\mathcal{A}^k)u_\mathcal{B}^{-n}) &= \Phi_\mathcal{B}(\sum_{k=-N}^{N} \pi(a_k)u_\mathcal{B}^{k-n}) \\ &= \pi(a_n) \\ &= \pi(\Phi_\mathcal{A}((\sum_{k=-N}^{N} a_k u_\mathcal{A}^k)u_\mathcal{A}^{-n})) \end{aligned}$$

(with $a_n = 0$ if $|n| > N$). Finally extend by continuity to obtain the desired result. ☐

Lemma 5.3.3. $\mathrm{Im}\Psi = \{x \in \mathcal{A} \rtimes_\alpha \mathbb{Z} \mid \forall n \in \mathbb{Z} : \pi(\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})) = 0\}$.

Proof. "$\subseteq$": If $x \in \mathrm{Im}(\Psi)$ and $n \in \mathbb{Z}$, then $\pi(\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})) = \Phi_\mathcal{B}(\Pi(x)u_\mathcal{B}^{-n}) = 0$ by the statements above.
"$\supseteq$": Suppose $x \in \mathcal{A} \rtimes_\alpha \mathbb{Z}$ satisfies $\forall n \in \mathbb{Z} : \pi(\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})) = 0$. Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ such that $||x - \sum_{n=-N}^{N}(1 - \frac{|n|}{N+1})\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})u_\mathcal{A}^n|| < \epsilon$ (see Theorem 2.1.4). By assumption we have

$$\forall n \in \mathbb{Z} : \Phi_\mathcal{A}(xu_\mathcal{A}^{-n}) \in \mathrm{Ker}(\pi) = \mathrm{Im}(\psi).$$

Therefore $\sum_{n=-N}^{N}(1 - \frac{|n|}{N+1})\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})u_\mathcal{A}^n$ must belong to $\mathrm{Im}(\Psi)$. Hence we conclude that $x \in \overline{\mathrm{Im}(\Psi)} = \mathrm{Im}(\Psi)$, since $\mathrm{Im}(\Psi)$ is a C*-algebra. ☐

Let us now return to the proof of the inclusion and assume that $\Pi(x) = 0$. Then it follows from Lemma 5.3.2 that $\pi(\Phi_\mathcal{A}(xu_\mathcal{A}^{-n})) = 0, n \in \mathbb{Z}$, whence Lemma 5.3.3 implies $x \in \mathrm{Im}\Psi$. ☐

We now show how to obtain a short exact sequence of crossed products from a dynamical system. First we need the following theorem:

Theorem 5.3.4. *Let X be a compact Hausdorff space, let Y be a closed subset of X, and put $Z = X \setminus Y$. Then there is a short exact sequence*

$$0 \longrightarrow C_0(Z) \xrightarrow{\psi} C(X) \xrightarrow{\pi} C(Y) \longrightarrow 0\,,$$

where π is the restriction map and ψ is given by

$$\psi(h)(x) = \begin{cases} h(x) & x \in Z \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let ψ be as above, and let us check that ψ is a well-defined map from $C_0(Z)$ to $C(X)$. To that end, let $h \in C_0(Z)$ and note that $f = \psi(h)$ is obviously continuous at every point in the open set Z. Now let $x_0 \in X \setminus Z$ and let $\epsilon > 0$. Put

$$K_\epsilon = \{x \in Z \mid |h(x)| \geq \epsilon\}.$$

K_ϵ is compact in Z by our assumption, and therefore also compact in X. Hence $X \setminus K_\epsilon$ is an open neighborhood of x_0 satisfying $f(X \setminus K_\epsilon) \subseteq D(0, \epsilon)$.

It is clear that ψ and π are *-homomorphisms, and that ψ is injective. Moreover, by Tietze's extension Theorem (cf. Pedersen [16], Proposition 1.5.8), π is surjective. The only remaining thing to prove is that $\mathrm{Im}\psi = \mathrm{Ker}\pi$.
"$\subseteq$": Obvious.
"$\supseteq$": Suppose $f \in \mathrm{Ker}\pi$. Put $h = f|_Z$, let $\epsilon > 0$, and put

$$K_\epsilon = \{x \in Z \mid |h(x)| \geq \epsilon\} = \{x \in X \mid |f(x)| \geq \epsilon\}.$$

K_ϵ is closed in X. Hence K_ϵ is compact in X, and therefore K_ϵ is also compact in Z. We conclude that $h \in C_0(Z)$, and it is now obvious that $\psi(h) = f$. □

Note that we may apply Theorem 5.3.1 to the short exact sequence from Theorem 5.3.4 to get the following corollary:

Corollary 5.3.5. *Let (X, σ) be a dynamical system, assume Y is a closed, strongly σ-invariant subset of X, and put $Z = X \setminus Y$. Let Ψ and Π be defined by*

$$\begin{aligned} \Psi(hu_Z^n) &= \psi(h)u_X^n,\ h \in C_0(Z),\ n \in \mathbb{Z},\ \textit{and} \\ \Pi(fu_X^n) &= \pi(f)u_Y^n,\ f \in C(X),\ n \in \mathbb{Z}, \end{aligned}$$

where ψ and π are defined as in Theorem 5.3.4, and u_X, u_Y and u_Z are the canonical unitaries from $C(X) \rtimes_\sigma \mathbb{Z}$, $C(Y) \rtimes_\sigma \mathbb{Z}$ and $C_0(Z) \rtimes_\sigma \mathbb{Z}$, respectively. Then the following sequence is exact:

$$0 \longrightarrow C_0(Z) \rtimes_\sigma \mathbb{Z} \xrightarrow{\Psi} C(X) \rtimes_\sigma \mathbb{Z} \xrightarrow{\Pi} C(Y) \rtimes_\sigma \mathbb{Z} \longrightarrow 0$$

Proof. Follows directly from Theorem 5.3.1 and Theorem 5.3.4. □

If the strongly σ-invariant set Y is in fact clopen, we will see that the short exact sequence from Corollary 5.3.5 may be expressed as a direct sum. First a lemma:

Lemma 5.3.6. *Assume we have a short exact sequence*

$$0 \longrightarrow \mathcal{I} \xrightarrow{\psi} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \longrightarrow 0,$$

where $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ are unital C-algebras. Then $\mathcal{A}$ is isomorphic to $\mathcal{B} \oplus \mathcal{I}$, and the isomorphism is given by $\varphi(a) = (\pi(a), \psi^{-1}(a\psi(1_{\mathcal{I}}))), a \in \mathcal{A}$, where $1_{\mathcal{I}}$ is the unit in $\mathcal{I}$.*

Proof. First note that φ is well-defined, since $\psi(\mathcal{I})$ is an ideal in $\mathcal{A}$.

Let us now prove that $\psi(1_{\mathcal{I}})$ commutes with $\mathcal{A}$. If $a \in \mathcal{A}$ then

$$\psi(1_{\mathcal{I}})a = \psi(1_{\mathcal{I}})\psi(1_{\mathcal{I}})a = \psi(1_{\mathcal{I}})a\psi(1_{\mathcal{I}}) = a\psi(1_{\mathcal{I}})\psi(1_{\mathcal{I}}) = a\psi(1_{\mathcal{I}}),$$

since $\psi(1_{\mathcal{I}})$ commutes with $\psi(\mathcal{I})$. It is now easy to check that φ is a *-homomorphism.

If $\varphi(a) = 0$ for some $a \in \mathcal{A}$, then $\pi(a) = 0$ implies $a \in \psi(\mathcal{I})$. Hence $\psi^{-1}(a) = \psi^{-1}(a\psi(1_{\mathcal{I}})) = 0$, whence $a = \psi(\psi^{-1}(a)) = 0$. Thus φ is injective.

Then suppose $(b, c) \in \mathcal{B} \oplus \mathcal{I}$, and choose $a \in \mathcal{A}$ such that $\pi(a) = b$. Then

$$\varphi(\psi(c) + a(1_{\mathcal{A}} - \psi(1_{\mathcal{I}}))) = (\pi(a), \psi^{-1}(\psi(c))) = (b, c)$$

proving that φ is surjective. We conclude that φ is a *-isomorphism. $\square$

Theorem 5.3.7. *Suppose the assumptions from Corollary 5.3.5 are satisfied, and Y is in fact clopen (whence Z is also clopen). Then*

$$C(X) \rtimes_\sigma \mathbb{Z} \cong C(Y) \rtimes_\sigma \mathbb{Z} \;\oplus\; C(Z) \rtimes_\sigma \mathbb{Z},$$

and the isomorphism is given by $\varphi(fu_X^n) = (f|_Y u_Y^n, f|_Z u_Z^n),\ f \in C(X), n \in \mathbb{Z}$.

Proof. Let φ denote the isomorphism obtained by combining Lemma 5.3.6 and Corollary 5.3.5. Then for $f \in C(X)$ and $n \in \mathbb{Z}$

$$\varphi(fu_X^n) = (\Pi(fu_X^n), \Psi^{-1}(fu_X^n 1_Z)) = (f|_Y u_Y^n, \Psi^{-1}(f1_Z u_X^n)) = (f|_Y u_Y^n, f|_Z u_Z^n)$$

as desired. $\square$

5.4 A lifting result and cancellation

In our proof of Theorem 5.5.6 below we will need a couple of results concerning general C*-algebras. The aim of this section is to obtain these results. We start with a lifting result.

It is well known, that it is not in general possible to lift invertible elements (cf. Rørdam et al. [22], Exercise 2.12). We will need that it is possible to lift invertible elements connected to the identity. First a lemma:

Lemma 5.4.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras, and $\pi : \mathcal{A} \to \mathcal{B}$ is surjective. Then*

$$\forall b \in \mathcal{B} : ||1_{\mathcal{B}} - b|| < 1 \Rightarrow b \in \pi(GL_1^0(\mathcal{A})).$$

Proof. Suppose $||1_{\mathcal{B}} - b|| < 1$, and pick $a \in \mathcal{A}$ such that $\pi(a) = 1_{\mathcal{B}} - b$. We have

$$||\pi(a)|| = \inf_{c \in ker(\pi)} ||a + c||,$$

and we may therefore assume $||a|| < 1$. Hence Rørdam et al. [22], Proposition 2.1.11, implies $1_{\mathcal{A}} - a \in GL_1^0(\mathcal{A})$, and moreover we have

$$\pi(1_{\mathcal{A}} - a) = 1_{\mathcal{B}} - (1_{\mathcal{B}} - b) = b$$

as desired. ☐

Theorem 5.4.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras, and $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective *-homomorphism (and thereby unital). Then $\pi(GL_1^0(\mathcal{A})) = GL_1^0(\mathcal{B})$.*

Proof. "$\subseteq$": Easy!
"$\supseteq$": $GL_1^0(\mathcal{B})$ is arcwise connected and thereby also connected (cf. Pedersen [16], E.1.4.13 and E.1.4.14). Thus it is sufficient to prove that $\pi(GL_1^0(\mathcal{A}))$ is nonempty and clopen relatively to $GL_1^0(\mathcal{B})$.
$\pi(GL_1^0(\mathcal{A})) \neq \emptyset$: Note that $1_{\mathcal{A}} \in GL_1^0(\mathcal{A})$.
$\pi(GL_1^0(\mathcal{A}))$ *is closed*: Suppose b belongs to the closure of $\pi(GL_1^0(\mathcal{A}))$, and pick an element a in $GL_1^0(\mathcal{A})$ such that $||\pi(a) - b|| < \frac{1}{||b^{-1}||}$. In particular

$$||\pi(a)b^{-1} - 1_{\mathcal{B}}|| < 1.$$

Therefore, according to Lemma 5.4.1, there exists an element $\tilde{a} \in GL_1^0(\mathcal{A})$ such that $\pi(\tilde{a}) = \pi(a)b^{-1}$. But then we have that $\pi(\tilde{a}^{-1}a) = b$, as desired.
$\pi(GL_1^0(\mathcal{A}))$ *is open*: By Rørdam et al. [22], Proposition 2.1.11, we have that $GL_1^0(\mathcal{A})$ is open. Hence $\pi(GL_1^0(\mathcal{A}))$ is open by the Open Mapping Theorem (cf. Rudin [24], Theorem 2.11). ☐

The following lemma and the subsequent theorem deals with *the cancellation property*. First we introduce a bit of notation (in harmony with the notation used in Rørdam et al. [22]).

Let $\mathcal{A}$ be a unital C*-algebra, and let $n \in \mathbb{N}$. We let $P_n(\mathcal{A})$ denote the projections in $M_n(\mathcal{A})$, the n-dimensional matrix algebra with entries from $\mathcal{A}$. The unit in $M_n(\mathcal{A})$ will be denoted by 1_n, and we have for two projections $p, q \in P_n(\mathcal{A})$ the relations $p \sim q$ (Murray-von Neumann equivalence) and $p \sim_u q$ (unitary equivalence).

We put $P_\infty(\mathcal{A}) = \cup_{n=1}^\infty P_n(\mathcal{A})$, and have for $p, q \in P_\infty(\mathcal{A})$ the relations $p \sim_0 q$ (generalized von Neumann equivalence) and $p \sim_s q$ (stable equivalence).

We may now introduce the cancellation property as follows (see [22], Definition 7.3.1):

Definition 5.4.3. *Let $\mathcal{A}$ be a (unital) C*-algebra. Then $\mathcal{A}$ has the cancellation property, if*

$$\forall p, q \in P_\infty(\mathcal{A}) : [p]_0 = [q]_0 \Rightarrow p \sim_0 q.$$

The following lemma gives an alternative criterion for a C*-algebra to have the cancellation property:

Lemma 5.4.4. *Suppose $\mathcal{A}$ is a unital C^*-algebra. Then the following conditions are equivalent:*

1. *$\mathcal{A}$ has the cancellation property,*

2. *$\forall n \in \mathbb{N}\ \forall p, q \in P_n(\mathcal{A}) : [p]_0 = [q]_0 \Rightarrow p \sim_u q$.*

Proof. "$1 \Rightarrow 2$": Assume $\mathcal{A}$ has the cancellation property. If $[p]_0 = [q]_0$ then we have that $[1_n - p]_0 = [1_n - q]_0$ (by [22], Proposition 3.1.7 (iv)), and hence $p \sim q$ and $1_n - p \sim 1_n - q$. It now follows from [22], Proposition 2.2.2, that $p \sim_u q$.
"$2 \Rightarrow 1$": Suppose $[p]_0 = [q]_0, p, q \in P_\infty(\mathcal{A})$. We must prove $p \sim_0 q$. By adding a suitable number of zeros we may assume p and q belong to the same matrix algebra: $p, q \in M_n(\mathcal{A})$ (see [22], Proposition 2.3.2 (i) and Proposition 3.1.7). Choose a unitary w such that $w^*pw = q$, and note that $u = pw$ is a partial isometry satisfying $uu^* = p$ and $u^*u = q$. $\square$

We have the following sufficient criterion for a C*-algebra to have the cancellation property:

Theorem 5.4.5. *Suppose $\mathcal{A}$ is a unital C^*-algebra and* $\mathrm{sr}(\mathcal{A}) = 1$. *Then $\mathcal{A}$ has the cancellation property.*

Proof. Let us first prove

$$\forall p, q \in P_n(\mathcal{A}) : p \sim q \Rightarrow p \sim_u q.$$

Note that $\mathrm{sr}(M_n(\mathcal{A})) = 1$ by Lemma 5.1.4. Now assume $p, q \in P_n(\mathcal{A})$ satisfies $p \sim q$, and choose a partial isometry $v \in M_n(\mathcal{A})$ such that $p = v^*v$ and $q = vv^*$. Choose $a \in GL_n(\mathcal{A})$ such that $||v - a|| < 1/6$, and note that $||a|| \leq ||v - a|| + ||v|| < 2$. Thus

$$\begin{aligned} ||p - a^*a|| &\leq ||v^*(v - a)|| + ||(v^* - a^*)a|| \\ &< \frac{1}{6} + 2\frac{1}{6} = \frac{1}{2} \end{aligned}$$

and similarly $||q - aa^*|| < \frac{1}{2}$. Write a as $a = u|a|$, where u is a unitary in $M_n(\mathcal{A})$ (cf. [22], Proposition 2.1.8), and note that $ua^*au^* = aa^*$. Thus

$$\begin{aligned} ||upu^* - q|| &\leq ||upu^* - aa^*|| + ||aa^* - q|| \\ &= ||u(p - a^*a)u^*|| + ||aa^* - q|| \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Hence (by [22], Proposition 2.2.4 and Proposition 2.2.7) we get that $p \sim_u q$.

Let us now use Lemma 5.4.4 to prove the theorem: Suppose $[p]_0 = [q]_0$ for some $p, q \in P_m(\mathcal{A})$, and choose $n \in \mathbb{N}$ such that $p \oplus 1_n \sim q \oplus 1_n$ in $M_{m+n}(\mathcal{A})$ ($p \oplus 1_n$

is the matrix having p and 1_n in the diagonal). The first part of the proof, and Proposition 2.2.2 in [22], now implies that

$$\begin{aligned} p \oplus 1_n \sim q \oplus 1_n &\Rightarrow p \oplus 1_n \sim_u q \oplus 1_n \\ &\Rightarrow (1_m - p) \oplus 0_n \sim (1_m - q) \oplus 0_n \\ &\Rightarrow 1_m - p \sim 1_m - q \\ &\Rightarrow 1_m - p \sim_u 1_m - q \\ &\Rightarrow p \sim q \\ &\Rightarrow p \sim_u q, \end{aligned}$$

as desired. □

5.5 The case of no clopen strongly invariant subsets

We now return to the discussion of Putnam systems, starting with a definition:

Definition 5.5.1. *Let (X, σ) be a Putnam system. If Y is a closed subset of X, we denote by $\mathcal{A}_Y$ the C^*-algebra $C^*(C(X), uC_0(X \setminus Y)) \subseteq C(X) \rtimes_\sigma \mathbb{Z}$.*

We will need a sufficient condition due to Poon [17] for $\mathcal{A}_Y$ to be an AF-algebra. First a lemma:

Lemma 5.5.2. *Let (X, σ) be a Putnam system, let Y be a clopen subset of X, and suppose that $\bigcup_{n \in \mathbb{Z}} \sigma^n(Y) = X$. Then*

$$\mathcal{A}_Y \cong \oplus_{k=1}^{K} M_{J_k}(C(Y_k))$$

for a partition $\{Y_k \mid 1 \leq k \leq K\}$ of Y and some positive integers $J_k, 1 \leq k \leq K$.

Proof. Choose a KR-partition $\{Y(k,j) \mid 1 \leq k \leq K,\ 1 \leq j \leq J_k\}$ of X with top floor Y (see Lemma 3.1.1) and put $Y_k = Y(k, J_k),\ 1 \leq k \leq K$. Put $f_{ij}^{(k)} = u^{i-j} 1_{Y(k,j)}$, and define $\psi_k : C(Y_k) \to C(X)$ by

$$\psi_k(g) = \sum_{l=1}^{J_k} \sigma^{l-J_k}(g),\ g \in C(Y_k) (\subseteq C(X)).$$

Note that $\sigma(C(Y(k,j))) = C(Y(k, j+1)), 1 \leq j < J_k$. Using this, it is easy to check that ψ_k is an injective *-homomorphism, and thus we may identify $C(Y_k)$ with $\psi_k(C(Y_k))$.

According to Lemma 3.2.2 and Lemma 3.2.3 we have to prove the following assertions:

1. $(f_{ij}^{(k)})$ form a set of matrix units,
2. The sets $\{f_{ij}^{(k)}, 1 \leq i,j \leq J_k\} \cup \psi_k(C(Y_k)),\ 1 \leq k \leq K$, are *-invariant and mutually orthogonal,
3. For fixed k, $\psi_k(C(Y_k))$ commutes with $(f_{ij}^{(k)})$.

4. $C^*(\bigcup_{k=1}^{K}(\{f_{ij}^{(k)}, 1 \leq i,j \leq J_k\} \cup \psi_k(C(Y_k)))) = \mathcal{A}_Y$

1: A computation identical to the one in the proof of Theorem 3.2.6.
2: It is clear that the sets are *-invariant. Considering orthogonality, note that, by disjointness of the sets in our partition, $f_{ij}^{(k)}\psi_{k'}(g) = 0$ whenever $k \neq k'$ and $1 \leq i,j \leq J_k$. Taking the adjoints on each side of the equation gives us that also $\psi_{k'}(g)f_{ij}^{(k)} = 0$ whenever $k \neq k'$ and $1 \leq i,j \leq J_k$. Thus the sets are mutually orthogonal.
3: Let k, i and j be fixed, and consider a function g in $C(Y_k)$. Then

$$\begin{aligned}
\psi_k(g)f_{ij}^{(k)} &= \sum_{l=1}^{J_k} \sigma^{l-J_k}(g)u^{i-j}1_{Y(k,j)} \\
&= \sum_{l=1}^{J_k} u^{i-j}\sigma^{l-J_k+j-i}(g)1_{Y(k,j)} \\
&= u^{i-j}\sigma^{j-J_k}(g)1_{Y(k,j)} \\
&= u^{i-j}1_{Y(k,j)}\sum_{l=1}^{J_k}\sigma^{l-J_k}(g) \\
&= f_{ij}^{(k)}\psi_k(g).
\end{aligned}$$

4: Let us prove each of the two inclusions:
"$\subseteq$": It is clear that $\psi_k(C(Y_k)) \subseteq \mathcal{A}_Y, 1 \leq k \leq K$. Moreover $f_{ij}^{(k)} = f_{i1}^{(k)}f_{j1}^{(k)*}$ for any $1 \leq k \leq K$ and $1 \leq i,j \leq J_k$, and

$$f_{i1}^{(k)} = f_{i,i-1}^{(k)} \cdots f_{21}^{(k)} = u1_{Y(k,i-1)} \cdots u1_{Y(k,1)} \in \mathcal{A}_Y, 2 \leq i \leq J_k.$$

"$\supseteq$": Let $g \in C(X)$. Since $g = g\sum_{j,k} 1_{Y(k,j)}$ we only have to consider $g1_{Y(k,j)}$ for some k and j. Now put $\tilde{g} = \sigma^{J_k-j}(g1_{Y(k,j)}) \in C(Y_k)$, and note that

$$\begin{aligned}
g1_{Y(k,j)} &= \sigma^{j-J_k}(\tilde{g}) \\
&= u^{j-J_k}1_{Y_k}\tilde{g}1_{Y_k}u^{J_k-j} \\
&= u^{j-J_k}1_{Y_k}\psi_k(\tilde{g})1_{Y_k}u^{J_k-j} \\
&= f_{jJ_k}^{(k)}\psi_k(\tilde{g})f_{J_kj}^{(k)} \in C^*(f_{ij}^{(k)}, \psi_k(C(Y_k))).
\end{aligned}$$

Moreover we have that

$$u1_{X\setminus Y} = u\sum_{k=1}^{K}\sum_{j=1}^{J_k-1} f_{jj}^{(k)} = \sum_{k=1}^{K}\sum_{j=1}^{J_k-1} f_{j+1,j}^{(k)} \in C^*(f_{ij}^{(k)}, \psi_k(C(Y_k))),$$

which proves the inclusion. □

If $\mathcal{A}$ and $\mathcal{B}$ are AF-algebras and $n \in \mathbb{N}$, then $\mathcal{A} \oplus \mathcal{B}$ and $M_n(\mathcal{A})$ are also AF-algebras (this follows for instance from the local characterization of AF-algebras, see Davidson [4], Theorem III.3.4).

In particular, $\mathcal{A}_Y$ is an AF-algebra if Y satisfies the assumptions from Lemma 5.5.2. In fact we get a more general result:

Theorem 5.5.3. *Suppose (X,σ) is a Putnam system, Y is a closed subset of X, and $\cup_{n\in\mathbb{Z}}\sigma^n(W) = X$ for every clopen subset W containing Y. Then $\mathcal{A}_Y$ is an AF-algebra.*

Proof. Using Theorem 1.1.7 it is not hard to see, that it is possible to find a decreasing sequence of clopen sets Y_n such that $\cap_{n=1}^{\infty} Y_n = Y$ (indeed let $\mathcal{P}_n$ be as in Theorem 1.1.7, let Y_n' be the union of sets $Z \in \mathcal{P}_n$ satisfying $Z \cap Y \neq \emptyset$, and put $Y_n = \cap_{k=1}^{n} Y_k'$).

It is now a consequence of Lemma 5.5.2, that $\mathcal{A}_{Y_1} \subseteq \mathcal{A}_{Y_2} \subseteq \dots$ is an increasing sequence of AF-algebras. We claim that

$$\overline{\cup_{n=1}^{\infty}\mathcal{A}_{Y_n}} = \mathcal{A}_Y.$$

"$\subseteq$": Obvious.
"$\supseteq$": Suppose $f \in C_0(X \setminus Y)$, and let $\epsilon > 0$. The set

$$K_\epsilon = \{x \in X \setminus Y \mid |f(x)| \geq \epsilon\}$$

is compact in $X \setminus Y = \cup_{n=1}^{\infty} X \setminus Y_n$. Thus it is possible to find $n \in \mathbb{N}$ such that $K_\epsilon \subseteq X \setminus Y_n$. Now put $g = f|_{X\setminus Y_n}$, and note that $ug \in \mathcal{A}_{Y_n}$ and $||uf - ug|| < \epsilon$.

Thus $\mathcal{A}_Y$ is the closure of an increasing sequence of AF-algebras; hence $\mathcal{A}_Y$ is also an AF-algebra by Lemma 5.5.4 below. ☐

Lemma 5.5.4. *Let $\mathcal{B}$ be a C^*-algebra, and let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ be an ascending sequence of AF-subalgebras of $\mathcal{B}$. Then $\mathcal{A} = \overline{\cup_{n=1}^{\infty}\mathcal{A}_n}$ is an AF-algebra.*

Proof. It is easy to check that $\mathcal{A}$ is a separable C*-algebra. Now let $\epsilon > 0$ and let $\{a_1, \dots, a_m\} \subseteq \mathcal{A}$. Choose $n \in \mathbb{N}$ and $a_1', \dots, a_m' \in \mathcal{A}_n$, such that $||a_i - a_i'|| < \frac{\epsilon}{2}$ for $1 \leq i \leq m$. Choose a finite dimensional C*-algebra $\mathcal{B} \subseteq \mathcal{A}_n$ and $b_1, \dots, b_m \in \mathcal{B}$, such that $||a_i' - b_i|| < \frac{\epsilon}{2}$ for $1 \leq i \leq m$. Now $||a_i - b_i|| < \epsilon, 1 \leq i \leq m$, and by Davidson [4], Theorem III.3.4, we conclude that $\mathcal{A}$ is an AF-algebra. ☐

We have the following remark, which connects Corollary 5.3.5 and Theorem 5.5.3:

Remark 5.5.5. *Let Y be a closed strongly σ-invariant subset of X and put*

$$\mathcal{I} = \overline{\operatorname{span}}\{fu^j \mid f \in C_0(X \setminus Y), j \in \mathbb{Z}\},$$

where u is the canonical unitary in $C(X) \rtimes_\sigma \mathbb{Z}$. Then $\mathcal{I}$ is a closed ideal in $\mathcal{A}_Y$. In particular, $\mathcal{I}$ is an AF-algebra if $\mathcal{A}_Y$ is an AF-algebra (see Murphy [15], Theorem 6.2.6).

We are now finally ready to generalize Theorem 5.2.1 and Corollary 5.2.4. We include the assumption that X has no non-trivial clopen strongly σ-invariant subsets; in the next section we will examine the general case.

Theorem 5.5.6. *Let (X,σ) be a Putnam system, and suppose that X has no non-trivial, clopen, strongly σ-invariant subsets. Then the following three conditions are equivalent:*

1. $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$,

2. $C(X) \rtimes_\sigma \mathbb{Z}$ *has the cancellation property,*

3. σ *has a unique minimal set.*

Proof. "$1 \Rightarrow 2$": See Theorem 5.4.5.
"$2 \Rightarrow 3$": Suppose 3 is false: Let Y_1 and Y_2 be two different minimal sets for σ (recall Theorem 1.2.4). Nonemptiness of $Y_1 \cap Y_2$ would contradict minimality of Y_1 and Y_2: Hence $Y_1 \cap Y_2 = \emptyset$. Consider the short exact sequence (see Corollary 5.3.5)

$$0 \longrightarrow C_0(X \setminus (Y_1 \cup Y_2)) \rtimes_\sigma \mathbb{Z} \xrightarrow{\Psi} C(X) \rtimes_\sigma \mathbb{Z} \xrightarrow{\Pi} C(Y_1 \cup Y_2) \rtimes_\sigma \mathbb{Z} \longrightarrow 0.$$

For brevity we shall denote the algebras by $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ respectively. Let us denote the canonical unitary in $C(X) \rtimes_\sigma \mathbb{Z}$ by u_X, the canonical unitary in $C(Y_1) \rtimes_\sigma \mathbb{Z}$ by u_{Y_1}, and so on. Moreover let $\iota_j : C(Y_j) \rtimes \mathbb{Z} \to C(Y_1 \cup Y_2) \rtimes_\sigma \mathbb{Z}, j = 1, 2$, be the canonical inclusion mappings from Theorem 5.3.7:

$$\iota_j(f u_{Y_j}^n) = f u_{Y_1 \cup Y_2}^n, f \in C(Y_j), n \in \mathbb{Z}.$$

By Theorem 5.3.7, Theorem 2.4.3, and Rørdam et al. [22], Proposition 8.2.6, we have that

$$K_1(\mathcal{B}) \cong K_1(C(Y_1) \rtimes_\sigma \mathbb{Z}) \oplus K_1(C(Y_2) \rtimes_\sigma \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

and $K_1(\mathcal{B})$ is generated by $K_1(\iota_1)([u_{Y_1}]_1)$ and $K_1(\iota_2)([u_{Y_2}]_1)$. By Exercise 8.5 in Rørdam et al. [22] we have that

$$K_1(\iota_1)([u_{Y_1}]_1) = [1_{Y_1} u_{Y_1 \cup Y_2} + 1_{Y_2}]_1$$

and

$$K_1(\iota_2)([u_{Y_2}]_1) = [1_{Y_2} u_{Y_1 \cup Y_2} + 1_{Y_1}]_1.$$

Put $u_1 = 1_{Y_1} u_{Y_1 \cup Y_2} + 1_{Y_2}$ and $u_2 = 1_{Y_2} u_{Y_1 \cup Y_2} + 1_{Y_1}$, and note that the strong σ-invariance of Y_1 and Y_2 implies that $u_1 u_2 = u_{Y_1 \cup Y_2}$. By Theorem 5.3.1, and Rørdam et al. [22], Exercise 8.5, we now have that

$$K_1(\Pi)([u_X]_1) = [u_{Y_1 \cup Y_2}]_1 = [u_1 u_2]_1 = [u_1]_1 + [u_2]_1.$$

If $[u_1]_1 \in \mathrm{Im} K_1(\Pi)$ then also $[u_2]_1 \in \mathrm{Im} K_1(\Pi)$. Hence $K_1(\Pi)$ is surjective, whence $\mathbb{Z} \oplus \mathbb{Z}$ is finite or isomorphic to $\mathbb{Z}$, a contradiction.

We conclude that $[u_1]_1 \notin \mathrm{Im} K_1(\Pi)$. Hence the index map $\delta_1 : K_1(\mathcal{B}) \to K_0(\mathcal{I})$ satisfies $\delta_1([u_1]_1) \neq 0$.

We will now compute $\delta_1([u_1]_1)$. Choose a clopen set E containing Y_1 and disjoint from Y_2, and put $F = X \setminus (E \cup \sigma(E))$. Note that since Y_2 is strongly σ-invariant, $\sigma(E)$ is also disjoint from Y_2, so $Y_2 \subseteq F$. Let $a = u_X 1_E + 1_F$. Then $a^* = 1_E u_X^* + 1_F$, and hence

$$\begin{aligned} a^* a &= 1_E + 1_E u_X^* 1_F + 1_F u_X 1_E + 1_F \\ &= 1_E + u_X^* 1_{\sigma(E)} 1_F + 1_F 1_{\sigma(E)} u_X + 1_F = 1_{E \cup F} \end{aligned}$$

and similarly $aa^* = 1_{\sigma(E)\cup F}$. Let us also note that

$$\Pi(a) = \Pi(u_X)\Pi(1_E) + \Pi(1_F) = u_{Y_1\cup Y_2}1_{Y_1} + 1_{Y_2} = u_1$$

by the definition of Π and the way we picked E and F. Hence by Rørdam et al. [22], Proposition 9.2.4, we have that

$$\delta_1([u_1]_1) = [1 - a^*a]_0 - [1 - aa^*]_0 = [1_{\sigma(E)\setminus E}]_0 - [1_{E\setminus\sigma(E)}]_0.$$

Put $p_1 = 1_{\sigma(E)\setminus E}$ and $p_2 = 1_{E\setminus\sigma(E)}$. They are both elements of $\mathcal{I}$, since strong σ-invariance of Y_1 and Y_2 implies that $\sigma(E)\setminus E$ and $E\setminus\sigma(E)$ are subsets of $X\setminus(Y_1\cup Y_2)$ (this also follows from Rørdam et al. [22], Proposition 9.2.4). We claim that p_1 and p_2 determines the same element in $K_0(\mathcal{A})$, and that they are not unitarily equivalent. Then Lemma 5.4.4 will imply that $\mathcal{A}$ fails to have the cancellation property.

As we have already observed, $[p_1]_0 - [p_2]_0 \neq 0$ in $K_0(\mathcal{I})$. However by the six-term short exact sequence, $[\Psi(p_1)]_0 - [\Psi(p_2)]_0 = K_0(\Psi)\delta_1([u_1]_1) = 0$ in $K_0(\mathcal{A})$.

If now $w\Psi(p_1)w^* = \Psi(p_2)$ for some unitary w in $\mathcal{A}$, then $q = \Psi^{-1}(w\Psi(p_1))$ is a well-defined partial isometry in $\mathcal{I}$ (since $\Psi(\mathcal{I})$ is an ideal and $p_1 \in \mathcal{I}$) with $qq^* = p_2$ and $q^*q = p_1$. Thus $[p_1]_0 = [p_2]_0$ in $K_0(\mathcal{I})$, a contradiction. We conclude that $\mathcal{A}$ fails to have the cancellation property.

"3 $\Rightarrow$ 1": Let us denote the unique minimal set by Y. By Theorem 5.2.1 and Corollary 5.2.4 we may assume Y is a proper subset of X. Again by Corollary 5.3.5 we have a short exact sequence:

$$0 \longrightarrow C_0(X\setminus Y)\rtimes_\sigma \mathbb{Z} \xrightarrow{\Psi} C(X)\rtimes_\sigma \mathbb{Z} \xrightarrow{\Pi} C(Y)\rtimes_\sigma \mathbb{Z} \longrightarrow 0.$$

For brevity, again we shall denote these C*-algebras by $\mathcal{I}$, $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $u_{\mathcal{A}}$ and $u_{\mathcal{B}}$ be the canonical unitaries in $\mathcal{A}$ and $\mathcal{B}$, and let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the units.

We claim, that $\mathcal{A}_Y$ is an AF-algebra. By Theorem 5.5.3 it is enough to prove that $X = \cup_{n\in\mathbb{Z}}\sigma^n(W)$ whenever W is a clopen set containing Y, but this follows from the fact that $X\setminus(\cup_{n\in\mathbb{Z}}\sigma^n(W))$ otherwise would be a closed strongly σ-invariant set disjoint from Y. Hence $\Psi(\mathcal{I})$ is also an AF-algebra by Remark 5.5.5. By Lemma 5.1.5 and Theorem 5.1.6, $\mathrm{sr}(\Psi(\mathcal{I}) + \mathbb{C}1_{\mathcal{A}}) = \mathrm{sr}(\widetilde{\Psi(\mathcal{I})}) = 1$.

By Theorem 5.2.1 and Corollary 5.2.4 we also have that $\mathrm{sr}(\mathcal{B}) = 1$.

Let us prove that $\Pi(GL_1(\mathcal{A})) = GL_1(\mathcal{B})$ ("$\subseteq$" is obvious). Let $b \in GL_1(\mathcal{B})$. By Theorem 5.2.5 we have that $[b|b|^{-1}]_1 = k[u_{\mathcal{B}}]_1$ for a certain $k \in \mathbb{Z}$. Hence $[b|b|^{-1}u_{\mathcal{B}}^{-k}]_1 = 0$, whence by Theorem 5.2.5 we have that $b|b|^{-1}u_{\mathcal{B}}^{-k} \sim_h 1_{\mathcal{B}}$ in $GL_1(\mathcal{B})$. Now $b|b|^{-1} \sim_h b$ by Rørdam et al. [22], Proposition 2.1.8, and we conclude that $bu_{\mathcal{B}}^{-k} \in GL_1^0(\mathcal{B})$. Finally $u_{\mathcal{B}}$ lifts to $u_{\mathcal{A}}$, and invoking Theorem 5.4.2 we get that $b \in \Pi(GL_1(\mathcal{A}))$.

Let us now show that $\mathrm{sr}(\mathcal{A}) = 1$. Let $a \in \mathcal{A}$ and $\epsilon > 0$. Choose $b \in GL_1(\mathcal{B})$ such that $||\Pi(a) - b|| < \epsilon/2$. Choose $c \in GL_1(\mathcal{A})$ such that $\Pi(c) = b$. Then $||\Pi(a) - \Pi(c)|| < \epsilon/2$, and

$$||\Pi(a) - \Pi(c)|| = \inf_{d\in\mathrm{Ker}\Pi}||a - c - d||.$$

Hence we may pick $d \in \mathrm{Ker}\Pi = \Psi(\mathcal{I})$ such that $||a - c - d|| < \epsilon/2$. Since $\Psi(\mathcal{I})$ is an ideal we have that $1_{\mathcal{A}} + c^{-1}d \in \Psi(\mathcal{I}) + \mathbb{C}1_{\mathcal{A}}$, and we may therefore choose

$e \in GL_1(\Psi(\mathcal{I}) + \mathbb{C}1_{\mathcal{A}})$ such that

$$||1_{\mathcal{A}} + c^{-1}d - e|| < \frac{\epsilon}{2||c||}.$$

Now $ce \in GL_1(\mathcal{A})$, and

$$\begin{aligned} ||a - ce|| &\leq ||a - (c + d)|| + ||c + d - ce|| \\ &\leq ||a - c - d|| + ||c||||1_{\mathcal{A}} + c^{-1}d - e|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

We will now give a couple of examples of applications of Theorem 5.5.6.

First note that the full n-shift from Example 1.2.9 provides an example of a system not having stable rank one (indeed this follows directly from Theorem 5.5.6 and Example 1.2.9).

Let us now consider a couple of examples of systems having stable rank one. First a remark:

Remark 5.5.7. *Let (X, σ) be a Putnam system, and suppose $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$ and σ has no clopen strongly invariant subsets. Then σ has a unique minimal set F by Theorem 5.5.6. It is clear that σ acts minimally on F, whence either F is finite or F is a Cantor set (see Remark 1.2.10). Suppose F contains an open set U. Then $X \setminus \cup_{n\in\mathbb{Z}}\sigma^n(U)$ is a closed strongly σ-invariant set, which does not contain F. Hence the set is empty and $F = X$.*

We therefore have the following three possibilities, each of which can occur according to Example 1.2.8 and the examples below:

1. *$F = X$, where X is the Cantor set,*
2. *F is a proper subset of X, without interior points and homeomorphic to the Cantor set,*
3. *F is finite.*

Example 5.5.8. *Let X be the Cantor set, and let σ be a minimal homeomorphism on X. Consider the Cantor set $\tilde{X} = (\mathbb{Z} \cup \{\infty\}) \times X$, where $\mathbb{Z} \cup \{\infty\}$ is the one-point compactification of $\mathbb{Z}$. Define $\tilde{\sigma} : \tilde{X} \to \tilde{X}$ by*

$$\tilde{\sigma}(n, x) = \begin{cases} (n + 1, \sigma(x)) & n \in \mathbb{Z} \\ (\infty, \sigma(x)) & n = \infty. \end{cases}$$

It can be checked that $\tilde{\sigma}$ is a homeomorphism; thus $(\tilde{X}, \tilde{\sigma})$ is a Putnam system. Suppose F is a nonempty, closed, strongly $\tilde{\sigma}$-invariant subset of $\tilde{X}$, and pick a point $(a, b) \in F$. Then either $a = \infty$ or $a \in \mathbb{Z}$. If $a \in \mathbb{Z}$, consider the sequence

$$\tilde{\sigma}^m(a, b) = (a + m, \sigma^m(b)), m \in \mathbb{N}.$$

Pick a convergent subsequence $(a + m_k, \sigma^{m_k}(b)) \to (a_0, b_0) \in F$. *Since every point in* $\mathbb{Z}$ *is isolated in* $\mathbb{Z} \cup \{\infty\}$, *we must have that* $(a_0, b_0) \in F \cap (\{\infty\} \times X)$. *Hence (by the fact, that* σ *acts minimally on* X*)* $\{\infty\} \times X \subseteq F$.

It follows that $\tilde{\sigma}$ *has no clopen, strongly* $\tilde{\sigma}$*-invariant, proper subsets (if* A *is a clopen, strongly* $\tilde{\sigma}$*-invariant, proper subset of* $\tilde{X}$, *then the same is true for* $\complement A$, *whence both* A *and* $\complement A$ *contain* $\{\infty\} \times X$*), and that* $\{\infty\} \times X$ *is the unique minimal set. Moreover it is clear that* $\{\infty\} \times X$ *is without interior points and homeomorphic to the Cantor set.*

Example 5.5.9. *Consider the Cantor set* $X \subseteq [0; 1]$, *defined by "the middlethird construction". Setting* $J_0 = X \cap [\frac{1}{6}; \frac{5}{6}]$, $J_{-1} = X \cap [\frac{1}{18}; \frac{1}{6}]$, $J_1 = X \cap [\frac{5}{6}; \frac{17}{18}]$ *etc., we get a disjoint union*

$$X = \{0\} \cup \bigcup_{n \in \mathbb{Z}} J_n \cup \{1\}.$$

Take $\tilde{X}$ *to be two copies of* X *"glued together" at* 0 *and* 1. *Then we have the disjoint union*

$$\tilde{X} = \{0, 1\} \cup \bigcup_{n \in \mathbb{Z}} J_n \cup \bigcup_{n \in \mathbb{Z}} K_n,$$

where J_n *and* K_n *are clopen sets. Define* σ *to be any homeomorphism sending* J_n *onto* K_{-n}, K_{-n} *onto* J_{n+1}, 0 *to* 1 *and* 1 *to* 0. *It follows by inspection, that this makes* σ *a homeomorphism on* $\tilde{X}$ *with* $F = \{0, 1\}$ *as the unique minimal set and with no clopen strongly invariant subsets (this example is easy to extend to an example of a minimal set with n points).*

5.6 The general case

In this section we will establish a theorem stating a general dynamic condition for the stable rank of $C(X) \rtimes_\sigma \mathbb{Z}$ to be one, in the case where (X, σ) is a Putnam system. We start with a couple of lemmas dealing with general C*-algebras. Recall that $\operatorname{dist}(a, \mathcal{B}) = \inf_{b \in \mathcal{B}} ||a - b||$, when $\mathcal{B}$ is a subset of a C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$. For a unital C*-algebra $\mathcal{A}$ we have that $\operatorname{sr}(\mathcal{A}) = 1$ if and only if

$$\forall a \in \mathcal{A} : \operatorname{dist}(a, GL_1(\mathcal{A})) = 0.$$

Lemma 5.6.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital C*-algebras and let* $\pi : \mathcal{A} \to \mathcal{B}$ *be a surjective *-homomorphism. Then*

$$\forall a \in \mathcal{A} : \operatorname{dist}(\pi(a), GL_1(\mathcal{B})) \leq \operatorname{dist}(a, GL_1(\mathcal{A})).$$

In particular, $\operatorname{sr}(\mathcal{A}) = 1$ *implies* $\operatorname{sr}(\mathcal{B}) = 1$.

Proof. Let $a' \in GL_1(\mathcal{A})$. Then $\pi(a') \in GL_1(\mathcal{B})$ and $||\pi(a) - \pi(a')|| \leq ||a - a'||$. □

Lemma 5.6.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots$ be an ascending sequence of ideals in $\mathcal{A}$. Put $\mathcal{I} = \overline{\cup_{n=1}^{\infty} \mathcal{I}_n}$. Then $\mathcal{I}$ is an ideal in $\mathcal{A}$. Let $\pi_n : \mathcal{A} \to \mathcal{A}/\mathcal{I}_n$ and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the quotient mappings. Then*

$$\operatorname{dist}(\pi_n(a), GL_1(\mathcal{A}/\mathcal{I}_n)) \searrow \operatorname{dist}(\pi(a), GL_1(\mathcal{A}/\mathcal{I})) \text{ as } n \to \infty.$$

Proof. It is clear that $\mathcal{I}$ is an ideal. Let us start by proving that for any $d \in \mathcal{A}$ $||\pi_n(d)|| \searrow ||\pi(d)||$. Since the sequence of ideals is ascending, we have that

$$||\pi_1(d)|| \geq ||\pi_2(d)|| \geq \cdots \geq ||\pi(d)||.$$

Now let $\epsilon > 0$, and choose $c \in \mathcal{I}$ such that $||\pi(d)|| \geq ||d + c|| - \epsilon/2$. Choose $N \in \mathbb{N}$ and $c' \in \mathcal{I}_N$ such that $||c' - c|| < \epsilon/2$. We now have

$$||\pi_n(d)|| \leq ||d + c'|| \leq ||d + c|| + ||c' - c|| < ||\pi(d)|| + \epsilon, n \geq N,$$

as desired.

Let us now put $t = \operatorname{dist}(\pi(a), GL_1(\mathcal{A}/\mathcal{I}))$. It follows from Lemma 5.6.1 that

$$\operatorname{dist}(\pi_1(a), GL_1(\mathcal{A}/\mathcal{I}_1)) \geq \operatorname{dist}(\pi_2(a), GL_1(\mathcal{A}/\mathcal{I}_2)) \geq \cdots \geq t.$$

Let $\epsilon > 0$, and choose $b \in \mathcal{A}$ such that $||\pi(a) - \pi(b)|| < t + \epsilon$ and

$$\pi(b)\pi(c) = \pi(c)\pi(b) = \pi(1_{\mathcal{A}})$$

for some $c \in \mathcal{A}$. In particular, $\pi(cb - 1_{\mathcal{A}}) = \pi(bc - 1_{\mathcal{A}}) = 0$. By the first part of the proof we may therefore pick $N' \in \mathbb{N}$ such that

$$n \geq N' \Rightarrow ||\pi_n(cb - 1_{\mathcal{A}})||, ||\pi_n(bc - 1_{\mathcal{A}})|| < 1.$$

In particular, $\pi_n(cb), \pi_n(bc) \in GL_1(\mathcal{A}/\mathcal{I}_n)$, whence $\pi_n(b) \in GL_1(\mathcal{A}/\mathcal{I}_n), n \geq N'$.

Finally choose $N'' \in \mathbb{N}$ such that

$$n \geq N'' \Rightarrow ||\pi_n(a) - \pi_n(b)|| < t + \epsilon,$$

and note that $n \geq \max\{N', N''\} \Rightarrow \operatorname{dist}(\pi_n(a), GL_1(\mathcal{A}/\mathcal{I}_n)) < t + \epsilon$. ☐

We now continue the study of Putnam systems. Recall that if (X, d) is a metric space, then for arbitrary subsets $A, B \subseteq X$ we have a distance from A to B:

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

We have the following remark:

Remark 5.6.3. *Let (X, σ) be a Putnam system, and let us for $x, y \in X$ define $x \sim y$ if and only if x and y belong to the same clopen, strongly σ-invariant subsets of X.*

It is easy to verify that $\sim$ is an equivalence relation. Let us write F_x for the equivalence class containing x. It is easy to see that each F_x is strongly σ-invariant. Moreover we have

$$F_x = \bigcap_{U \text{ clopen and str. } \sigma\text{-inv.}, x \in U} U$$

from the very definition of $\sim$. In particular, each F_x is closed.

Now given $\epsilon > 0$, consider the set $V = \{y \in X \mid d(y, F_x) < \epsilon\}$. V is open; hence $\complement V$ is compact, and since $F_x \subseteq V$ we have that $\complement V \subseteq \cup \complement U$ (taking the union over clopen, strongly σ-invariant sets U, which contain x). Choose a finite covering $\complement U_1, \ldots, \complement U_n$. We now have $F_x \subseteq \cap_{j=1}^n U_j \subseteq V$.

Since

$$F_x = \bigcap_{\epsilon > 0} \{y \in X \mid d(y, F_x) < \epsilon\},$$

it follows that to every $x \in X$ it is possible to find a descending sequence of clopen, strongly σ-invariant sets $U(n,x)$, such that $F_x = \cap_{n=1}^\infty U(n,x)$ and

$$F_x \subseteq U(n,x) \subseteq \{y \in X \mid d(y, F_x) < \frac{1}{n}\}.$$

Put

$$\mathcal{I}(n,x) = \overline{\operatorname{span}}\{fu^m \mid f \in C_0(X \setminus U(n,x)), m \in \mathbb{Z}\}$$

and

$$\mathcal{I}(\infty,x) = \overline{\operatorname{span}}\{fu^m \mid f \in C_0(X \setminus F_x), m \in \mathbb{Z}\},$$

where u is the canonical unitary in $C(X) \rtimes_\sigma \mathbb{Z}$. It is clear that $\mathcal{I}(n,x), n = 1, \ldots, \infty$, are ideals in $C(X) \rtimes_\sigma \mathbb{Z}$, and that $\mathcal{I}(1,x) \subseteq \mathcal{I}(2,x) \subseteq \cdots \subseteq \mathcal{I}(\infty,x)$.

The following lemma connects Lemma 5.6.2 and Remark 5.6.3:

Lemma 5.6.4. *Using the notation from Remark 5.6.3, we have that*

$$\mathcal{I}(\infty,x) = \overline{\cup_{n=1}^\infty \mathcal{I}(n,x)}, x \in X.$$

Proof. "$\supseteq$": Follows from the fact that $\mathcal{I}(1,x) \subseteq \mathcal{I}(2,x) \subseteq \cdots \subseteq \mathcal{I}(\infty,x)$.
"$\subseteq$": Let $f \in C_0(X \setminus F_x)$; we must prove that $fu^m \in \overline{\cup_{n=1}^\infty \mathcal{I}(n,x)}$. Hence let $\epsilon > 0$, and note that $\{y \in X \setminus F_x \mid |f(y)| \geq \epsilon\} \subseteq X \setminus F_x = \cup_{n=1}^\infty \complement U(n,x)$. We may therefore choose $n \in \mathbb{N}$ such that

$$\{y \in X \setminus F_x \mid |f(y)| \geq \epsilon\} \subseteq \complement U(n,x).$$

Letting $f' = f|_{\complement U(n,x)} \in C(X \setminus U(n,x))$ we get $||f'u^m - fu^m|| \leq ||f' - f|| < \epsilon$, as desired. $\square$

We will try to reduce the general case to a case, where the assumptions from Theorem 5.5.6 are satisfied. This is the motivation for the following lemma:

Lemma 5.6.5. *Using the notation from Remark 5.6.3, each of the sets F_x has no clopen, strongly σ-invariant, proper subsets in the relative topology.*

Proof. Assume $F_x = A_1 \sqcup A_2$, where A_1 and A_2 are clopen in the relative topology, non-empty, and strongly σ-invariant, and put $\epsilon = d(A_1, A_2) > 0$. σ is uniformly continuous on X, and we may therefore choose $\delta > 0$ such that

$$\forall x, y \in X : d(x, y) < \delta \Rightarrow d(\sigma(x), \sigma(y)) < \epsilon/2.$$

Now pick an integer $n > \max\{\frac{1}{\delta}, \frac{2}{\epsilon}\}$, and consider the clopen set

$$U(n, x) \subseteq \{y \in X \mid d(y, F_x) < 1/n\}$$

from Remark 5.6.3. We have that

$$U(n, x) \cap \{y \in X \mid d(y, A_1) < 1/n\} = U(n, x) \cap \{y \in X \mid d(y, A_1) \leq 1/n\}.$$

Indeed assume that $d(y, A_1) \leq 1/n$ and $d(y, F_x) < 1/n$. Since $d(y, F_x) = \min\{d(y, A_1), d(y, A_2)\}$ we have either $d(y, A_1) < 1/n$ or $d(y, A_2) < 1/n$, but $\frac{2}{n} < \epsilon = d(A_1, A_2) \leq d(y, A_1) + d(y, A_2)$ implies that we cannot have $d(y, A_2) < 1/n$: Hence $d(y, A_1) < 1/n$.

We thus conclude that the set

$$U_1 = U(n, x) \cap \{y \in X \mid d(y, A_1) < 1/n\}$$

is clopen in X. Similarly

$$U_2 = U(n, x) \cap \{y \in X \mid d(y, A_2) < 1/n\}$$

is clopen. Note that

$$U(n, x) = U_1 \cup U_2$$

("$\subseteq$" follows from the fact that $U(n, x) \subseteq \{y \in X \mid d(y, F_x) < 1/n\}$). Moreover U_1 is σ-invariant: Indeed assume $y \in U_1$. Since $d(y, A_1) < 1/n < \delta$, we have $d(\sigma(y), A_1) < \epsilon/2$. If now $\sigma(y) \in U_2$, we have $d(\sigma(y), A_2) < 1/n < \epsilon/2$, contradicting that $d(A_1, A_2) = \epsilon$.

Therefore $\sigma(U_1) \subseteq U_1$ and similarly $\sigma(U_2) \subseteq U_2$. Finally strong σ-invariance of $U(n, x)$ implies that U_1 and U_2 are in fact strongly σ-invariant, contradicting the way we constructed F_x. □

We are now ready to prove the following generalization of Theorem 5.5.6:

Theorem 5.6.6. *Let (X, σ) be a Putnam system. Then the following conditions are equivalent:*

1. $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$,
2. *For every pair of distinct minimal sets F_1 and F_2, there are disjoint, clopen, strongly σ-invariant sets A_1 and A_2 such that $F_1 \subseteq A_1$ and $F_2 \subseteq A_2$.*

Proof. Let us throughout the proof use the notation from Remark 5.6.3.
"$1 \Rightarrow 2$": Suppose F_1 and F_2 are minimal sets which are not separated by any pair of clopen, strongly σ-invariant sets. Then every clopen strongly σ-invariant set A must satisfy that either $F_1 \cup F_2 \subseteq A$ or $F_1 \cup F_2 \subseteq \complement A$. We conclude that $F_1 \cup F_2$ is a subset of F_x for some $x \in X$, and now Lemma 5.6.5 and Theorem 5.5.6 implies that $C(X) \rtimes_\sigma \mathbb{Z}$ has a quotient with stable rank different from one. Finally Lemma 5.6.1 implies that $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) \neq 1$.
"$2 \Rightarrow 1$": Let $a \in C(X) \rtimes_\sigma \mathbb{Z}$. We want to prove that $\mathrm{dist}(a, GL_1(C(X) \rtimes_\sigma \mathbb{Z})) = 0$. The assumption, Lemma 5.6.5, and Theorem 5.5.6 implies that each equivalence class F_x satisfies $\mathrm{sr}(C(F_x) \rtimes_\sigma \mathbb{Z}) = 1$. Indeed, if F_1 and F_2 are minimal sets in F_x, they are also minimal in X. Hence they are separated by clopen strongly σ-invariant sets, contradicting Lemma 5.6.5.

Thus Lemma 5.6.2 implies that

$$\mathrm{dist}(\pi_{n,x}(a), GL_1(C(U(n,x)) \rtimes_\sigma \mathbb{Z})) \to 0 \text{ as } n \to \infty,$$

where $\pi_{n,x} : C(X) \rtimes_\sigma \mathbb{Z} \to C(U(n,x)) \rtimes_\sigma \mathbb{Z}$ is the canonical surjective mapping.

Let $\epsilon > 0$, and choose to every $x \in X$ a positive integer N_x such that

$$n \geq N_x \Rightarrow \mathrm{dist}(\pi_{n,x}(a), GL_1(C(U(n,x)) \rtimes_\sigma \mathbb{Z})) < \epsilon.$$

We have a covering $X = \cup_{x \in X} U(N_x, x)$. Compactness of X therefore implies the existence of a finite covering:

$$X = \cup_{k=1}^{N} U(N_{x_k}, x_k).$$

Let us now put $U_j = U(N_{x_j}, x_j) \setminus (\cup_{k=1}^{j-1} U(N_{x_k}, x_k)), j = 1, \ldots, N$. Note that each U_j is strongly σ-invariant and clopen, and that $X = U_1 \sqcup \cdots \sqcup U_N$. By Theorem 5.3.7 we therefore have

$$C(X) \rtimes_\sigma \mathbb{Z} \cong \oplus_{j=1}^{N} C(U_j) \rtimes_\sigma \mathbb{Z},$$

and the *-isomorphism is given by $b \mapsto (\pi_1(b), \ldots, \pi_N(b)), b \in C(X) \rtimes_\sigma \mathbb{Z}$, where $\pi_j : C(X) \rtimes_\sigma \mathbb{Z} \to C(U_j) \rtimes_\sigma \mathbb{Z}$ is the canonical mapping. By Lemma 5.6.1 we now get that

$$\begin{aligned} \mathrm{dist}(\pi_j(a), GL_1(C(U_j) \rtimes_\sigma \mathbb{Z})) &\leq \mathrm{dist}(\pi_{N_{x_j}, x_j}(a), GL_1(C(U(N_{x_j}, x_j)) \rtimes_\sigma \mathbb{Z})) \\ &< \epsilon, \ j = 1, \ldots, N, \end{aligned}$$

whence $\mathrm{dist}(a, GL_1(C(X) \rtimes_\sigma \mathbb{Z})) < \epsilon$ as desired. □

We include an example of an application of Theorem 5.6.6:

Example 5.6.7. *Consider again the Cantor set $X \subseteq [0;1]$ defined by the middlethird construction. Put $I_1 = X \cap [\frac{1}{6}; 1]$ and*

$$I_n = X \cap [\frac{1}{2 \cdot 3^n}; \frac{1}{2 \cdot 3^{n-1}}], n \geq 2.$$

Then it is easy to see that each I_n is clopen and that we have a disjoint union:

$$X = \{0\} \cup \bigcup_{n=1}^{\infty} I_n.$$

Let σ be a homeomorphism on X defined by letting σ act minimally on each I_n and putting $\sigma(0) = 0$ (it is clear that this makes σ continuous). The minimal sets for σ are exactly the sets $\{0\}, I_1, I_2, \ldots,$ and it follows from Theorem 5.6.6 that $\mathrm{sr}(C(X) \rtimes_\sigma \mathbb{Z}) = 1$.

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