ON STABILITY AND PURE INFINITENESS OF C^*-ALGEBRAS

BY

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A Ph.D. THESIS

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Skønt er det vi ser, skønnere er det vi erkender, men skønnest er det vi ikke fatter.

Niels Stensen 1638-1686, Naturforsker og Helgen.

Abstract

Stability and pure infiniteness of C^*-algebras is considered. Let A be a σ-unital C^*-algebra, i.e. A admits a countable approximate unit. It is proved that A is stable if and only if for each positive element $a \in A$ and each $\varepsilon > 0$ there exists a positive element $b \in A$ such that $\|ab\| < \varepsilon$ and $x^*x = a$, $xx^* = b$ for some x in A.

Using this characterization it is proved among other things that the inductive limit of any sequence of σ-unital stable C^*-algebras is stable, and that the crossed product of a σ-unital stable C^*-algebra by a discrete group is again stable. It is discussed how stability is related to a structure problem for simple C^*-algebras. Also, stable C^*-algebras arising from directed graphs are characterized and a graph-theoretical analogue of stability is given.

Pure infiniteness is considered for (not necessarily simple) C^*-algebras arising from r-discrete groupoids, which again arise from certain dynamical systems. In particular, Cuntz-Krieger algebras and their generalizations, i.e., graph-algebras and $\mathcal{O}_B$ of an infinite matrix B, admit characterizations of pure infiniteness. As a consequence, these generalized Cuntz-Krieger algebras are traceless if and only if they are purely infinite. Also, a graph-theoretical analogue of pure infiniteness is given.

Abstract in Danish.

(Resume på dansk).

Stabilitet og uniform uendelighed ("pure infiniteness") af C^*-algebraer betragtes. Lad A være en σ-unital C^*-algebra, dvs. A har en tællelig approksimativ enhed. Der vises, at A er stabil hvis, og kun hvis, der for hvert positivt element $a \in A$ og for hvert $\varepsilon > 0$ findes et positivt element $b \in A$, således at $\|ab\| < \varepsilon$ og $x^*x = a$, $xx^* = b$ for et element x i A.

Ved brug af denne karakterisation vises blandt andet, at den induktive grænse af enhver følge af σ-unitale stabile C^*-algebraer er stabil, og at kryds-produktet af en σ-unital stabil C^*-algebra med en diskret gruppe igen er stabil. Stabilitet relateres til et struktur-problem for simple C^*-algebraer. Der gives en karakterisering af stabile C^*-algebraer som kommer fra orienterede grafer, og en graf-teoretisk analogi til stabilitet præsenteres.

Uniformt uendelige (og ikke nødvendigvis simple) C^*-algebraer fra r-diskrete groupoider betragtes. Specielt uniformt uendelige C^*-algebraer som kommer fra visse dynamiske systemer, herunder Cuntz-Krieger algebraer og generaliseringer af disse, dvs. graf-algebraer og $\mathcal{O}_B$ fra uendelige matriser B (Exel-Laca algebraer). Der gives en karakterisering af uniformt uendelighed, og en konsekvens er, at ovennævnte generaliserede Cuntz-Krieger algebraer er uden spor hvis, og kun hvis, de er uniformt uendelige. En graf-teoretisk analogi til uniform uendelighed præsenteres.

Preface

The content of this thesis is based on the papers [20] and [19]. Some material has been rewritten and a brief introduction has been added.

This work was initiated in 1995-1996 while my wife, Mette v.B. Hjelmborg, and I were visiting Toronto supported by the Andersen-grant (see below). Back then, George Elliott gave to me the extension problem for *stable* C^*-algebras; Is an extension of two stable C^*-algebras always stable? (see section 1.1). A general solution to the extension problem still remains to be found. However, from the attempt to solve this problem grew up a characterization theorem, Theorem 2.3, for stable C^*-algebras admitting a countable approximate unit consisting of projections, and later in joint work with Mikael Rørdam a generalization of this theorem was found in Theorem 2.4.

After looking into the paper [25, Graphs, groupoids and Cuntz-Krieger algebras], where C^*-algebras are associated to directed graphs, a question arose of characterizing stability by (topological) properties of the underlying graph (or groupoid). A characterization of stability for graph-algebras that among other things allowed to answer the extension problem in the affirmative was obtained.

In [22] E. Kirchberg and M. Rørdam generalize the notion of *pure infiniteness* of C^*-algebras to not necessarily simple C^*-algebras. Analogous to the above question for stability, one can ask if pure infiniteness can be characterized for C^*-algebras of groupoids. This is discussed in chapter 4.

I would like to thank Mikael Rørdam for many things, especially the paper [20] was a great boost. I am deeply grateful to Mikael Rørdam and Uffe Haagerup for teaching me operator algebra and for giving lots of inspiration. Since Mikael moved to Copenhagen in 1997 I have frequently visited the mathematics department there and I am thankful for kind hospitality. Multiple thanks to the operator algebra groups in Toronto, Orleans, Trondheim and Oslo for very kind hospitality. Also, I would like to thank the mathematics department in Odense for being a wonderful place and the operator algebra group there for good experiences and great entertainment. Especially thanks to Uffe for being a very kind person.

There are so many persons that I am grateful to and I can not mention all of them. However, I would like to thank George Elliott for filling me with interest and enthusiasm, Alex Kumjian and Jean Renault for good advice and Erling Størmer who among other things patiently taught me the basics of cross-country skiing which I enjoy very much.

At the moment my family; Mette, Rikke, Katrine and I enjoy a very pleasant stay in Oslo supported by the Andersen-grant. We, Mette and I, are very thankful to “Rejselegat for Matematikere” (the Andersen-grant) founded by Cand. Mag. Valdemar Andersen and wife, Else Andersen in 1964. In short, this grant covers all expenses related to one year of study in mathematics outside Denmark. It is won by lottery and requires the Danish degree Cand. Scient. in mathematics. We both won it and are very happy for the many good experiences it has given us.

I feel that I have been given a good education in mathematics. I regard mathematics as an eternal source of inspiration and hope that it will always play a role in my life.

Jacob von Bornemann Hjelmborg
Oslo, March 2000.

Contents

Chapter 1

INTRODUCTION

This chapter is intended to be a brief introduction. In sections 1.1 and 1.2 we introduce the notions of stability and pure infiniteness of C^*-algebras.

In section 1.3 we consider locally finite directed graphs which may have infinitely many vertices, but only finitely many edges in and out of each vertex. In [25] (and in Example 1.11) it is shown that to each such graph E one can associate a locally compact groupoid $\mathcal{G}_E$ such that its groupoid C^*-algebra $C^*(\mathcal{G}_E)$ is the universal C^*-algebra generated by families of partial isometries satisfying Cuntz-Krieger type relations determined by E. This class is widened in section 1.4 (original in [17]) to a class of C^*-algebras associated to infinite matrices. These C^*-algebras are named Exel-Laca algebras and may be viewed as generalized Cuntz-Krieger algebras. All AF-algebras resides in this class and are characterized in terms of the matrix (or underlying graph) in Theorem 1.8 of section 1.4.1.

In section 1.5 we consider C^*-algebras of r-discrete groupoids. In particular the groupoid of a singly generated dynamical system. Following J. Renault [32] a singly generated dynamical system, denoted (X, T), consists of a locally compact, second countable Hausdorff space X and a local homeomorphism T from an open subset of X onto an open subset of X. One can associate a locally compact r-discrete (amenable) Hausdorff groupoid, denoted $G(X, T)$. The full and reduced C^*-algebra of the convolution algebra of $G(X, T)$ coincide and is named $C^*(X, T)$. These C^*-algebras are of our interest since C^*-algebras of graphs and infinite matrices arise in this way, and a goal is to characterize singly generated dynamical systems (X, T) for which $C^*(X, T)$ is purely infinite. This is obtained for C^*-algebras of graphs and the generalized Cuntz-Krieger algebras associated to infinite matrices.

1.1 Stable C^*-algebras

A C^*-algebra A is *stable* if A is isomorphic to $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on an infinite dimensional separable Hilbert space. Equivalently, A is isomorphic to the C^*-algebra $\overline{\bigcup_{n=1}^{\infty} M_n(A)}$, where $M_n(A)$ is embedded in the upper left hand corner of $M_{n+1}(A)$.

Two C^*-algebras A and B are called *stably isomorphic* if $A \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{K}$. Isomorphic C^*-algebras are stably isomorphic and the converse holds for stable C^*-algebras, but is not true in general. Indeed, $\mathcal{K}$ is stably isomorphic to $\mathbb{C}$, the complex numbers. Stably isomorphic separable C^*-algebras are strongly Morita equivalent and vice versa as shown in [9]. Also, the K-functor is stable in the sense that the natural $*$-homomorphism of A into $A \otimes \mathcal{K}$ induces isomorphisms at the level of K-theory, i.e., $K_0(A) \cong K_0(A \otimes \mathcal{K})$ and $K_1(A) \cong K_1(A \otimes \mathcal{K})$ (see [38, Propositions 6.4.1 and 8.2.8]).

Suppose that A is a σ-unital C^*-algebra, that is, A admits a countable approximate unit. For instance A can be separable. In chapter 2 we characterize stability of A and give some applications as mentioned in the abstract. Our characterization result can be viewed as a generalization of a theorem of Shuang Zhang [40, Thm. 1.2], that every non-unital purely infinite simple C^*-algebra is stable (cf. Proposition 2.17). In section 2.4 we also discuss how stability of C^*-algebras is related to the structure problem for simple C^*-algebras: if every simple, unital C^*-algebra is either stably finite or purely infinite (cf. Proposition 2.18).

The original motivation for characterizing stability was an attempt to solve the extension problem: Given an extension of C^*-algebras:

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\pi} B \longrightarrow 0,$$

where I and B are stable C^*-algebras. Does it follow that A is stable? The converse holds by the proof of Corollary 3.17. A solution to this problem still remains to be found. However, the question will be answered in the affirmative for C^*-algebras arising from graphs (Corollary 3.17). All AF-algebras and Cuntz-Krieger algebras are in this class of C^*-algebras which will be introduced in section 1.3.

Another question of interest is whether one can conclude that A is stable if $M_2(A)$ is stable for a C^*-algebra A? This question was asked in [20]. Using techniques developed by J. Villadsen and the characterization theorem for stability (not to forget!), M. Rørdam has found a counterexample, that is, a C^*-algebra A such that $M_2(A)$ is stable, but A is not. We refer to section 2.5.

1.2 Purely infinite C^*-algebras

Recall that a projection p in a C^*-algebra A is *infinite* if it is equivalent to a proper subprojection of itself, and p is *finite* otherwise. A unital C^*-algebra A is *infinite* if the unit in A is infinite. If

A is a non-unital C^*-algebra, then A is infinite if A with a unit adjoined, $\tilde{A}$, is infinite. If there exist mutually orthogonal subprojections q and r of a projection p in a C^*-algebra A such that q and r are both equivalent to p, then p is said to be *properly infinite*. Equivalently, p is properly infinite if $p \oplus p \precsim p$ (and p is non-zero), where $p \oplus p$ denotes the element of $M_2(A)$ with p in the diagonal and zero elsewhere.

Recall that a simple C^*-algebra A is called purely infinite if every hereditary C^*-subalgebra of A contains an infinite projection. It is not clear what "pure infiniteness" should mean in case of non-simple C^*-algebras. An approach (in use) is simply to extend the definition above to non-simple C^*-algebras. But, as will be shown in Example 4.4, there are C^*-algebras which admit a (bounded) trace and every hereditary C^*-subalgebra contains an infinite projection. We will now consider the approach taken in [22].

By [22] a C^*-algebra A is *purely infinite* if there are no characters on A, and for every pair of positive elements a, b in A such that b lies in the closed two-sided ideal generated by a it follows that $b \precsim a$, i.e., there is a sequence $(r_j)_{j=1}^{\infty}$ in A such that $r_j a r_j^* \to b$ (cf. J. Cuntz' comparison theory for positive elements). An equivalent formulation of pure infiniteness is contained in [22]:

$$\forall a \in A^+ \colon a \oplus a \precsim a, \tag{1.1}$$

where $a \oplus a$ denotes the element of $M_2(A)$ with a in the diagonal and zero elsewhere.

The definition agrees with the usual definition by J. Cuntz when A is simple (see above), and in the non-simple case the two properties are neither weaker nor stronger than the other. Indeed, $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ is purely infinite ([22, Proposition 4.5]) but contains no projections, and in Example 4.4 a C^*-algebra A (of a graph) is given such that A is not purely infinite, but every hereditary C^*-subalgebra of A contains an infinite projection.

However, if every non-zero hereditary C^*-subalgebra in every quotient of A contains an infinite projection, then A is purely infinite, as shown in [22, Proposition 4.7]. This proposition and [1, Proposition 2.4], are used in Proposition 4.5 to obtain a sufficient condition on r-discrete essentially principal groupoids for which the associated reduced C^*-algebra is purely infinite. This condition is proven to be necessary for C^*-algebras of graphs, and more general for C^*-algebras of infinite matrices introduced in sections 1.3 and 1.4. The story of characterizing (r-discrete) groupoids for which the associated C^*-algebra is purely infinite is certainly not complete, as will be discussed in Remark 4.11.

A *trace* on a C^*-algebra A is a positive linear map $\tau\colon I \to \mathbb{C}$, where I is a (not necessarily closed or dense) two-sided ideal in A, such that $\tau(x^*x) = \tau(xx^*)$ for all $x \in I$. Notice that if τ is a bounded trace on I then it extends to a bounded trace on A.

It is an open question whether a C^*-algebra with no non-zero trace is purely infinite. The answer is affirmative for C^*-algebras of graphs and infinite matrices by Theorem 4.1 and Theorem 4.10.

1.3 C^*-algebras of directed graphs

A *directed graph* E is a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets and $r, s\colon E^1 \to E^0$ are maps. E^0 is the set of vertices and E^1 is the set of edges in the graph. The maps s and r are called the source and range maps and they give the direction of each edge. The graph E is *row-finite* if for every $v \in E^0$ the set $s^{-1}(v) \subseteq E^1$ is finite, i.e., only finitely many edges come out of each vertex, and if in addition $r^{-1}(v)$ is finite for all $v \in E^0$ then E is *locally finite*. A finite *path* in E is a sequence $\alpha = (e_1, e_2, \ldots, e_k)$ of edges in E with $s(e_{j+1}) = r(e_j)$ for $1 \leq j \leq k-1$. We write E^* for the finite path space of E, and E^∞ for the infinite path space. The range and source maps r, s extend naturally to E^*. Suppose α is a path in E. The subset $\alpha^0 \subseteq E^0$ given by

$$\alpha^0 = \{v \in E^0\colon v = s(e) \text{ or } v = r(e) \text{ for some } e \in \alpha\},$$

is the set of vertices of α. A *detour* of a path α is a path $\beta \in E^*$ such that $s(\beta) \in \alpha^0$ and $r(\beta) \in \alpha^0$. Trivially, subpaths of α are detours. A path $\alpha \in E^*$ of length $|\alpha| > 0$ is a cycle based at v if $s(\alpha) = v = r(\alpha)$. A vertex $v \in E^0$ that emits no edges is called a *sink*. We now consider sets of vertices that give rise to ideals in the associated C^*-algebra (consult [25] for details). Let H be a subset of E^0. H is *hereditary* if, whenever $v \in H$ and there is a path from v to w, then $w \in H$. H is *saturated* if $v \in H$ when every edge $e \in E^1$ with source $s(e) = v$ has range $r(e) \in H$.

Given a row-finite directed graph E, a *Cuntz-Krieger E-family* is given by a set of mutually orthogonal projections indexed by vertices, $\{P_v\colon v \in E^0\}$ and a family of partial isometries indexed by edges $\{S_e \colon e \in E^1\}$ subject to the following conditions:

$$S_e^* S_e = P_{r(e)} \tag{1.2}$$

for all $e \in E^1$ and

$$P_v = \sum_{\{e \colon s(e)=v\}} S_e S_e^* \tag{1.3}$$

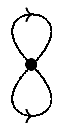
for all $v \in s(E^1)$. The universal C^*-algebra generated by a Cuntz-Krieger E-family $\{P_v, S_e\}$ is denoted by $C^*(E)$ (and there is such a C^*-algebra - see [24, Theorem 1.2]) . Note that the projections $\{P_v\}$ are the initial projections of the partial isometries S_e with $r(e) = v$, and note also that a projection P_v can be non-zero even if there are no edges coming out of v. If every vertex emits an edge then the family $\{S_e\}$ generates $C^*(E)$ and satisfy

$$S_e^* S_e = \sum_{\{f \in E^1\}} A_E(e,f) S_f S_f^*, \tag{1.4}$$

where A_E is the *edge matrix* of E defined by

$$A_E(e,f) = \begin{cases} 1 & \text{if } r(e) = s(f). \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.1 Consider the graph B given by:

$B\colon$ 

The Cuntz-Krieger B-family is given by the set $\{S_e, S_f \colon \text{e,f are the edges of } B\}$ of partial isometries subject to the relations

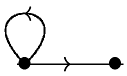
$$S_e^* S_e = S_f^* S_f = S_e S_e^* + S_f S_f^*$$

Hence the generators of $C^*(B)$ satisfies the Cuntz-relations for $\mathcal{O}_2$ and $C^*(B) \cong \mathcal{O}_2$.

The C^*-algebra $\mathcal{K}$ is represented by the following graph:

K: 

The Toeplietz C^*-algebra $\mathcal{T}$, i.e., the C^*-algebra generated by the unilateral shift S on the Hilbert space l^2, arises from the following graph:



The shift S is the sum of the two partial isometries associated to the edges. The unit 1 in $\mathcal{T}$ is infinite (since $1 = S^*S \sim SS^* < 1$) and $\mathcal{T}$ is infinite. However, 1 is not properly infinite (cf. section 1.2) as can be seen directly from the graph using Theorem 4.1, and $\mathcal{T}$ is not purely infinite.

Remark 1.2 If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a finite path, we define a non-zero partial isometry by $S_\alpha := S_{\alpha_1} \cdots S_{\alpha_n}$. Then $S_\alpha^* S_\alpha = P_{r(\alpha)}$ and $S_\alpha S_\alpha^* \leq P_{s(\alpha)}$ ($S_v = P_v$ for $v \in E^0$). It follows by [24, Lemma1.1] that

$$C^*(E) \cong C^*\{S_\alpha S_\beta^* \mid \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$$

By the universal property of $C^*(E)$ there is a strongly continuous canonical action of the torus $\mathbb{T}$, called the gauge action: $\alpha\colon \mathbb{T} \to \mathrm{Aut}(C^*(E))$ defined for $t \in \mathbb{T}$ and $S_e \in C^*(E)$ by

$$\alpha_t(S_e) = tS_e$$

A key result for analyzing $C^*(E)$ is the *gauge-invariant uniqueness theorem:*

Theorem 1.3 ([29] and [4]) *Let E be a row-finite directed graph, B a C^*-algebra, $\pi\colon C^*(E) \to B$ be a homomorphism and let $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be an action such that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbb{T}$. Then π is faithful if and only if $\pi(P_v) \neq 0$ for all vertices $v \in E^0$.*

A consequence of the theorem is that there is an isomorphism $C^*(E) \cong C^*(G(X_E, T_E))$, where $G(X_E, T_E)$ is the path-groupoid associated to the dynamical system (X_E, T_E) induced by E; In short, X_E is the infinite path space of E and $T_E\colon X_E \to X_E$ is the one-sided shift on X_E. Details are given in Example 1.11. The association of the path-groupoid to E making groupoid theory applicable in the analysis of $C^*(E)$ is done in [25]. Other routes have also been studied - see [4] and [17].

Remark 1.4 The crossed product $C^*(E) \rtimes_\alpha \mathbb{T}$, where α is the gauge action, is an AF-algebra as shown in [28, Proposition 2.6], and by the Takai duality theorem we see that $C^*(E)$ is stably isomorphic to the crossed product of an AF-algebra by a $\mathbb{Z}$-action. Hence $C^*(E)$ is nuclear (which also follows by Example 1.11) and in the bootstrap class to which the universal coefficient theorem applies. Thus, if $C^*(E)$ is simple and purely infinite it is determined up to isomorphism by its K-theory, as asserted by the Kirchberg-Phillips classification theorem ([21]).

We will now schematically list some of the many results obtained for $C^*(E)$. It is assumed that E is a locally finite directed graph without sinks although some of the results holds for more general graphs.

$\boldsymbol{E}$	-	$\boldsymbol{C^*(E)}$	**References**
E^0 is finite.	$\Leftrightarrow$	Unital	[24, Prop. 1.4], Remark 3.2.
E has no cycles.	$\Leftrightarrow$	AF-algebra	Theorem 1.8, [24, Thm. 2.4].
Every cycle has an exit.	$\Leftrightarrow$	Cuntz-Krieger uniqueness theorem	[24, Thm. 3.7], [4, Thm. 3.1].
$H \subseteq E^0$ is hereditary and saturated.	$\Rightarrow$	$C^*\{S_e \mid s(e) \in H\}$ is an ideal, $C^*\{S_e \mid r(e) \notin H\}$ the quotient.	[25, Thm. 6.6].
No vertex is base for precisely one cycle.	$\Leftrightarrow$	The lattice of her. and sat. subsets of E^0 is isomorphic to the lattice of ideals in $C^*(E)$.	[25, Thm. 6.6].
Every vertex connects to every infinite path and every cycle has an exit.	$\Leftrightarrow$	simple	[25, Cor. 6.8], [4, Prop. 5.1].
E admits no non-zero bounded graph trace and no left-finite cycles.	$\Leftrightarrow$	stable	Theorem 3.15.
Every infinite path admits a detour β such that there are at least two cycles based at some vertex of β^0.	$\Leftrightarrow$	purely infinite	Theorem 4.1.

For examples we refer to Example 1.1, 3.11, 4.3 and 4.4.

1.4 C^*-algebras of infinite matrices

In the following A will denote an $I \times I$-matrix with entries in $\{0, 1\}$ for an at most countable set I. The associated graph E_A to A is the directed graph with vertex set I and edge set the subset of $I \times I$ for which $A(i, j) = 1$. Paths in E_A consist of sequences of vertices $(i_k) \subseteq I^{\mathbb{N}}$ such that $A(i_k, i_{k+1}) = 1$. A cycle is a path $(i_1, \ldots, i_n)$ such that $A(i_n, i_1) = 1$, and a cycle has an exit if there is a vertex $j \in I$ such that $A(i_k, j) = 1$ and $j \neq i_{k+1}$ for some vertex i_k of the cycle (where $i_{n+1} = i_1$).

Assuming that A has no zero-rows, a generalized Cuntz-Krieger C^*-algebra $\mathcal{O}_A$ is constructed in [17] as follows: Suppose that $\{S_i \mid i \in I\}$ is a set of partial isometries indexed by the vertex set I such that the following conditions are satisfied

(i) $S_i^* S_i$ commutes with $S_j^* S_j$ for all i, j,

(ii) $S_i^* S_j = 0$ if $i \neq j$,

(iii) $(S_i^* S_i) S_j = A(i, j) S_j$ for all i, j,

(iv) For each pair of finite subsets $X, Y \subseteq I$ such that

$$A(X, Y, j) := \prod_{x \in X} A(x, j) \prod_{y \in Y} (1 - A(y, j))$$

vanishes for all but finitely many j's, the equation

$$\prod_{x \in X} S_x^* S_x \prod_{y \in Y} (1 - S_y^* S_y) = \sum_{j \in I} A(X, Y, j) S_j S_j^*$$

holds.

The universal C^*-algebra generated by such a set $\{S_i \mid i \in I\}$ is denoted by $\mathcal{O}_A$.

Remark 1.5 Note that for every pair $i, j \in I$, the element $S_i S_j$ is non-zero if and only if the entry $A(i, j)$ is non-zero, i.e., there is an edge from vertex i to vertex j. If $\alpha = (i_1, \ldots, i_n)$ is a path, we define a partial isometry by $S_\alpha := S_{i_1} \cdots S_{i_n}$. Using relation (iii) stated above it follows for finite paths α and β that

$$S_\beta^* S_\alpha = \begin{cases} S_{\beta'}^* & \text{if } \beta = \alpha\beta' \\ S_\beta^* S_\beta & \text{if } \beta = \alpha \\ S_{\alpha'} & \text{if } \alpha = \beta\alpha' \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

(c.f. [24, Lemma 1.1]). Set

$$M = \{S_\alpha (\prod_{i \in X} S_i^* S_i) S_\beta^* \mid \alpha, \beta \in E_A^*,\ X \subseteq I, \text{finite}\},$$

where $S_\emptyset = 1$ and note that $S_i = S_i S_i^* S_i \in M$ for all $i \in I$. A computation using (1.5) and that initial projections commute, then reveals that Span$\{M\}$ is an algebra. Also, Span$\{M\}$ is closed under taking adjoints and $\mathcal{O}_A$ is indeed the norm-closure of Span$\{M\}$.

In section 1.3 the partial isometries, associated to the edges of a row-finite graph E without sinks, satisfy relations determined by the edge matrix of E (see relation 1.4). Not every $\{0,1\}$-matrix is the edge matrix of a graph, however, given a $\{0,1\}$-matrix A one can always associate a graph E_A as above, such that A is then the *vertex matrix* of E_A. The vertex matrix of a graph associated to a matrix A, equals A, that is, $A_{E_A} \cong A$, and $E_{A_E} \cong E$ for a graph E.

The class of $\mathcal{O}_A$ for row-finite matrices A is obtained by the graph C^*-algebras of section 1.3. Indeed, if A is a row-finite, $I \times I$, $\{0,1\}$-matrix with no zero rows and E_A is the associated graph, then $\mathcal{O}_A$ is isomorphic to the C^*-algebra $C^*(E_A)$ (cf. [25, Prop. 4.1]). To see this, let $D = \{S_i S_j S_j^* \in \mathcal{O}_A \mid A(i,j) = 1 \text{ for } i,j \in I\}$. One can check that D consists of partial isometries that satisfy relations (i)-(iv) and that $S_i = \sum_{\{j\colon A(i,j)=1\}} S_i S_j S_j^*$. Since A is a $\{0,1\}$-matrix there is a map $\pi\colon C^*(E_A) \to \mathcal{O}_A$ defined on the generators by $\pi(S_e) = S_{s(e)} S_{r(e)} S_{r(e)}^*$. One observes that π is a surjective $*$-homomorphism. Since A has no zero rows, i.e., every vertex of E_A emits an edge, $\pi(P_v)$ is non-zero for every vertex $v \in E_A^0$. It is now an easy application of the gauge invariant uniqueness theorem, Theorem 1.3, to show that π is injective (noting that by universality of $\mathcal{O}_A$ there exists a canonical gauge action $\beta\colon \mathbb{T} \to \mathcal{O}_A$ defined on the generators by $\beta_t(S_i) = tS_i$ for $t \in \mathbb{T}$ and $i \in I$).

1.4.1 AF-algebras

The class of C^*-algebras $\mathcal{O}_A$ encompass all AF-algebras (see [16]). The theorem that follows below, characterizes AF-algebras among the C^*-algebras $\mathcal{O}_A$ and is a generalization of [24, Theorem 2.4]. First we state a lemma.

Lemma 1.6 *Let A be an $I \times I$-matrix with no zero rows and let E_A be the associated graph. If E_A has a cycle of period n with no exits, then there is a projection $P \in \mathcal{O}_A$ such that the corner $P\mathcal{O}_A P$ is $*$-isomorphic to $M_n(C(\mathbb{T}))$.*

Proof: Let $\alpha = (i_1, \ldots, i_n)$ be a cycle of period n with no exits, i.e., $i_{n+1} = i_1$, $A(i_n, i_1) = 1$ and if $A(i_k, j) = 1$ then $j = i_{k+1}$ for $j \in I$ and $k = 1, \ldots, n$. The associated partial isometries $S_{i_1}, \ldots, S_{i_n}$ satisfy that $S_{i_k}^* S_{i_k} = S_{i_{k+1}} S_{i_{k+1}}^*$, hence the initial projections $S_{i_k}^* S_{i_k}$ are mutually orthogonal and the partial isometry $S_\alpha := S_{i_1} \cdots S_{i_n}$ satisfies that $S_\alpha^* S_\alpha = S_\alpha S_\alpha^*$. Since for every set $\{c_{i_1}, \ldots, c_{i_n}\}$ of scalars of norm 1, the universal C^*-algebra generated by $\{c_{i_1} S_{i_1}, \ldots, c_{i_n} S_{i_n}\}$ is $*$-isomorphic to $C^*(S_{i_1}, \ldots, S_{i_n})$, it follows that $\mathbb{T}$ is contained in the spectrum of S_α. Hence S_α is a partial unitary with full spectrum, whence $C^*(S_\alpha) \cong C(\mathbb{T})$. There is a $*$-isomorphism of $C^*(S_{i_1}, \ldots, S_{i_n})$ onto $C^*(S_\alpha) \otimes M_n$ given by

$$S_{i_k} \mapsto \begin{cases} 1 \otimes e_{k,k+1} & \text{if } k \in \{1, \ldots, n-1\}, \\ S_\alpha \otimes e_{n,1} & k = n, \end{cases}$$

where 1 is the unit of $C^*(S_\alpha)$ and $e_{j,k}$ are matrix units (cf. [11, Remark 3.9]). Hence $C^*(S_{i_1}, \ldots, S_{i_n}) \cong C(\mathbb{T}) \otimes M_n$.

Set $P = \sum_{k=1}^n S_{i_k} S_{i_k}^*$ and note that $PS_j = S_j = S_j P$ if $j \in \{i_1, \ldots, i_n\}$ and $PS_j = 0 = S_j P$ otherwise. Also,

$$P(\prod_{i \in X} S_i^* S_i)P = \begin{cases} 0 & \text{if } A(i, i_k) = 0, i \in X, i_k \in \{i_1, \ldots, i_n\} \\ \sum_{k \in J}^n S_{i_k} S_{i_k}^* & \text{otherwise. } (J \subseteq \{1, \ldots, n\}). \end{cases}$$

Hence for every finite subset $X \subseteq I$ and every pair of finite paths $\alpha, \beta \in E_A^*$ (including the empty path) it follows that

$$PS_\alpha(\prod_{i\in X} S_i^* S_i)S_\beta^* P \in C^*(S_{i_1}, \dots, S_{i_n}).$$

This shows that the corner $P\mathcal{O}_A P$ is contained in $C^*(S_{i_1}, \dots, S_{i_n})$. Using once more that $PS_j = S_j = S_j P$ if $j \in \{i_1, \dots, i_n\}$, $P\mathcal{O}_A P$ equals $C^*(S_{i_1}, \dots, S_{i_n})$ and is thus $*$-isomorphic to $M_n(C(\mathbb{T}))$. □

Remark 1.7 In case of row-finite graphs E with no sinks, the argument given in the proof of Lemma 1.6 applies such that if E has a cycle of period n with no exit, then $C^*(E)$ contains a projection P such that the corner $PC^*(E)P$ is $*$-isomorphic to $M_n(C(\mathbb{T}))$.

Theorem 1.8 (cf. [24, Theorem 2.4]) *Let A be an $I \times I$-matrix with no zero rows and let E_A be the associated graph. Then $\mathcal{O}_A$ is an AF-algebra if and only if E_A has no cycles.*

Proof: Suppose that E_A has no cycles. We show that $\mathcal{O}_A$ has the local finiteness property, i.e., that every finite set of elements in $\mathcal{O}_A$ can be approximated by finitely many elements in a finite dimensional C^*-subalgebra of $\mathcal{O}_A$. Let $F \subseteq I$ be a finite subset and set $B = C^*(\{S_i \mid i \in F\})$. Then B is a C^*-subalgebra of $\mathcal{O}_A$ and we show that B is finite dimensional. Set

$$M_F = \{S_\alpha(\prod_{i\in F_0} S_i^* S_i)S_\beta^* \mid \alpha, \beta \in F^*, F_0 \subseteq F\},$$

By the argument given in remark 1.5 it follows that Span$\{M_F\}$ is a $*$-algebra containing S_i for all $i \in F$. Since F is finite and the graph E_A has no cycles, there can only be finitely many paths with vertices in F, hence Span$\{M\}$ is finite dimensional and equals B. This shows that $\mathcal{O}_A$ is an AF-algebra.

For the converse statement, suppose that α is a cycle in E_A of period n. If α has an exit then $\mathcal{O}_A$ contains an infinite projection by the argument given in the proof of [24, Theorem 2.4], hence $\mathcal{O}_A$ can not be an AF-algebra. If α has no exits then there is a projection $P \in \mathcal{O}_A$ such that the corner $P\mathcal{O}_A P$ is $*$-isomorphic to $M_n(C(\mathbb{T}))$ by Lemma 1.6. A corner of an AF-algebra is an AF-algebra, but $M_n(C(\mathbb{T}))$ is not AF (since $\mathbb{T}$ is not totally disconnected) and hence $\mathcal{O}_A$ is not an AF-algebra in this case either. This completes the proof. $\square$

Remark 1.9 Theorem 1.8 is inspired by [24, Theorem 2.4] and is easily translated to give a new proof in case of row-finite graphs E with no sinks such that $C^*(E)$ is an AF-algebra if and only if E has no cycles.

1.5 C^*-algebras of r-discrete groupoids

In this section we consider the theory of topological groupoids and their associated C^*-algebras. The main source is the book [31]. We will also refer to the paper [1]. For the purpose only to obtain a certain completeness of this text, we give the basic definitions of groupoid theory and sketch the construction of the reduced C^*-algebra of an r-discrete groupoid. In section 1.5.1 we consider groupoids associated to certain dynamical systems.

Basic definitions

Definition 1.10 *A groupoid is a set G with a subset $G^{(0)}$ (the set of units), maps d, r of G into $G^{(0)}$, an associative composition $(\gamma_1, \gamma_2) \in G^2 \mapsto \gamma_1\gamma_2 \in G$, where*

$$G^2 = \{(\gamma_1, \gamma_2) \in G \times G \colon d(\gamma_1) = r(\gamma_2)\},$$

and there is an involution $\gamma \mapsto \gamma^{-1}$ on G such that

(i) *For all $\gamma \in G$:* $\quad \gamma\gamma^{-1} = r(\gamma), \quad \gamma^{-1}\gamma = d(\gamma).$

(f) *For all $x \in G^{(0)}$:* $\quad d(x) = r(x) = x.$

(e) *For all $(\gamma_1, \gamma_2) \in G^2$:* $\quad d(\gamma_1\gamma_2) = d(\gamma_2), \quad r(\gamma_1\gamma_2) = r(\gamma_1).$

(u) *For all $\gamma \in G$:* $\quad \gamma d(\gamma) = \gamma, \quad r(\gamma)\gamma = \gamma.$

One may think of G as a set of "arrows", $G^{(0)}$ a subset of "units" or "endpoints", maps d, r as the "domain" and "range", the associative partially defined product $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$ as "composition of arrows" and the involution $\gamma \mapsto \gamma^{-1}$ on G as "reverse arrow".

We will consider a specific example in section 1.5.1 and refer to [31] for more examples. We will only consider groupoids G equipped with a topology such that the involuton and composition in G are continuous maps and G is locally compact.

G is called *r-discrete* if r is a local homeomorphism, that is, for every $\gamma \in G$ there is an open neighbourhood V such that r is, when restricted to V, a homeomorphism onto an open subset of $G^{(0)}$. For r-discrete groupoids, the fibers $r^{-1}(u)$ are discrete for every $u \in G^{(0)}$. A subset $S \subseteq G$ is a *bisection* if r and d are injective when restricted to S. We note that G is r-discrete if and only if it has a basis of open bisections.

The route to associate a C^*-algebra $C^*(G)$ (or $C_r^*(G)$) to a locally compact groupoid G, is to make the space $C_c(G)$ of continuous functions with compact support into a $*$-algebra, give to it the smallest C^*-norm making its representations continuous, and let $C^*(G)$ be the completion. Within this process one has to choose a left Haar system of measures on G (see [31, Def. I2.2]) (for instance to define a convolution product on $C_c(G)$). This choice is not canonical in the sense that $C^*(G)$ depends on it ([31, Cor. 2.11]). However, we will restrict ourselves to r-discrete groupoids, and for those, the counting measures form a left Haar system.

The C^*-algebra construction

We will now sketch the construction of the C^*-algebra of an r-discrete groupoid.

Let G be a second countable locally compact r-discrete groupoid. The space $C_c(G)$ of continuous functions with compact support is made into a (convolutive) $*$-algebra by imposing the following operations:

$$(fg)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}, \quad \gamma \in G, \quad f, g \in C_c(G).$$

Notice that for $\gamma \in G$ the fiber $r^{-1}(\gamma)$ is discrete since G is r-discrete, hence the sum above is finite for f and g with compact support.

$C_c(G)$ can be viewed as a $C_0(G^{(0)})$-module via the right multiplication

$$(\xi f)(\gamma) = \xi(\gamma) f(r(\gamma)), \quad \xi \in C_c(G), \quad f \in C_0(G^{(0)}),$$

and inner product with values in $C_0(G^{(0)})$

$$\langle \xi, \eta \rangle(u) = \sum_{r(\gamma)=u} \overline{\xi(\gamma)}\eta(\gamma), \quad \xi, \eta \in C_c(G), \quad u \in G^{(0)}.$$

Let E_G denote the Hilbert $C_0(G^{(0)})$-module obtained by completion of $C_c(G)$ with respect to the inner product norm. Let $\mathbb{B}(E_G)$ be the set of all adjointable maps in E_G (see [39, Def. 15.2.1]). When equipped with the operatornorm, $\mathbb{B}(E_G)$ is a C^*-algebra.

Let $\pi\colon C_c(G) \to \mathbb{B}(E_G)$ be defined by

$$(\pi(f)\xi)(\gamma) = \sum_{r(\beta)=r(\gamma)} f(\gamma^{-1}\beta)\xi(\beta), \quad f, \xi \in C_c(G).$$

Then π is a $*$-homomorphism and we define the *reduced* C^*-algebra, $C_r^*(G)$, of the groupoid G to be the closure of $\pi(C_c(G))$ in $\mathbb{B}(E_G)$.

Let $u \in G^{(0)}$ and let $l^2(r^{-1}(u))$ denote the usual Hilbert space. Define $\pi_u\colon C_c(G) \to l^2(r^{-1}(u))$ by

$$(\pi_u(f)\xi)(\gamma) = \sum_{r(\beta)=u} f(\gamma^{-1}\beta)\xi(\beta), \quad f \in C_c(G), \quad \xi \in l^2(r^{-1}(u)).$$

Then π_u is a representation of $C_c(G)$ and for $f \in C_c(G)$, the norm of f in $C_r^*(G)$ is given by

$$\|f\| = \sup_{u \in G^{(0)}} \|\pi_u(f)\|.$$

The class of groupoids treated in section 1.5.1 provides many examples.

1.5.1 Singly generated dynamical systems

Originally J. Renault was able to exhibit the Cuntz algebras of r-discrete groupoids associated to subshifts of finite type. We will now consider a more general class of r-discrete groupoids.

These groupoids model many interesting C^*-algebras, in particular the C^*-algebras obtained in sections 1.3 and 1.4.

Following [32] (cf. [3]) a singly generated dynamical system, denoted (X, T), consists of a locally compact, second countable Hausdorff space X and a local homeomorphism T from an open subset dom(T) of X onto an open subset ran(T) of X. The semi-direct product groupoid $G(X, T)$ attached to (X, T) is defined as follows. Let

$$G(X,T) = \{(x, m-n, y) \mid (x,y) \in \text{dom}(T^m)\times\text{dom}(T^n), T^m(x) = T^n(y) \text{ for } n, m \in \mathbb{N}\},$$

with the set of composable pairs

$$G^2 = \{((x, m-n, y), (z, k-l, v)) \mid y = z\},$$

groupoid operations

$$(x,k,y)(y,l,z) = (x, k+l, z), \quad (x,k,y)^{-1} = (y, -k, x)$$

and topology defined by the basic open sets

$$\mathcal{U}(U, m, n, V) = \{(x, m-n, y) \mid (x,y) \in U \times V \text{ such that } T^m(x) = T^n(y)\},$$

where U and V are open subsets of dom(T^m) resp. dom(T^n) on which T^m resp. T^n are injective.

The groupoid $G(X, T)$ of (X, T) is a locally compact Hausdorff groupoid, where the range and source maps, $r(x, m-n, y) = x$ and $d(x, m-n, y) = y$, are local homeomorphisms. The unit space is given by $G^0 = \{(x, 0, x) \mid x \in X\}$ and may be identified with X. The Haar system is given by the counting measures on the fibers $r^{-1}(x)$ for $x \in X$. (See [13, Theorem 1] for details on $G(X, T)$). The full and reduced C^*-algebra $C^*(X, T)$ of the convolution $*$-algebra $C_c(G(X, T))$ constructed above coincide and $C^*(X, T)$ is nuclear (since $G(X, T)$ is amenable by [32, Proposition 2.4], see [2]). For more information on the C^*-algebra of a singly generated dynamical system, we refer to [32], [13], [14] and [15].

Example 1.11 Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph without sinks, as defined in section 1.3. Let

$$X_E = \{(e_i)_{i=1}^{\infty} \in E^{\infty} \mid r(e_i) = s(e_{i+1}) \text{ for } i \in \mathbb{N}\}$$

be the infinite path space of E, and let $T_E\colon X_E \to X_E$ be the one-sided shift, i.e., $T_E((e_i)) = (e_{i+1})$. Then (X_E, T_E) is a singly generated dynamical system. The cylinder set

$$Z(\alpha) = \{(e_i)_{i=1}^{\infty} \in X_E \mid e_1 = \alpha_1, \dots e_{|\alpha|} = \alpha_{|\alpha|}\},$$

where $\alpha \in E^*$ is a finite path of length $|\alpha|$ and the set $\{Z(\alpha) \mid \alpha \in E^*\}$ generates a locally compact topology on X_E in which $Z(\alpha)$ is open and compact.

Let $G(X_E, T_E)$ be the associated groupoid as above. $G(X_E, T_E)$ is called the *path-groupoid* of E. Define for finite paths $\alpha, \beta \in E^*$ a subset of $G(X_E, T_E)$ by

$$Z(\alpha, \beta) = \{(x, |\alpha| - |\beta|, y) \mid x \in Z(\alpha), y \in Z(\beta) \text{ and } T_E^{|\alpha|}x = T_E^{|\beta|}y\}.$$

The set $\{Z(\alpha, \beta) \mid \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$ form a basis of open compact bisections which generate a locally compact topology on $G(X_E, T_E)$ by [25, Proposition 2.6].

Now, let $C^*(X_E, T_E)$ be the C^*-algebra associated to $G(X_E, T_E)$ as above.

Claim: There is an isomorphism $C^*(E) \cong C^*(X_E, T_E)$ such that $S_e \mapsto 1_{Z(e,r(e))}$ for every $e \in E^1$, where $1_{Z(e,r(e))}$ is the characteristic function of the subset

$$Z(e, r(e)) = \{(x, 1, y) \mid x_0 = e \text{ and } T_E(x) = y\}$$

of $G(X_E, T_E)$, for every $e \in E^1$.

The proof can be found in [25] and [29] and we give a sketch; Notice first that the family $\{1_{Z(e,r(e))} \mid e \in E^1\}$ is a $*$-representation of E, that is, a family of partial isometries in $C^*(X_E, T_E)$ satisfying the Cuntz-Krieger relations (1.2) and (1.3) determined by E. Hence by universality there is a $*$-homomorphism $\pi\colon C^*(E) \to C^*(X_E, T_E)$ such that $\pi(S_e) = 1_{Z(e,r(e))}$.

It is an observation that $C^*(X_E, T_E)$ is generated by $\{1_{Z(\alpha,\beta)} \mid \alpha, \beta \in E^*\}$ and since $\pi(S_\alpha S_\beta^*) = 1_{Z(\alpha,r(\alpha))} 1_{Z(\beta,r(\beta))^{-1}} = 1_{Z(\alpha,\beta)}$, it follows that π is surjective. Let $c\colon G(X_E, T_E) \to \mathbb{Z}$ be the one-cocycle $c(x, m-n, y) = m - n$ and let $\beta\colon \mathbb{T} \to \mathrm{Aut}(C^*(X_E, T_E))$ be the induced action defined by

$$(\beta_t f)(\gamma) = t^{c(\gamma)} f(\gamma), \quad f \in C_c(G(X_E, T_E)).$$

(See [31, Proposition 5.1]). Let $\alpha\colon \pi \to \mathrm{Aut}(C^*(E))$ be the gauge action defined in section (1.3). It is easily checked that $\pi \circ \alpha_t = \beta_t \circ \pi$ for every $t \in \mathbb{T}$. The graph E is assumed to be without sinks, hence $\pi(P_v) = 1_{Z(v,v)} \neq 0$ and it follows by the gauge invariant uniqueness theorem, Theorem 1.3, that π is injective. Hence $C^*(E) \cong C^*(X_E, T_E)$ as claimed.

In [32] it is shown that the generalized Cuntz-Krieger algebra $\mathcal{O}_A$, arising from a not necessarily row-finite matrix A and treated in section 1.4, is modelled by a singly generated dynamical system. The infinite path space is obtained by a compactification of X_E above, and the shift is only defined on a dense subset. We will refer to this example in section 4.2.1. For other examples of C^*-algebras of dynamical systems (X, T) we refer to [14].

Chapter 2

ON STABILITY OF C^*-ALGEBRAS

Joint work with Mikael Rørdam

2.1 Characterization of stability in terms of projections

The main theorem of this section, Theorem 2.3 below, gives a characterization of stability for C^*-algebras that admit a countable approximate unit consisting of projections. A more general version of this theorem was (later) found in Theorem 2.4 in which stability is characterized for σ-unital C^*-algebras. Theorem 2.3 is therefore a reformulation of Theorem 2.4 in the case where the C^*-algebra admits a countable approximate unit consisting of projections. However, relying only on the theory of projections in C^*-algebras, the proof of Theorem 2.3 is much more direct.

Lemma 2.1 *Let A be a C^*-algebra and let $(p_n)_{n=1}^\infty$ be an approximate unit for A consisting of projections. For every projection $q \in A$ there exist a sequence of projections $(\tilde{p}_n)_{n=1}^\infty$ in A such that $\tilde{p}_n \geq q$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\tilde{p}_n - p_n\| = 0$.*

Proof: Since $(p_n)_{n=1}^\infty$ is an approximate unit for A we get that $\|p_n q p_n - q\| \to 0$ as $n \to \infty$, and hence $\|(p_n q p_n)^2 - p_n q p_n\| \to 0$ as $n \to \infty$. By a continuous function calculus argument there is a sequence of projections $(\hat{p}_n)_{n=1}^\infty$ such that $\hat{p}_n \in p_n A p_n$ and $\|p_n q p_n - \hat{p}_n\| \to 0$ as $n \to \infty$. Hence $\|\hat{p}_n - q\| \to 0$ as $n \to \infty$ by the triangle inequality.

Find $n_0 \in \mathbb{N}$ such that $\|\hat{p}_n - q\| < 1$ for all $n \geq n_0$. For every $n \geq n_0$ there are unitaries $u_n \in \tilde{A}$ such that $\hat{p}_n = u_n q u_n^*$ and $\|u_n - 1\| \to 0$ as $n \to \infty$. Set $\tilde{p}_n = u_n^* p_n u_n$ for $n \geq n_0$, and set $\tilde{p}_n = q$ otherwise. Then $\tilde{p}_n \in A$, $\tilde{p}_n \geq q$ and $\|\tilde{p}_n - p_n\| \to 0$ as $n \to \infty$. $\square$

For a C^*-algebra A denote by $P(A)$ the set of projections of A.

Lemma 2.2 *Let A be a C^*-algebra and let $(p_n)_{n=1}^\infty$ be an approximate unit for A consisting of projections. The following statements are equivalent:*

(i) For all $p, q \in P(A)$ there exists $r \in P(A)$ such that $p \perp r$ and $q \sim r$.

(ii) For all $p \in P(A)$ there exists $q \in P(A)$ such that $p \perp q$ and $p \sim q$.

Proof: (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Let $p, q \in P(A)$. By Lemma 2.1 there is a sequence $(\tilde{p}_n)_{n=1}^\infty$ of projections satisfying

$$p \leq \tilde{p}_n, \qquad \lim_{n\to\infty} \|\tilde{p}_n - p_n\| = 0.$$

Since $\|p_nqp_n - q\| \to 0$ as $n \to \infty$ we obtain that $\|\tilde{p}_nq\tilde{p}_n - q\| < 1$ for some n, and this implies that q is equivalent to a subprojection of $\tilde{p}_n$. By assumption there is a projection p' with $p' \sim \tilde{p}_n$ and $p' \perp \tilde{p}_n$. Hence $p' \gtrsim q$ and since $p \leq \tilde{p}$ it follows that $p' \perp p$, and we may therefore take r to be a subprojection of p' with $r \sim q$. □

Theorem 2.3 *Let A be a C^*-algebra which admits a countable approximate unit consisting of projections. Then A is stable if and only if for each projection $p \in A$ there is a projection $q \in A$ such that $p \sim q$ and $p \perp q$.*

Proof: The "if-part". Note that A possess property (i) of Lemma 2.2. Let $(p_n)_{n=1}^\infty$ be an approximate unit for A consisting of projections. We first construct a system of projections $\{q_n^i\}$, where $n \in \mathbb{N}$ and $1 \leq i \leq n$ such that

$$\begin{array}{ccccccccc}
q_1^1 & \leq q_2^1 & \leq q_3^1 & \leq q_4^1 & \leq & \cdots \\
 & q_2^2 & \leq q_3^2 & \leq q_4^2 & \leq & \cdots \\
 & & q_3^3 & \leq q_4^3 & \leq & \cdots \\
 & & & q_4^4 & \leq & \cdots \\
 & & & & & \ddots
\end{array}$$

where the projections in each column are mutually orthogonal and equivalent, $q_{n+1}^i - q_n^i \sim q_{n+1}^j - q_n^j$ for $n \in \mathbb{N}$ and $i, j = 1, \ldots, n$, and such that the sequence $(q_n^1 + q_n^2 + \cdots + q_n^n)_{n=1}^\infty$ is an approximate unit for A.

In the initial step of the construction we set $q_1^1 = p_1$.

Assume now that $n \geq 1$, and that we have constructed $q_m^1, q_m^2, \ldots, q_m^m$ with the desired properties, and such that

$$\|p_m(q_m^1 + q_m^2 + \cdots + q_m^m)p_m - p_m\| \leq \tfrac{1}{m},$$

for all $m \leq n$. (Observe that if this inequality holds for all $m \in \mathbb{N}$, then $(q_n^1 + q_n^2 + \cdots + q_n^n)_{n=1}^\infty$ is an approximate unit for A.)

By Lemma 2.1 there is a projection $\tilde{p} \in A$ such that

$$\tilde{p} \geq q_n^1 + q_n^2 + \cdots + q_n^n, \qquad \|p_{n+1}\tilde{p}p_{n+1} - p_{n+1}\| \leq \tfrac{1}{n+1}.$$

Set

$$r^1 = \tilde{p} - (q_n^1 + q_n^2 + \cdots + q_n^n), \qquad q_{n+1}^1 = q_n^1 + r^1.$$

Successive applications of Lemma 2.2 yield projections $r^2, r^3, \ldots, r^n$ in A satisfying

$$r^i \sim r^1, \qquad r^i \perp (q_n^1 + q_n^2 + \cdots + q_n^n) + (r^1 + r^2 + \cdots + r^{i-1}).$$

Set $q_{n+1}^i = q_n^i + r^i$ for $2 \leq i \leq n$. One more application of Lemma 2.2 produces a projection q_{n+1}^{n+1} which is equivalent to q_{n+1}^1 and orthogonal to $q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^n$.

Since $\tilde{p} \leq q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^{n+1}$, we get

$$\|p_{n+1}(q_{n+1}^1 + q_{n+1}^2 + \cdots + q_{n+1}^{n+1})p_{n+1} - p_{n+1}\| \leq \|p_{n+1}\tilde{p}p_{n+1} - p_{n+1}\| \leq \tfrac{1}{n+1},$$

and this completes the construction of the system $\{q_n^i\}$.

By the properties of the system $\{q_n^i\}$ there exist partial isometries $\{v_n^i\}$ in A such that

$$(v_n^i)^* v_n^i = q_n^1, \qquad v_n^i (v_n^i)^* = q_n^i, \qquad v_n^i q_{n-1}^1 = v_{n-1}^i.$$

For each $n \in \mathbb{N}$ set $A_n = (q_n^1 + q_n^2 + \cdots + q_n^n)A(q_n^1 + q_n^2 + \cdots + q_n^n)$, $B_n = q_n^1 A q_n^1$ and let $\iota_{A_n} \colon A_n \to A_{n+1}$ be the inclusion map. Let $\{e_{ij}\}$ denote the matrix units in the C^*-algebra of $n \times n$ matrices, and define an isomorphism $\sigma_n \colon A_n \to M_n \otimes B_n$ by

$$\sigma_n(a) = \sum_{1 \leq i,j \leq n} e_{ij} \otimes (v_n^i)^* a v_n^j.$$

Let $\psi_n \colon M_n \to M_{n+1}$ be the embedding into the upper left-hand corner, let $\iota_{B_n} \colon B_n \to B_{n+1}$ be the inclusion map, and define $\varphi_n \colon M_n \otimes B_n \to M_{n+1} \otimes B_{n+1}$ by $\varphi_n = \psi_n \otimes \iota_{B_n}$. By the choice of the partial isometries (v_n^i) we have $\varphi_n \circ \sigma_n = \sigma_{n+1} \circ \iota_{A_n}$.

The inductive limit C^*-algebra $\lim_{n\to\infty}(A_n, \iota_{A_n})$ equals $\overline{\bigcup_{n=1}^\infty A_n}$, and this union is A, because $(q_n^1 + q_n^2 + \cdots + q_n^n)_{n=1}^\infty$ is an approximate unit for A. The inductive limit C^*-algebra $\lim_{n\to\infty}(M_n \otimes B_n, \varphi_n)$ is isomorphic to $\mathcal{K} \otimes B$, where $B = \lim_{n\to\infty} B_n$. We thus obtain the following commuting diagram:

$$\begin{array}{ccccccccccccc}
A_1 & \overset{\iota_{A_1}}{\hookrightarrow} & A_2 & \hookrightarrow & \cdots & \hookrightarrow & A_n & \overset{\iota_{A_n}}{\hookrightarrow} & A_{n+1} & \hookrightarrow & \cdots & \to & A \\
\| & & \sigma_2 \big\downarrow \cong & & & & \sigma_n \big\downarrow \cong & & \sigma_{n+1} \big\downarrow \cong & & & & \\
B_1 & \xrightarrow{\varphi_1} & M_2 \otimes B_2 & \longrightarrow & \cdots & \longrightarrow & M_n \otimes B_n & \xrightarrow{\varphi_n} & M_{n+1} \otimes B_{n+1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K} \otimes B
\end{array}$$

The intertwining induces an isomorphism between A and $\mathcal{K} \otimes B$, and A is therefore stable.

The "only if-part". Since A is stable, A is isomorphic to $\overline{\bigcup_{n=1}^{\infty} M_n(A)}$ $(= D)$. Let $p \in P(D)$. There is $n \in \mathbb{N}$ and $p' \in P(M_n(A))$ such that $\|p' - p\| < 1$. Hence $p = up'u^*$ for some unitary u in $\tilde{D}$. There is a projection q' in $M_{2n}(A)$ with $q' \sim p'$ and $q' \perp p'$. It follows that $(q =)\ uq'u^*$ is orthogonal and equivalent to p. □

2.2 Characterization of stable C^*-algebras

The main result of this section (Theorem 2.4 below) gives a characterization of stable C^*-algebras. The C^*-algebras in question are assumed to be σ-unital, i.e. they admit a countable approximate unit. Recall that an element a in a C^*-algebra A is *strictly positive* if $\varphi(a) > 0$ for every non-zero positive linear functional φ on A, and that A contains a strictly positive element if and only if A is σ-unital (see [30, Prop. 3.10.5]). Recall also that every separable C^*-algebra is σ-unital.

Denote by A^+ the positive cone of A. For positive elements a, b in A define:

$$\begin{aligned}
a \sim b \quad &\Leftrightarrow \quad \exists x \in A : x^*x = a \quad \text{and} \quad xx^* = b, \\
a \perp b \quad &\Leftrightarrow \quad ab = 0 \quad (= ba).
\end{aligned}$$

The following functions will be used throughout this paper. For each $\varepsilon > 0$ define (continuous) functions $h_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ and $f_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \varepsilon \\ t - \varepsilon & \text{if} \quad t \ge \varepsilon \end{cases}, \qquad f_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \varepsilon \\ \varepsilon^{-1}t - 1 & \text{if} \quad \varepsilon \le t \le 2\varepsilon \\ 1 & \text{if} \quad t \ge 2\varepsilon \end{cases}.$$

Notice that $h_\varepsilon(0) = f_\varepsilon(0) = 0$ for every $\varepsilon > 0$, hence $h_\varepsilon(a)$ and $f_\varepsilon(a)$ belongs to A for every positive element $a \in A$.

Let $F(A)$ denote the set of positive elements which have a multiplicative unit, i.e.

$$F(A) = \big\{ a \in A^+ \mid \exists\, b \in A^+ : ab = a \big\}.$$

Note that if $a \in A^+$ then $h_\varepsilon(a)$ and $f_\varepsilon(a)$ belong to $F(A)$ for all $\varepsilon > 0$ because $f_{\varepsilon/2}(a)$ is a multiplicative identity for these elements. For all $\varepsilon > 0$ the function h_ε has the property that $\|a - h_\varepsilon(a)\| \le \varepsilon$ for all $a \in A^+$, and hence $F(A)$ is dense in A^+.

Let $\mathcal{U}_0(\tilde{A})$ be the set of all unitaries u in $\tilde{A}$ such that u is homotopy equivalent to the unit, that is, there is a continuous function $v\colon [0,1] \to \mathcal{U}(\tilde{A})$ such that $v(0) = 1$ and $v(1) = u$. $\mathcal{U}_0(\tilde{A})$ is sometimes called the connected component of the group of unitary elements in $\tilde{A}$ that contains the unit. Let $M(A)$ denote the multiplier algebra of A.

Theorem 2.4 *Let A be a C^*-algebra which is σ-unital. The following six statements are equivalent:*

(a) A is stable.

(b) For all $a \in F(A)$ and all $\varepsilon > 0$ there are $b, c \in A^+$ such that $\|a - b\| < \varepsilon$, $b \sim c$ and $\|bc\| < \varepsilon$.

(c) For all $a \in F(A)$ there exists $b \in A^+$ such that $a \sim b$ and $a \perp b$.

(d) For all $a \in F(A)$ there exists a unitary $u \in \mathcal{U}_0(\tilde{A})$ such that $uau^* \perp a$.

(e) For all $a \in A^+$ and all $\varepsilon > 0$ there exists $b \in A^+$ such that $a \sim b$ and $\|ab\| < \varepsilon$.

(f) There is a sequence of mutually orthogonal and equivalent projections $(E_n)_{n=1}^\infty$ in $M(A)$ such that the infinite sum $\sum E_n$ converges to the unit 1 in the strict topology on $M(A)$.

We give the proof in the following order:

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b)

(c) $\Rightarrow$ (f) $\Rightarrow$ (a).

First we need some preliminary results.

Lemma 2.5 *If $a \in F(A)$, then there is $e \in F(A)$ with $ea = a$ $(= ae)$ and $\|e\| = 1$.*

Proof: Note that if $a, b \in A^+$ satisfy $ab = a$, then for each continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $f(1) = 1$ we have $f(b)a = a$ (this is clearly true, if f is a polynomial with $f(1) = 1$). Hence, $e = f_{\frac{1}{2}}(b)$ will have the desired properties. □

Lemma 2.6 *Let A be a C^*-algebra and assume $b, c \in A^+$ satisfy $b \sim c$. Then for each $\varepsilon > \|b^{\frac{1}{2}}c^{\frac{1}{2}}\|^{\frac{1}{4}}$ there exists a unitary $u \in \mathcal{U}(\tilde{A})$ such that $uf(b)u^* = f(c)$ for each continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ being zero on $[0, \varepsilon]$.*

Proof: Let $\varepsilon > 0$ be given. Assume $x \in A$ satisfies $x^*x = b$ and $xx^* = c$. Let $x = vb^{\frac{1}{2}}$ be the polar decomposition of x, where v is a partial isometry in A^{**}. Notice that $x = c^{\frac{1}{2}}v$. By assumption, $\|x^2\|^{\frac{1}{2}} = \|vb^{\frac{1}{2}}c^{\frac{1}{2}}v\|^{\frac{1}{2}} = \|b^{\frac{1}{2}}c^{\frac{1}{2}}\|^{\frac{1}{2}} < \varepsilon^2$. Since $\|x^2\|^{\frac{1}{2}} \geq \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(x)\}$, the spectral radius of x, we obtain that $\mathrm{dist}(x, \mathrm{GL}(A)) < \varepsilon^2$, i.e. the distance from x to the invertibles of A is less than ε^2.

For each $t \in \mathbb{R}^+$ set $E_t = 1_{[0,t]}(|x|) \in A^{**}$, the spectral projection corresponding to the interval $[0, t]$ for $|x|$. By [33, Thm. 2.2] there is a unitary u in $\mathcal{U}(\tilde{A})$ such that $v(1 - E_{\varepsilon^2}) = u(1 - E_{\varepsilon^2})$. By this identity we obtain for all continuous functions $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ being zero on $[0, \varepsilon^2]$ that $vg(|x|) = ug(|x|)$. Since $vg(|x|)v^* = g(|x^*|)$, we get that $g(|x^*|) = ug(|x|)u^*$.

Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be any continuous function which is zero on $[0, \varepsilon]$, and set $g(t) = f(t^2)$. Then $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, and g is zero on $[0, \varepsilon^2]$. It follows that

$$f(c) = f(|x^*|^2) = g(|x^*|) = ug(|x|)u^* = uf(|x|^2)u^* = uf(b)u^*,$$

as desired. □

Lemma 2.7 *For each $\varepsilon > 0$ and for each $K < \infty$ there is $\delta > 0$ such that for every C^*-algebra A and for every pair of positive elements b, c in A, if $\|b\| \leq K$, $\|c\| \leq K$, and $\|bc\| \leq \delta$, then $\|b^{1/2}c^{1/2}\| \leq \varepsilon$, and $\|b^{1/2}c\| \leq \varepsilon K^{1/2}$.*

Proof: Let b and c be positive elements in a C^*-algebra A. Approximate the square root of each of these by elements $p(b)$ and $p(c)$ for suitable polynomials p with vanishing constant term. Now, by considering the norm of $p(b)p(c)$ for such p, the assumption easily gives the desired result. □

Lemma 2.8 *If A is a C^*-algebra satisfying property (b) of Theorem 2.4, then for each $a \in F(A)$ and each $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(\tilde{A})$ such that $\|auau^*\| < \varepsilon$.*

Proof: Let $a \in F(A)$ and $\varepsilon > 0$ be given. We may without loss of generality assume that $\|a\| \leq 1$. Find $\delta > 0$ such that $7\delta + 4\delta^2 < \varepsilon$. By Lemma 2.7 and by the assumption that property (b) of Theorem 2.4 holds we can find $b, c \in A^+$ satisfying $\|b - a\| < \delta$, $b \sim c$, $\|bc\| < \delta$ and $\|b^{1/2}c^{1/2}\| < \delta^4$. By Lemma 2.6 there is a unitary $u \in \mathcal{U}(\tilde{A})$ such that $h_\delta(c) = uh_\delta(b)u^*$. Notice that $\|a - h_\delta(b)\| < 2\delta$ (because $\|a - b\| < \delta$ and $\|b - h_\delta(b)\| \leq \delta$) and notice also that $\|b\|$ and $\|c\|$ are less than $1 + \delta$. We can now make the following estimate:

$$
\begin{aligned}
& \|auau^*\| \\
\leq\ & \|auau^* - h_\delta(b)uau^*\| + \|h_\delta(b)uau^* - h_\delta(b)uh_\delta(b)u^*\| + \|h_\delta(b)h_\delta(c) - bc\| + \|bc\| \\
<\ & 2\delta + \|h_\delta(b)\|2\delta + \delta(\|b\| + \|h_\delta(c)\|) + \delta \\
\leq\ & 2\delta + (1+\delta)2\delta + \delta(2 + 2\delta) + \delta \ = \ 7\delta + 4\delta^2 \ < \ \varepsilon.
\end{aligned}
$$

□

For every strictly positive element a in A define

$$F_a(A) = \big\{ b \in A^+ \mid \exists\, \varepsilon > 0 : f_\varepsilon(a)b = b \big\}.$$

Notice that $F_a(A) \subseteq F(A)$.

Lemma 2.9 *Let A be a σ-unital C^*-algebra which satisfies property (c) of Theorem 2.4. For every strictly positive element $a \in A^+$ it follows that:*

(i) For all $b \in F_a(A)$ there exists $c \in F_a(A)$ with $b \sim c$ and $b \perp c$.

(ii) For all $\varepsilon > 0$ there is a projection $G \in M(A)$ satisfying $1 - G \perp f_\varepsilon(a)$, $G \sim 1$, and $1 - G \gtrsim 1$.

In order to prove Lemma 2.9 we need the some facts about properly infinite projections summarized in the remarks and in the lemma below. A C^*-subalgebra B of a C^*-algebra A is said to be *full* if it is not contained in any proper two-sided closed ideal of A. A projection p in A is *full* if the (hereditary) C^*-subalgebra pAp is full in A. A projection p is said to be *properly infinite* if there exist two projections p_1 and p_2, each Murray-von Neumann equivalent to p, such that $p_1 + p_2 \leq p$ (in particular $p_1 \perp p_2$).

If p is a properly infinite, full projection and if $p \leq q$, then q is properly infinite and full. If p and q are properly infinite, full projections, then $p \lesssim q$ and $q \lesssim p$. It can be deduced from [10, Section 1] that any two properly infinite full projections in a C^*-algebra A are Murray-von Neumann equivalent if they define the same element of $K_0(A)$, and that if A contains at least one properly infinite, full projection, then every element of $K_0(A)$ is represented by a properly infinite, full projection. The lemma below follows easily from these facts:

Lemma 2.10 *Let A be a unital C^*-algebra. If e and f are projections in A such that $f \leq e$ and $e - f$ dominates a properly infinite projection which is full in A, then there is a projection q in A such that*

(i) $f \leq q \leq e$, and

(ii) $q \sim 1$ and $e - q \gtrsim 1$.

Proof: By assumption there is a properly infinite full projection $p \in A$ such that $p \leq e - f$. Let p_1 and p_2 be two projections in A each Murray-von Neumann equivalent to p, such that $p_1 + p_2 \leq p$. Note that p_1 and p_2 are both properly infinite full projections that are mutually orthogonal.

Notice the above mentioned properties shared by properly infinite full projections. Containing a properly infinite full projection, $1 - f$ is itself a properly infinite full projection. Since $1 - f$ and p_1 are both properly infinite and full there is a subprojection $r \leq p_1$ such that $r \sim 1 - f$. Set $q = r + f$. Then $f \leq q \leq e$, $q \sim 1$ and since $p_2 \leq e - q$ it follows that $e - q \gtrsim 1$. □

Proof of Lemma 2.9:

(i). Suppose $a \in A^+$ is strictly positive. Let $b \in F_a(A)$ and find $\varepsilon > 0$ such that $bf_\varepsilon(a) = b = f_\varepsilon(a)b$. Since $f_\varepsilon(a) \in F(A)$ there exists $y \in A$ such that $f_\varepsilon(a) = yy^*$ and $f_\varepsilon(a) \perp y^*y$. We show the existence of an element $x \in A$ such that $b = x^*x$, $xx^* \in F_a(A)$ and $f_\varepsilon(a) \perp xx^*$. Because a is strictly positive, there exists a $\delta > 0$ such that $(\delta_0 =)\ \|yf_\delta(a)^2y^* - f_\varepsilon(a)\| < \frac{1}{2}$. By [34, Prop. 2.2] there is $r \in A$ such that $ryf_\delta(a)^2y^*r^* \geq f_{\frac{1}{2}}(f_\varepsilon(a))$ (one may take $r = (\frac{1}{2} - \delta_0)^{\frac{1}{2}} f_{\frac{1}{2}}(f_\varepsilon(a))^{\frac{1}{2}}$), and by [34, Lemma 2.3] there is an $s \in A$ such that

$$s(ryf_\delta(a)^2y^*r^*)^{\frac{1}{2}}s^* = f_{\frac{1}{2}}(f_\varepsilon(a)).$$

Observe that $f_{\frac{1}{2}}(f_\varepsilon(a))b = b = bf_{\frac{1}{2}}(f_\varepsilon(a))$. Let $v|ryf_\delta(a)|$ be the polar decomposition of $ryf_\delta(a)$, where $v \in A^{**}$ is a partial isometry, and put $x = b^{\frac{1}{2}}sv|ryf_\delta(a)|^{\frac{1}{2}} \in A$. Then

$$\begin{aligned} xx^* &= b^{\frac{1}{2}}sv|ryf_\delta(a)|v^*s^*b^{\frac{1}{2}} = b^{\frac{1}{2}}s|(ryf_\delta(a))^*|s^*b^{\frac{1}{2}} \\ &= b^{\frac{1}{2}}s(ryf_\delta(a)^2y^*r^*)^{\frac{1}{2}}s^*b^{\frac{1}{2}} = b^{\frac{1}{2}}f_{\frac{1}{2}}(f_\varepsilon(a))b^{\frac{1}{2}} = b. \end{aligned}$$

Since $f_{\delta/2}(a)f_\delta(a) = f_\delta(a)$ we see that $f_{\delta/2}(a)x^*x = x^*x$, and so $x^*x \in F_a(A)$. Also,

$$|ryf_\delta(a)|^2 f_\varepsilon(a) = f_\delta(a)y^*r^*ryf_\delta(a)f_\varepsilon(a) = f_\delta(a)y^*r^*ryf_\varepsilon(a)f_\delta(a) = 0,$$

from which we see that $x^*x \perp f_\varepsilon(a)$, and hence $x^*x \perp b$. We may therefore set $c = x^*x$.

(ii). Suppose $a \in A$ is a strictly positive element of norm 1. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$ let $g_n\colon [0,1] \to \mathbb{R}^+$ be piecewise linear functions satisfying

(α) g_1 is zero on $[0, \frac{1}{2}]$,

(β) g_n is zero outside the interval $[\frac{1}{n+1}, \frac{1}{n-1}]$ for all $n \geq 2$,

(γ) $\sum_{n=1}^\infty g_n(t) = 1$ for all $t \in (0,1]$.

Then the infinite sum $\sum g_n(a)$ converges strictly to the unit 1 in $M(A)$.

We shall inductively construct sequences $(b_n)_{n=1}^\infty$, $(c_n)_{n=1}^\infty$ in $F_a(A)$ and $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in A such that the set $\{b_n \mid n \in \mathbb{N}\} \cup \{c_n \mid n \in \mathbb{N}\}$ consists of mutually orthogonal elements,

$$b_n = x_n^*x_n, \quad c_n = y_n^*y_n, \quad g_n(a) = x_nx_n^* = y_ny_n^*,$$

and such that b_n is orthogonal to $f_\varepsilon(a)$, and b_n and c_n are orthogonal to $f_{1/n}(a)$.

Let $n \in \mathbb{N}$ be given, and suppose, if $n \geq 2$, that elements b_j, c_j, x_j, y_j, $j \leq n-1$, with the desired properties have been constructed. Choose δ with $0 < \delta < \frac{1}{2}\min\{\varepsilon, 1/(n+1)\}$ and such that $f_\delta(a)$ is a (two-sided) multiplicative identity for the elements $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}$ in $F_a(A)$. By (i), there are $d \in F_a(A)$ and $z \in A$ such that $d \perp f_\delta(a)$, $z^*z = d$ and $zz^* = f_\delta(a)$. Set $x_n = g_n(a)^{\frac{1}{2}}z$. Then $x_n^*x_n = z^*g_n(a)z$ and $x_nx_n^* = g_n(a)^{\frac{1}{2}}f_\delta(a)g_n(a)^{\frac{1}{2}} = g_n(a)$. By letting $b_n = x_n^*x_n$ we obtain that $b_n \in F_a(A)$ and $b_n \perp f_\delta(a)$, because b_n lies in the hereditary C^*-subalgebra of A generated by d. Hence b_n is orthogonal to $b_1, \dots, b_{n-1}, c_1, \dots, c_{n-1}$, $f_\varepsilon(a)$ and $f_{1/n}(a)$. By the same argument we can construct c_n in $F_a(A)$ and y_n in A.

Because a is strictly positive, it follows that $(f_{1/n}(a))_{n=1}^\infty$ is an approximate unit for A. Since each of x_n, y_n, b_n, c_n is orthogonal to $f_{1/(n-1)}(a)$, we see that the infinite sums $B = \sum b_n$, $C = \sum c_n$, $V = \sum x_n$ and $W = \sum y_n$ are strictly convergent, and therefore belong to $M(A)$. Since $b_i \perp c_j$ for all $i, j \in \mathbb{N}$, B and C are orthogonal. Also, since all b_n are orthogonal to $f_\varepsilon(a)$, so is B. By orthogonality of the sequence $(b_n)_{n=1}^\infty$ we see that $VV^* = \sum x_nx_n^* = \sum g_n(a) = 1$, and, similarly, $WW^* = 1$. It follows that $Z_1 = V^*V$ and $Z_2 = W^*W$ are projections in $M(A)$ that are equivalent to 1. Moreover, Z_1 lies in the hereditary subalgebra of $M(A)$ generated by B since $BZ_1B = \sum x_n^*g_n(a) \sum g_m(a)x_m = Z_1$, and Z_2 lies in the hereditary subalgebra of $M(A)$ generated by C. Hence $Z_1 \perp Z_2$ and $Z_1 \perp f_\varepsilon(a)$. We have thus shown that the unit 1 in $M(A)$ is properly infinite.

Now, Z_1, being equivalent to 1, is properly infinite and full in $M(A)$. Lemma 2.10 then provides a projection G such that $1 - Z_1 \leq G \leq 1$, $G \sim 1$ and $1 - G \gtrsim 1$. Since $Z_1 \perp f_\varepsilon(a)$ we obtain that $1 - G \perp f_\varepsilon(a)$.

□

Proof of Theorem 2.4:

(a) ⇒ (b): Assume A is stable. Then A is isomorphic to the C^*-algebra $B = \overline{\bigcup_{n=1}^\infty M_n(A)}$ where $M_n(A)$ is embedded in the upper left hand corner of $M_{n+1}(A)$. Let $a \in B^+$ and $\varepsilon > 0$ be given. Find $n \in \mathbb{N}$ and $c \in M_n(A)^+$ such that $\|a - c\| < \varepsilon$. Let $x \in M_{2n}(A)$ be defined by

$$x = \begin{pmatrix} 0 & c^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}.$$

Then $x^*x \perp xx^*$ and $\|a - xx^*\| < \varepsilon$, and we are done.

(b) $\Rightarrow$ (c): Let $a \in F(A)$ and find $e \in F(A)$ with $\|e\| = 1$ and $ea = ae = a$ as in Lemma 2.5. By Lemma 2.7 and Lemma 2.8 there is a unitary $u \in \mathcal{U}(\tilde{A})$ such that $(\delta =)\ \|(ueu^*)^{\frac{1}{2}} e (ueu^*)^{\frac{1}{2}}\| < 1$. Set $x = ue^{\frac{1}{2}}$, set $y = (1-e)^{\frac{1}{2}} x a^{\frac{1}{2}}$, and observe that $y \in A$, $y^*y = a - a^{\frac{1}{2}} x^* e x a^{\frac{1}{2}}$ and $y^*y \perp yy^*$. Let $v|x|$ be the polar decomposition for x, where v is a partial isometry in A^{**}.

Notice that $|x^*| = (xx^*)^{\frac{1}{2}} = (ueu^*)^{\frac{1}{2}}$. Hence $x^*ex = v^*|x^*|e|x^*|v = v^*(ueu^*)^{\frac{1}{2}} e (ueu^*)^{\frac{1}{2}} v$ which shows that $\|x^*ex\| = \delta$. Therefore $y^*y = a - a^{\frac{1}{2}} x^* e x a^{\frac{1}{2}} \geq (1-\delta)a$. By [34, Lemma 2.3] there is $r \in A$ such that $a = r^*(y^*y)^{\frac{1}{2}} r$. Let $w|y|$ be the polar decomposition of y and put $z = w|y|^{\frac{1}{2}} r$. Then $z^*z = r^*|y|^{\frac{1}{2}} w^* w |y|^{\frac{1}{2}} r = r^*|y|r = a$ and $zz^* = w|y|^{\frac{1}{2}} r r^* |y|^{\frac{1}{2}} w^* = |y^*|^{\frac{1}{2}} w r r^* w^* |y^*|^{\frac{1}{2}}$. Since $|y^*| \perp a$ it follows that $zz^* \perp a$, and we may set $b = zz^*$.

(c) $\Rightarrow$ (d): Let $a \in F(A)$ and find $e \in F(A)$ with $\|e\| = 1$ and $ea = ae = a$ as in Lemma 2.5. By (c) there are $f \in F(A)$ orthogonal to e and $x \in A$ such that $x^*x = e$ and $xx^* = f$. Notice that $x + x^*$ is a self-adjoint element of norm ≤ 1 (by orthogonality), and $(x + x^*)^2 = xx^* + x^*x$. Let

$$u = x + x^* + i(1 - xx^* - x^*x)^{1/2}.$$

Then u is a unitary and $uau^* = xax^* \perp a$. To see that $u \in \mathcal{U}_0(\tilde{A})$, let $v\colon [0,1] \to \mathcal{U}(\tilde{A})$ be a continuous function defined by

$$v(t) = a_t + i(1 - a_t^2)^{\frac{1}{2}}, \qquad a_t = (1-t)1 + t(x + x^*).$$

Since a_t is selfadjoint with $\|a_t\| \leq 1$ for all $t \in [0,1]$, $\{v(t)\}$ is the desired path of unitaries from 1 to u.

(d) $\Rightarrow$ (e): Let $a \in A^+$ and $\varepsilon > 0$ be given. Since $F(A)$ is dense in A^+ there is an element $a_0 \in F(A)$ with $\|a - a_0\|(\|a\| + \|a_0\|) < \varepsilon$. By assumption there is a unitary $u \in \tilde{A}$ such that

$ua_0u^* \perp a_0$. Set $x = ua^{1/2}$, and set $b = xx^* = uau^*$. Then $a = x^*x$, and

$$\begin{aligned}\|ab\| &\leq \|ab - a_0ua_0u^*\| + \|a_0ua_0u^*\| \\ &\leq \|ab - aua_0u^*\| + \|aua_0u^* - a_0ua_0u^*\| + 0 \\ &\leq \|a\|\|uau^* - ua_0u^*\| + \|a - a_0\|\|ua_0u^*\| \\ &= (\|a\| + \|a_0\|)\|a - a_0\| < \varepsilon.\end{aligned}$$

(e) ⇒ (b): Let $a \in F(A)$ and $\varepsilon > 0$ be given. Then there exist $x \in A$ and $c \in A^+$ such that $x^*x = a$, $xx^* = c$, and $\|ac\| < \varepsilon$. Setting $b = a$ we see that property (b) holds.

(c) ⇒ (f): Let $a \in A$ be strictly positive with $\|a\| = 1$. We construct inductively a sequence of mutually orthogonal projections $E_1, E_2, \dots \in M(A)$ so that

$$E_1 \sim E_2 \sim \cdots \sim E_n \sim 1, \quad 1-(E_1+E_2+\cdots+E_n) \gtrsim 1, \quad \|(1-(E_1+E_2+\cdots+E_n))a\| < 1/n$$

holds for each $n \in \mathbb{N}$. The infinite sum $\sum E_n$ will then converge strictly to 1, and the proof will be completed. We shall in the following use the fact that if $1 - E \perp f_\varepsilon(a)$ for $E \in M(A)$ then

$$\|(1-E)a\| \leq \|(1-E)a - (1-E)f_\varepsilon(a)a\| + \|(1-E)f_\varepsilon(a)a\| \leq \varepsilon.$$

The existence of E_1 follows from Lemma 2.9. Suppose $n \geq 1$ and that $E_1, E_2, \dots, E_n$ have been found. Set $E = E_1 + E_2 + \cdots + E_n$. Then $1 - E \gtrsim 1$ and $\|(1-E)a\| < 1/n$. We must find a multiplier projection E_{n+1} such that

$$E_{n+1} \perp E, \quad E_{n+1} \sim 1, \quad 1 - (E + E_{n+1}) \gtrsim 1, \quad \|(1 - E - E_{n+1})a\| < \tfrac{1}{n+1}.$$

Since $1 - E \gtrsim 1$ there is a projection $F \in M(A)$ such that $1 - E \geq F$ and $F \sim 1$. The latter implies that the C^*-algebra FAF is isomorphic to A.

We assert that FaF is a strictly positive element of FAF. Assume to the contrary that $\varphi(FaF) = 0$ for a non-zero positive functional φ on FAF. Let $\tilde{\varphi}$ be the positive functional on A defined by $\tilde{\varphi}(x) = \varphi(FxF)$ for $x \in A$. Since $\tilde{\varphi}(a) = 0$ and $a \in A$ is strictly positive it follows that $\tilde{\varphi} = 0$. But then $\varphi = 0$.

Notice that $M(FAF) = FM(A)F$. Lemma 2.9 (ii) provides for $\varepsilon > 0$, chosen such that $\varepsilon + \varepsilon^{\frac{1}{2}} < 1/(n+1)$, a projection $G \in FM(A)F$ satisfying

$$G \sim F \sim 1, \qquad F - G \gtrsim F \sim 1, \qquad \|FaF - G(FaF)\| \ (= \|(F-G)aF\|) \ \leq \varepsilon.$$

We proceed to show that $\|(F-G)a\| < 1/(n+1)$. From the inequality $\|paq\|^2 \leq \|pap\|\|qaq\|$, which holds when a is positive and p, q are projections, and since $\|a\| = 1$ we obtain that

$$\begin{aligned} \|(F-G)a(1-F)\| &\leq \|(F-G)a(F-G)\|^{\frac{1}{2}}\|(1-F)a(1-F)\|^{\frac{1}{2}} \\ &\leq \|(F-G)a(F-G)\|^{\frac{1}{2}} \\ &\leq \|(F-G)aF\|^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\|(F-G)a\| \leq \|(F-G)aF\| + \|(F-G)a(1-F)\| < \varepsilon + \varepsilon^{\frac{1}{2}} < \frac{1}{n+1}.$$

Lemma 2.10 provides a projection $E_{n+1} \in M(A)$ such that

$$1 - E - (F-G) \leq E_{n+1} \leq 1 - E, \qquad E_{n+1} \sim 1, \qquad 1 - E - E_{n+1} \gtrsim 1.$$

Since $\|(1 - E - E_{n+1})a\| \leq \|(F-G)a\| < 1/(n+1)$, the projection E_{n+1} is as wanted.

(f) ⇒ (a): Set $P_n = \sum_{j=1}^{n} E_j$, and let A_n be the hereditary C^*-subalgebra P_nAP_n of A. Since the projections E_j are mutually equivalent, there are partial isometries $V_1, V_2, V_3, \ldots$ in $M(A)$ such that $V_1 = E_1$, $V_j^*V_j = E_1$ and $V_jV_j^* = E_j$ for all $j \geq 2$. Let M_n denote the C^*-algebra of n by n matrices over $\mathbb{C}$, and let $\{e_{ij}\}$ be the standard system of matrix units for M_n. Define an isomorphism $\sigma_n \colon A_n \to M_n \otimes A_1$ by

$$\sigma_n(b) = \sum_{1 \leq i,j \leq n} e_{ij} \otimes V_i^* b V_j, \qquad b \in A_n.$$

For each $n \in \mathbb{N}$ let $\iota_{A_n} \colon A_n \to A_{n+1}$ be the inclusion map, let $\psi_n \colon M_n \to M_{n+1}$ be the embedding into the upper left-hand corner, and define $\varphi_n \colon M_n \otimes A_1 \to M_{n+1} \otimes A_1$ by $\varphi_n = \psi_n \otimes \mathrm{id}_{A_1}$. From the construction of σ_n we see that $\varphi_n \circ \sigma_n = \sigma_{n+1} \circ \iota_{A_1}$.

The inductive limit C^*-algebra $\lim_{n\to\infty}(A_n, \iota_{A_n})$ is equal to $\overline{\bigcup_{n=1}^\infty A_n} = \overline{\bigcup_{n=1}^\infty P_n A P_n}$. Since $(P_n)_{n=1}^\infty$ converges strictly to 1 this union is A. All in all we obtain the following commuting diagram:

$$\begin{tikzcd}
A_1 \arrow[r, hook, "\iota_{A_1}"] \arrow[d, equal] & A_2 \arrow[r, hook] \arrow[d, "\sigma_2"', "\cong"] & \cdots \arrow[r, hook] & A_n \arrow[r, hook, "\iota_{A_n}"] \arrow[d, "\sigma_n"', "\cong"] & A_{n+1} \arrow[r, hook] \arrow[d, "\sigma_{n+1}"', "\cong"] & \cdots \arrow[r] & A \\
A_1 \arrow[r, "\varphi_1"] & M_2 \otimes A_1 \arrow[r] & \cdots \arrow[r] & M_n \otimes A_1 \arrow[r, "\varphi_n"] & M_{n+1} \otimes A_1 \arrow[r] & \cdots \arrow[r] & \mathcal{K} \otimes A_1
\end{tikzcd}$$

This intertwining yields an isomorphism $A \to \mathcal{K} \otimes A_1$, and therefore A is stable. □

2.3 Some applications of the characterization theorem

In this section we present some corollaries to Theorem 2.4.

Corollary 2.11 *If A is the inductive limit of a sequence of stable σ-unital C^*-algebras, then A is stable.*

Proof: By assumption A is the inductive limit of a sequence

$$A_1 \to A_2 \to A_3 \to \cdots$$

of stable σ-unital C^*-algebras A_n. Let $\mu_n\colon A_n \to A$ be the associated homomorphisms. Notice that A is σ-unital because each A_n is σ-unital and the sequence is countable. We show that A satisfies property (b) of Theorem 2.4. Let $a \in F(A)$ and $\varepsilon > 0$ be given. Find $n \in \mathbb{N}$ and $b_0 \in F(A_n)$ such that $\|a - \mu_n(b_0)\| < \varepsilon$. Since A_n is stable there is by Theorem 2.4 an element c_0 in A_n^+ such that $b_0 \perp c_0$ and $b_0 \sim c_0$. Set $b = \mu_n(b_0)$ and $c = \mu_n(c_0)$. Then $b \perp c$, $b \sim c$, and $\|a - b\| < \varepsilon$ as desired. □

Lemma 2.12 *For each $\varepsilon > 0$ and for each $K < \infty$ there is a $\delta > 0$ so that the following holds: For every C^*-algebra A, and for every set of positive elements a_1, a_2, b_1, b_2 in A, if*

$$\|a_i\| \le K, \quad \|b_j\| \le K, \quad \|(a_1 + a_2)(b_1 + b_2)\| \le \delta,$$

then $\|a_i^{1/2}b_j\| \le \varepsilon$ *and* $\|a_ib_j\| \le \varepsilon K^{1/2}$

Proof: Choose $\delta_1 > 0$ such that $\delta_1^{1/2}(2K^3)^{1/4} < \varepsilon$. By Lemma 2.7 we can find $\delta > 0$ such that $\|(a_1+a_2)(b_1+b_2)\| \le \delta$, $\|a_i\| \le K$, and $\|b_j\| \le K$ implies $\|(a_1+a_2)^{1/2}(b_1+b_2)\| \le \delta_1$.

Notice that

$$0 \le (a_1+a_2)^{1/2}b_j(a_1+a_2)^{1/2} \le (a_1+a_2)^{1/2}(b_1+b_2)(a_1+a_2)^{1/2} \le \delta_1(2K)^{1/2}\cdot 1.$$

Set $x_j = (a_1+a_2)^{1/2}b_j^{1/2}$. Then $\|x_j\|^2 \le \delta_1(2K)^{1/2}$. Now,

$$0 \le b_j^{1/2}a_ib_j^{1/2} \le b_j^{1/2}(a_1+a_2)b_j^{1/2} = x_j^*x_j.$$

This shows that $\|a_i^{1/2}b_j^{1/2}\|^2 = \|b_j^{1/2}a_ib_j^{1/2}\| \le \|x_j^*x_j\| \le \delta_1(2K)^{1/2}$. Hence $\|a_i^{1/2}b_j\|^2 \le \delta_1(2K)^{1/2}K \le \varepsilon^2$, and $\|a_ib_j\| \le \varepsilon K^{1/2}$. □

Corollary 2.13 *Let A be a stable separable C^*-algebra. For each positive $a \in A$ of norm at most 1, the hereditary C^*-subalgebra $\overline{(1-a)A(1-a)}$ of A is stable.*

Proof: Set $B = \overline{(1-a)A(1-a)}$. Notice that B is σ-unital because A and hence B are separable. We show that B satisfies property (b) of Theorem 2.4. Let $b \in F(B)$ be given. Since A is stable, and using that $F(A)$ is dense in A^+ it follows from Theorem 2.4 (d) that there is a sequence $(u_n)_{n=1}^\infty$ of unitaries in $\tilde{A}$ such that $\|u_n(a+b)u_n^*(a+b)\|$ tends to zero. By Lemma 2.12 this implies that $\|u_nb^{1/2}u_n^*a\|$ and $\|u_nbu_n^*b\|$ tend to zero for large n.

Put $x_n = (1-a)u_nb^{1/2} \in B$ and $y_n = u_nb^{1/2} \in A$. Then

$$\|x_n - y_n\| = \|(x_n-y_n)^*\| = \|b^{1/2}u_n^*a\| = \|u_nb^{1/2}u_n^*a\| \to 0.$$

Since $y_n^*y_n = b$, and since $(y_ny_n^*)(y_n^*y_n) = u_nbu_n^*b$ tends to zero, we obtain that

$$\lim_{n\to\infty}\|x_n^*x_n - b\| = 0, \qquad \lim_{n\to\infty}\|(x_nx_n^*)(x_n^*x_n)\| = 0.$$

This shows that property (b) of Theorem 2.4 holds for B. □

We shall in the next two results consider inclusions $B \subseteq A$ of C^*-algebras with the property that B contains an approximate unit which is also an approximate unit for A. Notice that for such an inclusion $B \subseteq A$ necessarily *every* approximate unit for B is an approximate unit for A. This condition is again equivalent to the property that for each $a \in A$ and each $\varepsilon > 0$ there exists $e \in B$ with $0 \leq e \leq 1$ such that $\|a - ae\| < \varepsilon$. Since $F(B)$ is dense in B^+, we can in this case also find $e \in F(B)$ with $\|a - ae\| < \varepsilon$.

Proposition 2.14 *Let A be a σ-unital C^*-algebra and suppose $B \subseteq A$ is a C^*-subalgebra containing an approximate unit that is also an approximate unit for A. If B is stable, then so is A.*

Proof: Let $a \in F(A)$ and $\varepsilon > 0$ be given. By assumption there exists $e \in F(B)$ such that $2\|a\|\|a - ae\| < \varepsilon$. By Theorem 2.4 there is a unitary $w \in \tilde{B}$ such that $e \perp wew^*$. Now $a \sim waw^*$, and we have the following estimate:

$$\begin{aligned} \|awaw^*\| &\leq \|awaw^* - aewaw^*\| + \|aewaw^* - aeweaw^*\| + \|aeweaw^*\| \\ &\leq (\|a - ae\| + \|a - ea\|)\|a\| + 0 \;<\; \varepsilon. \end{aligned}$$

By Theorem 2.4 this shows that A is stable. □

In Proposition 2.14 one cannot conclude that B is stable if it is known that A is stable. For example, let $B = c_0(\mathbb{N})$ be embedded as the diagonal in $A = \mathcal{K}$.

Corollary 2.15 *Let A be a σ-unital C^*-algebra, let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. If A is stable then the crossed product $A \rtimes_\alpha G$ is stable.*

Proof: By Proposition 2.14 it suffices to show that A contains an approximate unit for the crossed product $A \rtimes_\alpha G$. Let $(e_n)_{n=1}^\infty$ be any (bounded) approximate unit for A. The set of elements $x \in A \rtimes_\alpha G$ for which $\|x - e_n x\|$ tends to zero is a norm-closed linear subspace of $A \rtimes_\alpha G$. Since the subalgebra of all finite sums $\sum_{\gamma \in G} a_\gamma u_\gamma$, with coefficients $a_\gamma \in A$ and unitaries u_γ that implement the action of G, is dense in $A \rtimes_\alpha G$, it suffices to show that $\|a_\gamma u_\gamma - e_n a_\gamma u_\gamma\|$ tends to zero for large n. This, however, is trivially the case because $\|a_\gamma - e_n a_\gamma\|$ tends to zero for all

a_γ in A. $\square$

It may happen that $A \rtimes_\alpha G$ is stable without A being stable. For example the compacts $\mathcal{K}$ is isomorphic to $c_0(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ where α acts by left translation. The following proposition is not contained in [20] (or anywhere else, as far as we know).

Proposition 2.16 *Let (G, α, A) be a C^*-dynamical system with G second countable. If A is separable and stable then the crossed product C^*-algebra $A \rtimes_\alpha G$ is stable.*

Proof: By Theorem 2.4 $M(A)$ contains a sequence $(E_n)_{n=1}^\infty$ of mutually othogonal and equivalent projections such that the infinite sum $\sum E_n$ converges strictly to the unit 1 in $M(A)$. We shall prove the existence of a strictly continuous unital $*$-homomorphism of $M(A)$ into $M(A \rtimes_\alpha G)$. Then by Theorem 2.4 $A \rtimes_\alpha G$ is stable. $A \rtimes_\alpha G$ is separable since G is second countable and A is separable.

By [30, Theorem 7.6.6] there exist injective morphisms

$$\begin{aligned} \varphi\colon \quad & A \to M(A \rtimes_\alpha G) \\ \nu\colon \quad & G \to M(A \rtimes_\alpha G) \\ \psi\colon \quad & A \rtimes_\alpha G \to M(A \rtimes_\alpha G), \end{aligned}$$

where for each $f \in C_c(G, A)$

$$\psi(f) = \int_G \varphi(f(s))\nu_s \, d\lambda(s),$$

with respect to Haar measure λ on G.

Let $(e_n)_{n=1}^\infty \subset A$ be an approximate unit for A, $f \in C_c(G, A)$ and let K denote the support of f. By the equation

$$\varphi(e_n)\psi(f) - \psi(f) = \int_K \varphi(e_n f(s) - f(s))\nu_s \, d\lambda(s),$$

we get that

$$\|\varphi(e_n)\psi(f) - \psi(f)\| \leq \int_K \|e_n f(s) - f(s)\| \, d\lambda(s) \to \ 0 \text{ as } n \to \infty,$$

since K is compact. Similarly $\|\psi(f)\varphi(e_n) - \psi(f)\| \to 0$ as $n \to \infty$. This shows that $\varphi(e_n) \to 1$ strictly in $M(A \rtimes_\alpha G)$. By [27, Lemma 1.1] φ has a unique strictly continuous extension $\bar{\varphi} : M(A) \to M(A \rtimes_\alpha G)$ which is unital. This completes the proof. □

2.4 Related remarks

One can characterize stable AF-algebras as being precisely those AF-algebras that do not admit a bounded densely defined trace. This can be seen by using the classification of AF-algebras by their ordered K_0-group. This criterion for stability is stronger and more useful than our characterization theorem (Theorem 2.4). For example, it is easy to see that if $0 \to I \to A \to B \to 0$ is an extension of C^*-algebras, and if I and B do not admit any bounded, densely defined trace, then A does not admit a bounded, densely defined trace.

The proposition below contains a related (partial) characterization of stability that holds more generally. In [20] it was asked if the statement would remain true without condition (iii). In fact it will not — see section 2.5.

Proposition 2.17 *Let A be a σ-unital C^*-algebra, and suppose that*

(i) A admits no bounded, densely defined trace,

(ii) no non-zero quotient of A is unital, and

(iii) for every full hereditary subalgebra B of A, such that B does not admit any bounded, densely defined trace, and for every $a \in F(A)$ there exists $b \in B^+$ with $a \sim b$.

It follows that A is stable.

Conversely, if A is stable, then (i) and (ii) hold.

Proof: Let $a \in F(A)$ be given, and let B be the hereditary subalgebra of A consisting of all elements in A that are orthogonal to a. Let $e \in F(A)$ be such that $ae = a$. If B were contained in a proper ideal I in A, then $e + I$ would be a unit for A/I, thus contradicting assumption (ii). Hence B is full.

The domain of every densely defined trace on A contains the Pedersen ideal of A, and the Pedersen ideal contains $F(A)$. Thus $\tau(e) < \infty$ for every densely defined trace τ on A. Assume that the restriction of τ to B were bounded. Then for every $x \in A^+$,

$$\tau(x) = \tau(e^{1/2}xe^{1/2}) + \tau((1-e)^{1/2}x(1-e)^{1/2}) \leq \|x\|\tau(e) + \|\tau|_B\| \cdot \|x\|,$$

which shows that τ is bounded, in contradiction with assumption (i).

It follows that B satisfies the conditions of (iii), and B therefore contains an element b which is equivalent to a. Hence (c) of Theorem 2.4 holds, and A must be stable.

The last statement is trivial. □

If it turns out that there exist stably finite C^*-algebras without traces (i.e. that quasi-traces need not be traces), then one should sharpen (i) to exclude the existence of bounded densely defined quasi-traces. (Recall that Uffe Haagerup has proved that quasi-traces on exact C^*-algebras are traces, [18].)

Condition (iii) is easily seen to be satisfied for all AF-algebras, and it follows from [7] that every exact approximately divisible (cf. [22, Definition 5.5]) simple C^*-algebra satisfies property (iii).

Condition (ii) follows from condition (i) for every exact C^*-algebra A with the strong finiteness property that every quotient of A is stably finite. Indeed, if A/I were unital for some proper ideal I, then A/I would admit a bounded trace, being exact, stably finite and unital (see [18]). Hence A would admit a bounded trace.

It follows that if A is an exact C^*-algebra such that every quotient of A is stably finite, and such that (iii) holds, then A is stable if and only if A admits no bounded densely defined trace. This can be applied to AF-algebras.

If it were true that every C^*-algebra that satisfies (i) and (ii) of Proposition 2.17 is stable, then it would also follow that every simple C^*-algebra is either stably finite or purely infinite. Indeed, if A is simple and not stably finite, then so is every non-zero hereditary C^*-subalgebra B of A (by Brown's theorem [8]), and therefore every non-unital hereditary C^*-subalgebra B of A would be stable. The claim now follows from the proposition below:

Proposition 2.18 *Let A be a simple C^*-algebra, not of type I, and with the property that if B is a hereditary C^*-subalgebra of A, then either B is unital or B is stable. It follows that A is purely infinite.*

Proof: We must show that every non-zero hereditary C^*-subalgebra B of A contains an infinite projection (cf. [10]). Assume that B is unital. Then, since A is assumed to be not of type I, B is infinite dimensional, and there is an $a \in F(B)$ such that a is non-invertible and 0 is not an isolated point of the spectrum of a. The hereditary subalgebra $\overline{aAa}$ of B is then non-unital. Upon replacing B with $\overline{aAa}$, we may assume that B is non-unital, and thus, by assumption, stable.

It follows from [6, Thm. 1.2] that either B admits a dimension function defined on its Pedersen ideal, or B contains an infinite projection. We proceed to show that B does not admit a dimension function.

Suppose, to the contrary, that φ is a dimension function defined on the Pedersen ideal of B. Choose (arguing as above) $a, e \in F(B)$ such that $ae = a = ea$, $\|e\| = 1$, and such that the hereditary C^*-subalgebra $(D =)$ $\overline{aAa}$ is non-unital (and non-zero). Then D is stable by our assumption. On the other hand, $ex = xe = x$, whence $\varphi(x) \leq \varphi(e) < \infty$, for every $x \in D$. Hence φ is bounded. One easily deduces from item (c) of Theorem 2.4, that no stable C^*-algebra admits a bounded (non-zero) dimension function.

□

2.5 Epilogue

Since the paper [20] was submitted some land has been conquered, as will be described now. The question asked in section 1.1 whether one can conclude that A is stable if $M_2(A)$ is stable for a C^*-algebra A has been answered in the negative. In [37] an example of a (simple, separable, stable rank one) C^*-algebra B admitting an approximate unit consisting of projections was found with the property that $M_2(B)$ is stable while B is not stable.

An important consequence is that condition (iii) of Proposition 2.17 above is not superflous. Indeed, if B is the C^*-algebra found in [37] with the property that $M_2(B)$ is stable while B is not stable, then notice that B admits no non-zero bounded trace and no non-zero quotient of B is unital. Hence conditions (i) and (ii) do not characterize stability in general. In cases where they do, every simple C^*-algebra is either stably finite or purely infinite, as remarked above. It will be shown in Theorem 3.15, that the conditions (i) and (ii) characterize stability for C^*-algebras of graphs (see also remark 3.19).

Proposition 2.17 is the key ingredient in the theorem below which is a generalization of Zhang's dichotomy [40, Thm. 1.2] that every σ-unital purely infinite simple C^*-algebra is either unital or stable. The proof can be found in [22, Theorem 4.24].

Theorem 2.19 (Kirchberg and Rørdam) *Let A be a σ-unital purely infinite C^*-algebra. Then A is stable if and only if no non-zero quotient of A is unital.*

By [22, Proposition 4.17] every non-zero hereditary C^*-subalgebra of a purely infinite C^*-algebra is purely infinite. Also, a purely infinite C^*-algebra is not of type I (by [22, Proposition 4.4]). It follows that the condition of Proposition 2.18 holds for purely infinite and simple C^*-algebras.

Chapter 3

STABLE C^*-ALGEBRAS OF GRAPHS

In this chapter we consider the C^*-algebras of directed graphs introduced in section 1.3. We characterize stability for these C^*-algebras.

3.1 Stability and projections

By Theorem 2.3 a C^*-algebra A that admits a countable approximate unit consisting of projections is stable if and only if for each projection $p \in A$ there is a projection $q \in A$ such that $p \sim q$ and $p \perp q$. Lemma 3.3 below is a reformulation suited for C^*-algebras of graphs.

Lemma 3.1 *Let A be a C^*-algebra equipped with an increasing approximate unit $(p_n)_{n=1}^{\infty}$ consisting of projections. The following statements are equivalent:*

(i) $\forall\, n \in \mathbb{N} \,\exists\, m > n : p_n \lesssim p_m - p_n$.

(ii) $\forall\, p \in P(A) \,\exists\, q \in P(A) : p \sim q$ *and* $p \perp q$.

Proof:

(i) ⇒ (ii): Let $p \in A$ be a projection. By Lemma 2.1 there is $k \in \mathbb{N}$, $\tilde{p}_k \in P(A)$ and a unitary $u \in \tilde{A}$ such that

$$p \le \tilde{p}_k, \quad \tilde{p}_k = u^* p_k u$$

From this and the assumption it follows that

$$upu^* \le u\tilde{p}_k u^* = p_k \lesssim p_m - p_k$$

for some $m > k$. Hence there is a projection $\tilde{q} \in P(A)$ such that

$$upu^* \sim \tilde{q}, \quad \tilde{q} \perp upu^*$$

Put $q = u^* \tilde{q} u$ and we are done.

(ii) ⇒ (i): Let $n \in \mathbb{N}$. By assumption there is $q \in P(A)$ satisfying $q \sim p_n$ and $q \perp p_n$. Note that $q \in (1 - p_n)A(1 - p_n)$. For each $k \ge n$ set $\hat{p}_k = p_k - p_n$. Then $(\hat{p}_k)_{k=n}^{\infty}$ is an increasing approximate unit for $(1 - p_n)A(1 - p_n)$. By Lemma 2.1 there is $m \in \mathbb{N}$ and a projection

$r \in (1-p_n)A(1-p_n)$ such that

$$q \le r, \quad r \sim \hat{p}_m$$

Hence

$$p_n \sim q \precsim \hat{p}_m = p_m - p_n$$

and the proof is complete. □

Remark 3.2 Let E be a directed graph and let $\{P_v, S_e\}$ be the associated Cuntz-Krieger E-family. The set of vertices E^0 is infinite if and only if $C^*(E)$ is non-unital ([24, Proposition 1.4]). Suppose that $E^0 = \{v_j : j \in \mathbb{N}\}$ is infinite and define $p_n = \sum_{j=1}^{n} P_{v_j}$ for $n \in \mathbb{N}$. Then $(p_n)_{n=1}^{\infty}$ is a strictly increasing approximate unit for $C^*(E)$ consisting of projections.

Lemma 3.3 *Let E be a row-finite directed graph. Then $C^*(E)$ is stable if and only if for each finite subset $F \subseteq E^0$ there is a finite subset $G \subseteq E^0$ such that $F \cap G = \emptyset$ and $\sum_{v \in F} P_v \precsim \sum_{v \in G} P_v$.*

Proof: If $C^*(E)$ is stable then it is non-unital, hence E^0 is infinite and $(p_n)_{n=1}^{\infty}$ defined above is a strictly increasing approximate unit for $C^*(E)$ consisting of projections. By Theorem 2.3 and Lemma 3.1 the desired property holds. Conversely, if for each finite subset $F \subseteq E^0$ there is a finite subset $G \subseteq E^0$ such that $F \cap G = \emptyset$ and $\sum_{v \in F} P_v \precsim \sum_{v \in G} P_v$ then $(p_n)_{n=1}^{\infty}$ satisfies statement (i) of Lemma 3.1 and $C^*(E)$ is stable by Theorem 2.3. □

3.2 Lifting

This section deals with the lifting of equivalent projections in a quotient of a C^*-algebra (cf. Lemma 3.6). See [22, Lemma 4.11] for a proof of the following Lemma.

Lemma 3.4 *Let $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ be an extension of C^*-algebras and let e, f be projections in A. Then $\pi(e) \precsim \pi(f)$ if and only if there exists $x \in I^+$ such that $e \precsim f \oplus x$.*

In the following A is a C^*-algebra, $\mathcal{P} = \{q_n\}_{n=1}^{\infty}$ is an increasing sequence of projections in A, and $I(\mathcal{P})$ denotes the ideal in A generated by $\mathcal{P}$.

Lemma 3.5 *$I(\mathcal{P}) \otimes \mathcal{K}$ has an approximate unit consisting of projections.*

Proof: Put $B = \overline{\bigcup_{n=1}^{\infty} q_n A q_n}$. Then B is a full hereditary subalgebra of $I(\mathcal{P})$. By Brown's Theorem [8], $B \otimes \mathcal{K} \cong I(\mathcal{P}) \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ clearly admits an approximate unit consisting of projections. □

Lemma 3.6 *Assume $\mathcal{P}$ satisfies*

$$\forall p \in \mathcal{P}\, \exists q \in \mathcal{P}\colon q \geq p \text{ and } p \precsim q - p. \tag{3.1}$$

Let $\pi\colon A \to A/I(\mathcal{P})$ be the quotient homomorphism. Let $e, f \in A$ be projections such that $\pi(e) \precsim \pi(f)$. Then $e \precsim f \oplus p$ for some projection $p \in \mathcal{P}$.

Proof: By Lemma 3.4 there exists $x \in I(\mathcal{P})^+$ such that $e \precsim f \oplus x$. Embed $I(\mathcal{P})$ into $I(\mathcal{P}) \otimes \mathcal{K}$. For each $\varepsilon > 0$ define $\varphi_\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\varphi_\varepsilon(t) = \begin{cases} 0 & \text{for} \quad t \leq \varepsilon, \\ \varepsilon^{-1}(t - \varepsilon) & \text{for} \quad \varepsilon \leq t \leq 2\varepsilon, \\ 1 & \text{for} \quad t \geq 2\varepsilon. \end{cases}$$

Hence φ_ε is the function f_ε on page 24. By [34, Prop. 2.4] there is $\delta \in \left]0, \frac{1}{2}\right[$ such that

$$e \precsim f \oplus \varphi_\delta(x).$$

By Lemma 3.5 there is a projection $q \in I(\mathcal{P}) \otimes \mathcal{K}$ such that $\|x - qxq\| < \delta$. By [34, Prop. 2.2], which generalizes the fact that close projections are equivalent, $\varphi_\delta(x) \precsim qxq$. Hence

$$e \precsim f \oplus \varphi_\delta(x) \precsim f \oplus qxq \precsim f \oplus q.$$

From the description of $I(\mathcal{P}) \otimes \mathcal{K}$ in Lemma 3.5 there exists $p' \in \mathcal{P}$ and $m \in \mathbb{N}$ such that $q \precsim p' \otimes 1_m$. By the assumed property, (3.1), there exists $p \in \mathcal{P}$ such that $p' \otimes 1_m \precsim p$. This proves that $e \precsim f \oplus p$. □

3.3 Graph-trace

Definition 3.7 *Let E be a directed graph. A map $\tau_E \colon E^0 \to \mathbb{R}^+$ defined on the set of vertices of E is said to be a graph-trace if*

$$\tau_E(v) = \sum_{\{e\colon s(e)=v\}} \tau_E(r(e)),$$

and it is bounded if

$$\sum_{v\in E^0} \tau_E(v) < \infty.$$

Lemma 3.8 *Let E be a locally finite directed graph with no cycles. Then every bounded graph-trace on E extends to a bounded trace on $C^*(E)$.*

Proof: Let τ_E be a bounded graph-trace on E. Since E has no cycles $C^*(E)$ is an AF-algebra by Theorem 1.8 (see Remark 1.9). Furthermore there is an increasing sequence $E_1 \subseteq E_2 \subseteq \cdots$ of finite subgraphs of E such that

$$C^*(E) = \varinjlim (C^*(E_n), \iota_n)$$

where for each $n \in \mathbb{N}$, $C^*(E_n)$ is finite dimensional and ι_n is the inclusion map (see [24, Remark 2.5]). Let $n \in \mathbb{N}$ and let $v_1, \ldots, v_{k_n}$ denote the sinks of E_n for $k_n \in \mathbb{N}$ (there is at least one sink). Let $m(v_i)$ denote the number of paths with range v_i for $i = 1, \ldots, k_n$. Then since E has no cycles it follows from [24, Corollary 2.3] that

$$C^*(E_n) \;\cong\; \bigoplus_{i=1}^{k_n} M_{m(v_i)}(\mathbb{C}), \tag{3.2}$$

where the isomorphism in (3.2) maps the projection $P_{v_i} \in C^*(E_n)$ to a minimal projection in $M_{n(v_i)}(\mathbb{C})$. Hence for each $n \in \mathbb{N}$ there is a unique trace $\tau_n \colon C^*(E_n) \to \mathbb{C}$ such that $\tau_n(P_{v_i}) = \tau_E(v_i)$ for $i = 1, \ldots, k_n$. Also from [24, Corollary 2.3]

$$K_0(C^*(E_n)) \;=\; \langle\{[P_{v_i}]\colon i = 1, \ldots, k_n\}\rangle \cong \mathbb{Z}^{k_n},$$

and for every $v \in E^0$

$$[P_v] = \sum_{i=1}^{k_n} n_{v_i}[P_{v_i}] \tag{3.3}$$

where n_{v_i} is the number of paths from v to v_i.

Let $v \in E_n^0$. Note that the graph-trace relation can be written as

$$\tau_E(v) = \sum_{\{e\colon s(e)=v\}} \tau_E(r(e)) = \sum_{w \in R(v)} n_v^w \tau_E(w), \tag{3.4}$$

where $R(v) = \{w \in E^0 \colon s(e) = v \text{ and } r(e) = w \text{ for some } e \in E^1\}$ and $n_v^w \in \mathbb{N}$ is the number of edges from v to w. Since E_n is finite with no cycles, we consider the tree starting in vertex v and ending in the sinks $v_1, \ldots, v_{k_n}$. Repeated use of the graph-trace relation (3.4) gives that

$$\tau_E(v) = \sum_{i=1}^{k_n} n_{v_i}\tau_E(v_i). \tag{3.5}$$

Hence from (3.3) and (3.5) we get that

$$\tau_n(P_v) = \sum_{i=1}^{k_n} n_{v_i}\tau_n(P_{v_i}) = \sum_{i=1}^{k_n} n_{v_i}\tau_E(v_i) = \tau_E(v),$$

for all $v \in E_n^0$. By uniqueness τ_{n+1} extends τ_n. Define τ on the $*$-subalgebra $\bigcup_{n=1}^\infty C^*(E_n)$ of $C^*(E)$ by setting $\tau(a) = \tau_n(a)$ if $a \in C^*(E_n)$. Put $k = \sum_{v\in E^0} \tau_E(v)$. If $a \in \bigcup_{n=1}^\infty C^*(E_n)$ then

$$\begin{aligned} |\tau(a)| &\le \sum_{v \in E^0} \tau(P_v)\|a\| \\ &= \sum_{v \in E^0} \tau_E(v)\|a\| \\ &= k\|a\|. \end{aligned}$$

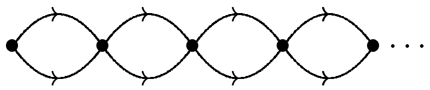
Hence $\|\tau\| < \infty$ and τ is a bounded linear function from $\bigcup_{n=1}^\infty C^*(E_n)$ to $\mathbb{C}$, so it extends to a bounded linear functional $\tilde{\tau}$ on $C^*(E)$. It is easily checked that $\tilde{\tau}$ is a trace on $C^*(E)$. This completes the proof. $\square$

Remark 3.9 Obviously the converse statement of Lemma 3.8 holds, i.e., a trace on $C^*(E)$ induces a graph-trace on E. The conclusion of Lemma 3.8 is likely to hold more general than for directed graphs with no cycles.

Lemma 3.10 *Let E be a locally finite directed graph with no cycles. Then $C^*(E)$ is stable if and only if E has no non-zero bounded graph-trace.*

Proof: Since E has no cycles, $C^*(E)$ is AF by Theorem 1.8 (Remark 1.9). Hence $C^*(E)$ is stable if and only if $C^*(E)$ admits no non-zero bounded trace by [5]. By Lemma 3.8, $C^*(E)$ admits no non-zero bounded trace if and only if E admits no non-zero bounded graph-trace. □

Example 3.11 Consider the graph D given by:



Represent D by

$$D^0 = \{0, 1, 2, \dots\}, D^1 = \{e_k, f_k \colon k \geq 0\}$$
$$s(e_k) = k, \quad r(e_k) = k+1, \quad s(f_k) = k, \quad r(f_k) = k+1$$

and assume that τ_D is a graph-trace on D. τ_D must satisfy $\tau_D(k) = \tau_D(k+1) + \tau_D(k+1)$ so

$$\sum_{k=0}^{\infty} \tau_D(k) = \sum_{k=0}^{\infty} 2^{-k} \tau_D(0) = 2\tau_D(0),$$

hence τ_D is a bounded graph-trace on D. Since D has no cycles it follows by Lemma 3.10 that $C^*(D)$ is not stable. By [24, Corollary 2.3] $C^*(D)$ is the inductive limit given by

$$M_3(\mathbb{C}) \to M_7(\mathbb{C}) \to M_{15}(\mathbb{C}) \to M_{31}(\mathbb{C}) \to \cdots \to C^*(D)$$

where the connecting maps are embeddings of multiplicity two into the upper left-hand corner.

3.4 Stability

Definition 3.12 *Let $E = (E^0, E^1, r, s)$ be a locally finite directed graph. A vertex $v \in E^0$ is left-infinite if the set*

$$L(v) = \{w \in E^0 \colon s(\alpha) = w \textit{ and } r(\alpha) = v \textit{ for some } \alpha \in E^*\}$$

is infinite. v is left-finite if $L(v)$ is finite. A subgraph $F \subseteq E$ is left-infinite (resp. left-finite) if every vertex in F^0 is left-infinite (resp. left-finite).

Lemma 3.13 *Let E be a locally finite directed graph. A vertex $v \in E^0$ is left-infinite if and only if there is an infinite path α ending at v such that all edges of α are distinct.*

Proof:

only if: This may be recognized by graph-theorists as König's infinity lemma [12]. Let $v \in E^0$ be left-infinite. Since $L(v)$ is infinite and only finitely many edges enter v there is an edge $e_1 \in E^1$ which is the last edge in infinitely many paths $\alpha \in E^*$ with $s(\alpha) \in L(v)$ and $r(\alpha) = v$. Furthermore we may assume that $s(e_1) \neq r(e_1)$. Now, since only finitely many edges enter $s(e_1)$ there is an edge $e_2 \in E^1$ with $r(e_2) = s(e_1)$, $s(e_2) \neq s(e_1)$, $s(e_2) \neq v$ and such that the path (e_2, e_1) can be concatenated to infinitely many paths $\alpha \in E^*$ with $s(\alpha) \in L(v)$ and $r(\alpha) = s(e_2)$. An obvious induction gives an infinite sequence $(e_i)_{i=1}^{\infty}$ of edges in E^1 with the desired properties.

if: This follows easily. □

Lemma 3.14 *Let E be a locally finite directed graph. If E is left-infinite then $C^*(E)$ is stable.*

Proof: Note that for every edge $e \in E^1$,

$$P_{r(e)} = S_e^* S_e \sim S_e S_e^* \leq P_{s(e)},$$

and $P_{r(e)} \perp P_{s(e)}$ if $r(e) \neq s(e)$. By Lemma 3.13 there is for every vertex $v \in E^0$ a sequence $\{e_i\}_{i=1}^{\infty}$ of edges such that the vertices $\{r(e_i)\}_{i=1}^{\infty}$ are mutually distinct, $r(e_{i+1}) = s(e_i)$ and

$r(e_1) = v$. Hence there is a sequence $(P_{s(e_i)})_{i=1}^{\infty}$ of mutually orthogonal projections satisfying

$$P_v \precsim P_{s(e_1)} \precsim P_{s(e_2)} \cdots ,$$

for all $v \in E^0$. This implies that for each finite subset $F \subseteq E^0$ there is a finite subset $G \subseteq E^0$ such that $F \cap G = \emptyset$ and $\sum_{v \in F} P_v \precsim \sum_{v \in G} P_v$, hence $C^*(E)$ is stable by Lemma 3.3. □

Theorem 3.15 *Let E be a locally finite directed graph with no sinks. The following five statements are equivalent*

(a) *$C^*(E)$ is stable.*

(b) *$C^*(E)$ admits no non-zero unital quotient and no non-zero bounded trace.*

(c) *No non-zero quotient of $C^*(E)$ is either unital or an AF-algebra with a non-zero bounded trace.*

(d) *E has no left-finite cycles and the subset $S^0 \subseteq E^0$ given by*

$$S^0 = \{v \colon v \in \gamma^0 \textit{ and } \gamma^0 \textit{ is left-finite for some infinite path } \gamma \in E^\infty\}$$

admits no non-zero bounded graph-trace (cf. section 3.3).

(e) *For every finite subset $V \subseteq E^0$ there exists a finite subset $W \subseteq E^0$ disjoint from V such that*

$$\sum_{v \in V} P_v \precsim \sum_{v \in W} P_v.$$

Proof:

(a) ⇒ (b): A stable C^*-algebra is non-unital with no non-zero bounded trace. Every quotient of a stable C^*-algebra is stable, hence non-unital (cf. Corollary 3.17).

(b) ⇒ (c): No non-zero quotient of $C^*(E)$ admits a bounded trace since it would lift.

(c) ⇒ (d): First we show that if E has a left-finite cycle, then $C^*(E)$ has a unital quotient. Consider a left-finite cycle α, i.e. a cycle with left-finite vertices, and let $v \in \alpha^0$. Define a subset

$L(v) \subseteq E^0$ by

$$L(v) = \{w \in E^0 \mid s(\alpha) = w \text{ and } r(\alpha) = v \text{ for some } \alpha \in E^*\},$$

i.e., $L(v)$ contains all vertices w for which there is a path connecting ω to v. It follows that $L(v)$ is finite since v is left-finite. Set $H^0 = E^0 \backslash L(v)$. If $H^0 = \emptyset$ then E^0 must be finite, hence $C^*(E)$ is unital. Suppose that H^0 is not empty. H^0 is a hereditary and saturated subset of E^0; H^0 is hereditary because if not we may assume there exist $u \in H^0$, $w \in L(v)$ and an edge with source u and range w. But then there is a path connecting u to v, hence $u \in L(v)$ and this is a contradiction. H^0 is saturated because for every $w \in L(v)$ there is an edge $e \in E^1$ such that $r(e) \in L(v)$ and $s(e) = w$, since $v \in \alpha^0$. Since E has no sinks and H^0 is a hereditary and saturated subset of E^0, there is by [25, Theorem 6.6] a two-sided closed ideal $I(H^0)$ of $C^*(E)$ such that the quotient C^*-algebra $C^*(E)/I(H^0)$ is naturally isomorphic to $C^*(F)$ of the directed graph $F = (L(v), \{e \in E^1 \colon r(e) \in L(v)\})$. $C^*(F)$ is unital since $L(v)$ is finite, hence $C^*(E)$ has a unital quotient.

Consider the subset S^0 defined in (d). We show that S^0 induces an AF-quotient of $C^*(E)$. Set $H^0 = E^0 \backslash S^0$. H^0 is a hereditary and saturated subset of E^0: Suppose H^0 is not hereditary, i.e., there exist vertices $v \in H^0$ and $w \in S^0$ and an edge with source v and range w. Since v is not in S^0, v must be left-infinite, which implies that w is left-infinite. This is a contradiction, so H^0 must be hereditary. H^0 is saturated because if $w \in S^0$ then there is an edge $e \in E^1$ such that $r(e) \in S^0$ and $s(e) = w$. Since E has no sinks and H^0 is hereditary and saturated we obtain by [25, Theorem 6.6] that

$$C^*(E)/I(H^0) \cong C^*(S),$$

where S is the subgraph of E given by $S = (S^0, \{e \in E^1 \colon r(e) \notin H^0\})$. As shown above E has no left-finite cycles, hence S has no cycles. Therefore $C^*(S)$ is an AF-algebra by Theorem 1.8 (Remark 1.9), and by Lemma 3.8 every bounded graph-trace on S comes from a bounded trace on $C^*(S)$. By assumption no such exist. This proves statement (d).

(d) $\Rightarrow$ (e): Suppose $S^0 \neq \emptyset$ and put $H^0 = E^0 \backslash S^0$. H^0 is a hereditary and saturated subset of E^0 by an argument similar to that given in the proof of statement (c) implies (d). We make the following

Claim: For every finite subset $K \subseteq H^0$ there is a finite subset $M \subseteq H^0$ such that $K \cap M = \emptyset$ and $\sum_{v \in K} P_v \lesssim \sum_{v \in M} P_v$.

Proof of claim: It is enough to show that for every vertex $v \in H^0$ there are left-infinite vertices $w_1, \ldots, w_n$ in $H^0 \backslash \{v\}$ for some $n \in \mathbb{N}$ such that $P_v \lesssim P_{w_1} + \cdots + P_{w_n}$. To see this, note that for each $j = 1, \ldots, n$, w_j is left-infinite and hence there is an infinite subset $\{w_j^{(i)} \colon i \in \mathbb{N}\}$ of $H^0 \backslash \{w_j\}$ such that $P_{w_j} \lesssim P_{w_j^{(1)}} \lesssim P_{w_j^{(2)}} \lesssim \cdots$, as shown in the proof of Lemma 3.14.

If $v \in H^0$ is left-infinite we are done. Suppose $v \in H^0$ is left-finite and

$$P_v = \sum_{e \in G_v} S_e S_e^*, \quad G_v = \{e \in E^1 \colon s(e) = v\}.$$

Note that G_v is finite since E is row-finite. For edges $e \in E^1$ let $m(r(e))$ denote the number of edges with source v and range $r(e)$. Since $P_{r(e)} = S_e^* S_e$ it follows that

$$P_v \sim \sum_{\{r(e) \colon e \in G_v\}} P_{r(e)} \otimes 1_{m(r(e))}. \tag{3.6}$$

Consider the vertices $r(e)$ for $e \in G_v$. If $r(e)$ is left-infinite then there is a subset $\{v_i \colon i \in \mathbb{N}\} \subseteq H^0$ of left-infinite vertices such that

$$P_{r(e)} \lesssim P_{v_1} \lesssim P_{v_2} \lesssim \cdots$$

It follows that there is a sequence $(F_j)_{j \in \mathbb{N}}$ of finite mutually disjoint subsets of H^0 such that

$$P_{r(e)} \otimes 1_{m(r(e))} \lesssim \sum_{w \in F_j} P_w \tag{3.7}$$

for each $j \in \mathbb{N}$. If every $r(e)$ is left-infinite for $e \in G_v$ then (3.6) and (3.7) gives the desired property. If $r(e)$ is not left-infinite then we repeat the argument with $r(e)$ in place of v. Recall that E has no left-finite cycles nor sinks. We continue repeating the argument until all vertices involved are left-infinite. The process stops since otherwise $v \in S^0$ which is a contradiction. It is now straightforward to show that there is a sum of projections, associated to left-infinite vertices of H^0, that contain a subprojection equivalent to P_v. This proves the claim.

Notice that H^0 is either an infinite set or the empty set. Suppose $H^0 = \{v_j \colon j \in \mathbb{N}\}$, $p_n = \sum_{j=1}^n P_{v_j}$ for $n \in \mathbb{N}$ and set $\mathcal{P} = \{p_n\}_{n=1}^\infty$. Let $I(H^0) = \text{ideal}(\mathcal{P})$ be the two-sided closed ideal of $C^*(E)$ generated by $\mathcal{P}$. H^0 is hereditary and saturated and since E has no sinks it follows by [25, Theorem 6.6], that the quotient C^*-algebra $C^*(E)/I(H^0)$ is isomorphic to $C^*(S)$, where $S = (S^0, \{e \in E^1 \colon r(e) \notin H^0\})$. Since S has no left-finite cycles and S^0 admits no non-zero bounded graph-trace, $C^*(S)$ is stable by Lemma 3.10.

We have shown that if S exists then $C^*(S)$ is stable and isomorphic to a quotient of $C^*(E)$. If $S = \emptyset$ then $E^0 = H^0$ and the claim above gives that statement (e) holds. If H^0 is the empty set (this occurs when E has no left-infinite vertices) then $C^*(E)$ equals $C^*(S)$ hence is stable and statement (e) follows by Lemma 3.3.

Let $V \subseteq E^0$ be a finite subset of E^0 and consider the decomposition

$$\sum_{v \in V} P_v = \sum_{v \in V \cap S^0} P_v + \sum_{v \in V \cap H^0} P_v. \tag{3.8}$$

Since $C^*(S)$ is stable there is a finite subset $W \subseteq S^0$ such that $W \cap V = \emptyset$ and

$$\pi\Big(\sum_{v \in V \cap S^0} P_v\Big) \precsim \pi\Big(\sum_{v \in W} P_v\Big),$$

where $\pi \colon C^*(E) \to C^*(S)$ is the homomorphism induced by the quotient map $C^*(E) \to C^*(E)/I(H^0)$. With $\mathcal{P}$ given above $C^*(E)$ satisfies property (3.1) of Lemma 3.6 by the claim. Hence by Lemma 3.6 there is a finite subset $M \subseteq H^0$ such that

$$\sum_{v \in V \cap S^0} P_v \precsim \sum_{v \in W} P_v + \sum_{v \in M} P_v. \tag{3.9}$$

By the claim there is a finite subset $K \subseteq H^0$ such that $K \cap (V \cup M) = \emptyset$ and

$$\sum_{v \in H^0 \cap (V \cup M)} P_v \precsim \sum_{v \in K} P_v. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we obtain that the subsets V, W and K are mutually disjoint and

$$\sum_{v\in V} P_v \lesssim \sum_{v\in W} P_v + \sum_{v\in K} P_v.$$

This proves statement (e).

(e) $\Rightarrow$ (a): This follows from Lemma 3.3 and the proof is complete. $\square$

Remark 3.16 We note that condition (d) above is equivalent to the following condition:

(d') E has no left-finite cycles and admits no non-zero bounded graph-trace (cf. section 3.3).

This follows from the claim given in the proof of (d) $\Rightarrow$ (e) above.

Corollary 3.17 *Given an extension of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\pi} B \longrightarrow 0.$$

If I, A and B are C^*-algebras of locally finite graphs without sinks, then A is stable if and only if I and B are stable.

The proof of Corollary 3.17 follows by Theorem 3.15 (description (b)) and the following lemma.

Lemma 3.18 *Given an extension*

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\pi} B \longrightarrow 0,$$

where I, A and B are C^*-algebras. Then

(i) If A is stable then I and B are stable.

(ii) If I and B admit no non-zero unital quotient and no non-zero bounded trace, then neither do A.

Proof: **(i):** Assume that A is stable and that I is a (two-sided closed) ideal of $A \otimes \mathcal{K}$. The ideal I equals $J \otimes \mathcal{K}$ for some ideal J of A. This can be seen as follows; Note that $A \otimes \mathcal{K}$ is *-isomorphic to the inductive limit C^*-algebra $(\bigcup M_n(A))^-$, where $M_n(A)$ is embedded into the upper left-hand corner of $M_{n+1}(A)$. A computation reveals that every ideal of $M_n(A)$ is of the form $M_n(J)$ for some ideal $J \subseteq A$ for every $n \in \mathbb{N}$. Hence every ideal of $(\bigcup M_n(A))^-$ is of the form $(\bigcup M_n(J))^-$ for some ideal $J \subseteq A$.

Let $\pi\colon A \to A/J$ be the quotient *-homomorphism. The *-homomorphism $\pi \otimes \mathrm{id}$ of $A \otimes \mathcal{K}$ into $(A/J) \otimes \mathcal{K}$ is surjective with $J \otimes \mathcal{K}$ as kernel. It follows that $B \cong A \otimes \mathcal{K}/I \cong (A/J) \otimes \mathcal{K}$, hence B is stable.

(ii): Assume now that I and B admit no non-zero bounded trace and no non-zero unital quotient. A admits no non-zero bounded trace either since if τ is such a trace, then τ must be zero on I, hence τ induces a non-zero bounded trace on B. Suppose to reach a contradiction that A contains a two-sided closed ideal J such that the quotient A/J is non-zero and unital. There is a commuting diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow \pi & & \downarrow \varphi & & \downarrow \psi & & \\ 0 & \longrightarrow & I/(I \cap J) & \longrightarrow & A/J & \longrightarrow & A/(I+J) & \longrightarrow & 0, \end{array}$$

where π, φ and ψ are surjective *-homomorphisms, and where each row is exact. It follows that the quotient $A/(I+J)$ is unital and if $A/(I+J)$ is zero then $I/(I \cap J)$ is *-isomorphic to A/J, hence non-zero and unital. This contradicts that I and B admit no non-zero unital quotients. $\square$

Remark 3.19 One could ask if statements (a) and (b) of Theorem 3.15 might be equivalent in general. In more detail, is a C^*-algebra A stable if and only if A admits no non-zero unital quotient and no non-zero bounded trace? The answer to the question is negative, as mentioned in section 2.5.

It would be interesting to determine the class of C^*-algebras for which stability can be characterized by the non-existence of unital quotients and bounded traces. It is noteworthy that for C^*-algebras in this class, the extension problem for stability can be solved affirmatively, i.e., the extension of two stable C^*-algebras is stable (cf. Corollary 3.17). Also, for C^*-algebras in this class it follows that every simple C^*-algebra is either stably finite or purely infinite, as shown in section 2.4.

We finish this chapter by posing the following problem:

Problem 3.20 Characterize (X, T) (cf. section 1.5.1) for which $C^*(X, T)$ is stable.

We expect that the reduced C^*-algebra of an r-discrete groupoid G is stable if and only if the unit space $G^{(0)}$ has no invariant compact subset and admits no finite invariant measure.

Chapter 4

PURELY INFINITE C^*-ALGEBRAS OF GROUPOIDS

In this chapter we consider pure infiniteness of C^*-algebras. Among other things we give a characterization of purely infinite C^*-algebras of graphs and infinite matrices (introduced in sections 1.3 and 1.4). We refer to section 1.2 for a brief introduction.

4.1 Purely infinite C^*-algebras of graphs

Theorem 4.1 *Let E be a locally finite directed graph with no sinks. The following six statements are equivalent:*

(a) $C^*(E)$ is purely infinite.

(b) $C^*(E)$ admits no non-trivial trace.

(c) No quotient of $C^*(E)$ contains a two-sided closed ideal that is an AF-algebra, nor does it contain a corner that is $*$-isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$.

(d) Every infinite path in E admits a detour β such that there are two or more cycles based at some vertex of β^0.

(e) Every vertex connects to a cycle with an exit in every subgraph $(F, \{e \mid r(e) \in F\})$, where F is the complement of a hereditary and saturated subset of vertices in E.

(f) For every vertex $v \in E^0$ the projection $P_v \in C^*(E)$ is properly infinite.

Proof:

(a) $\Rightarrow$ (b): This follows from [22, Proposition 5.1].

(b) $\Rightarrow$ (c): This follows by noting that an AF-algebra admits a densely defined trace and that a densely defined trace on a two-sided closed ideal of a quotient of a C^*-algebra A lift to a trace on a two-sided ideal of A. Also, if PAP is a corner in A for some projection $P \in A$ and if τ is a bounded everywhere defined trace on PAP, then the map $\sum_{i=1}^n x_i PaPy_i \mapsto \sum_{i=1}^n \tau(PaPy_ix_iP)$ extends τ to give a densely defined trace on the two-sided ideal $\{\sum_{i=1}^n x_iPaPy_i \mid n \in \mathbb{N}, a, x_i, y_i \in A\}$ of A generated by PAP.

(c) ⇒ (d): First we show that no vertex of E is base for precisely one cycle. Let $v \in E^0$ and suppose to reach a contradiction that precisely one cycle α is based at v. Set

$$L(v) = \{w \in E^0 \mid s(\gamma) = w \text{ and } r(\gamma) = v \text{ for some } \gamma \in E^*\},$$

and let $H^0 = E^0 \backslash L(v)$ denote the complement. Then H^0 is a saturated and hereditary subset of E^0 (see the proof of Theorem 3.15) and if F denotes the subgraph $(L(v), \{e \mid r(e) \in L(v)\})$ of E then $C^*(F)$ is isomorphic to a quotient of $C^*(E)$ by [25, Theorem 6.6]. The cycle α based at v is contained in F. Moreover, α has no exit in F, hence it follows by Lemma 1.6 that $C^*(F)$ contains a corner which is $*$-isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$.

We may assume that no vertex of E is base for precisely one cycle. Suppose now, to reach a contradiction, that there is an infinite path $\alpha \in E^\infty$ such that for every detour β of α, no cycle is based at any vertex of β^0. Set

$$K = \{w \in E^0 \mid w \in \beta^0 \text{ for a detour } \beta \text{ of } \alpha\}. \tag{4.1}$$

Note that α^0 is contained in K. Set

$$H = \{w \in E^0 \backslash K \mid s(\gamma) \in K \text{ and } r(\gamma) = w \text{ for some } \gamma \in E^*\}. \tag{4.2}$$

Let $\widetilde{H}$ be the smallest saturated subset containing the hereditary completion of H. We show that K and $\widetilde{H}$ are disjoint sets. Note first that K is disjoint to the hereditary completion of H, and next that the smallest saturation of this set does not contain any vertices of K, since for every vertex $w \in K$ there is an edge $e \in E^1$ such that $s(e) = w$, $r(e) \neq w$ and $r(e) \in K$. Hence $K \cap \widetilde{H} = \emptyset$. Let K' denote the smallest saturated subset containing K. Then $K' \cap \widetilde{H} = \emptyset$ because otherwise there will be a path from H to K contradicting that $K \cap \widetilde{H} = \emptyset$. K' is hereditary in $E^0 \backslash \widetilde{H}$. It follows that K' is a hereditary and saturated subset of $E^0 \backslash \widetilde{H}$. Moreover, no cycle is based at any vertex of K'.

Now, at the C^*-algebra-level we obtain that if F denotes the subgraph $(E^0 \backslash \widetilde{H}, \{e \mid r(e) \in E^0 \backslash \widetilde{H}\})$ then $C^*(F)$ is isomorphic to a quotient of $C^*(E)$. Let $I(K')$ be the two-sided closed ideal of $C^*(F)$ induced by K' as in [25, Theorem 6.6]. Since no cycle is based at any vertex of K' it follows that $I(K')$ is an AF-algebra by Theorem 1.8 or [24, Theorem 2.4], which contradicts our assumption. This shows that statement (c) implies (d).

(d) ⇒ (e): Let $H^0 \subseteq E^0$ be a hereditary and saturated subset and let $v \in E^0 \backslash H^0$. Since H^0 is saturated and E has no sinks there is an infinite path $\alpha \in E^\infty$ such that $v = s(\alpha)$ and $\alpha^0 \subseteq E^0 \backslash H^0$. By assumption there is a detour β of α such that two or more cycles are based at some vertex of β^0. Since H^0 is hereditary and $\alpha^0 \subseteq E^0 \backslash H^0$ it follows that $\beta^0 \subseteq E^0 \backslash H^0$. Hence v connects to a cycle with an exit in the subgraph $(E^0 \backslash H^0, \{e \mid r(e) \in E^0 \backslash H^0\})$.

(e) ⇒ (f): Let $v \in E^0$ be a vertex and let $P_v \in C^*(E)$ be the associated projection. By [22, Proposition 3.14] P_v is properly infinite when for every two-sided closed ideal I in $C^*(E)$, the projection $P_v + I$ is either zero or infinite.

No vertex in E^0 is base for precisely one cycle, since otherwise there is, as shown in "(c) implies (d)", a complement of a hereditary and saturated subset of E^0 that contains a vertex which is base for a cycle with no exit, and this contradicts the assumption. Hence there is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of ideals in $C^*(E)$ by [25, Theorem 6.6]. Let I be an ideal of $C^*(E)$ and let H^0 be a saturated hereditary subset of E^0 such that $I = I(H^0)$.

If $v \in H^0$ then $P_v \in I$ and we are done. Suppose $v \notin H^0$. By assumption there is a path α such that $\alpha^0 \cap H^0 = \emptyset$, $s(\alpha) = v$ and there is a cycle γ based at $r(\alpha)$. Set $w = r(\alpha)$ and note that there is another cycle δ based at w since no vertex in E^0 is base for precisely one cycle, as shown above. Since H^0 is hereditary and $w \notin H^0$, the subsets γ^0 and δ^0 are both disjoint to H^0. The projection P_w does not belong to I and P_w is properly infinite since

$$S_\gamma^* S_\gamma = S_\delta^* S_\delta = P_w \geq S_\gamma S_\gamma^* + S_\delta S_\delta^*.$$

Hence $P_v + I$ is (properly) infinite. Since v connects to w via α with $\alpha^0 \cap H^0 = \emptyset$, it follows that $P_w + I \precsim P_v + I$ and $P_v + I$ must be infinite.

(f) ⇒ (a): We aim to show that for every closed two-sided ideal I and every hereditary C^*-subalgebra B in the quotient by I, there is an infinite projection in B. Then $C^*(E)$ is purely infinite by [22, Proposition 4.7].

Let I be a two-sided closed ideal of $C^*(E)$. The assumption implies that no vertex of E is base for precisely one cycle (see "(c) implies (d)"). Hence $I = \text{Ideal}(H^0)$ for some hereditary saturated subset $H^0 \subseteq E^0$ (cf. [25, Theorem 6.6]) and if F denotes the subgraph $(E^0 \backslash H^0, \{e \mid$

$r(e) \in E^0 \backslash H^0\}$) then $C^*(F) \cong C^*(E)/I$ by [25, Theorem 6.6]. Notice that every projection associated to a vertex of F is properly infinite, since this property is preserved by the quotient-map.

Let B be a hereditary C^*-subalgebra of $C^*(F)$. For each $n \in \mathbb{N}$ the projections

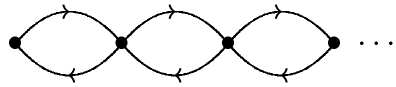
$$\{S_\alpha S_\alpha^*\colon |\alpha| = n \text{ for some } \alpha \in F^*\}$$

are mutually orthogonal and span a C^*-subalgebra A_n of $C^*(F)$. Moreover, the projections are properly infinite since $S_\alpha^* S_\alpha = P_{r(\alpha)}$. Set $D = (\cup A_n)^-$. By [24, Lemma 3.5] $C^*(F)$ contains a partial isometry v such that $v^*v \in D$ and $vv^* \in B$. Hence there is a (properly) infinite projection in A_n for some n which is equivalent to a subprojection of B. This shows that $C^*(E)$ is purely infinite and the proof is complete. □

Remark 4.2 For a graph without non-trivial hereditary and saturated subsets of the set of vertices, statement (e) of Theorem 4.1 is identical to the hypothesis of [24, Theorem 3.9] and in this case the associated C^*-algebra is simple.

Example 4.3 (cf. [23]) Consider the graph E given by:

E:

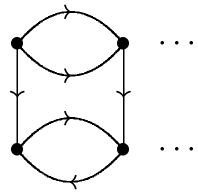


$C^*(E)$ is purely infinite by Theorem 4.1 and stable by Theorem 3.15. Using the Kirchberg–Phillips classification theorem it can be shown that $C^*(E) \cong \mathcal{O}_\infty \otimes \mathcal{K}$. I dont know whether this isomorphism can be given explicit. Analoguesly, the isomorphism of $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$ is implicit (cf. [35]).

Example 4.4 Using [24, Theorem 3.9], [25, Theorem 6.6] and Lemma 3.8 one can construct examples of graph-algebras which admit a non-zero bounded trace and such that every non-zero hereditary C^*-subalgebra contains an infinite projection. For instance, consider the graph F

given by:

F:



By [25, Theorem 6.6] the lower row of F forms a hereditary and saturated subset (cf. 1.3) which induces an ideal Morita equivalent to $C^*(E)$ of Example 4.3 and the quotient, induced by the upper row of F, is isomorphic to $C^*(D)$ of Example 3.11. It follows that $C^*(F)$ admits a (bounded) trace and is not purely infinite by Theorem 4.1. By [24, Theorem 3.9] every hereditary C^*-subalgebra of $C^*(F)$ contains an infinite projection.

4.2 Purely infinite C^*-algebras of groupoids

In this section, G is a second countable, locally compact and r-discrete groupoid. In [1] (see also [26]) C. Anantharaman-Delaroche introduces the notion of locally contracting r-discrete groupoids; G is *locally contracting* if for every non-empty open subset U of the unit space of G, there exist an open subset V in U and an open bisection S (i.e., r and d are one to one on S) with $\overline{V} \subseteq d(S)$ and $\alpha_{S^{-1}}(\overline{V})$ strictly contained in V, where α_S is the G-map associated with S, i.e., the homeomorphism of $r(S)$ onto $d(S)$ given by $\alpha_S(x) = d(xS)$ for all $x \in r(S)$ (cf. [1, Definition 2.1]). A subset V of the unit space $G^{(0)}$ is *invariant* if $V = r(d^{-1}(V))$, and for a unit $u \in G^{(0)}$ the *isotropy subgroup* of G at u is the group $r^{-1}(u) \cap d^{-1}(u)$. G is said to be *essentially principal* if for every closed invariant subset F of the unit space the elements of F for which the isotropy subgroup is trivial, are dense in F (cf. [31, Definition 4.3]).

In [1, Proposition 2.4] it is shown that every non-zero hereditary C^*-subalgebra of the reduced C^*-algebra of a locally contracting and essentially principal groupoid G contains an infinite projection.

Proposition 4.5 *Let G be an r-discrete and essentially principal groupoid. Assume for every closed invariant subset F of the unit space that the reduced subgroupoid G_F is locally contracting. Then $C_r^*(G)$ is purely infinite.*

Proof: $C_r^*(G)$ is purely infinite if every non-zero hereditary C^*-subalgebra in every quotient contains an infinite projection (cf. [22, Proposition 4.7]). Let I be a two-sided closed ideal of

$C_r^*(G)$. By [31, Proposition 4.6] I is the closure of the $*$-algebra of continuous functions with compact support that vanishes outside U for some open invariant subset $U \subseteq G^0$, and

$$C_r^*(G_F) \cong C_r^*(G)/I,$$

where $F = G^0 \backslash U$ is a closed invariant subset of G^0. By assumption the elements of G_F^0 (=F) with trivial isotropy are dense and G_F is locally contracting. It follows from [1, Proposition 2.4] that every non-zero hereditary C^*-subalgebra of $C_r^*(G_F)$ contains an infinite projection, hence $C_r^*(G)$ is purely infinite. □

Corollary 4.6 *Let (X, T) be a singly generated dynamical system. Assume for every closed invariant subset $F \subseteq X$ and for every open subset $U \subseteq F$ that*

(i) There is an element $u \in U$ such that $T^n(u) \neq T^m(u)$ for all distinct numbers $n, m \in \mathbb{N}$.

(ii) There is an open set $V \subseteq U$ and distinct numbers $n, m \in \mathbb{N}$ such that T^n and T^m are injective when restricted to $\overline{V}$ and $T^n(\overline{V})$ is strictly contained in $T^m(V)$.

Then $C^*(X, T)$ is purely infinite.

Proof: Let U and F be as in the corollary. It follows from condition (i) that every open subset of F contains an element with trivial isotropy, hence $G(X, T)$ is essentially principal.

Let $V \subseteq U$ and $n, m \in \mathbb{N}$ be as in condition (ii). Since T^n and T^m are injective when restricted to $\overline{V}$, one defines a bisection in $G(X, T)_F$ by letting

$$S = \{(x, m - n, y) \mid T^m(x) = T^n(y) \text{ for } x, y \in \overline{V}\}.$$

It is easily checked that the composed map $T^{-m} \circ T^n$ is, when restricted to $\overline{V}$, a homeomorphism of $\overline{V}$ onto a proper subset of V which equals the bisection-map (or G-map) associated with S. Hence $G(X, T)_F$ is locally contracting. It follows from Proposition 4.5 that $C^*(X, T)$ is purely infinite. □

Remark 4.7 The converse statement to Corollary 4.6 does not hold in general, as shown in Section 4.3.

For the locally finite graphs E (without sinks) of section 1.3, Example 1.11 reveals that $C^*(E)$ comes from a dynamical system (X_{A_E}, T_{A_E}), where T_{A_E} is the one-sided shift on the infinite path space X_{A_E} associated to the edge-matrix A_E of the graph E. By noting this, Corollary 4.6 gives an alternative proof of "(e) implies (a)" of Theorem 4.1.

4.2.1 A characterization theorem

Theorem 4.10 of this section is a characterization of pure infiniteness for the C^*-algebras $\mathcal{O}_A$ of section 1.4. It may be viewed as a (slight) generalization of Theorem 4.1.

In [32, Section 4] J. Renault constructs a groupoid model for $\mathcal{O}_A$. More precisely, there is a singly generated dynamical system (X_A, T_A), called a Markov shift (cf. [32, Definition 4.1]), such that $C^*(X_A, T_A) = \mathcal{O}_A$. In [17] one distinguishes between $\mathcal{O}_A$ and $\tilde{\mathcal{O}}_A$, where the latter is the unital C^*-algebra generated by $\{S_i \mid i \in I\}$ and 1. Either $\mathcal{O}_A = \tilde{\mathcal{O}}_A$ or $\tilde{\mathcal{O}}_A$ is $\mathcal{O}_A$ with a unit adjoined. Both C^*-algebras come from a Markov shift as shown in [32, Proposition 4.8]; $\mathcal{O}_A$ comes from (X_A, T_A), where either one point is removed from X_A or no point is removed.

We refer to [32, Section 4] for the construction of X_A, the compact and totally disconnected Hausdorff space consisting of *terminal paths* associated to an infinite matrix A, where a terminal path is either an infinite path $(i_0, i_1, \ldots)$ or a path with finite exit time (cf. [32, Proposition 4.1]). The partial homeomorphism T_A is the one-sided shift with domain a dense open subset $U \subseteq X_A$ and range an open subset of X_A. The domain U admits a partition $\{U_i \mid i \in I\}$ of pairwise disjoint compact open subsets U_i. For each $i \in I$, U_i is the set of all terminal paths of lenght ≥ 1 starting with i.

Remark 4.8 A terminal path in X_A is called *periodic* if it, after some iterations by T_A, repeats indefinitely a cycle. Moreover, if the cycle has no exit, then the path is called *isolated periodic*. From Lemma 1.6 it follows for a Markov shift (X_A, T_A) of an $I \times I$-matrix A with no zero rows that if X_A has an isolated periodic path, then there is a corner in $C^*(X_A, T_A)$ which is $*$-isomorphic to $M_n(C(\mathbb{T}))$. We express Theorem 1.8 in terms of Markov shifts:

Corollary 4.9 *Let A be an $I \times I$-matrix with no zero rows and let (X_A, T_A) be the associated Markov shift. Then $C^*(X_A, T_A)$ is an* AF*-algebra if and only if X_A has no periodic paths.*

Theorem 4.10 *Let A be an $I \times I$-matrix with no zero rows and let (X_A, T_A) be the associated Markov shift. The following five statements are equivalent.*

(a) $C^*(X_A, T_A)$ is purely infinite.

(b) $C^*(X_A, T_A)$ admits no non-zero trace.

(c) No quotient of $C^*(X_A, T_A)$ contains a two-sided closed ideal that is an AF-algebra, nor does it contain a corner that is $*$-isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$.

(d) For every closed invariant subset $F \subseteq X_A$, every vertex $k \in I$ of every terminal path in F connects to a cycle with an exit, i.e., there are subpaths $(i_1, \ldots, i_n)$ and $\beta = (j_1, \ldots, j_m)$ of paths in F such that $k = i_1$, $i_n = j_1 = j_m$ for some $m, n \in \mathbb{N}$ and β has an exit in F.

(e) For every closed invariant subset $F \subseteq X_A$ and for every open subset $V \subseteq F$ the following conditions hold;

(i) There is a terminal path $x \in V$ such that $T_A^n(x) \neq T_A^m(x)$ for all distinct numbers $n, m \in \mathbb{N}$.

(ii) There is an open set $W \subseteq V$ and distinct numbers $n, m \in \mathbb{N}$ such that T_A^n and T_A^m are injective when restricted to $\overline{W}$ and $T_A^n(\overline{W})$ is strictly contained in $T_A^m(W)$.

Proof:

(a) $\Rightarrow$ (b): This follows from [22, Proposition 5.1].

(b) $\Rightarrow$ (c): See (b) implies (c) of Theorem 4.1.

(c) $\Rightarrow$ (d): Let $F \subseteq X_A$ be a closed invariant subset and let $G(X_A, T_A)_F$ denote the reduced subgroupoid of $G(X_A, T_A)$. Suppose to reach a contradiction that there is a vertex of a path in F which connects to a cycle with no exit (in sense of (d)). Hence F contains an isolated periodic path, whence the quotient $C^*(G(X_A, T_A)_F)$ of $C^*(X_A, T_A)$ contains a corner which is $*$-isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ (cf. Lemma 1.6 and Remark 4.8). This contradicts statement (c) and we may assume that every cycle has an exit.

Suppose now that there is a vertex $k \in I$ of some path in F such that k does not connect to a cycle. Let V_k denote the open subset of F consisting of all terminal paths in F starting with k, and let $O(V_k)$ denote the union of all orbits $T_A^n(V_k)$ of V_k. Set

$$\tilde{O}(V_k) = \{x \in F \mid T_A^m(x) \in O(V_k) \text{ for some } m \in \mathbb{N}\}$$

Then $\tilde{O}(V_k)$ is an open invariant subset of F without periodic paths. By [31, Proposition 4.5], $\tilde{O}(V_k)$ induce a two-sided closed ideal of the quotient $C^*(G(X_A, T_A)_F)$, and this ideal is an AF-algebra by Corollary 4.9. This contradicts the assumption and statement (d) holds.

(d) ⇒ (e): Let $F \subseteq X_A$ be a closed invariant subset and let V be an open subset of F. Let $k \in I$ be a vertex of some (terminal) path in V and let $\kappa = (k, k_1, \ldots, k_l)$ be a finite path of a path in F such that the cylinderset $Z(\kappa) \subseteq V$ of all terminal paths starting with $(k, k_1, \ldots, k_l)$ is an open subset of V. Since every cycle has an exit, there is a path in $Z(\kappa)$ which is not periodic, hence condition (i) of statement (e) holds.

By assumption there are finite subpaths $\alpha = (i_1, \ldots, i_n)$, $\beta = (j_1, \ldots, j_m)$ and $\gamma = (l_1, \ldots, l_p)$ of paths in F such that $k_l = i_1$, $i_n = j_1 = j_m = l_1$ and $l_2 \neq j_2$. Notice that β is a cycle (when repeated) and γ is an exit of this cycle. Set $W = Z(\kappa\alpha\beta\beta)$ and let m and n denote the lenght of the paths $\kappa\alpha\beta$ and $\kappa\alpha$, respectively. Notice that $W = \overline{W}$, $W \subseteq V$, $T_A^m(W) = Z(\beta)$ and $T_A^n(W) = Z(\beta\beta)$. Clearly T_A^m and T_A^n are injective when restricted to W. Since paths starting with $\beta\gamma$ do not belong to $T_A^n(W)$, it follows that $T_A^n(\overline{W})$ is strictly contained in $T_A^m(W)$. Hence condition (ii) holds.

(e) ⇒ (a): This follows from Corollary 4.6 and the proof is complete.

□

4.3 Example of $\mathcal{O}_2$ arising from a principal groupoid

An example, originally due to A. Kumjian but modified here, of a locally compact, étale (i.e., r and d are local homeomorphisms), amenable and principal Hausdorff groupoid G such that G does not satisfy the conditions of Corollary 4.6 and $C^*(G)$ is simple and purely infinite is presented. In fact, $C^*(G)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$ which is thus realized as the C^*-algebra of a principal groupoid. In [31, page 138] it is asked whether such a realization is possible.

Let $X = \{0,1\}^{\mathbb{N}}$ and let $T : X \to X$ denote the one-sided shift on X. Let $Y = X \times \mathbb{T}$, where $\mathbb{T}$ is the circle. Let λ be an irrational number and define a map $\sigma : Y \to Y$ by

$$\sigma(x, \eta) = (Tx, e^{2i\pi\lambda}\eta)$$

for all $(x, \eta) \in Y$. The map σ is continuous, surjective and two-to-one. However, σ is a local homeomorphism.

Let $G(Y, \sigma)$ denote the associated groupoid (as in section 1.5.1). Then $G(Y, \sigma)$ is a locally compact, étale, amenable Hausdorff groupoid. Furthermore, $G(Y, \sigma)$ is principal since there are no elements of Y for which σ^m and σ^n agree for distinct natural numbers n and m: If $\sigma^m(x, \eta) = \sigma^n(x, \eta)$ then

$$(T^m x, e^{2i\pi m\lambda}\eta) = (T^n x, e^{2i\pi n\lambda}\eta),$$

hence $m = n$ since λ is irrational.

$G(Y, \sigma)$ does not satisfy the conditions of Corollary 4.6. There is an open subset $U \subseteq Y$ such that for every open subset $V \subseteq U$ and every pair of distinct numbers $n, m \in \mathbb{N}$, $\sigma^m(\overline{V})$ is not strictly contained in $\sigma^n(V)$. To see this, let α be a finite path in X and let $W \subseteq \mathbb{T}$ be an open proper subset of $\mathbb{T}$. Set $U = Z(\alpha) \times W$. Let $V \subseteq U$ be an open subset of U. Then

$$V = \bigcup_{j=1}^{\infty} Z(\beta_j) \times W_j,$$

for paths β_j (extending α) of finite lenght and open subsets $W_j \subseteq \mathbb{T}$ such that $W_j \subseteq W$ for each $j \in \mathbb{N}$.

Suppose $\sigma^m(V)$ is contained in $\sigma^n(V)$ for some $n, m \in \mathbb{N}$, i.e.,

$$\bigcup_j T^m(Z(\beta_j)) \times e^{2i\pi m\lambda} W_j \subseteq \bigcup_j T^n(Z(\beta_j)) \times e^{2i\pi n\lambda} W_j.$$

Let $\Pi_{\mathbb{T}}$ denote the projection of Y onto $\mathbb{T}$. It follows that

$$\Pi_{\mathbb{T}}(\sigma^m(V)) = \cup_j e^{2i\pi m\lambda} W_j = e^{2i\pi m\lambda}(\cup_j W_j)$$
$$\Pi_{\mathbb{T}}(\sigma^n(V)) = \cup_j e^{2i\pi n\lambda} W_j = e^{2i\pi n\lambda}(\cup_j W_j)$$

Hence $\sigma^m(V) \subseteq \sigma^n(V)$ implies that

$$e^{2i\pi m\lambda}(\cup_j W_j) \subseteq e^{2i\pi n\lambda}(\cup_j W_j),$$

which forces $m = n$. This shows that $G(Y, \sigma)$ does not satisfy the conditions of Corollary 4.6.

It is noteworthy that $G(Y, \sigma)$ fulfills the simplicity condition of [32, Proposition 2.5]; Let $(x, \eta) \in Y$ and let $Z(\alpha) \times W$ be a basic open subset of Y (as above). Notice that if $m \geq |\alpha|$, where $|\alpha|$ denotes the lenght of α, then

$$\sigma^m(Z(\alpha) \times W) = X \times e^{2i\pi m\lambda} W.$$

Since λ is irrational there is $k \geq m$ such that

$$\sigma^k(x, \eta) = (T^k x, e^{2i\pi k\lambda}\eta) \in X \times e^{2i\pi m\lambda} W.$$

It follows that Y has no non-trivial open invariant subsets, hence $C^*(Y, \sigma)$ is simple.

We may identify $G(Y, \sigma)$ with the skew-product groupoid $G(X, T) \times_c \mathbb{T}$ via the isomorphism

$$(x, m-n, y; \eta) \mapsto ((x, \eta), m-n, (y, \eta c(x, m-n, y)))$$

of $G(X, T) \times_c \mathbb{T}$ onto $G(Y, \sigma)$, where $c : G(X, T) \to \mathbb{T}$ is the continuous homomorphism defined by

$$c(x, m-n, y) = e^{2i\pi m\lambda}/e^{2i\pi n\lambda}.$$

The map c induces an action $\alpha : \mathbb{Z} \to \mathrm{Aut}(C^*(G(X, T)))$ by $k \mapsto \alpha_k$, where

$$\alpha_k(f)(x, m-n, y) = e^{ikc(x,m-n,y)} f(x, m-n, y)$$

for all $f \in C_c(G(X,T))$. It follows that

$$\begin{aligned} C^*(Y,\sigma) &\cong C^*(G(X,T) \times_c \mathbb{T}) \\ &\cong C^*(G(X,T)) \rtimes_\alpha \mathbb{Z} \\ &\cong \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}, \end{aligned}$$

where the latter isomorphism follows by [31, III.2]. Since α^n is outer for all $n \in \mathbb{Z}\backslash\{0\}$ (by [31, Proposition 5.3, second part of proof]) it follows from [36, Corollary 6.4] that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_2$, hence $C^*(Y,\sigma) \cong \mathcal{O}_2$.

Remark 4.11 We do not know whether the groupoid $G(Y,\sigma)$ treated in the previous example is locally contracting in sense of [1, Definition 2.1] (stated in section 4.2).

Unfinished mathematics that will not go for a journal can, in the opinion of the author, easily be placed in a thesis. So, let us consider a possible weakening of the conditions of Corollary 4.6.

Consider the following condition.

(ulc) For every open subset $U \subseteq Y$ there exist $r \in \mathbb{N}$, open mutually disjoint subsets $V_1, \dots, V_r \subseteq U$ and numbers $m_1, \dots, m_r, n_1 \dots, n_r \in \mathbb{N}$ such that $\cup_j^r \sigma^{m_j}(\overline{V_j}) \subsetneqq \cup_j^r \sigma^{n_j}(V_j)$ and σ^{m_j}, σ^{n_j} are injective when restricted to $\overline{V_j}$ for $j = 1, \dots, r$.

The previous example satisfies the condition (ulc): Assume that U is a basic open subset of Y such that $U = Z(\beta) \times W$ for some finite path β of lenght l and some open subset $W \subseteq \mathbb{T}$. We make the following claim (and omit the proof):

Claim: There exists $r \in \mathbb{N}$ such that for all $m_0 \in \mathbb{N}$ there exist numbers $n_1, \dots, n_r \in \mathbb{N}$ such that $n_j \geq m_0$ for $j = 1, \dots, r$ and $\cup_j^r e^{2i\pi n_j \lambda} W = \mathbb{T}$

Let $r \in \mathbb{N}$ be as in the claim and suppose $n_1, \dots, n_r \in \mathbb{N}$ such that $n_j = r + l$ for $j = 1, \dots, r$ and $\cup_j^r e^{2i\pi n_j \lambda} W = \mathbb{T}$. Choose r distinct paths $\alpha_1, \dots, \alpha_r$ each of lenght r (there are 2^r such paths). Set $\gamma_j = \beta\alpha_j$ and $V_j = Z(\gamma_j) \times W$ for $j = 1, \dots, r$. Then the sets $V_j \subseteq U$ are mutually disjoint and

$$\sigma^{n_j}(V_j) = X \times e^{2i\pi n_j \lambda} W, \qquad j = 1, \dots, r.$$

Hence

$$\bigcup_j^r \sigma^{n_j}(V_j) = X \times \mathbb{T}.$$

This shows that condition (ulc) is satisfied.

We finish by stating the following conjecture.

Conjecture: Let (X, σ) be a dynamical system. Assume for every closed invariant subset $F \subseteq X$ and for every open subset $U \subseteq F$ that

(i) There is an element $u \in U$ such that $\sigma^n(u) \neq \sigma^m(u)$ for all distinct numbers $n, m \in \mathbb{N}$.

(ii) There exist $r \in \mathbb{N}$, open mutually disjoint subsets $V_1, \ldots, V_r \subseteq U$ and numbers $m_1, \ldots, m_r$, $n_1 \ldots, n_r \in \mathbb{N}$ such that $\cup_j^r \sigma^{m_j}(\overline{V_j}) \subsetneqq \cup_j^r \sigma^{n_j}(V_j)$ and σ^{m_j}, σ^{n_j} are injective when restricted to $\overline{V_j}$ for $j = 1, \ldots, r$.

Then $C^*(X, \sigma)$ is purely infinite.

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