

Thesis for the Master degree in Mathematics. Department of Mathematical Sciences, University
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On Almost Commuting Operators and Matrices

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30. april 2012

Abstract

This master thesis is about almost commuting matrices, and a Brown-Douglas-Fillmore Theorem. The main results are:

In the part about almost commuting matrices it is shown that almost commuting self-adjoint matrices can be uniformly approximated by exactly commuting self-adjoint matrices (Lin's theorem), and some non-trivial counterexamples are given about almost commuting unitaries, and almost commuting pairs consisting of a self-adjoint and a normal matrix. Other simpler related questions are also examined.

In the part about the Brown-Douglas-Fillmore theorem it is shown that an essentially normal operator is a compact perturbation of a normal operator if and only if it has trivial index function.

In this connection a short introduction to Fredholm operators is also given.

Resume: Dette speciale handler om næsten kommuterende matricer, og et Brown-Douglas-Fillmore Theorem. Hovedresultaterne er:

I den del der handler om næsten kommuterende matricer vises det at næskommuterende selvadjungerede matricer kan approksimeres uniformt med exact kommuterende selvadjungerede matrices (Lins sætning), og nogle ikke-trivielle modeksempler gives vedrørende næsten kommuterende unitære matricer, og næsten kommuterende par bestående af en selvadjungeret og normal matrix. Andre enklere relaterede spørgsmål undersøges også.

I den del der handler om Brown-Douglas-Fillmore teoremet vises det at en essentielt normal operator er en kompakt perturbation a en normal operator hvis og kun hvis den har triviel index funktion.

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Chapter 1

Introduction

The topics which will be discussed in this master thesis are some natural topics one might consider when recalling the following classical theorem about operators on a finite dimensional Hilbert space.

Theorem 1.1. *Let N be a normal operator on a finite dimensional Hilbert space. Then there exists an orthonormal basis where N is diagonal.*

This theorem, which says that two self-adjoint operators on a finite dimensional Hilbert space can be simultaneously diagonalized, has many direct generalizations, to several operators and compact operators on an infinite-dimensional Hilbert space (For instance any family of mutually commuting normal compact operators can be simultaneously diagonalized).

The main topic of of this text can loosely be described as an attempt to answer the following (vague) question: If an operator is not normal, but in some sense close to being normal, is it then close to a normal operator? Of course we have to specify what we mean by ‘close to being normal’, and ‘close to a normal operator’. We will think of this in primarily two different ways:

1. Loosely speaking we will call an operator whose self-commutant has small norm ‘almost normal’, and we will say two operators are close if the norm of their difference is small. First we will ask whether operators (mostly on a finite dimensional Hilbert space), which are almost normal are close to normal operators. We will formulate this more precisely in section 2.2. This section is really about when two almost commuting operators of some species (say self-adjoint) are close to commuting operators of the same species. The main result here is Lin’s Theorem, which will be proved in chapter 3. We will also focus on some interesting non-trivial counterexamples in chapter 4.

2. Secondly we will ask when operators with compact self-commutant, are compact perturbations of normal operators. Operators with compact self-commutant will be called essentially normal. To answer this precisely we need the notion of Fredholm index, and the main result is a so-called

Brown-Douglas-Fillmore (BDF) theorem. BDF-theory is a theory about classification of essentially normal operators. We will only touch upon one of the theorems from BDF-theory (however the most famous one). The precise statement will be given in 2.1, and the proof will be given in chapter 5, and uses Lin's theorem.

In section 2.3 we will show some elementary things about quasicompact operators, and use Lin's theorem to give a short proof of the Weyl-von Neumann-Berg Theorem for one normal operator on a Hilbert space. The theorem can be seen as answer to a natural question one can ask in connection with Theorem 1.1: Is a normal operator, which is not necessarily compact, diagonalizable, or at least close to being diagonal in some orthonormal basis?

Some remarks on notation, approximation, lifting problems, corner algebras, and polar decomposition

Let H be a Hilbert Space, $B(H)$, $B_K(H)$, and $B_f(H)$ be respectively the bounded, compact, and finite rank operators on H . From now on we will assume, although it is not always necessary, H to be *separable*.

We will extensively use the theory of C^* -algebras to prove our results. In particular in connection with almost commuting matrices we will introduce the following two C^* -algebras. Let (n_k) be a sequence of natural numbers and define:

$$M = \{(A_k) \mid A_k \in M_{n_k}(\mathbb{C}) \text{ and } \sup_{k \in \mathbb{N}} \|A_k\| < \infty\}, \quad (1.1)$$

and

$$A = \{(A_k) \mid A_k \in M_{n_k}(\mathbb{C}) \text{ and } \lim_{k \rightarrow \infty} \|A_k\| = 0\}. \quad (1.2)$$

With coordinate-wise definitions of addition, multiplication and adjoints M and A are C^* -algebras. Moreover M is a von Neumann algebra, so we can apply Borel function calculus and stay within it. A is obviously a closed self-adjoint two-sided ideal in M so we can consider the quotient algebra M/A which is also a C^* -algebra. We denote the quotient mapping by π , which is a continuous $*$ -homomorphism.

In connection with the BDF-theorem we shall use the Calkin algebra, $B(H)/B_K(H)$, which we will denote $Q(H)$. The compact operators in a Hilbert space is a self-adjoint two-sided ideal in $B(H)$, so $Q(H)$ is a C^* -algebra (we assume these results to be already established). We again use π for the canonical homomorphism from $B(H)$ into $Q(H)$ - this should not lead to any confusion although we already used it for M/A , since they will appear in different contexts.

One main part of the problem in both cases is approximating normal elements in these quotient algebras with normal elements with *finite* spectrum. This is not something which is possible in general in C^* -algebras, however in a von Neumann algebra we have the following easy result (which we will use later).

Lemma 1.2. *In a von Neumann algebra the set of normal elements with finite spectrum is dense in the set of normal elements*

Proof. Let t be normal with spectrum $\sigma(t)$ and $\epsilon > 0$. Since $\sigma(t)$ is compact we can choose s to be a simple borel measurable function on $\sigma(t)$ such that $\|s - \text{id}_{\sigma(t)}\|_\infty < \epsilon$. Apply s to the operator in question - we now have $\|s(t) - t\| = \|s - \text{id}_{\sigma(t)}\|_\infty < \epsilon$. \square

In general we will often approximate by normal elements with some specific structure on the spectrum, as an example consider the following nice lemma which also will be useful later.

Lemma 1.3. *Let a be a self-adjoint element in a C^* -algebra, satisfying $\|a^2 - a\| \leq \epsilon < 1/4$, and let $f : \mathbb{R} \setminus \{1/2\} \rightarrow \{0, 1\}$ be defined such that $f(x)$ is 0 when $x < 1/2$, and $f(x) = 1$, when $x > 1/2$. Then the projection $f(a)$ is well-defined, and $\|f(a) - a\| \leq \epsilon + O(\epsilon^2)$.*

Proof. First notice that since $\|a^2 - a\| = \sup\{|t^2 - t| : t \in \sigma(a)\}$, $1/2$ is not in $\sigma(a)$, so $f(a)$ is well-defined since f is continuous on $\sigma(a)$. In general the spectrum of a is contained in the set $\{t : |t^2 - t| < \epsilon\}$ - i.e: $\sigma(a) \subset [1/2 - \sqrt{1/4 + \epsilon}, 1/2 - \sqrt{1/4 - \epsilon}] \cup [1/2 + \sqrt{1/4 - \epsilon}, 1/2 + \sqrt{1/4 + \epsilon}]$. So $\|f(a) - a\| \leq \epsilon + O(\epsilon^2)$. \square

If X is a C^* -algebra and Y is a self-adjoint closed two-sided ideal in X , and $b \in X/Y$, and $a \in X$ satisfies $\pi(a) = b$ (π is the quotient mapping), we say a is a lift of b . Many of our problems will be formulated as lifting problems: If I have $b \in X/Y$ with a certain property, does there exist a lift $a \in X$ with the same property? For instance one can always lift a self-adjoint element to a self-adjoint element. To see this let $b \in X/Y$ be self-adjoint, and let x be any element in X such that $\pi(x) = b$, and put $a = \frac{1}{2}(x + x^*)$. Then $\pi(a) = y$, and a is clearly self-adjoint.

In general it is not possible to lift normal elements to normal elements, however if the normal element has finite spectrum it can be lifted, which follows from the following result:

Lemma 1.4. *Let b be a normal element in X/Y with spectrum homeomorphic to some subset of the real line. Then there exists a normal element $a \in X$ such that $\pi(a) = b$.*

Proof. Let $b \in X/Y$ be normal, and let f be a homeomorphism which maps $\sigma(b)$ to a subset of the real line. Lift $f(b)$ to a self-adjoint element $x \in X$, as remarked above. Let \tilde{f} be a continuous function from \mathbb{R} into \mathbb{C} which is identical to f^{-1} on $\sigma(f(b))$. Put $a = \tilde{f}(x)$ which is normal by continuous function calculus, and we have $\pi(a) = \pi(\tilde{f}(x)) = \tilde{f}(\pi(x)) = (\tilde{f} \circ f)(b) = b$. \square

We will also need the concept of corner algebras: Let p be a projection in a C^* -algebra A , then the set pAp is stable under multiplication, addition, scalar multiplication, and taking adjoints, and we can regard it as a C^* -subalgebra of A . It is called a corner algebra because in matrices, in appropriate basis, everything takes place in a corner. Notice, that unless p is the identity pAp does not inherit the same unit as A but it however has p as a unit. When we calculate spectrum in the corner algebra we mean with respect to this new unit, so the spectrum of an element in the corner algebra is not the same as the spectrum in A . Now if $x \in A$ commutes with p we have $pxp = xp$, and if we want to apply a continuous function ϕ inside the algebra we have to use the new unit p and will get $\phi(xp) = \phi(x)p$ (to see this start with polynomials). Moreover we have that $\sigma_A(x) = \sigma_{pAp}(xp) \cup \sigma_{(1-p)A(1-p)}(x(1-p))$. To see this realize A as a C^* -subalgebra of $B(H')$ (where H' is some Hilbert space of appropriate dimension). Projections will go to projections and p and x will still commute in $B(H')$, so in an appropriate basis x will consist of two blocks, and p will be diagonal.

Finally we will mention a few things about polar decomposition. Let us recall that if $T \in B(H)$, and it has polar decomposition $T = V|T|$ where V is a partial isometry, when we speak of *the* polar decomposition we mean *the unique* decomposition of a partial isometry times a positive operator that satisfies $\ker V = \ker T$. Moreover we recall that V can be ‘extended’ to a unitary U such that $T = U|T|$ if and only if $\dim \ker T = \dim \ker T^*$. Such a decomposition we will refer to as a unitary polar decomposition. In particular such a unitary, U , exists if T is normal, and it is straightforward to check that U then commutes with T (We get $|T|^2U = U|T|^2$, and since the square root function is a continuous we also have $[U, |T|] = 0$), and also the partial isometry V will commute with T . The converse is also true for the unitary part: If the factors of a unitary polar decomposition commute, then T is normal (To see this: Let $T = U|T|$, where U is unitary and $[U, |T|] = 0$, then $[U^*, |T|] = 0$ and hence $T^*T = |T|^2 = TT^*$).¹

In a C^* -algebra we do not in general have a polar decomposition (clearly the absolute value would be in the C^* -algebra but not necessarily the partial isometry). However if an element has unitary polar decomposition where the

¹In general if $T = V|T|$, where V is a partial isometry which commutes with $|T|$ one cannot say that T is normal. Take for instance the unilateral shift.

composants commute then the element is indeed normal - the proof is in the same way as before.

Let us also note that for the adjoint we have a polar decomposition by:

$$T^* = |T|V^* = (V^*V)|T|V^* = V^*(|T|V^*)$$

Since $|T|$ is positive $TV^* = V|T|V^*$ is positive. Moreover $\ker(TV^*) = \ker V^*$: Clearly $\ker V^* \subset \ker(TV^*)$. To see the other inclusion assume $x \in \ker(TV^*)$ but not in $\ker V^*$ to obtain a contradiction. We then have $V^*x \in \ker T = \ker V$. Hence $V^*x \in V^*(H)^\perp$, which means $V^*x = 0$, since V^* is partial isometry. This establishes the contradiction. From the uniqueness of the polar decomposition we must therefore have $|T^*| = V|T|V^*$, so:

$$T^* = V^*|T^*|$$

Finally by taking adjoints we also have:

$$T = |T^*|V.$$

Chapter 2

Some Questions, and Some Answers

In this chapter we make a precise formulation of the two main theorems we want to prove, and consider some related questions.

2.1 Index Theory and a Brown-Douglas Fillmore Theorem

We want to investigate when an essentially normal operator is a compact perturbation of a normal operator. In order to formulate the result we need to introduce the Fredholm index, and prove some fundamental results.

Recall that an operator A is *essentially normal* if its self-commutant $[A, A^*]$ is compact, or equivalently that $\pi(A)$ is normal in the Calkin algebra. The essential spectrum of A we define to be the spectrum of $\pi(A)$ and we denote it $\sigma_{ess}(A)$.

Fredholm Operators and Index

It is worth recalling that $\ker T^* = (T(H))^\perp$, which we will use constantly in what follows.

Definition 2.1. An operator $T \in B(H)$ is called a Fredholm operator if and only if T has closed range, and the kernels of T and T^* are both finite-dimensional. The collection of Fredholm operators is denoted by $F(H)$. Moreover for a Fredholm operator T we define its index as the integer $\dim \ker T - \dim \ker T^*$. For $n \in \mathbb{Z}$ we let $F_n(H)$ denote the class of Fredholm operators with index n .

The function which takes each Fredholm operator to its index we will denote just by index. Moreover, in this connection, we will view \mathbb{Z} as a group with respect to addition, and equip it with the discrete topology.

We notice that any invertible operator A is Fredholm, and has index 0, moreover if $T \in F_n(H)$ then $AT \in F_n(H)$, and $T^* \in F_{-n}(H)$.

If T is a normal Fredholm operator then the index of T is zero since T^* and T have identical kernels ($\|Tx\| = 0$ is equivalent to $\|T^*x\| = 0$ because T is normal).

Let U_+ be the unilateral shift - we see that index $U_+^n = -n$ and index $U_+^{*n} = n$, so no Fredholm classes are empty.

To each Fredholm operator T there exists a partial inverse in the following sense:

Proposition 2.2. *Let $T \in F(H)$. There exists a unique $S \in F(H)$ with $\ker S = \ker T^*$ and $\ker S^* = \ker T$, and such that ST is the projection onto $\ker T^\perp$, and TS is the projection onto $(\ker T^*)^\perp$.*

Proof. Let $\tilde{T} : \ker T^\perp \rightarrow T(H)$ be equal to the restriction of T . Since \tilde{T} is bijective and $T(H)$ is closed by assumption it has a bounded inverse \tilde{S} . Let $S : H \rightarrow H$ be the linear extension of \tilde{S} by letting $S = 0$ on $T(H)^\perp = \ker T^*$. S has the desired properties. \square

The following fundamental result is often called Atkinson's Theorem.

Theorem 2.3. *An operator $T \in B(H)$ is a Fredholm operator if and only if $\pi(T)$ is invertible in the Calkin algebra.*

Proof. That $\pi(T)$ is invertible in the Calkin algebra means that there exists an operator $S \in B(H)$ such that both $ST - I$ and $TS - I$ are compact, and we see that the 'only if' part of the proof follows from the existence of a partial inverse which we have already established.

It remains to be shown that if $\pi(T)$ is invertible in the Calkin algebra then T is a Fredholm operator. Since $\pi(T)$ is invertible there exists $S \in B(H)$ such that $ST = I + K$, and $TS = I + K'$, where K , and K' are compact operators. We now have:

$$\ker T \subset \ker(ST) = \ker(I + K) = \{\text{eigenvectors of } K \text{ with eigenvalue } -1\}$$

Since K' is compact it can have at most finite dimension of eigenspaces which are not associated with eigenvalue 0. Hence $\ker T$ has finite dimension.

By using $T^*S^* = I + K^*$, we similarly show that the dimension of $\ker T^*$ is finite.

To see that $T(H)$ is closed it is enough to show that T restricted to $(\ker T)^\perp$ is bounded from below. Assume it is not bounded from below. Then there exists a sequence of unit vectors $(x_n) \in (\ker T)^\perp$ such that $\|Tx_n\| < 1/n$.

Hence $Tx_n \rightarrow 0$.

Hence $STx_n \rightarrow 0$.

Hence $(1 + K)x_n \rightarrow 0$.

Since K is compact there exists a subsequence (y_n) of (x_n) such that (Ky_n) converges to some $y \in H$, and hence $(y_n) \rightarrow -y$.

We have $y \in (\ker T)^\perp$, since $(\ker T)^\perp$ is closed, but we also have $Ty_n \rightarrow 0 = -Ty$, so $y \in \ker T$. Hence $Ty = 0$.

This established a contradiction since (y_n) is a sequence of unit vectors. \square

We now continue by showing some fundamental properties of the index.

Theorem 2.4. *The path-connected components of $F(H)$ are precisely the classes $F_n(H)$, $n \in \mathbb{Z}$, and the function $\text{index} : F(H) \rightarrow \mathbb{Z}$ is a continuous homomorphism which is invariant under compact perturbations.¹*

To prove Theorem 2.4 we will start with a few lemmas:

Lemma 2.5. *Let $A \in B_f(H)$, then $I + A \in F_0(H)$.*

Proof. $I + A \in F(H)$, since $(I + A) - I$ is compact. Let R be the partial inverse of $I + A$, and put $P = I - R(I + A)$, and $Q = I - (I + A)R$, which are both of finite rank, and satisfy $\text{index}(I + A) = \text{rank } P - \text{rank } Q$. Let E the projection onto the finite dimensional space spanned by the images of P, Q, A , and A^* . Then E is a unit for P, Q , and A (Clearly $PA = A$, and we have $AP = (PA^*)^* = (A^*)^* = A$). Now $P - Q = AR - RA = A(ERE) - (ERE)A$ and restricting to $E(H)$ we get a linear map from $E(H)$ into $E(H)$, so it is meaningful to take the trace, and we obtain

$$\text{rank } P - \text{rank } Q = \text{tr}(P - Q) = \text{tr}(A(ERE) - (ERE)A) = 0$$

\square

Lemma 2.6. *If $T \in F_0(H)$, there exists a partial isometry V of finite rank such that $T + V$ is invertible.*

Proof. Let V be the partial isometry with initial space equal to $\ker T$ and final space equal to $\ker T^* = T(H)^\perp$ (this is possible since $\dim \ker T = \dim \ker T^*$ and both kernels are closed, by the assumption that $T \in F_0(H)$). It now easily follows that $T + V$ is injective (kernel is $\{0\}$), and surjective ($T(H)$ is closed since it is a Fredholm operator, so $H = T(H) \oplus V(H)$). Hence $T + V$ is invertible. \square

Lemma 2.7. *If $T \in F_0(H)$, and $A \in B_K(H)$ then $T + A \in F_0(H)$.*

¹That index is a continuous homomorphism which is invariant under compact perturbations translates into: Each Fredholm class is open, and if $S \in F_n(H)$ and $T \in F_m(H)$ then $ST \in F_{n+m}(H)$. Moreover if K is compact then $T + K \in F_n(H)$.

Proof. By Lemma 2.6 Let V be a partial isometry of finite rank such that $T + V$ is invertible

Since the $B_f(H)$ is dense in $B_K(H)$ we can choose a finite rank operator F such $\|F - A\| < \|(T + V)^{-1}\|^{-1}$.

Put $R = T + V + A - F = (T + V)(I + (T + V)^{-1}(A - F))$.

We have $I + (T + V)^{-1}(A - F)$ is invertible since $\|(T + V)^{-1}(A - F)\| < 1$, and hence R is invertible. We now have:

$$\text{index}(T + A) = \text{index}(R(I + R^{-1}(F - V))) = \text{index}(I + R^{-1}(F - V)) = 0,$$

where we in the second last inequality used that multiplication by an invertible operator does not change the Fredholm class, and in the last inequality we used that $R^{-1}(F - V) \in B_f(H)$ and Lemma 2.5 \square

We now interlude our search for a proof of 2.4 with proving the following theorem for compact operators known as the Fredholm Alternative:

Theorem 2.8. *Let $A \in B_K(H)$, and let $\lambda \in \sigma(A) \setminus \{0\}$, then λ is an eigenvalue for A with finite multiplicity. Moreover $\bar{\lambda}$ is an eigenvalue for K^* with the same multiplicity.*

Proof. Let $\lambda \in \sigma(A) \setminus \{0\}$, and assume that λ is not an eigenvalue of A with finite multiplicity (i.e. either an eigenvalue with infinite multiplicity or not an eigenvalue at all). Put $T = I - \lambda^{-1}A$. By Lemma 2.7 $T \in F_0(H)$, since A is compact. Since T is Fredholm $\ker T$ is finite-dimensional, and λ therefore cannot be an eigenvalue with infinite multiplicity. Hence λ is not an eigenvalue, and $\ker T$ must be $\{0\}$. Moreover since T is Fredholm $T(H)$ is closed. Hence $T(H) = (\ker T^*)^\perp$, but since $T \in F_0(H)$ we have $\ker T^* = \{0\}$ (since $\ker T = 0$), and hence $T(H) = H$. So T is invertible contradicting $\lambda \in \sigma(A)$.

The last part about K^* follows from $\dim \ker T^* = \dim \ker T$, and $T^* = I - \bar{\lambda}^{-1}A^*$. \square

Lemma 2.9. *$F_0(H)$ is open in $B(H)$.*

Proof. Assume the result is not true. Then for some $T \in F_0(H)$ there exists a sequence (T_n) in $B(H)$ converging to T , where each $T_n \notin F_0(H)$. By Lemma 2.6 there exist a partial isometry V of finite rank such that $T + V$ is invertible. Since the collection of invertible elements is open in $B(H)$ there exists a positive integer N such that $T_N + V$ is invertible, and hence $T_N + V \in F_0(H)$. Since $-V$ is compact $T_N = (T_N + V) - V \in F_0(H)$ by Lemma 2.7, contradicting the way (T_n) was constructed. \square

Before we proceed, we remark the following with respect to direct sums: Let $H = H_1 \oplus H_2$, where H_1 and H_2 are Hilbert spaces. Let $S \in F(H_1)$ and

let $T \in F(H_2)$, then $\text{index}(S \oplus T) = \text{index}(S) + \text{index}(T)$.

Proof that compact perturbations do not change index:

Let $T \in F_n(H)$ for some integer n , and let $A \in B_K(H)$. We now have: $(T + A) \oplus T^* = T \oplus T^* + A \oplus 0$, and since $\text{index}(T \oplus T^*) = 0$ and $A \oplus 0$ is compact we have from Lemma 2.7 and $\text{index}((T + A) \oplus T^*) = \text{index}(T \oplus T^* + A \oplus 0) = 0$. By Atkinson's theorem $T + A$ is a Fredholm operator since A is compact. Hence $\text{index}(T + A) = n$, as desired.

Proof that each Fredholm class is open:

Let $T \in F_n(H)$. We have $T \oplus T^* \in F_0(H \oplus H)$. By Lemma 2.9 there exists an open neighborhood U of $T \oplus T^*$ contained in $F_0(H \oplus H)$, for which we have $U \cap B(H) \oplus T^* \subset F_n(H) \oplus T^*$. This shows $U \cap B(H) \oplus T^*$ is an open subset in the induced topology on $B(H) \oplus T^*$ contained in $F_n(H) \oplus T^*$, which shows what we want since T was arbitrary.

Proof that index is a homomorphism:

Let $T, S \in F(H)$. Since $S \oplus S^* \in F_0(H \oplus H)$ there exists a partial isometry V such that $S \oplus S^* + V$ is invertible, by Proposition 2.6. Now we have:

$$\begin{aligned} \text{index}(T) &= \text{index}(T \oplus I) = \text{index}((T \oplus I)(S \oplus S^* + V)) \\ &= \text{index}(TS \oplus S^* + (T \oplus I)V) = \text{index}(TS \oplus S^*) \\ &= \text{index}(TS) - \text{index}(S), \end{aligned}$$

where we in the second equality used that multiplying by an invertible operator does not change the index, and in the second last equality we used the result proved above that compact perturbations do not change the index.

To show that the Fredholm classes are path-connected we need the following two results:

Proposition 2.10. *In $B(H)$ the collection of unitary operators are path-connected.*

Proof. Let $U \in B(H)$ be unitary. We will show there exists a continuous path connecting U to the identity. Let $\text{Arg} : \mathbb{C} \setminus \{0\} \rightarrow [0; 2\pi[$ be the argument function on the complex plane. This is a bounded borel function into the real numbers, and hence $A := \text{Arg}(U)$ is selfadjoint. Let now the continuous path be defined by $t \mapsto e^{tiA}$, for $t \in [0, 1]$. Since A is selfadjoint e^{tiA} is unitary for all $t \in [0, 1]$. It is straightforward to see that the path is continuous. \square

Proposition 2.11. *In $B(H)$ the collection of invertible operators are path-connected.*

Proof. Let $T \in B(H)$ be an invertible operator, with polar decomposition $T = UP$, where U is unitary and P is strictly positive. By Proposition 2.10 there exists a continuous path from $[0, 1] \ni t \mapsto U_t$ connecting the identity to U . Together with the continuous path $[0, 1] \ni t \mapsto P_t$, where $P_t = (1-t)I + tP$ is an invertible positive operator (To see the invertibility look at $\sigma(P_t)$), the result follows. \square

Proof that each Fredholm class is path-connected:

Assume $n \geq 0$ (Since $F_n(H)^* = F_{-n}(H)$ and the adjoint map is continuous this is without loss of generality).

First we notice that a Fredholm operator T with index n , can be connected by a path in $F_n(H)$ to an operator in $F_n(H)$ which kernel has dimension n (and hence is surjective). To see this let V be a partial isometry with initial space contained in $\ker T$ and image equal to $\ker T^*$ (This is possible since $n \geq 0$). Define a continuous path by $t \mapsto T + tV$, where $t \in [0, 1]$. For any $t > 0$ it is clear that $T + tV$ is surjective and the dimension of its kernel is n .

Secondly we notice that if $R, T \in F_n(H)$ and their kernels have equal dimension there exists a continuous path in $F_n(H)$ connecting R to $\tilde{R} \in F_n(H)$ with same kernel as T . To see this let U be a unitary such the $\ker RU = \ker T$, and put $\tilde{R} = RU$, and let $t \mapsto U_t$ be a path of unitary operators from I to U (using Proposition 2.10). Now since the the unitaries are invertible $t \mapsto RU_t$ is a map into $F_n(H)$ since multiplication by an invertible operator does not change the index.

We thus only have to show that if $R, T \in F_n(H)$ are surjective operators with identical kernel, then they can be connected by a continuous path. To see this consider the restrictions of R and T to the orthogonal complement of their kernels. These restrictions are now invertible, and we can use Proposition 2.11 to path-connect the restricted operators with invertible operators. Extending the operators in this path to H by putting them equal to 0 on the kernel of R and T we obtain the desired path in $F_n(H)$.

This completes the proof of Theorem 2.4

A Brown-Douglas-Fillmore Theorem

Because of Atkinson's theorem, and since the index is invariant under compact perturbations, the following definition makes sense:

Definition 2.12. Let t be an invertible operator in $Q(H)$, we define the index of t as the index of any of its preimages.

Moreover if $T \in B(H)$ is an essentially normal operator we define *the index function* of T as function:

$$i_T : \mathbb{C} \setminus \sigma_{ess}(T) \rightarrow \mathbb{Z}, \quad \text{by} \quad i_T(\lambda) = \text{index}(T - \lambda I)$$

We speak of Fredholm classes in $Q(H)$ in the same way we speak of Fredholm classes in $B(H)$ - in $Q(H)$ the Fredholm classes is a partition of the invertible elements, and they share similar properties to the Fredholm classes in $B(H)$:

Theorem 2.13. *In $Q(H)$ the Fredholm classes are open and path-connected.*

In particular the index function of an essentially normal operator is continuous, and constant on the connected components of its domain, and 0 on the unbounded component.

Proof. To see that each Fredholm class in $Q(H)$ is open let $x = \pi(X) \in Q(H)$ have index n . Then X has index n . Choose r such that $B(X, r) \subset F_n(H)$. We will show that $B(x, r)$ consists only of elements in $Q(H)$ of index n . Let $y = \pi(Y) \in B(x, r)$. By definition of the quotient norm there exists a $K \in B_K(H)$ such that $Y + K \in B(X, r)$. Hence $Y + K \in F_n(H)$. Hence $y = \pi(Y + K)$ has index n in the Calkin algebra. Combining that the Fredholm classes in $Q(H)$ are open with the fact that path-connected sets are mapped to path-connected sets by π (since π is continuous) it also follows that the path-connected components of the invertible elements in $Q(H)$ are exactly the Fredholm classes.

From this it follows that the index function is continuous and constant on the path-connected components. To see that the index function is 0 on its unbounded component just consider any element, λ , in the unbounded component of $\mathbb{C} \setminus \sigma(T)$,² then $\text{index}(T - \lambda) = 0$, since $T - \lambda$ is invertible. \square

Remark: Since for any invertible element $a \in Q(H)$, the ball $B_{\frac{1}{\|a^{-1}\|}}(a)$ consists of invertible elements and since the ball is path connected, Theorem 2.13 implies that for any $b \in B_{\frac{1}{\|a^{-1}\|}}(a)$,

$$\text{index}(b) = \text{index}(a).$$

We can now state the theorem we want to prove:

Theorem 2.14. *Let $T \in B(H)$ be an essentially normal operator. Then T is a compact perturbation of a normal operator if and only if it has trivial index function.*

The proof will be given in chapter 5. The original proof appeared in [7]. The proof we will give is much shorter and has similarities to the proof of Lin's theorem and is from [8].

²The spectrum of *any* bounded operator is compact, so there is an unbounded component.

2.2 Almost Commuting Matrices

Our starting point will be the following simple question: If two self-adjoint matrices almost commute do there exist self-adjoint exactly commuting matrices which are close to the given almost commuting matrices? More precisely: Given $n \in \mathbb{N}$, and $\epsilon > 0$, does there exist a $\delta > 0$ such that for any self-adjoint matrices $A, B \in M_n(\mathbb{C})$ with $\|A\|, \|B\| \leq 1$ satisfying $\|[A, B]\| < \delta$ (we say (A, B) δ -commute) there exist self-adjoint matrices A' and B' such that $[A', B'] = 0$ and $\|A - A'\|, \|B - B'\| < \epsilon$? This question is treated in the following theorem which depends on a straightforward compactness argument:

Theorem 2.15. *Let \mathcal{K} be a compact set of operators on a finite-dimensional vector space with respect to the supremum-norm. If $r > 0$, let \mathcal{K}_r consist of all pairs of r -commuting matrices in \mathcal{K} .*

Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\forall (A, B) \in \mathcal{K}_\delta \exists A', B' \in \mathcal{K} : [A', B'] = 0 \quad \text{and} \quad \|A - A'\|, \|B - B'\| < \epsilon \quad (2.1)$$

Proof. Assume the theorem is not true.

Then there exist $\epsilon > 0$, and sequences $(A_n), (B_n)$ in \mathcal{K} such that $\|[A_n, B_n]\| \rightarrow 0$ as $n \rightarrow \infty$ while for all n and commuting $R, S \in \mathcal{K}$ we have $\|R - A_n\|, \|S - B_n\| \geq \epsilon$.

By compactness there exists sequence of naturals (n_k) such that (A_{n_k}) converges to say A' and (B_{n_k}) converges to say B' . By continuity of the elementary operations and the norm we must have $[A', B'] = 0$.

Now just choose K so large that $\|A_{n_K} - A'\| < \epsilon$ and $\|B_{n_K} - B'\| < \epsilon$. Since A' and B' commute, this is in contradiction with how we constructed the sequences. \square

Since the self-adjoint matrices form a closed subset of $M_n(\mathbb{C})$ we have our result as a corollary since the the unit ball in $M_n(\mathbb{C})$ is compact. Similarly the set of respectively normal, unitaries, and (orthogonal) projections are closed, and hence almost commuting normals can be approximated by commuting normals, and so on...

One can ask to what extend the assumption that the norm of matrices have some upper bound is necessary. In an article by Shields and Percy,[5], they prove the following theorem relating to this situation:

Theorem 2.16. *Let $A, B \in M_n(\mathbb{C})$ with A self-adjoint. Let $\epsilon > 0$. If*

$$\|[A, B]\| \leq \frac{2\epsilon^2}{n-1}, \quad (2.2)$$

then there exists $A', B' \in M_n(\mathbb{C})$ with A' self-adjoint, such that:

$$[A', B'] = 0 \quad \text{and} \quad \|A - A'\| < \epsilon \quad \|B - B'\| \leq \epsilon \quad (2.3)$$

Furthermore if B is self-adjoint, B' can also be chosen to be self-adjoint.

We omit the proof, and instead go in a different direction. Notice how the δ in Theorem 2.15 depends not only on ϵ but also on the dimension of the Hilbert Space which the operators act on. We will now ask whether it is possible to choose the δ in the Theorem 2.15 independently on the dimension of the underlying Hilbert Space - we say then that the approximation is *uniform*. In this case we cannot use the compactness argument, and the problem becomes much harder. We will be interested in the following natural questions: Are contractive almost commuting projections/self-adjoints/unitaries/normals uniformly close to exactly commuting projections/self-adjoints/unitaries/normals?

Projections

We start with simplest case of projections, and give a proof of the following theorem:

Theorem 2.17. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for all positive integers n and any at most countable collection $A_i \in M_n(\mathbb{C})$ of pairwise δ -commuting projections there exists a pairwise commuting collection $A'_i \in M_n(\mathbb{C})$ of projections such that $\|A_i - A'_i\| < \epsilon$.*

Proof. Let us first reformulate the problem into a lifting problem. Let us just for notational clarity consider the case of two almost commuting projections - the reformulation for countably many almost commuting projections is analogous.

Assume the theorem is not true. Then there exist some $\epsilon > 0$, and sequence of positive integers, and sequences of matrices $(P_k), (Q_k) \in M$ such that $[\pi(P_k), \pi(Q_k)] = 0$, and for any commuting pair $(P'_k), (Q'_k) \in M$ we have $\|P'_k - P_k\| > \epsilon$ and $\|Q'_k - Q_k\| > \epsilon$ for all $k \in \mathbb{N}$. In particular this means that $\pi(P'_k) \neq \pi(P_k)$ and $\pi(Q'_k) \neq \pi(Q_k)$.

In other words if the theorem is not true it establishes the existence of an at most countable family of mutually commuting projections in M/A which cannot be lifted to commuting projections in M .

However this is exactly what we will show, that we can do, in a few steps below:

Step 1 - A projection in M/A can be lifted to a projection in M : Let $p = (p_k)$ be a projection in M/A . Since p is self-adjoint it lifts to a self-adjoint $s = (s_k) \in M$. We have $\pi(s^2 - s) = 0$, and hence $\|(s_k^2 - s_k)\| \rightarrow 0$ as $k \rightarrow \infty$.

Choose N such that $\|(s_k^2 - s_k)\| < 1/4$ for $k > N$. Choose f as in Lemma 1.3.

Let $s' = (s'_k)$ be defined by $s'_k = s_k$ for $k > N$, and $s_k = 0$ for $k \leq N$.

Let $P = f(s')$. This is clearly a projection, and

$$\pi(P) = f(\pi(s)) = f(p) = p$$

Step 2 - Let $p, x \in M/A$, where p is a projection and x is self-adjoint, for which $[p, x] = 0$. Then p lifts to a projection P , and x lifts to a self-adjoint X , such that $[P, X] = 0$:

Lift p to a projection P , and lift x to a self-adjoint Y , and put $X = (1 - P)Y(1 - P) + PYP$. Then:

$$\pi(X) = (1 - p)x(1 - p) + pxp = x(1 - p) + xp = x,$$

and

$$[P, X] = PXP - PXP = 0.$$

Step 3 - Let $p, q \in M/A$ be projections such that $[p, q] = 0$. Then p and q lift to projections P and Q respectively such that $[P, Q] = 0$:

First lift p and q to a projection P and a self-adjoint X respectively, such that $[P, X] = 0$. Now use the procedure as in step 1 to X to obtain a projection Q which is a lift of q . Since this Q will be a continuous function of element with which P commutes, P will also commute with Q .

Step 4 - Let $p_1, \dots, p_N \in M/A$, be a finite family of mutually commuting projections which can be lifted to mutually commuting projections $P_1, \dots, P_N \in M$, and let q be a projection which commutes with all p_1, \dots, p_N , then q can be lifted to a projection $Q \in M$ which commutes with all P_1, \dots, P_N .

Lift q to a selfadjoint Y , and now put

$$X_1 = (1 - P_1)Y(1 - P_1) + P_1YP_1, \text{ and}$$

$$X_k = (1 - P_k)X_{k-1}(1 - P_k) + P_kX_{k-1}P_k, \text{ for } 2 \leq k \leq N. \text{ Let } X := X_N.$$

By induction it is easy to see that each $\pi(X_k) = Q$, and $[X_k, P_l] = 0$ for $1 \leq l \leq k \leq N$ (but it is tedious to write down). Proceeding with X as in step 3, we obtain our desired projection Q .

□

In the proof above we showed the nice result that projections can be lifted to projections from M/A . This is not in general true for other quotient algebras.

The question remains whether it is always possible to lift an uncountable family of commuting projections to a family of commuting projections.³

Self-adjoints

In the case of self-adjoint matrices we have Lin's Theorem [1]:

³The answer is not known to me at the time of writing this (EDIT: Now it is known to me - it is in general not possible to lift an uncountable family of commuting projections to an uncountable family of commuting projections).

Theorem 2.18. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for all positive integers n and any pair $A, B \in M_n(\mathbb{C})$ of self-adjoint δ -commuting matrices with $\|A\|, \|B\| \leq 1$ there exists a commuting pair $A', B' \in M_n(\mathbb{C})$ of self-adjoint matrices which are ϵ -close to A' and B' .*

We give a proof of the theorem in chapter 3. The proof given there is based on a proof in an article, [2], by Friis and Rørdam which is significantly shorter than Lin's original proof appearing in [1].

It is now prudent to ask whether Lin's theorem holds for several almost commuting selfadjoint matrices. It turns out to not be the case - even for 3 almost commuting matrices one cannot find 3 exactly commuting matrices in general. We will show this in chapter 4, as a part of a stronger result, based on a proof in [4].

Let us say an operator A is δ -normal if $\|[A, A^*]\| < \delta$. Write $A = X + iY$ with X, Y self-adjoint, and notice that A δ -commutes if and only if X, Y $\frac{\delta}{2}$ -commutes. It follows that Lin's theorem is equivalent to saying that almost normal matrices can be uniformly approximated by normal matrices.

Moreover recalling that compact operators in arbitrary Hilbert spaces can be approximated by finite rank operators it also follows from Lin's theorem that almost normal compact operators are close to normal compact operators.

Let us now give an example of almost normal operators in $B(H)$ which are not close to normal operators.

Let e_1, e_2, \dots be an orthonormal basis in H . For each $n \in \mathbb{N}$ we define a weighted shift operator T_n by:

$$T_n e_k = \begin{cases} \frac{k}{2n^2} e_{k+1}, & 1 \leq k \leq 2n^2 \\ e_{k+1}, & k > 2n^2. \end{cases}$$

Then

$$T_n^* e_k = \begin{cases} \frac{k-1}{2n^2} e_{k-1}, & 1 \leq k \leq 2n^2 \\ e_{k-1}, & k > 2n^2 \end{cases}$$

for $k > 1$, and for $k = 0$, $T_n e_1 = 0$. We have

$$T_n T_n^* e_k = \begin{cases} \frac{(k-1)^2}{(2n^2)^2} e_k, & 1 \leq k \leq 2n^2 \\ e_k, & k > 2n^2, \end{cases}$$

$$T_n^* T_n e_k = \begin{cases} \frac{k^2}{(2n^2)^2} e_k, & 1 \leq k < 2n^2 \\ e_k, & k \geq 2n^2. \end{cases}$$

Hence

$$[T_n, T_n^*] e_k = \begin{cases} \frac{(k-1)^2 - k^2}{(2n^2)^2} e_k, & 1 < k < 2n^2 \\ \frac{(2n^2-1)^2}{(2n^2)^2} - 1, & k = 2n^2 \\ 0, & k > 2n^2. \end{cases}$$

Since

$$\left| \frac{(2n^2 - 1)^2}{(2n^2)^2} - 1 \right| < \frac{1}{n^2}$$

and

$$\left| \frac{(k - 1)^2 - k^2}{(2n^2)^2} \right| < \frac{1}{n^2}$$

for all $k < 2n^2$, we obtain

$$\|[T_n, T_n^*]\| < \frac{1}{n^2} \rightarrow 0$$

as $n \rightarrow \infty$. Thus the operators T_n are almost normal. We claim that they are not close to any normal operators. Suppose they were. Then there would exist normal operators N_n such that

$$T_n - N_n \rightarrow 0. \tag{2.4}$$

Since all T_n are compact perturbations of the unilateral shift, we have

$$\text{index } T_n = -1$$

for all n .

From the remark after Theorem 2.13, (2.4) it follows that for n large enough N_n is Fredholm of index -1 . But any normal Fredholm operator has index 0. This gives a contradiction.

Unitaries and Normals

It is interesting that the same question for unitary matrices comes with a negative answer:

Theorem 2.19. *There exist pairs $\{(A_n, B_n)\}_{n \in \mathbb{N}}$ of unitary operators, $A_n, B_n \in M_n(\mathbb{C})$ such that $\|[A_n, B_n]\| \rightarrow 0$ as $n \rightarrow \infty$ and such that $\max\{\|A' - A_n\|, \|B' - B_n\|\} \geq c$ for some positive c for all positive integers n and all commuting matrices A' , and B' (of the appropriate dimension).*

In chapter 4 we give an explicit example (called Voiculescu's pair) based on the short article, [3], by Loring and Exel. The argument depends on a topological obstruction involving winding numbers. Since unitaries are normal the question is already answered in the negative above with the Voiculescu pair. Notice that the Voiculescu pair cannot be approximated by any commuting matrices (they need not be unitary, not even normal).

2.3 Quasidiagonal Operators and The Weyl-von Neumann-Berg Theorem

Now let us consider the validity of Theorem 1.1 for non-compact normal operators. We will show the following theorem relating this question:

Theorem 2.20. *Let H be a separable Hilbert space, and let A be a normal operator on H . Given $\epsilon > 0$. There exists a diagonal operator, D , and a compact operator, K such that $A = D + K$, where $\|K\| < \epsilon$.*

We will use Lin's Theorem to prove this result. Before this we will however need to introduce quasidiagonal operators and discuss some fundamental properties of these. This will also be useful in chapter 5.

Quasidiagonal Operators

We define the following:

Definition 2.21. We say $T \in B(H)$ is blockdiagonal if and only if there exists an increasing sequence of finite rank projections (P_n) which converges to I in the strong operator topology such that $[T, P_n] = 0$ for all $n \in \mathbb{N}$.

In particular if the rank of $P_n - P_{n-1}$ is 1 for positive integers n , T in the definition is diagonal (We here define block diagonal and diagonal without reference to any particular basis - thus diagonalizable, and blockdiagonalizable might be more appropriate words).

Definition 2.22. We say an operator $T \in B(H)$ is quasidiagonal if and only if there exist an increasing sequence of finite rank projections (P_n) which converges to I in the strong operator topology such that $\|[T, P_n]\| \rightarrow 0$ for $n \rightarrow \infty$.

Actually what we call quasidiagonal would make more sense if we called it quasi-block-diagonal. We also have an equivalent so called local definition:

Local Definition 2.23. *An operator $T \in B(H)$ is quasidiagonal if and only if for any $\epsilon > 0$, and for any finite rank projection E there exists a finite rank projection $F \geq E$ such that $\|[F, T]\| < \epsilon$.*

Proof. First we prove the if part of the theorem. Let (E_n) be any increasing sequence of finite rank projections which converge to I in SOT. Choose a finite rank projection F_1 such that $F_1 \geq E_1$. For $n > 1$ let \tilde{F}_n be the finite rank projection onto the subspace spanned by the ranges of E_n and F_{n-1} and choose a finite rank projection F_n such that $F_n \geq \tilde{F}_n$ and $\|[F_n, T]\| < 1/n$. By construction (F_n) is an increasing sequence of finite rank projections - we only need to show they converge strongly to I . Let $x \in H$. Then:

$$\|F_n x - x\| \leq \|F_n x - F_n E_n x\| + \|F_n E_n x - x\| = \|F_n(1 - E_n)x\| + \|E_n x - x\|$$

Since E_n converges strongly to 0, $\|E_n x - x\| \rightarrow 0$, and since $\|F_n(1 - E_n)x\| \leq \|(1 - E_n)x\|$, $\|F_n(1 - E_n)x\| \rightarrow 0$.

To see the only if part, let $\epsilon > 0$, and E be a finite rank projection. Let (e_i) , $i = 1 \dots, N$, be an orthonormal basis for the range of E . Since there are finitely many e_i there exist a finite rank projection P such that $\|Pe_i - e_i\| < \epsilon/N$ for all $1 \leq i \leq N$ and $\|[P, T]\| < \epsilon$. Let $x \in H$ be a unit vector, then $\|Ex\| \leq 1$. Writing Ex as a linear combination of the e_i we see that $\|PEx - Ex\| \leq \epsilon$. Hence $\|PE - E\| \leq \epsilon$.

Put $Q = 1 - P$. Then Q is a projection with finite codimension, such that $\|[T, Q]\| < \epsilon$ and $\|QE\| < \epsilon$. Put $Y = (1 - E)Q(1 - E)$. Then Y is selfadjoint, and $YE = 0$. We now have:

$$\|Y - Q\| = \|-QE - EQ(1 - E)\| \leq 2\epsilon$$

And we therefore have $\|Y\| \leq 1 + 2\epsilon$, and hence also:

$$\begin{aligned} \|Y^2 - Q^2\| &\leq \|Y^2 - YQ\| + \|YQ - Q^2\| \\ &\leq \|Y\|\|Y - Q\| + \|Y - Q\|\|Q\| \\ &\leq 4\epsilon + 4\epsilon^2 \end{aligned}$$

So:

$$\|Y^2 - Y\| = \|Y^2 - Q^2 + Q - Y\| \leq 6\epsilon + 4\epsilon^2$$

Without loss of generality assume that ϵ is smaller than $1/4$. Choose f as in Lemma 1.3 then $f(Y)$ is a projection satisfying $\|f(Y) - Q\| \leq \|f(Y) - Y\| + \|Y - Q\| < \epsilon'$, where ϵ' depends on ϵ in such a way it tends to 0 as ϵ tends to 0.

Let $F = I - f(Y)$. We must show F has the desired properties

That F is of finite rank follows from $f(Y)$ and Q having same codimension, since $\|f(Y) - Q\| < 1$.

Since $f(0) = 0$ we can find a sequence of polynomials (p_n) converging uniformly to f such $p_n(0) = 0$ for all n , Hence we can write $p_n(t) = tq_n(t)$ for appropriate q_n . Hence $f(Y)E = \lim_{n \rightarrow \infty} q_n(Y)YE = 0$. So $F \geq E$.

Finally we have:

$$\|[T, F]\| = \|[T, f(Y)]\| \leq \|[T, f(Y) - Q]\| + \|[T, Q]\| \leq 2\|T\|\epsilon' + \epsilon$$

□

Let us show the following fundamental result:

Theorem 2.24. *An operator $T \in B(H)$ is quasidiagonal if and only if it is a compact perturbation of a blockdiagonal operator.*

Proof. To see the only if part, let T be quasidiagonal, and let (P_n) be an increasing sequence of finite rank projections such that $\|[T, P_n]\| \rightarrow 0$. Choose a

subsequence (Q_n) of (P_n) such that $\|[T, Q_n]\| < 2^{-n}$, then $\sum_{n=1}^{\infty} \|[Q_n, T]\| < \infty$. Let $Q_0 = 0$, and let S be the blockdiagonal operator $\sum_{n=1}^{\infty} (Q_n - Q_{n-1})T(Q_n - Q_{n-1})$. Now since $T = \sum_{n=1}^{\infty} (Q_n - Q_{n-1})T$, we have:

$$T - S = \sum_{n=1}^{\infty} -(Q_n - Q_{n-1})[T, Q_n] + \sum_{n=1}^{\infty} (Q_n - Q_{n-1})[T, Q_{n-1}]$$

Since each of sums converge in norm (since $\sum_{n=1}^{\infty} \|[Q_n, T]\| < \infty$) and the terms are compact operators, the limit must be compact since the set of compact operators is closed.

The if part of the theorem is obvious. \square

Remark: It follows from the proof above that if we start the subsequence (Q_n) such that $\|[T, Q_n]\| < C$ for all n , then for the compact perturbation $T - S$ we get $\|T - S\| < 4C$. Hence we can choose the compact perturbation to be arbitrary small.

It now follows:

Corollary 2.25. *The set of quasidiagonal operators is closed under taking compact perturbations.*

We also have:

Theorem 2.26. *The set of quasidiagonal operators is closed in the norm topology.*

Proof. We will use the local definition of a quasidiagonality described in Theorem 2.23.

Let (T_n) be a sequence of quasidiagonal operators that converges to an operator T . We must show that T is quasidiagonal. Let $\epsilon > 0$ and let E be a finite rank projection.

Choose N such that $\|T_N - T\| < \epsilon$. Since T_N is quasidiagonal there exists a finite rank projection $F \geq E$ such that $\|[T_N, F]\| < \epsilon$. We now have:

$$\|[T, F]\| \leq \|[T - T_N, F]\| + \|[T_N, F]\| \leq 3\epsilon$$

\square

Lemma 2.27. *Every normal operator in $B(H)$ is quasidiagonal.*

Proof. Every normal operator in $B(H)$ can be approximated by a normal operator with finite spectrum by Lemma 1.2. Since every normal operator with finite spectrum is quasidiagonal (by spectral theorem), and since the set of quasidiagonal operators is closed by Theorem 2.26, we conclude that normal operators in $B(H)$ are quasidiagonal. \square

Proof of the Weyl-von Neumann-Berg Theorem using Lin's Theorem

Let N be a normal operator. We want to write it as a diagonal plus an arbitrary small compact operator. Let $\epsilon > 0$.

By Lemma 2.27 N is quasideagonal, and by Theorem 2.24 we can write $N = B + K$ where B is blockdiagonal, and K is compact, and by the remark after Theorem 2.24 we can choose $\|K\| < \epsilon$.

Write $B = \sum_{n=1}^{\infty} B_n$, where $B_n = (P_n - P_{n-1})B(P_n - P_{n-1})$, and (P_n) is an increasing sequence of finite rank projections converging strongly to I , and $P_0 = 0$.

Since N is normal we have $0 = [N, N^*] = [B, B^*] + K'$, where K' is a compact operator (since the compact operators form a two-sided ideal), with $\|K'\| < \epsilon'$ where ϵ' tends to zero as ϵ tends to zero.

Now $\|[B_n, B_n^*]\| \rightarrow 0$, since $[B, B^*] = -K'$ is compact.⁴ Moreover each $\|[B_n, B_n^*]\| \leq \|K'\|$. By Lin's Theorem we can choose a sequence of normal operators D_n satisfying $D_n = (P_n - P_{n-1})D_n(P_n - P_{n-1})$, such that $\|B_n - D_n\| \rightarrow 0$, and $\|B_n - D_n\| < \epsilon''$ where ϵ'' tends to zero as ϵ tends to zero.

Put $K'' = \sum_{n=1}^{\infty} (B_n - D_n)$ which is clearly compact.

Put $D = \sum_{n=1}^{\infty} D_n$. Since each D_n can be considered as a normal operator on a finite dimensional Hilbert space, D is clearly diagonal.

We now have:

$$N = D + K'' + K, \text{ where } \|K'' + K\| \leq \epsilon + \epsilon''.$$

This completes the proof.

⁴To see this: Assume the supremum norm of the blocks do not tend to 0. Then we can choose a sequence of unit vectors all belonging to different blocks such that their images are mutually orthogonal and has norm greater than some fixed positive real number. This sequence clearly does not have a convergent subsequence which contradicts with the image of the unit ball being precompact.

Chapter 3

Lin's Theorem

We want to prove Lin's theorem:

Theorem 3.1. *Given $\epsilon > 0$, there exists a $\delta > 0$, such that for all positive integers n and all δ -normal contractions $A \in M_n(\mathbb{C})$ there exists a normal contraction $A' \in M_n(\mathbb{C})$ such that $\|A - A'\| < \epsilon$*

We will prove the theorem using the following theorem which will be proved in several steps in the next section, and contains the essential part of theorem.

Theorem 3.2. *In M/A the set of normal elements with finite spectrum is dense in the set of normal elements.*

Proof of Lin's Theorem using Theorem 3.2 Assume the theorem is false. Then there would exist an $\epsilon > 0$ and a sequence of natural numbers (n_k) and a sequence of matrices $A_k \in M_{n_k}(\mathbb{C})$ with $\|A_k\| \leq 1$ such that $\|[A_k, A_k^*]\| \rightarrow 0$ and such that the distance to any normal element in $M_{n_k}(\mathbb{C})$ for each A_k is greater than or equal to ϵ .

Put $x = (A_k)$ - since the $\|A_k\| \leq 1$ we have $x \in M$ with the proper identification.

Put $y = \pi(x)$. We have

$$[y, y^*] = \pi(x)\pi(x)^* - \pi(x)^*\pi(x) = \pi(xx^* - x^*x) = 0,$$

since $xx^* - x^*x \in A$. So y is normal.

By Theorem 3.2 there exists a normal element $y' \in M/A$ with finite spectrum such that $\|y - y'\| < \epsilon/4$.

By Lemma 1.4 there exists a normal element $x' \in M$ such that $\pi(x') = y'$.

There exists an $a = (a_k) \in A$ (by definition of the quotient norm) such that

$$\|x - x' - a\| \leq \|y - y'\| + \epsilon/4 \leq \epsilon/2.$$

Now choose K such that $\|a_K\| \leq \epsilon/2$. We have

$$\|x_K - x'_K - a_K\| \leq \|x - x' - a\|,$$

and by the reverse triangle inequality we get

$$\|x_K - x'_K\| < \epsilon,$$

which is a contradiction.

3.1 Proof that the set of normal elements with finite spectrum is dense in the set of normal elements in M/A

We will prove the theorem through a sequence of lemmas, starting with:

Lemma 3.3. *Each element $x \in M/A$ has a polar decomposition, $x = u|x|$, where u is unitary.*

Proof. Let $y = (y_k) \in M$ be any lift of x . For each positive integer n we have $y_k = u_k|y_k|$ with u_k unitary, since \mathbb{C}^{n_k} is finite dimensional. Put $u = (u_k)$, which is unitary. Now $y = u|y|$, and $x = \pi(u)|x|$, where $\pi(u)$ is unitary since π is a *-homomorphism. \square

Using unitary polar decomposition we will show the following:

Lemma 3.4. *In M/A the set of invertible normal elements is dense in the set of normal elements.*

Proof. Let $x \in M/A$ be normal. Let $\epsilon > 0$.

By Lemma 3.3 we can write $x = u|x|$ where $u \in M/A$ is unitary.

Since x is normal we have $|x|^2 = u|x|^2u^*$, hence $|x|^2$ and u commute, and hence $|x|$ and u commute. Moreover $|x| + \epsilon 1$ is invertible since it has strictly positive spectrum. It follows that $y = u(|x| + \epsilon 1)$ is normal and invertible, and moreover $\|x - y\| = \epsilon$ as desired. \square

Lemma 3.5. *Let $F \subset \mathbb{C}$ be an at most countable set. The set of normal elements in M/A which have spectrum disjoint with F is dense in the set of normal elements in M/A .*

Proof. Let H_λ be the set of normal elements in M/A which do *not* have λ in the spectrum. The mapping $x \mapsto x - \lambda$ defined on the normal elements in M/A onto the normal elements is a homeomorphism for any $\lambda \in \mathbb{C}$, which in particular maps H_λ onto the set of invertible normal elements. By Lemma 3.4 it follows that H_λ is dense in the set of normal elements.

Moreover H_λ is relatively open because the set of invertible normal elements is relative open (and the set of invertible normal elements is relatively open since the set of invertible elements is open).

By the Baire Category theorem the set $\bigcap_{\lambda \in F} H_\lambda$ is dense in the set of normal elements, which proves the lemma. \square

The results in Lemma 3.4 and 3.5 only depended on the existense of a unitary polar decomposition of normal elements.

For the next lemma we will introduce the following subsets of the complex plane (An ϵ -grid, and its center-points):

$$\Gamma_\epsilon := \{x + iy \in \mathbb{C} \mid x \in \epsilon\mathbb{Z} \text{ or } y \in \epsilon\mathbb{Z}\}$$

$$\Sigma_\epsilon := \{x + iy \in \mathbb{C} \mid x \in \epsilon(\mathbb{Z} + 1/2) \text{ and } y \in \epsilon(\mathbb{Z} + 1/2)\}$$

Lemma 3.6. *For any normal element $x \in M/A$, and for any $\epsilon > 0$ there exists normal element $y \in M/A$ with $\sigma(y) \subset \Gamma_\epsilon$ and $\|x - y\| < \epsilon$.*

Proof. Let $x \in M/A$ By Lemma 3.5, since Σ_ϵ is countable, there exists an $x' \in M/A$ such that $\|x - x'\| < \epsilon(1 - \frac{1}{\sqrt{2}})$, and $\sigma(x') \cap \Sigma_\epsilon = \emptyset$.

Let $r : \mathbb{C} \setminus \Sigma_\epsilon \rightarrow \Gamma_\epsilon$ be the continuous retraction defined as follows: For each $c \in \Sigma_\epsilon$, consider all lines from c until they intersect with the grid Γ_ϵ - every point on such a line segment is mapped to where the line intersects with the grid. We have: $|r(z) - z| < \frac{1}{\sqrt{2}}\epsilon$.

Now take $y = r(x')$ - we have $\|y - x\| = \|f(x') - x' + x' - x\| \leq \|f(x') - x'\| + \|x' - x\| < \epsilon$.

□

Lemma 3.7. *Let $u \in M/A$ be unitary. Then u can be lifted to a unitary in M .*

Proof. Let $\pi(a) = u$. Since in matrix algebras all isometries are unitaries, we can write $a = v|a|$, where v is unitary. Now we have

$$u = \pi(a) = \pi(v|a|) = \pi(v)\pi(|a|) = \pi(v)|\pi(a)| = \pi(v).$$

□

Lemma 3.8. *Let $x \in M/A$ be a normal element. Suppose V is a relatively open subset of $\sigma(x)$, which is homeomorphic to an open interval. Then for any $\lambda_0 \in V$ and any $\epsilon > 0$, there exists a normal element $y \in M/A$ such that $\sigma(y) \subset \sigma(x) \setminus \{\lambda_0\}$ and $\|x - y\| \leq \epsilon$.*

Proof. Let $x \in M/A$ be a normal element. Let U be a relatively open subset of V (V as in the theorem statement), which satisfy: $\lambda_0 \in U \subset \bar{U} \subset V$, and $\text{diam}(U) < \epsilon$.

Let f_0 be a homeomorphism from V onto $\mathbb{T} \setminus \{-1\}$, and extend it to a continuous function $f : \sigma(x) \rightarrow \mathbb{T}$ by putting it equal to -1 everywhere else. Put $u = f(x)$ which is unitary since its spectrum is contained in the unit circle. By the Lemma 3.7 there exists a unitary $v \in M$ such that $\pi(v) = u$.

Let $W = f(U)$ and let 1_W be the characteristic function on W . Since M is a von Neumann algebra $1_W(v)$ is in M . Let $e := \pi(1_W(v))$ which is a projection in M/A .

Let $\varphi : \sigma(x) \rightarrow \mathbb{C}$ be a continuous function which is 0 on $\sigma(x) \setminus V$, and let $\hat{\varphi} : \mathbb{T} \rightarrow \mathbb{C}$ be a (the) continuous function such that $\varphi = \hat{\varphi} \circ f$.

Since $1_W(v)$ commutes with v , $e = \pi(1_W(v))$ commutes with $u = \pi(v)$, and hence e commutes with $\varphi(x)$, since $\varphi(x) = \hat{\varphi}(u)$.

If $\varphi = 1$ on U , then $\hat{\varphi} = 1$ on W , and then

$$1_W(v)\hat{\varphi}(v) = (1_W\hat{\varphi})(v) = 1_W(v),$$

meaning that

$$e\varphi(x) = e\hat{\varphi}(u) = \pi(1_W(v)\hat{\varphi}(\pi(v))) = \pi(1_W(v)\hat{\varphi}(v)) = e.$$

If $\varphi = 0$ on $\sigma(x) \setminus U$, then $\hat{\varphi} = 0$ on $\mathbb{T} \setminus W$, and then

$$1_W(v)\hat{\varphi}(v) = (1_W\hat{\varphi})(v) = \hat{\varphi}(v),$$

meaning that

$$e\varphi(x) = \varphi(x).$$

Now let $h : \sigma(x) \rightarrow [0, 1]$ be continuous with $h|_{\bar{U}} = 1$ and $h|_{\sigma(x) \setminus V} = 0$. From the above considerations we have $h(x)e = eh(x) = e$, and since $z \mapsto zh(z)$ vanishes on $\sigma(x) \setminus V$ we also have $xh(x)e = exh(x)$. Hence

$$xe = xh(x)e = exh(x) = eh(x)x = ex$$

so x and e commute.

We want to show that:

$$\sigma_{e(M/A)e}(exe) \subset \bar{U} \quad \text{and} \quad \sigma_{(1-e)(M/A)(1-e)}((1-e)x(1-e)) \subset \sigma(x) \setminus U \quad (3.1)$$

First we show $\sigma_{e(M/A)e}(exe) \subset \bar{U}$. Let $\phi, \psi : \sigma(x) \rightarrow \mathbb{C}$ be any continuous functions such that ϕ vanish on \bar{U} and is equal to 1 on $\sigma(x) \setminus V$, and ψ vanish on $\sigma(x) \setminus U$. We now have (in the corner algebra eM/Ae)

$$\phi(exe) = \phi(xe) = \phi(x)e = e - (1 - \phi(x))e = e - e = 0,$$

where we in the second last equality used that $1 - \phi$ is equal to 1 on U . And we also have (in the corner algebra $(1 - e)M/A(1 - e)$)

$$\psi(x(1 - e)) = \psi(x)(1 - e) = \psi(x) - \psi(x)e = \psi(x) - \psi(x) = 0,$$

where we in the second last equality used that ψ vanishes on $\sigma(x) \setminus U$. This shows 3.1.¹

¹Why? For instance assume some $\alpha \in U$ is in $\sigma_{(1-e)(M/A)(1-e)}((1-e)x(1-e))$. Then there exists a continuous function ψ_α which is 1 at α , and which is 0 on $\sigma(x) \setminus U$ (for instance by Urysohn's lemma). Then $\psi_\alpha(xe)$ would have 1 in its spectrum and then $\psi_\alpha(xe) \neq 0$.

Choose $\lambda_1 \in U \setminus \{\lambda_0\}$ and put $y = \lambda_1 e + (1-e)x$. We have: $y^* = \bar{\lambda}_1 e + (1-e)x^*$, and we get: $y^*y = yy^*$ since x is normal and commutes with e - so y is normal. Moreover we have, since $\sigma_{(1-e)(M/A)(1-e)}((1-e)x(1-e)) \subset \sigma(x) \setminus U$ that

$$\sigma(y) \subset \{\lambda_1\} \cup (\sigma(x) \setminus U) \subset \sigma(x) \setminus \{\lambda_0\}$$

Moreover

$$\|x - y\| = \|(x - \lambda_1)e\| \leq \text{diam}(U) \leq \epsilon$$

as desired - in the second last inequality we used that $\sigma_{e(M/A)e}(xe) \subset \bar{U}$. \square

Lemma 3.9. *Let $\epsilon > 0$, and let $x \in M/A$ be a normal element whose spectrum is contained in some grid Γ_δ . Then there exists a normal element $y \in M/A$ with finite spectrum and with $\|x - y\| < \epsilon$.*

Proof. Let x be a normal element with its spectrum contained in some grid. Since the spectrum is compact, and because of the grid structure, there is only a finite number of connected components with diameter greater than $\epsilon/2$. To all those components apply Lemma 3.8 to obtain an element $x_1 \in M/A$ with $\|x - x_1\| < \epsilon/2$, where all components have diameter $< \epsilon/2$.

To each connected component of $\sigma(x_1)$ choose an open neighborhood of diameter $< \epsilon/2$ containing the component, such that *any* other connected component is entirely contained in the neighborhood, or entirely contained in the complement (This is possible because of the grid structure).

Now, by compactness we reduce the number of neighborhoods to finitely many. Since finite intersections are open and since no connected components touch the boundary of the neighborhoods we can make a finite partition of spectrum into relatively clopen sets V_1, \dots, V_n . Select $\lambda_i \in V_i$, and let f be a continuous function taking every element of V_i to λ_i such that $|f(z) - z| < \epsilon/2$ for all $z \in \sigma(x_1)$. Put $y = f(x_1)$. Then $\|y - x\| < \epsilon$, and y is normal with finite spectrum. \square

Theorem 3.2 now follows from Lemma 3.6, and Lemma 3.9.

3.2 Constructing Almost Commuting Matrices

Let $\delta > 0$, and let A and B be self-adjoint matrices with $\|A\|, \|B\| \leq 1$ such that $[A, B] < \delta$. Hastings in [6] outline a procedure which constructs self-adjoint matrices A' and B' such that $[A', B'] = 0$, and $\sup\{\|A - A'\|, \|B - B'\|\} \leq E(1/\delta)\delta^{1/5}$ where E grows slower than any power - in particular this tends to zero as δ tends to zero, and the function E is the same for all matrices, regardless of dimension, and only depends on δ . This gives a procedure to construct the matrices which Lin's Theorem shows the existence of - moreover it gives bounds on how the ϵ depends of the δ in Lin's Theorem.

We will outline some of Hasting's procedure, and show two theorems which are used, which are interesting in themselves. We will however skip the most involved part of the algorithm.²

STEP 1 Put $\Delta = \delta^{4/5}$. Construct a matrix H such that $\| [H, B] \| \leq \delta$ with the following properties:

(1A) $\| A - H \| \leq \epsilon_1 := c_0 \delta / \Delta$, where c_0 is a constant specified in Theorem 3.10.

(1B) For any eigenvectors v_1, v_2 of B with eigenvalues x_1 and x_2 satisfying $|x_1 - x_2| \geq \Delta$, we have $(v_1, H v_2) = 0$.

The construction is given below in Theorem 3.10 as well as proof of the claims.

STEP 2 Choose $c \geq -1$ such that $[-1, 1]$ has no point with distance greater than $\Delta/2$ to the set $S_c := (c + \Delta \mathbb{N}_0) \cap [-1, 1]$. Let

$$Q : [-1, 1] \rightarrow (c + \Delta \mathbb{N}_0) \cap [-1, 1]$$

be defined by letting $Q(x)$ be the number in S_c closest to x (taking the smallest in case there are two possibilities with same distance), and put $X = Q(B)$. We clearly have:

(2A) $\| X - B \| \leq \epsilon_2 := \Delta/2$

STEP 3 Change to an (ordered) orthonormal basis O such that B is diagonal with increasing eigenvalues in this basis. In this basis impose a block structure such that to each element i in $\{N \in \mathbb{N}_0 | c + N\Delta \in S_c\}$ corresponds the block leaving eigenspace of X with eigenvalue $c + \Delta i$ invariant - we here admit blocks to be of dimension 0 if no such eigenspace exists. In this block structure $[X]_O$ is blockdiagonal with each block equal to a scalar (We shall call this a 'block identity matrix' using Hastings terminology), and $[H]_O$ is block tridiagonal due to property (1B).

STEP 4 Put $n_{cut} = \Delta^{-1/4}$. For $0 \leq i \leq n_{cut} - 1$, define $I_i = [-1 + 2(j-1)/n_{cut}[,$ for $i = 1, \dots, n_{cut} - 1$, and $I_i = [-1 + 2(j-1)/n_{cut}]$ for $i = n_{cut}$. Group the blocks defined in step 3 into n_{cut} superblocks letting the i th superblock consist of those blocks corresponding to eigenvalues of X in the interval I_i . Let J_i correspond to the matrix representing the

²Hasting's procedure is not our main goal in this text, and the steps are not always so easy to follow, so we skip a lot here. The purpose of this section is just to mention this relatively new result, and emphasize the difference in having a constructive proof and not having it. This entire section can be read as an (very) early attempt at making an exposition of Hasting's article.

i -th superblock of H . That is: J_i is the matrix obtained by projecting H to the subspace \mathcal{B}_i consisting of the eigenspaces of X corresponding to eigenvalues in the interval I_i . Each J_i consist of L_i blocks where $L_i \geq \lfloor \frac{2}{n_{cut}} - 1 \rfloor$, and is block tridiagonal. Label the eigenspaces of X contained in \mathcal{B}_i by $\mathcal{V}_{i,j}$ in order of increasing eigenvalues (So each eigenspace corresponds to a block).

STEP 5 For each $1 \leq i \leq n_{cut}$ construct a subspace \mathcal{W}_i of \mathcal{B}_i , which explicit construction we will skip.

The subspace \mathcal{W}_i will have the following properties.

- (5A) The projection of any normalized vector $v \in \mathcal{V}_{i,1}$ onto \mathcal{W}_i^\perp has norm bounded by $\epsilon_3 := \frac{1}{L_i^{1/3}} f_3(L_i)$, where f_3 is a function growing slower than any power.
- (5B) For any normalized vector $w \in \mathcal{W}_i$, the projection of $J_i w$ onto \mathcal{W}_i^\perp has norm bounded by $\epsilon_4 := \frac{1}{L_i^{1/3}} f_4(L_i)$, where f_4 is a function growing slower than any power.
- (5C) The projection of any normalized vector $v \in \mathcal{V}_{i,L}$ onto \mathcal{W}_i has norm bounded by $\epsilon_5 := f_5(L_i)$, where f_5 is a function decaying faster than any power.

STEP 6 Let D_{B_i} be the dimension of \mathcal{B}_i , and D_{W_i} be the dimension of \mathcal{W}_i . For $1 \leq i \leq L_i$ create orthonormalbasis $p_{i,k}$, $1 \leq k \leq D_{W_i}$ of \mathcal{W}_i , and orthonormalbasis $q_{i,l}$, $1 \leq l \leq D_{B_i} - D_{W_i}$ for \mathcal{W}_i^\perp . Change from old basis O into a new basis N with a $n_{cut} + 1$ blocks indexed with integer $0 \leq m \leq n_{cut}$, where the m th block consists of basis vectors of \mathcal{W}_m^\perp and then \mathcal{W}_{m+1} in the given order (Here \mathcal{W}_0^\perp and $\mathcal{W}_{n_{cut}+1}$ are empty). In the new basis $[H]_N$ will be block pentadiagonal, and $[X]_N$ will be block diagonal.

STEP 7 In the new basis choose $[A']_N = [H]_N$ on the blocks in the diagonal, and 0 outside. And choose B' in the in the new basis such that it is a block identity matrix with the m th block equal to $-1 + \frac{2m}{n_{cut}}$ times the identity.

Now A' and B' commute, and the off diagonal blocks of $[A']_N - [H]_N$ are bounded by $2(\epsilon_3 + \epsilon_4 + \epsilon_5)$ (blocks in the diagonals straight above and below the main diagonal), and $\epsilon_3\epsilon_5$ (blocks in the diagonals two above and two below the main diagonal). Moreover $\|B' - B\| \leq 2/n_{cut}$.

The matrices obtained in step 7 are the desired matrices. The part skipped, was the subspace constructions in step 5, which admittedly also is the hardest part. Let us also remark that the proof that the algorithm works actually depends on Lin's Theorem (or rather a corollary to Lin's theorem). The Theorem below deals with step 1, and is interesting in itself:

Theorem 3.10. Given self-adjoint matrices $A, B \in M_n(\mathbb{C})$, such that $\|[A, B]\| \leq \delta$. Given $\Delta > 0$. Let f be a Schwartz-function, such that its Fourier transform \hat{f} is supported in $[-1, 1]$, and is even, and has $\hat{f}(0) = 1$.

Put $H = \Delta \int_{-\infty}^{\infty} \exp(iBt)A \exp(-iBt)f(\Delta t)dt$.

Then H is self-adjoint, $\|H - A\| \leq \frac{\delta}{\Delta} \int_{-\infty}^{\infty} |tf(t)|$, $\|H - B\| \leq \delta$, and for any eigenvectors v, w of B with respective eigenvalues r and s , which satisfy $|r - s| \geq \Delta$, we have $(v, Hw) = 0$.

Proof. Since \hat{f} is real and even f is real, and H is self-adjoint.

Since, $1 = \hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt = \Delta \int_{-\infty}^{\infty} f(\Delta t)dt$, we have:

$\|H - A\| = \|\Delta \int_{-\infty}^{\infty} (\exp(iBt)A \exp(-iBt) - A)f(\Delta t)dt\|$, which is less than or equal to: $\Delta \int_{-\infty}^{\infty} \|(\exp(iBt)A \exp(-iBt) - A)f(\Delta t)dt\|$.³ Now, since

$\|(\exp(iBt)A \exp(-iBt) - A)'(t)\| = \|[A, B]\|$, we have have:

$\|(\exp(iBt)A \exp(-iBt) - A)\| \leq |t|\|[A, B]\|$.⁴ Hence we have:

$\|H - A\| \leq \Delta^{-1}\delta \int_0^t |t|f(t)|dt$

$\|[H, B]\| \leq \delta$, since $\int_{-\infty}^{\infty} |f(t)|dt = 1$ (???)

Let v, w be eigenvectors of B with eigenvalues r and s respectively, such that $|r - s| \geq \Delta$. We have:

$$\begin{aligned} (v, Hw) &= (v, \Delta \int_{-\infty}^{\infty} \exp(iBt)A \exp(-iBt)f(\Delta t)dtw) \\ &= \int_{-\infty}^{\infty} \Delta f(\Delta t)(v, \exp(iBt)A \exp(-iBt)w)dt \\ &= \int_{-\infty}^{\infty} \Delta f(\Delta t)(\exp(-iBt)v, A \exp(-iBt)w)dt \\ &= \int_{-\infty}^{\infty} \Delta f(\Delta t)e^{i(s-r)t}(v, Aw)dt \\ &= \int_{-\infty}^{\infty} f(t)e^{i\frac{s-r}{\Delta}t}dt(v, Aw), \\ &= \hat{f}\left(\frac{r-s}{\Delta}\right)(v, Aw), \end{aligned}$$

which shows the last property, since $|s - r|/\Delta \geq 1$, since \hat{f} is zero outside $[-\Delta, \Delta]$. \square

Let S be a finite set of real numbers, and B a self-adjoint operator. We say that a vector is supported on the set S for B if it is a linear combination of eigenvectors of B with eigenvalues in S . By $P(S, B)$ we mean the

³Of course we should be careful those elementary identities and inequalities actually hold for matrices. So far everything we have used can justified straightforwardly by looking at Riemann sums of matrices.

⁴Too see this, let $v(t) = A(t)e$, with $A(0) = 0$, where e is a unit vector. Now $v(t) = v(0) + \int_0^t v'(s)ds$. So $\|v(t)\| \leq \int_0^t \|A'(s)\|ds$. This holds for all unit vectors, so $\|A(t)\| \leq \int_0^t \|A'(s)\|ds$.

orthogonal projection onto the subspace spanned by eigenvectors of B with eigenvalues in S . If T is another finite set of real numbers $\text{dist}(S, T)$ denotes $\min_{s \in S, t \in T} |s - t|$.

We now end this section by showing the following theorem which is an example of a so-called Lieb-Robinson bound (and is used in Hastings proof).

Theorem 3.11. *Let H be a hermitian matrix with $\|H\| \leq 1$, and let B be a self-adjoint matrix, and $\Delta > 0$ such that for any two eigenvectors v, w of B , with respective eigenvalues r and s satisfying $|r - s| \geq \Delta$, we have $(v, Hw) = 0$.*

Let S_1 and S_2 be finite sets of real numbers, and let $|t| \leq \text{dist}(S_1, S_2)/v_{LR}$, where $v_{LR} = e^2 \Delta$.⁵

Then for any v supported on set S_1 for B , we have:

$$\|P(S_2, B) \exp(-iHt)v\| \leq e^{-\text{dist}(S_1, S_2)/\Delta} \|v\|$$

Proof. First notice if v is supported on S_1 for B , then Hv is supported on a set with distance less than or equal to Δ to S_1 , because of the way H and B are constructed. Thus applying H^n to v is 0, when n is a positive integer less than $m = \text{dist}(S_1, S_2)/\Delta$. Thus expressing $\exp(iHt)$ by its powerseries and applying it to v , we see that the first terms of $P(S_2, B) \exp(-iHt)v$ vanish. The result will now follow from the following inequalities, which will be explained below:

$$\begin{aligned} \|P(S_2, B) \exp(-iHt)v\| &\leq \left\| \sum_{n \geq m} (-it)^n (H^n/n!)v \right\| \\ &\leq \sum_{n \geq m} (|t|^n/n!) \|v\| \\ &\leq \frac{1}{e} \sum_{n \geq m} (e|t|)^n \|v\| \\ &\leq \frac{1}{e} (e|t|)^m \frac{1}{1 - e|t|} \|v\| \\ &\leq e^{-\text{dist}(S_1, S_2)/\Delta} \end{aligned}$$

First inequality is obvious from the previous discussion.

Second inequality follows from $\|H\| \leq 1$, and the triangle inequality.

Third inequality is the Stirling approximation, which gives: $n! < \sqrt{2\pi n} (\frac{n}{e})^n$.

Fourth (in-)equality is just a geometric series.

In the last inequality notice $0 \leq e|t|/m \leq 1/e < 1$, and hence $\frac{1}{e} (e|t|/m)^m \leq e^{-m-1} \leq e^{-\text{dist}(S_1, S_2)/\Delta}$, and $\frac{1}{1 - e|t|/m} \leq 1$. □

⁵LR stands for Lieb-Robinson: v_{LR} is a so-called Lieb-Robinson bound, which is something that arises in the theory of many-body systems.

Chapter 4

Two Counterexamples

In this chapter we give two non-trivial counter examples of failures in approximating almost commuting matrices.

4.1 A Topological Obstruction

We will now give an example of almost commuting unitary matrices which cannot be uniformly approximated by commuting unitaries. The example is a so-called Voiculescu pair S_n, Ω_n defined by:

$$S_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

and

$$\Omega_n = \begin{pmatrix} \omega_n & & & & \\ & \omega_n^2 & & & \\ & & \omega_n^3 & & \\ & & & \ddots & \\ & & & & \omega_n^n \end{pmatrix}$$

where ω_n is the n th unit root $e^{2\pi i/n}$.

We see immediately that the Voiculescu pair has the following elementary properties which are straightforward to verify:

Proposition 4.1. *Let Ω_n and S_n be defined as above, then*

- (a) $\|[\Omega_n, S_n]\| = |1 - \omega_n|$, which tends to 0 as n tends to ∞ .
- (b) $\det(\Omega_n) = \det(S_n) = (-1)^{n+1}$

$$(c) S_n \Omega_n S_n^* = \bar{\omega}_n \Omega_n$$

We will prove that although $\|[\Omega_n, S_n]\| \rightarrow 0$ one has:

Theorem 4.2. *For any $n \geq 7$ and any pair of commuting matrices $X, Y \in M_n(\mathbb{C})$ we have: $\max\{\|X - \Omega_n\|, \|Y - S_n\|\} \geq \sqrt{2 - |1 - \omega_n|} - 1$.*

The argument makes use of a topological obstruction (the winding number) and is reasonably short - it is based on [3]. The number $n \geq 7$ is chosen to insure $\sqrt{2 - |1 - \omega_n|} - 1$ is positive.

Proof. First let us note that if a matrix is non-invertible then the distance to any unitary is at least 1,¹ so we will limit our attention to invertibles from now on:

Let $X, Y \in M_n(\mathbb{C})$ be commuting invertibles and put

$$d = \max\{\|X - \Omega_n\|, \|Y - S_n\|\}.$$

We will assume $d < \sqrt{2 - |1 - \omega_n|} - 1$ and obtain a contradiction.

Let $A_t = \Omega_n + t(X - \Omega_n)$, and $B_t = S_n + t(Y - S_n)$ for $t \in [0, 1]$.

For each $t \in [0, 1]$ let γ_t be the *closed* curve in the complex plane defined by

$$\gamma_t(r) = \det((1 - r)A_t B_t + rB_t A_t),$$

$r \in [0, 1]$.

First we will show that $\gamma_t(r)$ is never 0. To do this we will show that $(1 - r)A_t B_t + rB_t A_t$ is invertible for all r and t in the unit interval, by showing $\|(1 - r)A_t B_t + rB_t A_t - \Omega_n S_n\| < 1$. Since $\Omega_n S_n$ is unitary this shows that $\|(1 - r)A_t B_t + rB_t A_t\|$ is invertible. We find (by several times using the trick of adding and subtracting the same term):

$$\begin{aligned} & \|(1 - r)A_t B_t + rB_t A_t - \Omega_n S_n\| \\ & \leq (1 - r)\|A_t B_t - \Omega_n S_n\| + r\|B_t A_t - \Omega_n S_n\| \\ & \leq (1 - r)(\|A_t B_t - A_t S_n\| + \|A_t S_n - \Omega_n S_n\|) \\ & \quad + r(\|B_t A_t - S_n A_t\| + \|S_n A_t - S_n \Omega_n\| + \|S_n \Omega_n - \Omega_n S_n\|) \\ & \leq (1 - r)(\|A_t\| \|B_t - S_n\| + \|A_t - \Omega_n\|) \\ & \quad + r(\|B_t - S_n\| \|A_t\| + \|A_t - \Omega_n\| + |1 - \omega_n|) \\ & \leq (1 - r)((1 + d)d + d) + r((1 + d)d + d + |1 - \omega_n|) = d^2 + 2d + r|1 - \omega_n| \\ & \leq d^2 + 2d + |1 - \omega_n| < 1. \end{aligned}$$

Since this complex curve is never 0, we will now look at the winding number around 0 for different t . Since the winding number is a homotopy invariant it should be the same for all t . Now, for $t = 0$ we have $A_t = \Omega_n$ and $B_t = S_n$ and hence:

¹Let $\|U - X\| < 1$ with U unitary. We have $\|U - X\| = \|U(1 - U^*X)\| = \|1 - U^*X\|$. By a standard theorem in Banach algebras U^*X is invertible - hence X is invertible, since U is.

$$\begin{aligned}
\gamma_0(r) &= \det((1-r)\Omega_n S_n + r S_n \Omega_n) \\
&= \det(S_n((1-r)\Omega_n S_n + r S_n \Omega_n) S_n^*) \\
&= \det(S_n) \det((1-r)\Omega_n + r \bar{\omega}_n \Omega_n) \\
&= (-1)^{n+1} \det((1-r + r \bar{\omega}_n)\Omega_n) \\
&= (-1)^{n+1} (1-r + r \bar{\omega}_n)^n (-1)^{n+1} \\
&= (1-r + r \bar{\omega}_n)^n
\end{aligned}$$

As r goes from 0 to 1, $(1-r + r \bar{\omega}_n)$ goes from 1 to $\bar{\omega}_n$ along the straight line segment connecting those two points. Thinking about polar coordinates we see that curve, γ_0 winds once in the clockwise direction around 0 in the complex plane.

For $t = 1$ we have $A_t = X$ and $B_t = Y$ which commute, so γ_t is constant and non-zero - and the winding number is 0.

Since the winding number is a homotopy invariant we have obtained a contradiction. □

4.2 Three Almost Commuting Matrices

In this section we give an example constructed by Davidson of 3 almost commuting self-adjoints which are not close to exactly commuting self-adjoints (Actually the counterexample is even stronger). Namely we will construct two sequences $(A_n), (B_n)$ of matrices which are respectively self-adjoint and normal, such that

$$[A_n, B_n] \rightarrow 0$$

while there are no matrices $A'_n = A_n^*$ and B'_n (B'_n not necessarily normal!) such that

$$[A'_n, B'_n] = 0, A_n - A'_n \rightarrow 0, B_n - B'_n \rightarrow 0.$$

We will need three auxiliary results. For the first one (Theorem 4.4 below), in [4] it is given a reference to a paper of Voiculescu where it is called a folk theorem.²

Lemma 4.3. *Let T and S be self-adjoint operators in $B(H)$. Suppose C_1, C_2 are closed intervals such that $\sigma(S) \subseteq C_1, \sigma(T) \subseteq C_2$ and*

$$\text{dist}(C_1, C_2) = a$$

Let L_S, R_T be the operators of left multiplication by S and right multiplication by T respectively. Then

$$\|(L_S - R_T)^{-1}\| \leq 1/a$$

²Being unable to find this paper, we give here our own proof.

Proof. Without loss of generality we can assume that $z_1 - z_2 \geq a$ for all $z_1 \in \sigma(S), z_2 \in \sigma(T)$. By shifting to a constant, we can assume that $\sigma(T) \subset (-\infty, -a/2], \sigma(S) \subset [a/2, \infty)$.

Let us begin with the formula:

$$\int_0^\infty \exp(-tz)dt = 1/z$$

which is valid for all $z > 0$ and even for all z with $\Re(z) > 0$.

Now we may apply both analytic functions (in left-hand side and in right-hand side) to any operator X with $\sigma(X) \subset \{z : \Re(z) > 0\}$

$$\int_0^\infty \exp(-tX)dt = X^{-1}. \quad (4.1)$$

Let $X = L_S - R_T$. Since $\sigma(R_T) = \sigma(T)$, $\sigma(L_S) = \sigma(S)$, and since left multiplications and right multiplications commute we have

$$\sigma(X) \subseteq \{z_1 - z_2 \mid z_1 \in \sigma(S), z_2 \in \sigma(T)\} \subset (0, \infty).$$

Hence the formula (4.1) is valid for our X .

Now

$$(L_S - R_T)^{-1} = \int_0^\infty \exp(-t(L_S - R_T))dt = \int_0^\infty \exp(-tL_S) \exp(tR_T)dt$$

(we used here that L_S and R_T commute).

Applying this equality to an operator $V \in B(H)$ and taking into account that

$$\begin{aligned} \|\exp(-tL_S) \exp(tR_T)(V)\| &= \|\exp(-tS)V \exp(tT)\| \\ &\leq \|V\| \exp(-ta/2) \exp(-ta/2) = \|V\| \exp(-ta), \end{aligned}$$

we get

$$\|(L_S - R_T)^{-1}(V)\| \leq \|V\| \int_0^\infty \exp(-ta)dt = \|V\|/a$$

We proved that

$$\|(L_S - R_T)^{-1}\| \leq 1/a$$

□

For a self adjoint operator T , let $E_T(C)$ denote the spectral projection for T corresponding to the set C .

Theorem 4.4. *Let ϵ and a be positive constants, and let C_1 and C_2 be closed intervals with $\text{dist}(C_1, C_2) \geq a$. For any pair of self-adjoint operators A and B satisfying $\|A - B\| < a\epsilon$, one has*

$$\|E_A(C_1)E_B(C_2)\| < \epsilon.$$

Proof. Let us denote $P = E_A(C_1)$ and $Q = E_B(C_2)$ for short. Write A with respect to the decomposition $H = PH \oplus (1 - P)H$ as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and B with respect to the decomposition $H = QH \oplus (1 - Q)H$ as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Let $\lambda \in C_1$ and $\mu \in C_2$. Define S with respect to the decomposition $H = PH \oplus (1 - P)H$ as

$$S = \begin{pmatrix} A_{11} & \\ & \lambda \end{pmatrix}$$

and T with respect to the decomposition $H = QH \oplus (1 - Q)H$ as

$$T = \begin{pmatrix} B_{11} & \\ & \mu \end{pmatrix}$$

Then

$$SP = \begin{pmatrix} A_{11} & \\ & \lambda \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & \\ & 0 \end{pmatrix} = PAP,$$

$$QT = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} B_{11} & \\ & \mu \end{pmatrix} = \begin{pmatrix} B_{11} & \\ & 0 \end{pmatrix} = QBQ,$$

and we have

$$\|(L_S - R_T)(PQ)\| = \|PAPQ - PQBQ\| = \|PAQ - PBQ\| = \|P(A - B)Q\| \leq \|A - B\| \leq a\epsilon.$$

Since $\sigma(S) \subseteq C_1$ and $\sigma(T) \subseteq C_2$, by Lemma 4.3

$$\|(L_S - R_T)^{-1}\| \leq 1/a$$

Hence

$$\|PQ\| \leq \|(L_S - R_T)^{-1}\| \|(L_S - R_T)(PQ)\| < (1/a)a\epsilon = \epsilon$$

□

The second auxiliary result can be considered as stability of the relation $E \leq F \leq G$ (E, F, G are projections) under small perturbations of the middle part F .

For a projection F we use notation $F^\perp = 1 - F$.

Lemma 4.5. *Let $\epsilon > 0$. If E, F' and G are projections with $E \leq G$, $\|EF'^\perp\| < \epsilon$, and $\|F'G^\perp\| < \epsilon$, then there is a projection F such that $E \leq F \leq G$ and $\|F' - F\| \leq 3\epsilon$.*

Proof. Since the distance between any two projections is not larger than 1, we can assume $\epsilon \leq 1/3$. Let us decompose the Hilbert space as

$$H = EH \oplus (G - E)H \oplus G^\perp H.$$

With respect to this decomposition we will write F' as

$$F' = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Since $\|EF'^\perp\| < \epsilon$ and

$$EF'^\perp = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 - F_{11} & -F_{12} & -F_{13} \\ -F_{21} & 1 - F_{22} & -F_{23} \\ -F_{31} & -F_{32} & 1 - F_{33} \end{pmatrix} = \begin{pmatrix} 1 - F_{11} & -F_{12} & -F_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get

$$\|1 - F_{11}\| < \epsilon, \|F_{12}\| < \epsilon, \|F_{13}\| < \epsilon. \quad (4.2)$$

Since $\|F'G^\perp\| < \epsilon$ and

$$F'G^\perp = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & F_{13} \\ 0 & 0 & F_{23} \\ 0 & 0 & F_{33} \end{pmatrix}$$

we get

$$\|F_{13}\| < \epsilon, \|F_{23}\| < \epsilon, \|F_{33}\| < \epsilon. \quad (4.3)$$

Since F' is a projection, we have $F' = F'^*$ which implies

$$F_{ij} = F_{ji}^* \quad (4.4)$$

and $F' = F'^2$ which implies

$$F_{22} = (F'^2)_{22} = F_{21}F_{12} + F_{22}^2 + F_{23}F_{32} \quad (4.5)$$

From (4.4) and (4.5) we obtain that $F_{22} - F_{22}^2 \geq 0$ and $\|F_{22} - F_{22}^2\| \leq 2\epsilon^2$. Since $F_{22} \geq 0$, we conclude that $\sigma(F_{22}) \subset [0, 4\epsilon^2] \cup [1 - 4\epsilon^2, 1]$.

Since $\epsilon < 1/3$, we have $4\epsilon^2 < 1/2$ and hence there is a continuous function f on \mathbb{R} such that $f|_{[0, 4\epsilon^2]} = 0$ and $f|_{[1 - 4\epsilon^2, 1]} = 1$. Then $P = f(F_{22})$ is a projection and

$$\|F_{22} - P\| \leq \sup_{t \in [0, 4\epsilon^2] \cup [1 - 4\epsilon^2, 1]} |f(t) - t| \leq 4\epsilon^2 < \epsilon \quad (4.6)$$

Finally we define F as

$$F = \begin{pmatrix} 1 & & \\ & P & \\ & & 0 \end{pmatrix}.$$

Clearly F is a projection satisfying $E \leq F \leq G$ and, by (4.2), (4.3), (4.4), (4.6),

$$\begin{aligned} \|F' - F\| &= \left\| \begin{pmatrix} F_{11} - 1 & F_{12} & F_{13} \\ F_{21} & F_{22} - P & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} F_{11} - 1 & & \\ & F_{22} - P & \\ & & F_{33} \end{pmatrix} \right\| + \\ &\quad \left\| \begin{pmatrix} & F_{13} \\ F_{21} & \\ & F_{32} \end{pmatrix} \right\| + \left\| \begin{pmatrix} & F_{12} \\ & & F_{23} \\ F_{31} & & \end{pmatrix} \right\| \leq 3\epsilon \quad (4.7) \end{aligned}$$

□

Let e_k , $0 \leq k \leq n^2$, be an orthonormal basis in $(n^2 + 1)$ -dimensional Hilbert space. The third auxiliary result concerns the shift operator $S_n \in M_{n^2+1}$ defined by

$$S_n e_k = \begin{cases} e_{k+1}, & k < n^2 \\ 0, & k = n^2. \end{cases}$$

Proposition 4.6. *If $P \in M_{n^2+1}$ is a projection with*

$$\ker S_n^* \subseteq \text{Ran} P \subseteq (\ker S_n)^\perp,$$

then $\|[P, S_n]\| = 1$.

Proof. Since $\ker S_n^* = \text{span}\{e_0\}$, $(\ker S_n)^\perp = \text{span}\{e_0, \dots, e_{n^2+1}\}$, we have

$$\text{span}\{e_0\} \subseteq \text{Ran} P \subseteq \text{span}\{e_0, \dots, e_{n^2+1}\}.$$

Since S_n is a partial isometry, it is an isometry on $(\ker S_n)^\perp = \text{span}\{e_0, \dots, e_{n^2+1}\}$. Hence S_n is an isometry on $\text{Ran} P$. It follows that

$$\dim \text{Ran} S_n P = \dim \text{Ran} P.$$

Let us denote this dimension by k . Since

$$\text{Ran} S_n P \subseteq \text{Ran} S_n \subseteq \text{span}\{e_1, \dots, e_{n^2}\},$$

e_0 is orthogonal to $\text{Ran} S_n P$ and we get

$$\dim \text{span}\{e_0, \text{Ran} S_n P\} = k + 1.$$

Thus

$$\dim(\text{Ran} P)^\perp + \dim \text{span}\{e_0, \text{Ran} S_n P\} = n^2 + 1 - k + k + 1 > n^2 + 1$$

whence

$$(\text{Ran} P)^\perp \cap \text{span}\{e_0, \text{Ran} S_n P\} \neq \emptyset.$$

Hence there exists $x \in \text{span}\{e_0, \text{Ran}S_nP\}$ orthogonal to $\text{Ran}P$. Since $e_0 \in \text{Ran}P$, we have $x \in \text{Ran}S_nP$. It follows that $x = S_nPy$, for some vector y . So S_nPy is orthogonal to $\text{Ran}P$, hence $S_nPy \in \text{Ran}(1 - P)$ and

$$(1 - P)S_nPy = S_nPy.$$

Hence

$$\|(1 - P)S_nPPy\| = \|S_nPy\| = \|Py\|$$

(here the last equality holds since S_n is an isometry on $\text{Ran}P$). Thus

$$\|(1 - P)S_nP\| \geq 1. \quad (4.8)$$

Now writing S_n with respect to the decomposition $H = PH \oplus (1 - P)H$

$$S_n = \begin{pmatrix} PS_nP & PS_n(1 - P) \\ (1 - P)S_nP & (1 - P)S_n(1 - P) \end{pmatrix}$$

we obtain

$$[S_n, P] = \begin{pmatrix} 0 & PS_n(1 - P) \\ (1 - P)S_nP & 0 \end{pmatrix}.$$

By (4.8)

$$\|[S_n, P]\| = \max\{\|PS_n(1 - P)\|, \|(1 - P)S_nP\|\} \geq 1.$$

The opposite inequality is obvious. \square

Theorem 4.7. *There exist finite rank matrices A_n, B_n , $n \geq 1$, of norm 1 such that A_n is self-adjoint, B_n is normal, and $\lim_{n \rightarrow \infty} \|[A_n, B_n]\| = 0$, yet there are no commuting pairs A'_n, B'_n such that A'_n is self-adjoint and $\lim_{n \rightarrow \infty} \|A_n - A'_n\| + \|B_n - B'_n\| = 0$.*

Proof. Define A_n and \tilde{B}_n in M_{n^2+1} as follows. Let e_k , $0 \leq k \leq n^2$ be an orthonormal basis and let

$$A_n e_k = \frac{k}{n^2} e_k, \quad \tilde{B}_n e_k = \begin{cases} \frac{k+1}{n} e_{k+1}, & 0 \leq k < n \\ e_{k+1}, & n \leq k \leq n^2 - n \\ \frac{n^2 - k}{n} e_{k+1}, & n^2 - n < k \leq n^2 \end{cases}$$

(it is meant that $B_n e_{n^2} = 0$).

Now $A_n = A_n^*$, \tilde{B}_n is a weighted shift and it is straightforward to check that

$$\|[A_n, \tilde{B}_n]\| = 1/n^2 \rightarrow 0.$$

Also one checks that $\lim_{n \rightarrow \infty} \|\tilde{B}_n, \tilde{B}_n^*\| = 0$ and hence by Lin's theorem there are normal matrices $B_n \in M_{n^2+1}$ such that $\lim_{n \rightarrow \infty} \|\tilde{B}_n - B_n\| = 0$. It means that we can use \tilde{B}_n for constructing the example.

In order to obtain a contradiction, we assume the existence of commuting pairs A'_n, B'_n such that A'_n is self-adjoint and

$$\lim_{n \rightarrow \infty} \|A_n - A'_n\| + \|\tilde{B}_n - B'_n\| = 0. \quad (4.9)$$

Let F'_n be the spectral projection of A'_n for the interval $[0, 1/2]$ and let E_n and G_n be the spectral projections for A_n corresponding to the interval $[0, 1/3]$ and $[0, 2/3]$ respectively. Then from (4.9) and Theorem 4.4 it follows that

$$\lim_{n \rightarrow \infty} \|E_n F_n'^{\perp}\| = 0 = \lim_{n \rightarrow \infty} \|F_n' G_n^{\perp}\|$$

By Lemma 4.5 there exist projections F_n such that $E_n \leq F_n \leq G_n$ and

$$\lim_{n \rightarrow \infty} \|F_n - F_n'\| = 0. \quad (4.10)$$

Since F_n' is a function of A'_n and A'_n commutes with B'_n , we get $[F_n', B'_n] = 0$. Now (4.9) and (4.10) imply that

$$\lim_{n \rightarrow \infty} \|[F_n, \tilde{B}_n]\| = 0. \quad (4.11)$$

Now we notice that on $\text{span}\{e_k, n \leq k \leq n^2 - n\}$ \tilde{B}_n acts as the shift operator S_n defined before Proposition 4.6. Since $G_n - E_n$ is the spectral projection for A_n corresponding to the interval $(1/3, 2/3]$ and since for the eigenvalues k/n^2 of A_n lying in the interval $(1/3, 2/3]$ we have $n < k < n^2 - n$ for all $n \geq 4$, we conclude that

$$S_n(G_n - E_n) = \tilde{B}_n(G_n - E_n) \quad (4.12)$$

and

$$(G_n - E_n)S_n = (G_n - E_n)\tilde{B}_n. \quad (4.13)$$

Let us decompose the Hilbert space $H = \mathbb{C}^{n^2+1}$ as

$$H = E_n H \oplus (G_n - E_n)H \oplus G_n^{\perp} H$$

and write S_n and F_n with respect to this decomposition. Then (4.12) and (4.13) mean that

$$(S_n)_{12} = (\tilde{B}_n)_{12}, (S_n)_{21} = (\tilde{B}_n)_{21}, (S_n)_{22} = (\tilde{B}_n)_{22}, (S_n)_{23} = (\tilde{B}_n)_{23}, (S_n)_{32} = (\tilde{B}_n)_{32} \quad (4.14)$$

We notice also that by their constructions \tilde{B}_n and S_n do not have anything in the right upper and left lower corners, so that

$$(\tilde{B}_n)_{13} = (S_n)_{13} = (\tilde{B}_n)_{31} = (S_n)_{31} = 0. \quad (4.15)$$

Since $E_n \leq F_n \leq G_n$, we write F_n as

$$F_n = \begin{pmatrix} 1 & & \\ & P & \\ & & 0 \end{pmatrix},$$

where P is a projection. Now using (4.14) and (4.15) we get

$$[F_n, \tilde{B}_n] = \begin{pmatrix} 0 & (\tilde{B}_n)_{12}(1-P) & 0 \\ (P-1)(\tilde{B}_n)_{21} & [P, (\tilde{B}_n)_{22}] & P(\tilde{B}_n)_{23} \\ 0 & -(\tilde{B}_n)_{32}P & 0 \end{pmatrix} = \begin{pmatrix} 0 & (S_n)_{12}(1-P) & 0 \\ (P-1)(S_n)_{21} & [P, (S_n)_{22}] & P(S_n)_{23} \\ 0 & -(S_n)_{32}P & 0 \end{pmatrix} = [F_n, S_n]. \quad (4.16)$$

This and (4.11) imply

$$\lim_{n \rightarrow \infty} [F_n, S_n] = 0.$$

But it is impossible because for any projection Q with

$$\ker S_n^* \subseteq \text{ran} Q \subseteq (\ker S_n)^\perp$$

(in particular for F_n) one has $\|[Q, S_n]\| \geq 1$. We obtained a contradiction. \square

Chapter 5

A BDF-Theorem

The goal is to prove, we recall:

Theorem 5.1. *Let $T \in B(H)$ be an essentially normal operator. Then T is a compact perturbation of a normal operator if and only if T has trivial index function.*

The proof of 5.1 uses the following two results.

Theorem 5.2. *Let $A \in B(H)$. A is a compact perturbation of a normal operator if and only if A is quasidiagonal and essentially normal.*

Proof. By Lemma 2.27 every normal operator in $B(H)$ is quasidiagonal. By corollary 2.25 compact perturbations of a normal operator is then also quasidiagonal, and clearly also essentially normal.

To prove the converse let $T \in B(H)$ be quasidiagonal and essentially normal. By Theorem 2.24 T is a compact perturbation of a blockdiagonal operator $S = \sum_{n=1}^{\infty} S_n$, where $S_n = P_n S P_n$, $P_n = Q_n - Q_{n-1}$, where (Q_n) increasing sequence of finite rank projections which converges to I in SOT, and $Q_0 = 0$.

Since S is a compact perturbation of an essential normal operator S is itself essentially normal and we therefore have $\sum_{n=1}^{\infty} [S_n, S_n^*]$ is compact.

Hence $\|[S_n, S_n^*]\| \rightarrow 0$ since for any blockdiagonal compact operator the norms of the individual blocks must tend to 0,¹ moreover $\|S_n\| \leq \|S\|$, so we can use Lin's theorem to construct a sequence R_n of normal operators, where $P_n R_n P_n = R_n$ and $\|R_n - S_n\| \rightarrow 0$. Putting $R = \sum_{n=1}^{\infty} R_n$, it follows $R - S$ is compact, and R is normal. This shows the desired result. \square

Corollary 5.3. *The set of compact perturbations of normal operators on H is closed with respect to the norm topology.*

¹As in the proof of the Weyl-von Neumann-Berg theorem.

Proof. The set of compact perturbations of normal operators is the intersection of two closed sets by Theorem 5.2. Namely, the set of quasideagonal operators (Theorem 2.26) and the set of essentially normal operators (The preimage of the closed set of normal elements in the Calkin-algebra). \square

Theorem 5.4. *Let $T \in Q(H)$ be normal. T is the norm limit of a sequence of normal elements in $Q(H)$ with finite spectrum if and only if T has trivial index function.*

The proof of this theorem will be given in section 5.1.

Proof of Theorem 5.1:

Since a normal operator has index 0 and the index is invariant under compact perturbations, every compact perturbation of a normal operator has trivial index function.

To see the other direction let T be an essentially normal with trivial index function. Hence $S := \pi(T)$ is normal with trivial index function. By Theorem 5.4 there exists a sequence of normals with finite spectrum $(S_n) \in Q(H)$ which converges to S in the norm topology, and lift (S_n) to (T_n) such that T_n converges to T in norm topology in $B(H)$ (This is possible because of the way the quotient norm is defined). Now, since every normal element in the Calkin algebra with finite spectrum can be lifted to a normal element in $B(H)$ by Lemma 1.4, we can choose a sequence of normal operators $R_n \in B(H)$, such that $\pi(R_n) = S_n$. Hence each T_n is a compact perturbation of a normal operator, and hence T is a compact perturbation of a normal operator by corollary 5.3.

5.1 Proof of Theorem 5.4

We will prove Theorem 5.4 through a sequence of lemmas in a similar way we proved Lin's theorem. The new thing is that the quotient is now the Calkin algebra, and we have to take into account the index.

Lemma 5.5. *Let $t \in Q(H)$ be a normal element, let $\lambda \in \sigma(t)$, and let $\epsilon > 0$. Then there exists a normal element $s \in Q(H)$ satisfying: $\|t - s\| \leq 2\epsilon$, $\lambda \notin \sigma(s)$, $\text{index}(s - \lambda) = 0$, and $\sigma(s) \setminus B(\lambda, \epsilon) = \sigma(t) \setminus B(\lambda, \epsilon)$.*

Proof. Step 1 - Constructing s : Without loss of generality assume $\lambda = 0$. Let $t \in Q(H)$ be normal and non-invertible, and let R be any lift of t such that $\pi(R) = t$. Then R has polar decomposition $R = V|R|$, and R^* has polar decomposition $R^* = V^*|R^*|$.

Let $f_\epsilon : [0; \infty[\rightarrow [0, \infty[$ be the continuous function defined by $f_\epsilon(x) = 0$, when $x \leq \epsilon$, and $f_\epsilon(x) = x - \epsilon$, when $x > \epsilon$.

Consider $Vf_\epsilon(|R|)$ and its adjoint $V^*f_\epsilon(|R^*|)$.² We want to show they both have infinite-dimensional kernels. Assume the spectral projection $1_{[0,\epsilon]}(|R|)$ has finite rank. Then since $(1_{[0,\epsilon]} + \text{id})(|R|)$ is invertible, we get $\pi(1_{[0,\epsilon]} + \text{id})(|R|) = |t|$ is invertible, and since t is normal this implies t invertible, which is a contradiction. So $1_{[0,\epsilon]}(|R|)$ must have infinite-dimensional range. It now follows that $f_\epsilon(|R|) = f_\epsilon(|R|)(1 - 1_{[0,\epsilon]})(|R|)$ has infinite-dimensional kernel. A similar argument shows $V^*f_\epsilon(|R^*|)$ has infinite-dimensional kernel. Let $Wf_\epsilon(|R|)$ be the polar decomposition of $Vf_\epsilon(|R|)$. Extend W to a unitary U such that $Vf_\epsilon(|R|) = Uf_\epsilon(|R|)$, which is possible since the kernels of W and W^* have same cardinality (because they are infinite-dimensional and H is assumed to be separable).

Let $g_\epsilon = f_\epsilon + \epsilon$, and put

$$s = \pi(Ug_\epsilon(|R|)) = \pi(V)f_\epsilon(|t|) + \pi(U)\epsilon.$$

Step 2 - Showing s has the desired properties: Since $g_\epsilon(|t|)$ and U are invertible, s is invertible and $\text{index}(s) = 0$ because s lifts to an invertible element in $B(H)$.

We also have

$$\|t - s\| = \|\pi(V)(|t| - f_\epsilon(|t|)) - \pi(U)\epsilon\| \leq 2\epsilon$$

To see s is normal, let $\beta : [0, \infty[\rightarrow [0, \infty[$ be a continuous function such that $f_\epsilon(x) = x\beta(x)$, and notice that

$$\pi(V)f_\epsilon(|t|) = \pi(V)|t|\beta(|t|) = t\beta(|t|)$$

is normal because it is a continuous function of t . We therefore have that $\pi(U)f_\epsilon(|t|) = \pi(V)f_\epsilon(|t|)$ is normal. It is now straightforward to see $[\pi(U), f_\epsilon(|t|)] = 0$, and from this it is straightforward to see that s is normal.

Concerning the spectra, embed $Q(H)$ into $B(H')$ for some Hilbert space H' . Write T for the image of t , and S for the image of s . Now we can use Borel function calculus inside $B(H')$ to compare the spectra of s and t .

Put $E = 1_{[0,\epsilon]}(|T|)$. Then E clearly commutes with T , moreover E commutes with S , since $S = \gamma(|T|)$, where γ is chosen such that $\gamma \circ g_\epsilon = 1_{[0,\epsilon]}$.

We now have $T = TE + T(1 - E)$ and $S = SE + S(1 - E)$, so we have

$$\sigma(T) = \sigma_{TE}(TE) \cup \sigma_{T(1-E)}(T(1 - E))$$

and

$$\sigma(S) = \sigma_{SE}(SE) \cup \sigma_{S(1-E)}(S(1 - E)).$$

We have $\sigma(TE) = \sigma(T) \cap B(0, \epsilon)$, and $\sigma(SE) = \sigma(S) \cap B(0, \epsilon)$. So if we can show $T(1 - E) = S(1 - E)$ we are done.

²To see that $f_\epsilon(|R|)V^* = V^*f_\epsilon(|R^*|)$, show first this holds for polynomials, using $V|R| = V^*|R^*|$. Then take sequence of polynomials converging uniformly to f_ϵ , and take limit on both sides of the identity for the polynomials.

First notice that by expressing $|S|$ and $|E|$ as functions of $|T|$ we have $|S|(1 - E) = |T|(1 - E)$. Since $|T|$ and $1 - E$ commute it is enough to show $K(1 - E) = 0$, where K is the image of $\pi(V) - \pi(U)$ in $B(H')$. Choose a sequence of bounded borel functions (h_n) such that $f_\epsilon h_n$ converges monotonically and pointwise to $1 - 1_{[0, \epsilon]}$. Now since $K f_\epsilon(|T|) = 0$ we have $K f_\epsilon h_n(|T|) = 0$,³ and hence $K(1 - E) = 0$ as desired. \square

Lemma 5.6. *Let F be a finite subset of \mathbb{C} . The set $L_F \subset Q(H)$ of elements, S , for which $S - \lambda$ is invertible and $\text{index}(S - \lambda) = 0$ for all $\lambda \in F$, is open.*

Proof. Let $\lambda \in F$, then $L_{\{\lambda\}}$ is open since the elements of index 0 constitute an open set in the Calkin algebra. Hence L_F is open since it is a finite intersection of open sets. \square

Lemma 5.7. *Let $F \subset \mathbb{C}$ be finite. Then each normal element of $Q(H)$ with trivial index function can be approximated by a normal element, $b \in Q(H)$, for which $b - \lambda$ is invertible, and $\text{index}(b - \lambda) = 0$, for all $\lambda \in F$.*

Proof. Let $a \in Q(H)$ be normal with trivial index function. Enumerate the elements in F , such that $F = \{\lambda_1, \dots, \lambda_N\}$, and $F \cap \sigma(a) = \{\lambda_i \in F \mid i \geq K + 1\}$. Given $\kappa > 0$, we want to find $b \in Q(H)$ such that $\|a - b\| \leq \kappa$, $\sigma(b) \cap F = \emptyset$, and $\text{index}(b - \lambda_i) = 0$ for all $1 \leq i \leq N$.

Inductively choose elements a_K, a_{K+1}, \dots, a_N by letting $a_K = a$, and choose a_k , for $K + 1 \leq k \leq N$, to be the normal element s obtained in 5.6 with $t = a_{k-1}$, $\lambda = \lambda_k$ and $\epsilon = \frac{1}{2} \min\{\delta, d, \frac{\kappa}{N-K}\}$, where the choice of δ and d will be specified below.

By Lemma 5.6 we choose δ such that $\text{index}(a_k - \lambda_i) = 0$ for all $1 \leq i \leq k - 1$ (Notice that T having trivial index function implies that $\text{index}(a_K - \lambda_i) = 0$ for $1 \leq i \leq K$). Lemma 5.5 makes sure that we also have $\text{index}(a_k - \lambda_k) = 0$.

Let d be the minimum distance between any points in F , it then follows that $\lambda_k, \dots, \lambda_N \in \sigma(a_k)$.

Finally putting $b = a_N$ we have $\|b - a\| \leq \kappa$. \square

Lemma 5.8. *Let $t \in Q(H)$ be a normal element with trivial index function, and let $\epsilon > 0$. Then there exists a normal element $s \in Q(H)$ with trivial index function, $\sigma(s) \subset \Gamma_\epsilon$, and such that $\|s - t\| \leq \epsilon$.*

Proof. Tactically choose $N \in \mathbb{N}$ such that $N\epsilon \geq \|t\| + \epsilon/4$, and let $X = \{x + iy \in \mathbb{C} \mid |x|, |y| \leq N\epsilon\}$. Loosely speaking X is a quadratic box made up of ϵ -cells which boundaries are in the grid Γ_ϵ , and contains the spectrum of any element with norm less than or $\|t\| + \epsilon/4$. Applying Lemma 5.7 on t with $F = \Sigma_\epsilon \cap X$ we obtain a normal $a \in Q(H)$ such that: $\|a - t\| \leq \epsilon/4$, $\sigma(a) \cap (\Sigma_\epsilon \cap X) = \emptyset$, $\text{index}(a - \lambda) = 0$ for all $\lambda \in \Sigma_\epsilon \cap X$. By the choice of N we also have $\sigma(a) \subset X \setminus \Sigma_\epsilon$.

³See for instance [9] or [10]

Put $Y = X \setminus \Sigma_\epsilon$. Let r be as in the proof of Lemma 3.6 but restricted to Y . Put $s = r(a)$, which is normal with spectrum $\sigma(s) \subset \Gamma_\epsilon \cap X$, and has $\|t - s\| \leq \epsilon/4 + \frac{1}{\sqrt{2}} \leq \epsilon$.

It remains to be shown that if $\lambda \in \mathbb{C} \setminus \sigma(s)$, then $\text{index}(s - \lambda) = 0$. If λ is in the unbounded component this is obvious, otherwise λ will be in the same component as some $\mu \in \Sigma_\epsilon \cap X$, and we have $\text{index}(s - \lambda) = \text{index}(s - \mu)$. Moreover there exists a continuous path of continuous functions $f_t : Y \rightarrow Y$ such that f_0 is the identity, and $f_1 = r$ (For instance a deformation retraction along the line segments discussed in the definition of r in Lemma 3.6), so we have $\text{index}(s - \mu) = \text{index}(a - \mu) = 0$. \square

Lemma 5.9. *Let X be a compact subset of Γ_ϵ for some $\epsilon > 0$. Every continuous map $f : X \rightarrow \mathbb{C} \setminus \{0\}$ is homotopic inside $C(X, \mathbb{C} \setminus \{0\})$ to a map of the form*

$$z \mapsto (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}, \quad (5.1)$$

where $\lambda_i \in \mathbb{C} \setminus X$ and $n_i \in \mathbb{Z}$.

Proof. Choose $N \in \mathbb{N}$ such that $X \subset [-N\epsilon, N\epsilon]^2$, and put $Y = \Gamma_\epsilon \cap [-N\epsilon, N\epsilon]^2$. Extend f to a continuous function $g : Y \rightarrow \mathbb{C} \setminus \{0\}$.⁴

Let C_1, \dots, C_{4N^2} be the the bounded connected components of $\mathbb{C} \setminus Y$. For $1 \leq i \leq 4N^2$, let γ_i be the simple closed curve with image $B_i = \bar{C}_i \setminus C_i$ with positive orientation, let n_i be the winding number of h with respect to γ_i , choose $\lambda_i \in C_i$, and define:

$$h : Y \rightarrow \mathbb{C} \setminus \{0\} \text{ by } h(z) = (z - \lambda_1)^{-n_1} \cdots (z - \lambda_{4N^2})^{n_{4N^2}} g(z)$$

The winding number of h around γ_i is now 0. This means that $h|_{B_i}$ can be written as $r(z)e^{i\theta(z)}$, where r is a continuous positive real function which is never 0, and θ is a *continuous* real function.⁵ This means we can extend h_{B_i} to a continuous function on the whole of \bar{C}_i which has image contained in $\mathbb{C} \setminus \{0\}$.⁶ Repeating this for all cells we get a continuous extension $\tilde{h} : [N\epsilon, N\epsilon]^2 \rightarrow \mathbb{C} \setminus \{0\}$ of h .

Since $[N\epsilon, N\epsilon]^2$ is contractible, \tilde{h} is homotopic to the constant function 1 inside $C([N\epsilon, N\epsilon]^2, \mathbb{C} \setminus \{0\})$. Restricting the functions in the homotopy to X we obtain the desired result. \square

⁴Start by putting $\tilde{g}(x) = 1$ when x is a crosshairpoint of Y which is not in X . To see \tilde{g} can be extended to the whole Y , consider a closed subset, C of $[0, 1]$ which includes the endpoints where a continuous function h is defined and non-zero. The complement is then a collection of disjoint open intervals (a_i, b_i) where $h(a_i)$ and $h(b_i)$ are not 0. To extend h continuously to the whole $[0, 1]$ so the extension is different from 0 everywhere we just choose for each i a continuous curve in \mathbb{C} connecting $h(a_i)$ and $h(b_i)$ which does not go through 0.

⁵ θ can be chosen continuous since since the winding number is zero.

⁶To see this, let for instance r_0 and θ_0 be the minimums of r , and θ on B_i respectively. Set the function equal to $r_0 e^{i\theta_0}$ in the centerpoint of C_i . On the ‘radial’ line segments from the center to the boundary make r and θ straight lines.

Lemma 5.10. *Suppose $u \in Q(H)$ is unitary with $\text{index}(u) = 0$. Then u can be lifted to a unitary $U \in B(H)$.*

Proof. Let V be any lift of u . Then $\text{index} V = 0$ and hence $\ker V$ and $\ker V^*$ have the same (finite) dimensions. Hence in its polar decomposition, the partial isometry can be extended to unitary. So $V = U|V|$ where U is unitary. Applying π on both sides we have $u = \pi(U)|u| = \pi(U)$. \square

Lemma 5.11. *Let $t \in Q(H)$ be normal with trivial index function, and with $\sigma(t) \subset \Gamma_\epsilon$ for some $\epsilon > 0$. Let V be a relatively open subset of $\sigma(t)$ which is homeomorphic to the open interval $]0; 1[$. Given $\delta > 0$ and $\lambda_0 \in V$ there exists a normal element $s \in Q(H)$ with trivial index function such that $\|s - t\| < \delta$, and $\sigma(s) \subset \sigma(t) \setminus \{\lambda_0\}$.*

Proof. Let F be a finite set which consists of one point from each bounded component of $\mathbb{C} \setminus \sigma(t)$. Since t has trivial index function $\text{index}(t - \lambda) = 0$ for all $\lambda \in F$. Notice that if $s \in Q(H)$ satisfies $\sigma(s) \subset \sigma(t)$ then for each bounded component of $\mathbb{C} \setminus \sigma(s)$ there exists a $\lambda \in F$ such that λ belongs to this component. Hence s will have trivial index function. By Lemma 5.6 there exists a δ_0 such that $\text{index}(s - \lambda) = 0$ for all $\lambda \in F$ if $\|s - t\| < \delta_0$. Let us therefore from now on assume $\delta < \delta_0$ without loss of generality, and we will not have to worry about s having trivial index function.

The proof now proceed as the proof for Lemma 3.8 with $x = t$, and $f : \sigma(t) \rightarrow \mathbb{C}$ as in that proof. The only thing we need to consider is if we can lift the unitary element $f(t) \in Q(H)$ to a unitary in $B(H)$. To see this we first notice that by Lemma 5.9 f is homotopic to some $g : \sigma(t) \rightarrow \mathbb{C} \setminus \{0\}$ inside $C(\sigma(t), \mathbb{C} \setminus \{0\})$, where $g(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$, with $\lambda_i \in \mathbb{C} \setminus X$ and $n_i \in \mathbb{Z}$.

Hence we have a continuous path of invertible elements connecting $f(t)$ and $g(t)$, so it follows $\text{index}(f(t)) = \text{index}(g(t))$. But $\text{index}(g(t)) = \sum_i n_i \text{index}(t - \lambda_i) = 0$, since t was normal. Hence $f(t)$ is unitary with index 0, and by Lemma 5.10 $f(t)$ can be lifted to a unitary in $B(H)$.

The rest of the proof is now as the proof of Lemma 3.8. \square

Proof of Theorem 5.4 : Let $t \in Q(H)$ be normal with trivial index function, and let $\epsilon > 0$. By Lemma 5.8 there exists an $s \in Q(H)$ such that $\|s - t\| \leq \epsilon$, s is normal with trivial index function, and $\sigma(s) \subset \Gamma_\epsilon$.

Applying Lemma 5.11 a finite number of times we obtain a normal element $r \in Q(H)$ such that $\|r - s\| < \epsilon$, and $\sigma(r) \subset \Gamma_\epsilon$, and the components have diameters less than ϵ (similar to the proof of lemma 3.9).

As in the proof of lemma 3.9 partition the spectrum of r into a finite number of clopen sets, and let f be a function on $\sigma(r)$ which is constant on the clopen sets such that $\|f(r) - r\| < \epsilon$.

$f(r)$ is now a normal element in $Q(H)$ with finite spectrum and $\|f(r) - t\| < 3\epsilon$. This shows the if part of the theorem.

Conversely assume (t_n) is a sequence of normal elements with finite spectrum converging in norm to a normal t in the Calkin algebra. Clearly each t_n has trivial index function since the complement of their spectrum is all the unbounded component. Let $\lambda \in \mathbb{C} \setminus \sigma(t)$. Then $t - \lambda$ is invertible. Since the set invertible elements is open $t_n - \lambda$ is invertible for large enough n , and hence $\lambda \in \mathbb{C} \setminus \sigma(t_n)$. By continuity of index we have $\text{index}(t_n - \lambda) \rightarrow \text{index}(t - \lambda)$. Since each t_n has trivial index function it follows that t also has.

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