The University of Edinburgh School of Mathematics

Cohomology of Quiver Moduli

by

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Abstract

We present an exposition of the paper [Rei03], focusing on the chain of arguments leading up to an explicit formula for calculating the Betti numbers of a certain class of quiver moduli. This is supplemented by an introduction to the theory of quiver representations, Hall algebras, and the associated algebraic geometry, as well as numerous calculated examples, and clarifications and motivation concerning the proofs.
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.
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Chapter 1

Prologue

1.1 Introduction

A quiver is simply a finite oriented graph, and a quiver representation is obtained by interpreting the vertices as vector spaces, and the edges as corresponding linear maps. The theory of quivers and their representations has been an active area of research for many years, as links have been identified with several other areas of inquiry such as algebras, Lie theory, algebraic geometry, and even physics.

As with other representation theories, the main problem lies in the classification of the representations of a given quiver, up to isomorphism. This problem can be completely solved in a very special case, for quivers of Dynkin type, since by Gabriel’s theorem, they admit only finitely many isomorphism classes of indecomposable representations. In general, it is very difficult to find a classification result for arbitrary quivers.

Interpreting the question geometrically, the problem translates into the study of the orbit space of a certain affine space, the quiver variety, under the action of a reductive group. Thus one hopes to consider the quotient and obtain interesting moduli spaces that could tell us something about the isomorphism classes of the representations. However as we shall see, simply considering the quotient obtained by “classical” invariant theory does not always lead to interesting moduli spaces: the quotient is trivial for quivers without oriented cycles.

To go around this inconvenience, King described in [Kin94] a construction that yields interesting moduli spaces, and that has since become a standard technique of constructing moduli in representation theory. The idea is to pick an open
subset that contains enough closed orbits, and to construct the quotient of that subset via Mumford’s Geometric Invariant Theory (GIT). The choice of the open set depends on a notion of stability, which has both algebraic and geometric interpretations.

We are interested in calculating cohomological data of the quiver moduli, in particular, its Betti numbers (i.e. the ranks of the ordinary singular cohomology groups), or equivalently its Poincaré polynomial, which is the generating function of the Betti numbers. In [Rei03], Reineke studies the moduli problem for quiver representations by adapting methods of Harder and Narasimhan from the theory of moduli of vector bundles (See for example [Sch06]). Thus he obtains a stratification of the quiver variety, and then by associating a Hall algebra to the quiver, finds an analogue of the Harder-Narasimhan (HN) recursion inside the quantized enveloping algebra of a Kac-Moody algebra. He then resolves the recursion explicitly, and with the help of Deligne’s solution to the Weil conjectures, and motivated by techniques such as in [Kir84] and [Goe94], writes down an explicit formula for the Poincaré polynomial of the quiver moduli.

In this thesis we present a detailed exposition of Reineke’s aforementioned paper, supplemented by introductory background material on quivers and their representations, the associated algebraic geometry, and Hall algebras, as well numerous examples to illustrate these ideas. We focus on the chain of reasoning leading up to the explicit formula for the Poincaré polynomial, and we also supply a Mathematica program which performs the calculation.

1.2 Structure of the thesis

Chapter 2 starts by introducing the theory of quiver representations, and deals with the pure algebraic aspect of the subject. The exposition in this chapter is mainly based on chapter 1 of the lecture notes by Crawley-Boevey ([CB92]).

Chapter 3 introduces the geometry associated to quivers and their representations. Section 1 is based on Chapter 3 of [CB92], while sections 2 and 3 are drawn from Reineke’s paper in question ([Rei03]). Section 4 incorporates material from King’s paper ([Kin94]) as well as a later survey by Reineke ([Rei08]).

Chapter 4 describes the Hall algebra associated to quivers. Sections 1 and 2 have some introductory material on Hall algebras taken from Schiffmann’s lectures ([Sch06]). Section 3 describes how to relate the Hall algebra to the HN stratifi-
cation and derives the HN recursion, as done in [Rei03].

Chapter 5 addresses the main goal of the thesis, which is the cohomology of the quiver moduli. Section 1 derives the explicit solution of the HN recursion, as in [Rei03]. Section 2 uses this solution, as well as material from [Kir84],[Goe94],[Rei08] to obtain the explicit formula for the Poincaré polynomial.

In conclusion, chapter 6 describes some interesting, familiar examples of quiver moduli, and uses the formula obtained in chapter 6 to calculate their Betti numbers.

Appendix A contains, mostly with proof, a summary of the combinatorics of quiver representations, roots and the Weyl group, as well as statements of Gabriel’s and Kac’s theorems, with application. The Mathematica code for the calculations can be found in the Appendix B.
Chapter 2

Quivers and Their Representations

2.1 Quivers, morphisms, representations

We begin by presenting the main definitions of quivers, their morphisms and representations. Throughout the following, \( k \) denotes a fixed arbitrary field. This exposition mainly follows [CB92].

**Definition 2.1.1.** A quiver \( Q \) is an oriented (finite) graph (where multiple arrows between two vertices, and loops on vertices are allowed). We denote by \( Q_0 \) the set of vertices, and \( Q_1 \) the set of arrows. For an arrow \( \rho \in Q_1, \rho : i \rightarrow j \) we have maps \( s,t : Q_1 \rightarrow Q_0 \) given by \( s(\rho) = i \) and \( t(\rho) = j \).

**Definition 2.1.2.** Given a quiver \( Q \), a **quiver representation** \( X \) of \( Q \) is the assignment to each vertex \( i \in Q_0 \) of a (finite dimensional) \( k \)-vector space \( X_i \), and to each arrow \( \rho \in Q_1, \rho : i \rightarrow j \) a linear map \( X_\rho : X_i \rightarrow X_j \). If we let \( Q_0 = \{1,2,\cdots,n\} \), the **dimension type** of \( X \), is the element \( \text{dim} \in \mathbb{Z}^{Q_0} \) of the free abelian group generated by \( Q_0 \), given by the ordered tuple \( (\text{dim}_k X_1, \text{dim}_k X_2, \cdots, \text{dim}_k X_n) \).

**Definition 2.1.3.** Given two representations \( X,Y \) of a quiver \( Q \), a **morphism** \( \varphi : X \rightarrow Y \) is a collection of linear maps \( \varphi_i : X_i \rightarrow Y_i \) for each vertex \( i \in Q_0 \),
such that for each arrow $\rho : i \rightarrow j$, the following diagram commutes:

![Diagram of arrows and compositions](image)

that is, $Y_\rho \circ \varphi_i = \varphi_j \circ X_\rho$. If each $\varphi_i$ is an isomorphism, we say that $\varphi$ is an isomorphism of $X$ and $Y$, and that $X$ and $Y$ are isomorphic. We write $X \cong Y$.

**Definition 2.1.4.** Given two representations $X, Y$ of the quiver $Q$, their direct sum $W = X \oplus Y$ is defined to be the representation consisting of assigning to each vertex $i \in Q_0$ the vector space $W_i = X_i \oplus Y_i$, and to each arrow $\rho : i \rightarrow j$ the linear map $W_\rho = X_\rho \oplus Y_\rho$ (where the sum is carried component-wise). A representation $U$ of $Q$ is said to be decomposable if there exist non-zero representations $V, W$ of $Q$ such that $U \cong V \oplus W$. A representation $U$ of $Q$ is said to be indecomposable if it is not decomposable.

**Definition 2.1.5.** Given a representation $X$ of a quiver $Q$, a subrepresentation $U$ of $X$ is a representation of $X$ such that for each $i, j \in Q_0$, $\rho : i \rightarrow j \in Q_1$ we have $U_i \subseteq X_i$ and $U_\rho = X_\rho|_{U_i} : U_i \rightarrow U_j$. A non-zero representation of $Q$ with no proper non-zero subrepresentations is said to be simple. A semisimple representation of $Q$ is a representation which is the direct sum of simple subrepresentations.

**Remark 2.1.6.** It is clear that simple representations are indecomposable. For a quiver $Q$ and a vertex $i \in Q_0$, let $S(i)$ be the representation consisting of the 1-dimensional vector space $k$ at the vertex $i$, and the trivial space $0$ at every other vertex. Then it is clear that $S(i)$ is simple. If $Q$ has no oriented cycles, then we know exactly what the simple, and therefore semisimple, representations are, as shown in the following lemma.

**Lemma 2.1.7.** If $Q$ has no oriented cycles, then the only simple representations are $\{S(i) : i \in Q_0\}$.

**Proof.** Assume $Q$ has no oriented cycles. Then $Q$ must have a sink, i.e. a vertex $i$ such that $i \neq s(\rho)$ for any $\rho \in Q_1$; the following is an example of a sink:

![Diagram of a sink](image)
For suppose not, then for each $i \in Q_0$ there exists $\rho_i \in Q_1$ such that $s(\rho_i) = i$. If $t(\rho_i) = j$, there must exist $\rho_j$ such that $s(\rho_j) = j$. Repeating this process and following the arrows, we must obtain a cycle since $Q$ is a finite quiver, a contradiction.

Now we proceed by induction on the number of vertices. If there is one vertex $i$, then $S(i)$ is the only simple representation. If there is more than one vertex, let $i$ be a sink, and let $X$ be a simple representation. If $X_i \neq 0$, then $S(i)$ is clearly a subrepresentation of $X$, and thus $X = S(i)$. If $X_i = 0$, let $Q'$ be the quiver obtained from $Q$ by deleting the vertex $i$ and every arrow $\rho \in Q_1$ such that $t(\rho) = i$, and let $X'$ be the representation obtained by appropriately restricting $X$ to $Q'$, i.e. $X'_j = X_j$ for all $j \neq i$ and $X'_{\rho} = X_{\rho}$ for every $\rho$ such that $t(\rho) \neq i$. Note that $Q'$ also has no oriented cycles. The representation $X'$ must also be simple, for any proper subrepresentation of $X'$ gives rise to a proper subrepresentation of $X$ by adding the missing vertex and arrows. By induction, $X' = S(j)$ for some $j \in Q_0$, and hence $X = S(j)$.

The following result is then immediate:

**Corollary 2.1.8.** Let $d \in \mathbb{Z}^{Q_0}$ be a dimension vector. If $Q$ has no oriented cycles, then there exists a unique semisimple representation of dimension type $d$.

**Example 2.1.9.** We give some basic examples of quivers and quiver representations to illustrate the above definitions. Many of these examples will be recurring throughout this exposition.

1. The simplest quiver is made up of one vertex and no arrows, i.e., $Q_0 = \{1\}$ and $Q_1 = \emptyset$:

   \[ 1 \]

   Any representation of this quiver can be considered simply as a finite dimensional vector space.

2. The *Jordan quiver* is given by one vertex and one arrow forming a loop. A representation of it looks like the following:

   \[ k^n \tria \alpha \]

9
where $n \in \mathbb{N}$ and $\alpha \in \text{End}_k(k^n)$. The direct sum of two such representations is given by:

$$
\begin{align*}
    k^n & \oplus \alpha \oplus \beta \cong k^{m+n} \oplus \alpha \oplus \beta
\end{align*}
$$

The reason for calling this quiver after Jordan becomes clear when we look for its indecomposable representations: assuming that $k$ is algebraically closed, we can find a basis of $k^n$ that decomposes it into $\alpha$-cyclic subspaces, and such that the matrix of $\alpha$ is given by a direct sum of Jordan blocks. Thus, the representation $(k^n, \alpha)$ is indecomposable if and only if the $\alpha$ is similar to a Jordan block. Note that in this example we have infinitely many indecomposable representations, parametrized by a discrete quantity $(n)$ and a continuous quantity $\lambda$ which is the eigenvalue of the Jordan block.

3. Consider the $A_2$ quiver:

\[1 \longrightarrow 2\]

Then the simple representations are:

- $S(1) : k \xrightarrow{0} 0$
- $S(2) : 0 \xrightarrow{0} k$

Their direct sum is:

- $S(1) \oplus S(2) : k \xrightarrow{0} k$

and hence this representation is decomposable (and in fact, it is semisimple). Consider the representation:

- $M : k \xrightarrow{1} k$

where 1 denotes the identity map on $k$. Then this representation is indecomposable, since it has only one non-zero proper subrepresentation, which is $S(2)$. In fact, we can show by elementary linear algebra that the only indecomposable representations of this quiver are $S(1), S(2),$ and $M$. For
suppose that $X$ is an indecomposable representation such that $X \neq S(1)$ and $X \neq S(2)$. Then $X$ must be of the form:

$$X : \quad \begin{array}{c}
k^n \\
\downarrow \alpha \\
k^m
\end{array}$$

with $\alpha \neq 0$. If $\ker \alpha \neq 0$, then we have a decomposition of $X$ as:

$$X : \quad \begin{array}{c}
\ker \alpha \oplus A \\
\downarrow 0 \oplus \alpha \\
0 \oplus k^m
\end{array}$$

For some subspace $A \subset k^n$. Similarly, if $\im \alpha \neq k^m$, we can find a decomposition of $X$ as:

$$X : \quad \begin{array}{c}
k^n \oplus 0 \\
\downarrow \alpha \oplus 0 \\
\im \alpha \oplus B \\
k^m
\end{array}$$

for some $B \subset k^m$. Thus $\alpha$ must be an isomorphism, $n = m$ and we can choose (different!) bases for both vector spaces so that the matrix of $\alpha$ is the identity matrix. Therefore, $X \cong \bigoplus_{i=1}^n M$. Since $X$ is indecomposable, it follows that $n = 1$ and $X \cong M$.

4. Consider the following two representations of the same quiver:

$$X : \quad \begin{array}{c}
k \\
\downarrow 1
\end{array} \quad \begin{array}{c}
k \\
\downarrow 1
\end{array} \quad \begin{array}{c}
k \\
\downarrow 1
\end{array}$$

$$Y : \quad \begin{array}{c}
k \\
\downarrow 1
\end{array} \quad \begin{array}{c}
k \\
\downarrow 0
\end{array} \quad \begin{array}{c}
0
\end{array}$$

Then $\Hom_k(X, Y) = k$ while $\Hom_k(Y, X) = 0$. Indeed, consider the commutative diagram:

$$\begin{array}{ccc}
k & \xrightarrow{1} & k \\
\downarrow f & & \downarrow g \\
k & \xrightarrow{1} & 0 \\
\downarrow h & & \downarrow 0
\end{array}$$

Then $h = 0$, and $f = g$ can be represented by any scalar. On the other hand:

$$\begin{array}{ccc}
k & \xrightarrow{1} & k \\
\downarrow f & & \downarrow g \\
k & \xrightarrow{1} & 0 \\
\downarrow h & & \downarrow 0
\end{array}$$

and in this case we have $f = g = h = 0$. 11
2.2 Path algebras

We now investigate the category of representations of a quiver, some of its homological properties, and some connections with the representation theory of finite dimensional algebras.

Definition 2.2.1. Given a quiver $Q$, a path is a sequence $\rho_1, \rho_2, \ldots, \rho_r$ of arrows such that $t(\rho_{i+1}) = s(\rho_i)$ for $1 \leq i \leq r$. The path algebra $kQ$ is the $k$-algebra with basis the paths of $Q$, and with the product given by the concatenation of paths (by the path $xy$ we mean traveling along $y$ first then along $x$). If two paths cannot be concatenated (i.e. their endpoints don’t match), then their product is defined to be 0. If for $i \in Q_0$ we let $e_i$ be the trivial path with $s(e_i) = t(e_i) = i$, then $\sum_{i=1}^n e_i$ is the multiplicative identity of $kQ$.

Example 2.2.2. We give some examples of path algebras for familiar quivers.

1. The path algebra of the Jordan quiver is the polynomial ring $k[T]$.
2. The path algebra of the following $A_3$ quiver:

\[
\begin{array}{c}
1 \\
\alpha \\
2 \quad \beta \\
3
\end{array}
\]

consists of the algebra of lower triangular $3 \times 3$ matrices with entries in $k$. To see this, notice that the paths are given by $e_1, e_2, e_3, \alpha, \beta,$ and $\beta \alpha$. Thus define a map $f : kQ \to M_{3 \times 3}(k)$ by:

\[
a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 \alpha + a_5 \beta + a_6 \beta \alpha \mapsto \begin{pmatrix} a_1 & 0 & 0 \\ a_4 & a_2 & 0 \\ a_6 & a_5 & a_3 \end{pmatrix}
\]

Then it’s easy to verify that $f$ is a $k$-algebra isomorphism onto the lower triangular matrices.

Definition 2.2.3. With the definitions of the previous section, for a given quiver $Q$ we can associate the category of representations $\text{Rep}_k(Q)$ (with composition of morphisms defined in the obvious way).

Lemma 2.2.4. The category $\text{Rep}_k(Q)$ is equivalent to the category (finite dimensional) $kQ$-mod (left modules).
Proof. We just describe the construction. Let $\mathcal{X}$ be a $kQ$-module. Then we can define a representation:

$$X_i = e_i \mathcal{X}$$

$$X_\rho(x) = \rho x = e_{t(\rho)} \rho x \in X_{t(\rho)} \quad \text{for} \ x \in X_s(\rho)$$

Conversely, given a representation $X$, we can construct a $kQ$-module $\mathcal{X}$ via:

$$\mathcal{X} = \bigoplus_{i \in Q_0} X_i$$

with

$$X_i \xrightarrow{i_i} \mathcal{X} \xrightarrow{\pi_i} X_i$$

being the canonical maps,

$$\rho_1 \cdots \rho_m x = i_1(\rho_1)X_{\rho_1} \cdots X_{\rho_m} \pi_s(\rho_m)(x)$$

$$e_i x = i_i \pi_i(x)$$

Remark 2.2.5.

1. Because of the above lemma, we shall use the notation $\text{mod}_k Q$ interchangeably with $\text{Rep}_k(Q)$.

2. There is a link between the representation theory of finite dimensional algebras and the representation theory of quivers. If $A$ is a finite dimensional $k$-algebra, then the category of representations of $A$ is equivalent to the category of representations of the algebra $kQ/I$ for some quiver $Q$ and some two-sided ideal $I$ of $kQ$ ([Sav06, 4.1]).

We can now prove a key property of quiver representations: the category $\text{Rep}_k(Q)$ has the Krull-Schmidt property.

Lemma 2.2.6. (Krull-Schmidt property) Given a representation $X$ of $Q$, there exist indecomposable representations $X_1, X_2, \cdots, X_n$ such that $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Moreover, this decomposition is unique up to reordering of the summands.
Proof. Since vector spaces have the ACC and DCC properties, we see that the
$kQ$-modules that are finite dimensional over $k$ have ACC and DCC. It follows by
the Krull-Schmidt theorem for modules that this category has the Krull-Schmidt
property. By the equivalence of categories (Lemma 2.2.4), the result extends
immediately to $\text{Rep}_k(Q)$.

Thus in order to understand the representation theory of a quiver, we need only
understand its indecomposable representations. However, as we have seen in
previous examples, some (indeed, most) quivers have infinitely many indecom-
posable representations. It is natural to ask for which quivers do we have only a
finite number of indecomposable representations, and this question is answered
by Gabriel’s theorem (see Appendix A).

We list some properties of the path algebras of quivers.

**Lemma 2.2.7.** Let $A = kQ$.

1. The elements $e_i \in A$ are orthogonal idempotents. That is, $e_i e_j = \delta_{ij} e_i$.

2. The spaces $A e_i$ and $e_j A$ have as bases respectively those paths starting at
   $i$, and those ending at $j$. The space $e_j A e_i$ has as bases the paths starting
   at $i$ and ending at $j$.

3. $A = \bigoplus_{i \in Q_0} e_i A$, and so each $e_i A$ is a projective right $A$-module. Similarly,
   each $A e_i$ is a projective left $A$-module.

4. If $X$ is a left $A$-module, then $\text{Hom}_A (A e_i, X) \cong e_i X$.

5. If $0 \neq f \in A e_i$ and $0 \neq g \in e_i A$, then $f g \neq 0$.

   *Proof.* Let $x$ be path of greatest possible length and appearing in $f$, simi-
larly let $y$ be a path of greatest length in $g$, then in $f g$ the coefficient of $xy$
cannot be $0$. \qed

6. The $e_i$ are *primitive* idempotents, i.e. $A e_i$ is an indecomposable module.

   *Proof.* By (4), $\text{End}_A (A e_i) \cong e_i A e_i$. Suppose $f \in \text{End}_A (A e_i)$ is an idem-
po-tent, then $f^2 = f = f e_i$, hence $f(f - e_i) = 0$. Now by (5), either $f = 0$ or
$f = e_i$. \qed
7. if $e_i \in Ae_j A$, then $i = j$, since $Ae_j A$ has as a basis the paths passing through $j$.

8. if $i \neq j$ then $Ae_i \not \equiv Ae_j$.

Proof. By (4), $\text{Hom}_A(Ae_i, Ae_j) \cong e_i Ae_j$, hence if $Ae_i \cong Ae_j$, we can find elements $f \in e_i Ae_j$ and $g \in e_j Ae_i$ such that $fg = e_i$ and $gf = e_j$. But $e_i Ae_j$ has basis the paths starting at $j$ and ending at $i$, while $e_j Ae_i$ has basis the paths starting at $i$ and ending at $j$, and so the element $fg$ can be expressed as a linear combination of paths going through $j$, therefore $e_i \in Ae_j A$ and so by (7) we must have $i = j$. \qed

9. $A$ is finite dimensional (over $k$) if and only if $A$ has no oriented cycles.

2.3 The standard resolution and homological properties

In this section we investigate some homological properties of the category $\text{Rep}_k(Q)$, and we introduce some key definitions.

Definition 2.3.1. The Euler form* is a bilinear form defined on $\mathbb{Z}^Q_0$ by:

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\rho \in Q_1} \alpha_{s(\rho)} \beta_{t(\rho)}$$

for all $\alpha, \beta \in \mathbb{Z}^Q_0$. The symmetric Euler form is given by $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. The Tits form is the quadratic form $q$ given by $q(\alpha) = \langle \alpha, \alpha \rangle$.

The following proposition is the key to understand the homological properties of $\text{Rep}_k(Q)$.

Proposition 2.3.2. (The standard resolution) Let $A = kQ$. If $X$ is a left $kQ$-module, then there is an exact sequence:

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)} X \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes_k e_i X \xrightarrow{g} X \longrightarrow 0$$

where:

$$g(a \otimes x) = ax \quad \text{for} \ a \in Ae_i, \ x \in e_i X$$

---

*The name is related to the fact that one can define this form in general for a certain type of categories satisfying certain finiteness conditions, in which case the Euler form corresponds to the Euler characteristic of a certain chain complex. See [Sch06], or chapter 4.
\[ f(a \otimes x) = a \rho \otimes x - a \otimes \rho x \quad \text{for } a \in Ae_{t(\rho)}, x \in e_{s(\rho)} X \]

This short exact sequence is called the **standard resolution**.

**Proof.** The proof is detailed in [CB92, §1].

**Corollary 2.3.3.**

1. If \( X \) is a left \( A \)-module, then \( pd(X) \leq 1 \), thus \( \text{Ext}^i_A(X, Y) = 0 \ \forall Y, i \geq 2 \).

   **Proof.** In the standard resolution, \( f \) and \( g \) are \( A \)-module maps, and each \( Ae_i \otimes V \) is isomorphic to the direct sum of \( \dim V \) copies of \( Ae_i \), which is projective (check the previous section). Therefore the standard resolution is a projective resolution.

2. \( A \) is **hereditary**, i.e. if \( X \subseteq P \) and \( P \) is projective, then \( X \) is projective.

   **Proof.** Consider the short exact sequence:

   \[ 0 \rightarrow X \rightarrow P \rightarrow P/X \rightarrow 0 \]

   with natural maps. If \( Y \) is another \( A \)-module, then by applying the \( \text{Hom}_A(-, Y) \) functor to the above short exact sequence, we obtain an exact sequence:

   \[ \cdots \rightarrow 0 \rightarrow \text{Ext}^1_A(X, Y) \rightarrow \text{Ext}^2_A(P/X, Y) \rightarrow 0 \]

   Thus \( \text{Ext}^1_A(X, Y) \cong \text{Ext}^2_A(P/X, Y) = 0 \).

3. If \( X, Y \) are finite dimensional \( A \)-modules, then:

   \[ (\dim X, \dim Y) = \dim \text{Hom}_A(X, Y) - \dim \text{Ext}^1_A(X, Y) \]

   In particular, \( q(\dim X) = \dim \text{End}_A(X) - \dim \text{Ext}^1_A(X, X) \).

   **Proof.** Applying \( \text{Hom}_A(-, Y) \) to the standard resolution, we get a short exact sequence:

   \[ 0 \rightarrow \text{Hom}_A(X, Y) \rightarrow \text{Hom}_A(\bigoplus_{i \in Q_0} Ae_i \otimes e_i X, Y) \rightarrow \text{Hom}_A(\bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes e_{s(\rho)} X, Y) \rightarrow \text{Ext}^1_A(X, Y) \rightarrow 0 \]

   But \( \dim \text{Hom}_A(Ae_i \otimes e_j X, Y) = (\dim e_j X)(\dim \text{Hom}_A(Ae_i, Y)) = (\dim X)_j(\dim Y)_i \), so the result follows.
Chapter 3

The Geometry of Quivers

3.1 The quiver variety

Let \( Q \) be a quiver, and \( k \) an algebraically closed field (say, \( k = \mathbb{C} \)). We define the quiver variety and state some of its properties, following [CB92, §3].

**Definition 3.1.1.** For the dimension vector \( d \in \mathbb{Z}^{Q_0} \), the corresponding quiver variety of representations is defined as:

\[
R_d = \bigoplus_{\rho \in Q_0} \text{Hom}_k(k^{d_{s}(\rho)}, k^{d_{t}(\rho)}).
\]

Note that this is an affine variety isomorphic to \( \mathbb{A}^r \) where \( r = \sum_{\rho \in Q_1} d_{t}(\rho)d_{s}(\rho) \).

On this variety, there is a natural action by the following group:

\[
G_d = \prod_{i \in Q_0} \text{GL}(d_i, k)
\]

given by base change, i.e. for \( x \in R_d, g \in G_d \), we have:

\[
(g \cdot x)_{\rho} = g_{t(\rho)}x_{\rho}g_{s(\rho)}^{-1}.
\]

Note that \( G_d \) is an algebraic group and it’s an open (and non-empty, hence dense) subset of \( \mathbb{A}^s \) where \( s = \sum_{i \in Q_0} d_i^2 \).

**Lemma 3.1.2.** The following observations are immediate:

1. There is a one-to-one correspondence between points \( x \in R_d \) and representations of \( Q \) of dimension type \( d \).
2. To a point \( x \in R_d \) we denote the corresponding representation by \( R(x) \).
   Then for \( x, y \in R_d \), the set of isomorphisms \( R(x) \to R(y) \) can be identified with \( \{ g \in G_d : g \cdot x = y \} \).

3. For each \( x \in R_d \), \( \text{Stab}_{G_d}(x) \cong \text{Aut}_{kQ}(R(x)) \).

4. For any dimension type \( d \), there is a one-to-one correspondence between
   the isomorphism classes of representations of \( Q \) of dimension type \( d \) and
   the orbits in \( R_d \) under the action of \( G_d \). In other words, \( \{ y \in R_d : R(x) \cong R(y) \} = \{ g \cdot x : g \in G_d \} = \mathcal{O}_x \).

From this point on we will identify the points of \( R_d \) with representations of \( Q \) of dimension type \( d \).
For \( X \in R_d \), we denote by \( \mathcal{O}_X \) the orbit of \( X \) under the action of \( G_d \). We have
the following properties from algebraic geometry.

**Lemma 3.1.3.** Let \( X \in R_d \). Then:

1. The orbits \( \mathcal{O}_X \) are locally closed, that is, \( \mathcal{O}_X \) is an open subset of its closure \( \overline{\mathcal{O}_X} \). Moreover, \( \dim \mathcal{O}_X = \dim \overline{\mathcal{O}_X} \).

2. \( \overline{\mathcal{O}_X} \setminus \mathcal{O}_X \) is a union of orbits of dimension strictly smaller than \( \dim \mathcal{O}_X \).

3. \( \dim \mathcal{O}_X = \dim G_d - \dim \text{Stab}_{G_d}(X) \).

Combining these facts with the ideas from the previous chapter, we arrive at
the following result.

**Lemma 3.1.4.** Let \( X \in R_d \). Then:

\[
\dim R_d - \dim \mathcal{O}_X = \dim \text{End}_{kQ}(X) - q(d) = \dim \text{Ext}^1(X, X)
\]

where \( q \) is the Tits form defined in 2.3.1.

**Proof.** By Lemmas 3.1.2 and 3.1.3,

\[
\dim \mathcal{O}_X = \dim G_d - \dim \text{Stab}_{G_d}(X) = \dim G_d - \dim \text{Aut}_{kQ}(X).
\]

The assertion follows from the fact that both \( G_d \) and \( \text{Aut}_{kQ}(X) \) are non-empty,
open sets, therefore dense. \( \square \)
Corollary 3.1.5.

1. If $d \neq 0$ and $q(d) \leq 0$ then there are infinitely many orbits in $R_d$.

   Proof. If $0 \neq d$ and $q(d) \leq 0$, then for any $X \in R_d$, $\dim \text{End}_k Q(X) > 0$ and so $\dim R_d > \dim \mathcal{O}_X$, hence $R_d$ cannot be the union of finitely many orbits. \hfill \Box

2. Let $X \in R_d$. Then $\mathcal{O}_X$ is open if and only if $\text{Ext}^1(X, X) = 0$.

   Proof. By Lemma 3.1.4, $\text{Ext}^1(X, X) = 0 \iff \dim R_d = \dim \mathcal{O}_X \iff \dim R_d = \dim \overline{\mathcal{O}}_X$.

   Suppose $\dim \overline{\mathcal{O}}_X = \dim R_d$. Then $\overline{\mathcal{O}}_X = R_d$, since $R_d$ is irreducible, and otherwise the closed subset $\overline{\mathcal{O}}_X$ would have strictly smaller dimension. Since $\mathcal{O}_X$ is locally closed, it is then open in $R_d$.

   Conversely, suppose that $\mathcal{O}_X$ is open in $R_d$. Then it’s dense since $R_d$ is irreducible, and therefore $\overline{\mathcal{O}}_X = R_d$, so they certainly have the same dimension. \hfill \Box

3. Let $d \in \mathbb{Z}^{Q_0}$ be a dimension vector. Then there is at most one representation of $Q$ up to isomorphism without self-extensions and of dimension type $d$.

   Proof. Suppose $X, Y \in R_d$ are both without self-extensions. Then by the previous part, $\mathcal{O}_X$ and $\mathcal{O}_Y$ are both open, hence dense since they are non-empty and $R_d$ is irreducible. Thus $\mathcal{O}_X \cap \mathcal{O}_Y \neq \emptyset$, hence $\mathcal{O}_X = \mathcal{O}_Y$ since the orbits form a partition of $R_d$. By Lemma 3.1.2, part (4), we have $X \cong Y$. \hfill \Box

Lemma 3.1.6. if $\xi : 0 \to U \to X \to V \to 0$ is a non-split exact sequence, then $\mathcal{O}_{U \oplus V} \subset \overline{\mathcal{O}}_X \setminus \mathcal{O}_X$.

Proof. For each $i \in Q_0$, identify $U_i$ with a subset of $X_i$. Thus for each $\rho \in Q_1$, we have:

$$X_\rho = \begin{pmatrix} U_\rho & W_\rho \\ 0 & V_\rho \end{pmatrix}$$

by choosing bases for the $U_i$ and extending to bases of $X_i$ (note that this is not a $2 \times 2$ matrix, but represented by blocks). For $\lambda \in k^*$, we have an element
$g_{\lambda} \in G_d$, with
$$g_{\lambda} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$  
(Note here also that the right hand side is not a $2 \times 2$ matrix, but a block representation, with $\lambda$ standing for a diagonal block matrix with $\lambda$ on the diagonal).
Thus:
$$(g_{\lambda} \cdot x)_{\rho} = \begin{pmatrix} U_{\rho} & \lambda W_{\rho} \\ 0 & V_{\rho} \end{pmatrix}$$
and so $\mathcal{O}_X$ contains the points with matrices:
$$\begin{pmatrix} U_{\rho} & 0 \\ 0 & V_{\rho} \end{pmatrix}$$
which correspond to $U \oplus V$, hence $\mathcal{O}_{U \oplus V} \subset \mathcal{O}_X$.
Applying the functor $\text{Hom}(-,U)$ to $\xi$ yields the exact sequence:
$$0 \rightarrow \text{Hom}(V,U) \rightarrow \text{Hom}(X,U) \rightarrow \text{Hom}(U,U) \rightarrow \text{Ext}^1(V,U) \rightarrow \cdots$$
and so: $\dim \text{Hom}(V,U) - \dim \text{Hom}(X,U) + \dim \text{Hom}(U,U) - \dim \text{im } f = 0$. However by definition, $f(id_U) = \xi \neq 0$, and so $\dim \text{im } f \neq 0$, which gives $\dim \text{Hom}(V,U) + \dim \text{Hom}(U,U) \neq \dim \text{Hom}(X,U)$, which means that $X \nsubseteq U \oplus V$, and so $\mathcal{O}_{U \oplus V} \nsubseteq \mathcal{O}_X$. \hfill \Box

**Corollary 3.1.7.**

1. If $\mathcal{O}_X$ is of maximal dimension, and $X = U \oplus V$, then $\text{Ext}^1_{kQ}(V,U) = 0$.

*Proof.* Suppose that $\text{Ext}^1_{kQ}(V,U) \neq 0$. Then there exists a non-split short exact sequence $0 \rightarrow U \rightarrow E \rightarrow V \rightarrow 0$. By the lemma, $\mathcal{O}_X \subset \mathcal{O}_E \setminus \mathcal{O}_E$, hence by Lemma 3.1.3, part (2) we have $\dim \mathcal{O}_X < \dim \mathcal{O}_E$, a contradiction. \hfill \Box

2. If $\mathcal{O}_X$ is closed then $X$ is semisimple. If $Q$ has no oriented cycles, then there is a unique closed orbit consisting of the single point 0. Moreover, 0 is in the closure of every orbit*.

*This was stated in [CB92] without a proof. I have supplied a proof.
Proof. Assume $\mathcal{O}_X$ is closed, and suppose there exists a non-split short exact sequence $0 \rightarrow U \rightarrow X \rightarrow V \rightarrow 0$. Then by the lemma, $\mathcal{O}_{U\oplus V} \subset \overline{\mathcal{O}_X \setminus \mathcal{O}_X} = \emptyset$, a contradiction. Thus every such short exact sequence splits, and hence $X$ is semisimple.

If $Q$ has no oriented cycles, it follows from Corollary 2.1.8 that there is a unique semisimple representation $X$, and so $\mathcal{O}_X$ is the unique closed orbit. Also if $Q$ has no oriented cycles, we claim then for all elements $\lambda \in k^*$, representations $Y$ and $\rho \in Q_1$ we can find $g \in G_d$ and a positive integer $t_\rho$ such that $(g \cdot Y)_\rho = \lambda^{l_\rho} Y_\rho$. To see this, note that for vertices $i, j$ and arrow $\rho : i \rightarrow j$, one can choose $g \in G_d$ so that $g_i = \mu_1 \cdot id_{Y_i}$ and $g_j = \mu_2 \cdot id_{Y_j}$ for any $\mu_1, \mu_2 \in k^*$, so that $(g \cdot Y)_\rho = \frac{\mu_2}{\mu_1} Y_\rho$. Now recall the proof of Lemma 2.1.7, and that $Q$ must have a sink. The same argument can be used to show that it must also have a source. Without loss of generality we can assume $Q$ is connected. Let $g \in G_d$ be such that $g_i = \lambda^{l_i} \cdot id_{Y_i}$ where $l_i$ is the length of a longest path from a source to $i$. This $g$ acts on $Y$ in the desired way. This establishes the claim and shows that 0 is in the closure of every orbit.

Example 3.1.8.

1. Consider the representations of the $A_2$ quiver of the dimension type $d = (1, 1)$:

$$
\begin{array}{c}
k \\
\alpha
\end{array} \rightarrow
\begin{array}{c}
k
\end{array}
$$

Then $R_d \cong \mathbb{A}, G_d \cong \mathbb{C}^* \times \mathbb{C}^*$, with $(\lambda, \mu) \cdot \alpha = \frac{\mu}{\lambda}\alpha$ for $(\lambda, \mu) \in G_d$. Thus we have two orbits, $\{0\}$ (corresponding to the semisimple representation) and $\mathbb{A} \setminus \{0\}$ (corresponding to the indecomposable representations).

2. Consider the representations of the $A_2$ quiver of the dimension type $d = (1, 2)$:

$$
\begin{array}{c}
k \\
\alpha
\end{array} \rightarrow
\begin{array}{c}
k^2
\end{array}
$$

Then $R_d \cong \mathbb{A}^2, G_d \cong \mathbb{C}^* \times GL(2, \mathbb{C})$, and there are two orbits, $\{0\}$ (corresponding to the semisimple representation) and $\mathbb{A}^2 \setminus \{0\}$. Note that $\mathbb{A}^2 \setminus \{0\}$ is not an affine variety†.

†Since Spec($k[\mathbb{A}^2 \setminus \{0\}]$) = Spec($k[x, y]$).
3. Consider the representations of the $A_3$ quiver of dimension type $d = (1,1,1)$:

\[ \begin{array}{c}
  k & \xrightarrow{\alpha} & k & \xrightarrow{\beta} & k
\end{array} \]

Then $R_d \cong \mathbb{A}^2$, $G_d \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, with $(\lambda, \mu, \eta) \cdot (\alpha, \beta) = (\frac{\mu}{\lambda} \alpha, \frac{\eta}{\mu} \beta)$ for $(\lambda, \mu, \eta) \in G_d$. We have 4 orbits: $\{0\}$ (the semisimple representation), $\{x = 0, y \neq 0\}$, $\{x \neq 0, y = 0\}$, and $\mathbb{A}^2 \setminus \{xy = 0\}$ (the indecomposable representations). Note that $\mathbb{A}^2 \setminus \{xy = 0\} = V_{xy}$ is a basic open set hence an affine variety.

4. Consider the representations of the 2-Kronecker quiver of dimension type $d = (1,1)$:

\[ \begin{array}{c}
  k & \xrightarrow{\alpha} & k & \xrightarrow{\beta} & k
\end{array} \]

Then $R_d \cong \mathbb{A}^2$ and $G_d \cong \mathbb{C}^* \times \mathbb{C}^*$. If a point $x = (\alpha, \beta) \neq 0$, then under the action of $G_d$, it moves along a line through the origin (but minus the origin). Thus we see that $(R_d \setminus \{0\})/G_d = \mathbb{P}^1$.

5. Consider the representations of the Jordan quiver, and 2-dimensional representations:

\[ k^2 \cdot \bullet \alpha \]

then we have the following cases:

a) $\alpha$ is diagonalizable and has distinct eigenvalues, that is,

\[ [\alpha] = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \]

for some $s \neq t$. In this case, the orbit of $\alpha$ corresponds to diagonalizable matrices of trace $s + t$ and determinant $st$, i.e. of the form:

\[ \begin{pmatrix} x & y \\ z & s + t - x \end{pmatrix} \]

\[ x^2 - (s + t)x - zy = st. \]
This corresponds to a smooth closed subvariety:

Note that this shows that the Jordan quiver possesses infinitely many semisimple representations (at least over \(\mathbb{C}\)).

b) \(\alpha\) is diagonalizable and has a unique eigenvalue, so that:
\[
[\alpha] = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}
\]
so the orbit of \(\alpha\) is just the singleton \(\{\alpha\}\).

c) \(\alpha\) is not diagonalizable, so it is similar to a Jordan block of the form:
\[
[\alpha] = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}.
\]

Then the orbit of \(\alpha\) corresponds to non-diagonalizable matrices of trace \(2s\) and determinant \(s^2\), so matrices of the form:
\[
\begin{pmatrix} x & z \\ y & 2s-x \end{pmatrix}
\]
\((x, y, z) \neq (s, 0, 0)\)
\((x-s)^2 + yz = 0.\)

This corresponds to a (conic) singular subvariety with a point missing:

3.2 Stability of representations

We will define a notion of stability in \(\text{mod}_k(Q)\).

**Definition 3.2.1.** Let \(\Theta = \sum_{i \in Q_0} \Theta_i \delta^i \in \mathbb{Z}^{Q_0} \to \mathbb{Z}\) be a given, fixed linear form (here, \(\delta^i\) is the dual to the vertex \(i\)). We call \(\Theta\) a *weight* for \(Q\). Let \(\dim : \mathbb{Z}^{Q_0} \to \mathbb{Z}\)
be the linear form given by \( \dim(i) = 1 \) for each \( i \in Q_0 \). We define the slope function \( \mu \) on \( \mathbb{Z}^{Q_0} \setminus \{0\} \) by \( \mu = \frac{\Theta(d)}{\dim(d)} \). For a representation \( 0 \neq X \in \text{mod}_k(Q) \), we let \( \mu(X) = \mu(\dim(X)) \).

We give the definition of (semi)stability following [Rei03], and then following [Kin94], and we show how they are related.

**Definition 3.2.2.** ([Rei03, 2.1]) A representation \( X \in \text{mod}_k(Q) \) is said to be \( \mu \)-semistable if for all proper subrepresentations \( 0 \neq U \subset X \) we have \( \mu(U) \leq \mu(X) \). It is called \( \mu \)-stable if for all proper \( 0 \neq U \subset X \) we have \( \mu(U) < \mu(X) \).

**Definition 3.2.3.** ([Kin94, 1.1]) Let \( \sigma : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z} \) be a linear form. A representation \( X \in \text{mod}_k(Q) \) is said to be \( \sigma \)-semistable if \( 0 = \sigma(X) = \sigma(\dim M) \) and for all proper subrepresentations \( 0 \neq U \subset X \) we have \( \sigma(U) \geq 0 \). It is called \( \sigma \)-stable if \( \sigma(X) = 0 \) and for all proper \( 0 \neq U \subset X \) we have \( \sigma(U) > 0 \).

**Lemma 3.2.4.** Let \( \Theta \) be a weight and \( \mu \) the corresponding slope. Consider all representations \( X \) such that \( \mu(X) = \frac{p}{q} \), with \( p, q \in \mathbb{Z} \). Then there exists a linear form \( \sigma : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z} \) such that \( X \) (with slope \( \frac{p}{q} \)) is \( \mu \)-(semi)stable if and only if it is \( \sigma \)-(semi)stable.

**Proof.** Define \( \sigma \) by \( \sigma(Y) = -q\Theta(Y) + p\dim(Y) \). Then for any \( Y \subset X \) it’s straightforward to check that:

\[
\begin{align*}
\sigma(Y) \geq 0 & \Leftrightarrow \frac{p}{q} = \mu(X) \geq \mu(Y) \\
\sigma(Y) > 0 & \Leftrightarrow \frac{p}{q} = \mu(X) > \mu(Y)
\end{align*}
\]

From this point on we will refer to \( \mu \)-(semi)stability simply as (semi)stability.

**Lemma 3.2.5.** (cf. [Rei08, 4.1]) Let \( 0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0 \) be a short exact sequence in \( \text{mod}_k(Q) \). Then we have the following equivalences:

1. \( \mu(M) \leq \mu(X) \Leftrightarrow \mu(X) \leq \mu(N) \Leftrightarrow \mu(M) \leq \mu(N) \)
2. \( \mu(M) < \mu(X) \Leftrightarrow \mu(X) < \mu(N) \Leftrightarrow \mu(M) < \mu(N) \)
3. \( \mu(M) \geq \mu(X) \Leftrightarrow \mu(X) \geq \mu(N) \Leftrightarrow \mu(M) \geq \mu(N) \)
4. \( \mu(M) > \mu(X) \Leftrightarrow \mu(X) > \mu(N) \Leftrightarrow \mu(M) > \mu(N) \)
Thus we have the following inequality:

\[
\min(\mu(M), \mu(N)) \leq \mu(X) \leq \max(\mu(M), \mu(N))
\]

**Proof.** We prove only the equivalence (1), since the rest are proved similarly. Let \( d = \dim M, e = \dim N \). Then:

\[
\mu(X) = \frac{\Theta(d) + \Theta(e)}{\dim d + \dim e}
\]

From simple arithmetic manipulations we have:

\[
\frac{\Theta(d)}{\dim d} \leq \frac{\Theta(d) + \Theta(e)}{\dim d + \dim e} \iff \frac{\Theta(e)}{\dim e} \leq \frac{\Theta(d) + \Theta(e)}{\dim d + \dim e} \iff \frac{\Theta(d) + \Theta(e)}{\dim d + \dim e} \leq \frac{\Theta(e)}{\dim e}
\]

which establishes (1). The last inequality follows immediately from the equivalences (1), (2), (3), and (4). \( \square \)

**Lemma 3.2.6.** Let \( 0 \to M \xrightarrow{i} X \xrightarrow{\pi} N \to 0 \) be a short exact sequence in \( \mod_k Q \). If \( \mu(M) = \mu(X) = \mu(N) = \mu \), then \( X \) is semistable if and only if \( M \) and \( N \) are semistable.

**Proof.** (cf. [Rei08, 4.2]) Suppose \( M \) and \( N \) are semistable. Let \( U \subset X \) be a subrepresentation of \( X \). Then there exists a short exact sequence:

\[
0 \to U \cap M \to U \to U + \frac{M}{M} \to 0
\]

where the maps are the natural inclusion and quotient maps. Note that \( U \cap M \) is a subrepresentation of \( M \) and \( \frac{U + M}{M} \) can be regarded as a subrepresentation of \( N \). Hence by semistability we have:

\[
\mu(U \cap M) \leq \mu(M) = \mu
\]

\[
\mu\left( \frac{U + M}{M} \right) \leq \mu(N) = \mu
\]

and by the result of the previous lemma,

\[
\mu(U) \leq \max \left( \mu(U \cap M), \mu\left( \frac{U + M}{M} \right) \right) \leq \mu = \mu(X)
\]

showing that \( X \) is semistable.

Conversely, suppose \( X \) is semistable. Let \( U \subset M \) be a subrepresentation of \( M \). Since \( M \) can be regarded as a subrepresentation of \( X \), we can write \( U \subset X \). By semistability of \( X \), we have that \( \mu(U) \leq \mu(X) = \mu(M) \), showing that \( M \) is semistable.
Let $U \subset N$ now be a subrepresentation of $N$. Let $V = \pi^{-1}(N)$. Then we get a short exact sequence:

$$0 \longrightarrow M \overset{i}{\longrightarrow} V \overset{\pi}{\longrightarrow} U \longrightarrow 0$$

By semistability, $\mu(V) \leq \mu(X) = \mu = \mu(M)$. Thus by the equivalences of the previous lemma, $\mu(U) \leq \mu(M) = \mu(N)$, showing that $N$ is semistable.

Putting these lemmas together, we have the following result:

**Lemma 3.2.7.** The semistable representations of $Q$ with slope $\mu$ form a subcategory $\text{mod}^\mu_kQ$ of $\text{mod}_kQ$. For all $\mu \in \mathbb{Q}$, $\text{mod}^\mu_kQ$ is an (extension closed) abelian category whose simple objects are the stable representations of $Q$ of slope $\mu$. Moreover, $\text{Hom}_Q(\text{mod}^\mu_kQ, \text{mod}_kQ) = 0$ whenever $\mu > \nu$.

**Proof.** (cf. [Rei08, 4.2]) Let $X, Y \in \text{mod}^\mu_kQ$, and $f : X \rightarrow Y$ a morphism of representations of $Q$. We will just show that $\ker f$, $\text{im} f$ and $\text{coker} f$ have slope $\mu$.

There is a short exact sequence with natural maps:

$$0 \longrightarrow \ker f \longrightarrow X \overset{f}{\longrightarrow} \text{im} f \longrightarrow 0$$

By semistability of $X$, $\mu(\ker f) \leq \mu(X)$, and hence by Lemma 3.2.5, $\mu(\ker f) \leq \mu(\text{im} f)$ and $\mu(X) \leq \mu(\text{im} f)$. By semistability of $Y$, $\mu(\text{im} f) \leq \mu(Y) = \mu(X)$, so $\mu(\text{im} f) = \mu(X) = \mu(\ker f) = \mu$.

For the cokernel, consider the short exact sequence:

$$0 \longrightarrow \text{im} f \longrightarrow Y \longrightarrow \text{coker} f \longrightarrow 0$$

Since $\mu(\text{im} f) = \mu(Y)$, it follows from Lemma 3.2.5 that $\mu(\text{coker} f) = \mu(Y) = \mu$.

Since $\ker f$, $\text{im} f$, and $\text{coker} f$ have all been shown to have slope $\mu$, applying Lemma 3.2.6 to the two short exact sequences shown above, we see that they are all semistable.

Suppose now that $\mu > \nu$, $X \in \text{mod}^\mu_kQ$, $Y \in \text{mod}^\nu_kQ$, and $f : X \rightarrow Y$ is a morphism of representations of $Q$. We have a short exact sequence:

$$0 \longrightarrow \ker f \longrightarrow X \overset{f}{\longrightarrow} \text{im} f \longrightarrow 0$$

Assume first that $\ker f$ and $\text{im} f$ are non-zero. By semistability of $Y$, $\mu(\text{im} f) \leq \nu < \mu = \mu(X)$, hence by Lemma 3.2.5, $\mu(X) < \mu(\ker f)$. However, by semistability of $X$, we have $\mu(\ker f) \leq \mu(X)$. There’s a contradiction, so either $\ker f = 0$ or
$\text{im } f = 0$. However, $f$ cannot be injective, since that would mean that $X \cong \text{im } f$ which is impossible since they have different slopes. We conclude that $\text{im } f = 0$ hence $\text{Hom}_Q(\text{mod}_f^0 Q, \text{mod}_f^0 Q) = 0$. 

We end this section by giving several remarks and examples of the concepts explained above.

**Example 3.2.8.**

1. It is clear from the definition that any stable representation is also semistable. From Lemma 3.2.7, we see that if $X$ is decomposed into the direct sum of indecomposable representations $X_1, \ldots, X_n$ then $\mu(X) = \mu(X_1) = \cdots = \mu(X_n)$. Thus stable representations must be indecomposable, otherwise they would contain a proper non-zero subrepresentation of the same slope.

2. Note that substracting an integer multiple of $\dim$ from the weight $\Theta$ does not affect stability. Similarly, multiplying $\Theta$ by positive integer does not affect stability.

3. If $Q$ is any quiver, then choosing the weight $\Theta = 0$, we see that every representation is semistable, and the stable representations are exactly the simple ones.

4. Let $Q$ be the $A_n$ quiver $i_1 \to i_2 \to \cdots \to i_n$, and consider the weight $\Theta(i_k) = -k$. Let $X$ be an indecomposable representation. Then $X$ must have dimension type $I_p^q = \sum_{s=p}^{q} i_s$ for $p \leq q$. We have $\mu(I_p^q) = -\frac{p+q}{2}$. Any subrepresentation has dimension type $I_s^q$ for $q \geq s \geq p$. Thus every indecomposable representation is stable, and hence (by (1)) the stable representations are precisely the indecomposable ones. It’s easy to see from Lemma 3.2.7 and basic calculation that the semistable representations are direct sums of indecomposables of the same slope (i.e. with dimension vectors $I_{p+s}^{q-s}$ for some fixed $p$ and $q$).\footnotemark

5. Let $Q$ be the 2-Kronecker quiver:

\[
i \quad \longrightarrow \quad j
\]

\footnotetext{In [Rei03, 2, A], it says that the semistables are the powers of the stable representations, but that is incorrect. A simple counterexample is $k \to k^2 \to k$ which is semistable but not a power of an indecomposable. The correct statement is as mentioned in the above example.}
A weight for this quiver has the form $\Theta = ai^* + bj^*$. By (2), the only really distinct cases are then:

a) $\Theta = 0$, hence the slope $\mu$ is constant for all representations, and that’s the same case as in (3).

b) $\Theta = j^*$, then clearly for any $a, b \in \mathbb{N}$, the representations $S(i)^b$ and $S(j)^b$ are semistable. These are in fact the only semistable representations: For if $X$ is semistable with dimension vector $ai + bj$, with $a \neq 0$ and $b \neq 0$, then $X$ has slope $\frac{b}{a+b} < 1$. But the simple representation $S(j)$ would be a subrepresentation of $X$ of slope 1, a contradiction.

c) $\Theta = i^*$, this is then the only interesting choice of stability. A straightforward calculation shows that the semistable (resp. stable) representations correspond exactly to pairs $(\alpha, \beta)$ of linear maps $k^a \rightarrow k^b$ such that for any non-zero proper subspace $U \subset k^a$, $\dim (\alpha(k^a) + \beta(k^b)) \geq (\text{resp. } >) \frac{b}{a} \dim U$ ([Rei03, 2, D]). Consider the case with dimension vector $i+j$. Then a representation $X$:

$$
\begin{array}{c}
\text{k} \\
\alpha \downarrow \beta \\
\text{k}
\end{array}
$$

is semistable if and only if $(\alpha, \beta) \neq (0,0)$, and the semistable representations are also the stable representations.

### 3.3 The HN stratification

**Definition 3.3.1.** Let $X$ be a representation of $Q$. A Harder-Narasimhan filtration (HN filtration) of $X$ is a sequence of representations $0 = X_0 \subset X_1 \subset \cdots \subset X_s = X$ such that the quotients $X_k/X_{k-1}$ are semistable for $k = 1 \cdots s$, and $\mu(X_1/X_0) > \mu(X_2/X_1) > \cdots > \mu(X_s/X_{s-1})$.

**Definition 3.3.2.** Let $X \in \text{mod}_k Q$. A subrepresentation $U \subset X$ is called scss (strongly contradicting semistability) if its slope is maximal among the subrepresentations of $X$, and is of maximal dimension among those with this property.

**Lemma 3.3.3.** Let $X \in \text{mod}_k Q$. Then $X$ has a unique scss subrepresentation.

**Proof.** (cf. [Rei08, 4.4]) It’s clear that $X$ has a scss subrepresentation, since it has finitely many subrepresentations. It’s also clear that any scss subrepresentation
of $X$ must be semistable. If $U, V \subset X$ are scss subrepresentations, then by definition they must have the same slope $\mu$. Consider the short exact sequence:

$$0 \rightarrow U \cap V \rightarrow U \oplus V \rightarrow U + V \rightarrow 0.$$  

Then $\mu(U \cap V) \leq \mu(U \oplus V) = \mu$ (cf. Lemma 3.2.7), and hence by Lemma 3.2.5, $\mu \leq \mu(U + V)$. Since $\mu$ is maximal by our definition of scss, we have $\mu(U + V) = \mu$. By maximality of the dimension of $U$ and $V$, we have $\dim(U + V) \leq \dim U, \dim V$ and hence $U = V$. \hfill \square

**Proposition 3.3.4.** Any $X \in \text{mod}_k \mathbb{Q}$ has a unique HN filtration.

**Proof.** (cf. [Rei08, 4.7]) We proceed by induction on $\dim(\dim(X))$. The base case is clear. Let $X_1 \subset X$ be a subrepresentation with maximal dimension among subrepresentations with maximal slope, i.e. a scss subrepresentation. Thus by Lemma 3.3.3, $X_1$ has a unique HN filtration.

$$0 = Y_0 \subset Y_1 \subset \cdots \subset Y_{s-1} = X/X_1$$

by induction, which we can lift to one of $X$ via the projection $\pi : X \rightarrow X/X_1$. Let $X_i = \pi^{-1}(Y_{i-1})$, with $X_0 = 0$. Now $X_1/X_0$ is semistable by definition, and $X_{i+1}/X_i \cong Y_i/Y_{i-1}$ is semistable by the choice of the $Y_i$’s. We also have $\mu(X_2/X_1) > \mu(X_3/X_2) > \cdots > \mu(X_s/X_{s-1})$.

Since $X_2$ is a subrepresentation of $X$ of strictly larger dimension than $X_1$, we have $\mu(X_1) > \mu(X_2)$, and thus $\mu(X_1/X_0) = \mu(X_1) > \mu(X_2/X_1)$.

To prove uniqueness, suppose that $0 = X_0' \subset \cdots \subset X_t' = X$, let $t$ be minimal such that $X_1 \subset X_t'$. Thus we have a non-zero map $f : X_1 \rightarrow X_t'/X_{t-1}$. Since $X_1$ is semistable, by Lemma 3.2.7 we have that $\mu(X_1) = \mu(\text{im } f)$. Since $X_t'/X_{t-1}$ is also semistable, it follows that $\mu(X_1) \leq \mu(X_t'/X_{t-1})$. By the choice of $X_1$ (scss), we have $\mu(X_1') \leq \mu(X_1)$. By the defining property of HN filtrations, $\mu(X_t'/X_{t-1}) \leq \mu(X_t'/X_0) = \mu(X_1')$. Thus we have shown that $\mu(X_1') \leq \mu(X_1) \leq \mu(X_t'/X_{t-1}) \leq \mu(X_1')$, and so $\mu(X_1) = \mu(X_1')$ and, by the properties of HN filtrations, $t = 1$, which means $X_1 \subset X_1'$. By the choice of $X_1$, we conclude that $X_1 = X_1'$. By induction, we find that these two filtrations of $X$ are the same. \hfill \square

**Definition 3.3.5.** A dimension vector $d \in \mathbb{Z}^{Q_0}$ is called semistable if there exists a semistable representation with dimension vector $d$. A tuple $d^* = (d^1, \cdots, d^s)$ is called a HN type if for each $k$, $d^k$ is semistable, and $\mu(d^1) > \cdots > \mu(d^s)$. The weight of a HN type $d^*$ is $|d^*| = \sum_{k=1}^s d^k \in \mathbb{Z}^{Q_0}$. The length of $d^*$ is $l(d^*) = s$. 29
**Definition 3.3.6.** For a HN type $d^*$, we define the HN stratum $R_{d^*}^{HN} \subset R_d$ to be the subset of representations with HN filtration of type $d^*$. We let $R_d^{d^*}$ be the set of representations in $R_d$ with some filtration of type $d^*$ (i.e. $\dim(X_k/X_{k-1}) = d^k$).

The main use of the Harder-Narasimhan filtration in our context can be seen from the fact that it allows us to construct a stratification of the representation space $R_d$, that is, a decomposition of $R_d$ into nice simpler pieces, as we see in the following proposition.

**Proposition 3.3.7.** The HN strata for HN type $d^*$ of weight $d$ partition $R_d$ into irreducible, disjoint, locally closed subvarieties. The codimension of $R_{d^*}^{HN}$ in $R_d$ is $-\sum_{1 \leq k \leq s} \langle d^k, d^l \rangle$.

**Proof.** The proof is given in [Rei03, 3.4].

### 3.4 Quiver moduli

In this section we describe the construction of the moduli space of quiver representations. Our first guess for the moduli space is the orbit space obtained as the quotient of the quiver variety $R_d$ by the action of the group $G_d$. We want this space to inherit a natural geometric structure.

The quotient defined by classical invariant theory is given by:

$$R_d/G_d = \text{Spec}(k[R_d]^{G_d}).$$

However as we have seen in Corollary 3.1.7, when $Q$ has no oriented cycles, there is a unique closed orbit, which is $\{0\}$, and it’s in the closure of every orbit. Thus the ring of invariants $k[R_d]^{G_d}$ reduces to scalars, for if $f \in k[R_d]^{G_d}$, then $f$ is constant on orbits, and thus by continuity must be equal to $f(0)$ everywhere. Thus in this case this quotient is trivial. We will describe King’s construction ([Kin94]) for using GIT to obtain more interesting quotients.

First we need a couple of definitions:

**Definition 3.4.1.** Let $\chi : G_d \to \mathbb{C}^*$ be a linear character. A function $f \in k[R_d]$ is said to be semi-invariant of weight $\chi$ if:

$$f(g \cdot x) = \chi(g)f(x)$$

for all $x \in R_d, g \in G_d$. The set of all semi-invariant functions on $R_d$ of weight $\chi$ is denoted by $k[R_d]^{G_d, \chi}$. 

30
Remark 3.4.2. When $\chi$ is the trivial character, then the semi-invariant functions of weight $\chi$ are precisely those that are invariant under the action of $G_d$.

We have the following result:

**Theorem 3.4.3.** [Kin94] Let $k$ be an algebraically closed field. For a dimension vector $d \in \mathbb{Z}^{Q_0}$, we denote by $R_d^{ss} \subset R_d$ the subset consisting of the semistable representations, and by $R_d^s \subset R_d$ the subset consisting of the stable representations. Then $R_d^{ss}$ is a Zariski open subvariety, and admits a categorical GIT quotient $\mathcal{M}_d^{ss} = R_d^{ss} \sslash G_d$, which is a projective variety. The quotient $\mathcal{M}_d^{ss}$ contains an open subvariety $\mathcal{M}_d^s$, which is a geometric quotient by $G_d$ of $R_d^s \subset R_d^{ss}$.

**Proof.** We will very briefly sketch King’s argument ([Kin94]) showing that $R_d^{ss}$ is an open subvariety, and describe the construction of $\mathcal{M}_d^{ss}$.

Recall King’s definition of $\sigma$-semistability, given in Definition 3.2.3. For such a $\sigma$, define a character:

$$\chi : G_d \to \mathbb{C}^*$$

$$\chi(g) = \prod_{i \in Q_0} \det(g_i)^{\sigma(i)}$$

We have the following definition:

**Definition 3.4.4.** A point $x \in R_d$ is said to be $\chi$-semistable if there exists $n \geq 1$ and $f \in k[R_d]^G \cdot x^n$ such that $f(x) \neq 0$.

It’s clear then that the set of $\chi$-semistable points is Zariski-open. King then shows that a point is $\sigma$-semistable if and only if it is $\chi$-semistable, proving that $R_d^{ss}$ is open. The moduli space is thus given by:

$$\mathcal{M}_d^{ss} = \text{Proj} \left( \bigoplus_{k \in \mathbb{N}} k[R_d]^G \cdot x^n \right).$$

**Remark 3.4.5.**

1. By the construction above, $\mathcal{M}_d^{ss}$ is projective over $\text{Spec}(k[R_d]^G)$. When $Q$ has no oriented cycles, $k[R_d]^G = k$ and thus $\mathcal{M}_d^{ss}$ is a projective variety.

2. $\mathcal{M}_d^s$ is a smooth variety (cf. [Rei08, 3.5]).
**Example 3.4.6.** Consider the representations of the 2-Kronecker quiver \( i \rightarrow j \) of dimension type \( d = (1, 1) \):

\[
\begin{array}{ccc}
  & \alpha & \\
\cdots & \rightarrow & \cdots \\
  k & \beta & k
\end{array}
\]

and let \( \Theta = i^* \). Then by Lemma 3.2.4, we may choose \( \sigma = -i^* + j^* \). Thus the character is given by:

\[
\chi(\lambda, \mu) = \mu \lambda^{-1}.
\]

Now \( G_d \) acts on \( k[R_d] = \mathbb{A}^2 \) as follows:

\[
(\lambda, \mu) \cdot f(x, y) = f\left(\frac{\mu}{\lambda} x, \frac{\mu}{\lambda} y\right)
\]

and so it is clear that \( k[R_d]^{G_d, \chi} \) consists of precisely the homogeneous polynomials of degree \( n \). Thus,

\[
\mathcal{M}_d^{\chi^\circ} = \text{Proj}(k[x, y]) = \mathbb{P}^1
\]

which agrees with our previous results concerning this example.
Chapter 4

The HN Recursion

4.1 Finitary categories

We’ll briefly describe finitary categories as in [Sch06].

Definition 4.1.1. A small abelian category $\mathcal{A}$ is called finitary if the following conditions hold:

1. $|\text{Hom}(M, N)| < \infty$
2. $|\text{Ext}^1(M, N)| < \infty$

for every two objects $M$ and $N$ in $\mathcal{A}$.

Example 4.1.2. Let $k = \mathbb{F}_q$ be a finite field, $Q$ a quiver, and consider the category $\text{mod}_k Q$ of finite dimensional $k$-representations of $Q$. Then this is a finitary category, since any object in it is finite as a set, and therefore there are only finitely many set maps between any two objects, hence finitely many morphisms and short exact sequences.

Definition 4.1.3. Let $\mathcal{A}$ be a small abelian category. On the free abelian group $F$ with basis the isomorphism classes of objects in $\mathcal{A}$, let $R$ be the relation: $[X] = [Y] + [Z]$ whenever there exists a short exact sequence $0 \to Y \to X \to Z \to 0$. The Grothendieck group of $\mathcal{A}$ is given by $K(\mathcal{A}) = F/R$.

Lemma 4.1.4. For a field $k$ and a quiver $Q$ with no oriented cycles, $K(\text{mod}_k Q) = \mathbb{Z}^{Q_0}$. The class of a representation is simply its dimension type.
Proof. Recall that by Lemma 2.1.7, the simple representations \( Q \) are precisely those of the form \( S(i) \) for \( i \in \mathbb{Z}Q_0 \). Thus every representation \( X \) of \( Q \) admits a composition series with each of the factors isomorphic to \( S(i) \) for some \( i \in \mathbb{Z}Q_0 \). Thus the class of a representation \( X \) in \( K(\text{mod}_k Q) \) can be identified with \( \dim(X) \).

If we have a short exact sequence \( 0 \to X \to Y \to Z \to 0 \) of representations of \( Q \), then \( \dim(Y) = \dim(X) + \dim(Z) \). Therefore, \( K(\text{mod}_k Q) \cong \mathbb{Z}Q_0 \) as required.  

Definition 4.1.5. Let \( \mathcal{A} \) be a finitary \( k \)-linear abelian category (i.e. \( \text{Hom}(M, N) \) is a \( k \)-vector space for every \( M \) and \( N \)), with finite global dimension. Let:

\[
\langle M, N \rangle_m = \left( \prod_{i=0}^{\infty} |\text{Ext}^i(M, N)|(-1)^i \right)^{1/2} \\
(M, N)_m = (M, N)_m \cdot (N, M)_m \\
\langle M, N \rangle_a = \sum_{i=0}^{\infty} (-1)^i \dim_k \text{Ext}^i(M, N) \\
(M, N)_a = \langle M, N \rangle_a + \langle N, M \rangle_a
\]

be respectively the multiplicative Euler form, the symmetric multiplicative Euler form, the additive Euler form, and the symmetric additive Euler form.

Remark 4.1.6. The Euler forms depend only on the classes in the Grothendieck group of the input objects [Sch06, §1.2]. Let \( Q \) be a quiver with no oriented cycles. By Corollary 2.3.3, part (3), together with Lemma 4.1.4, and the fact just stated in this remark, we see that the additive Euler form for \( Q \) defined in this chapter agrees with the Euler form defined on the category of quiver representations in section 2.3. From this it follows that for \( i, j \in Q_0 \), \( \dim \text{Ext}^1(S(i), S(j)) = c_{ij} \) which is the number of arrows from \( i \) to \( j \). Thus if \( k = \mathbb{F}_q \), \( \text{Ext}^1(S(i), S(j)) \cong k^{c_{ij}} \).

We also have \( \text{Hom}(S(i), S(j)) \cong k^{δ_{ij}} \) and so we have for the multiplicative Euler form that \( \langle S(i), S(j) \rangle_m = q^{1/2(δ_{ij} - c_{ij})} \). Since the Euler forms depend only on the dimension types, we have that \( \langle M, N \rangle_m = q^{1/2(\dim M \cdot \dim N)} \).

4.2 Hall algebras

We define Hall algebras as in [Sch06].

Let \( \mathcal{A} \) be a finitary category with finite global dimension, and let \( k \) be a field.
Let \( \mathcal{H}_A \) be the \( \mathbb{C} \)-vector space with basis the isomorphism classes of objects in \( A \). We define a product on \( \mathcal{H}_A \):

\[
[M] \cdot [N] = \langle M, N \rangle \sum_{R \in \text{Ob}(A)} \frac{1}{a_M a_N} P_{M,N}^R
\]

where \( a_M = |\text{Aut}(M)| \), and \( P_{M,N}^R \) is the number of short exact sequences \( 0 \to N \to R \to M \to 0 \) (called the Hall numbers).

**Proposition 4.2.1.** The above product turns \( \mathcal{H}_A \) into an associative algebra.

**Proof.** The proof can be found in detail in [Sch06, Proposition 1.1]. The main idea is to reinterpret \( \mathcal{H}_A \) as the set of arbitrary \( \mathbb{C} \)-valued functions on the set isomorphism classes of objects in \( A \). The class \([M]\) is then identified with the map \( 1_{[M]} = \delta_{[M],[N]} \) that maps \([M]\) to 1 and every other class to 0. The product is reinterpreted as:

\[
(f \cdot g)(X) = \sum_{Q \subset R} \langle R/Q, Q \rangle f(R/Q)g(Q).
\]

This idea will be useful when we consider Hall algebras of quivers. \(\square\)

The following lemmas will be useful later.

**Lemma 4.2.2.** (cf. [Sch06, Lemma 1.2]) For any three objects \( M, N, R \) of \( A \) we have:

\[
\frac{1}{a_M a_N} P_{M,N}^R = |\{ L \subset R : L \cong N \text{ and } R/L \cong M \}|.
\]

**Proof.** The group \( \text{Aut}(M) \times \text{Aut}(N) \) acts on the set \( \mathcal{P}_{M,N}^R \) of short exact sequences \( 0 \to N \to R \to M \to 0 \) by:

\[
\begin{array}{ccccccccc}
0 & \to & N & \stackrel{f}{\to} & R & \stackrel{g}{\to} & M & \to & 0 \\
\wedge & & \phi & & \eta & & \wedge & & \\
0 & \to & N & \stackrel{f \phi^{-1}}{\to} & R & \stackrel{\eta g}{\to} & M & \to & 0
\end{array}
\]

That is, \((\phi, \eta) \cdot (f, g) = (f \phi^{-1}, \eta g)\). It is easy to check that this is in fact a group action. In fact, since \( f \) is injective and \( g \) is surjective, this action is free. The quotient \( \mathcal{P}_{M,N}^R / (\text{Aut}(M) \times \text{Aut}(N)) \) can be identified with the set \( \{ L \subset R : L \cong N \text{ and } R/L \cong M \} \). Thus by Burnside’s lemma:

\[
\frac{\left| \mathcal{P}_{M,N}^R \right|}{|\text{Aut}(M) \times \text{Aut}(N)|} = \frac{1}{a_M a_N} P_{M,N}^R
\]

as required. \(\square\)
Lemma 4.2.3. (cf. [Sch06, Proposition 1.5]) Assume that $gldim(\mathcal{A}) \leq 1$. For any fixed objects $M, N, R$,

$$
\frac{1}{|\text{Ext}^1(M, N)|} \{ \xi \in \text{Ext}^1(M, N) : \text{X}_\xi \cong R \} = \langle M, N \rangle_1^2 \frac{1}{a_R} \mathcal{P}_{M, N}^R
$$

where $X_\xi$ is the middle term of the extension of $M$ by $N$ which is associated to $\xi$.

Proof. The group $\text{Aut}(R)$ acts on the set $\mathcal{P}_{M, N}^R$ of short exact sequences $0 \rightarrow N \rightarrow R \rightarrow M \rightarrow 0$ as shown in the following diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & N & \xrightarrow{f} & R & \xrightarrow{g} & M & \rightarrow & 0 \\
0 & \rightarrow & N & \xrightarrow{\phi f} & R & \xrightarrow{g \phi^{-1}} & M & \rightarrow & 0 \\
\end{array}
$$

We claim that there is a one-to-one correspondence:

$$
\{ \phi \in \text{Aut}(R) : \phi \cdot (f, g) = (f, g) \} \longleftrightarrow \{ \eta \in \text{Hom}(R, \text{im } f) : \eta|_{\text{im } f} = 0 \}.
$$

Suppose $\phi \in \text{Aut}(R)$ such that $\phi \cdot (f, g) = (f, g)$. Then clearly $\phi|_{\text{im } f}$ is the identity $id|_{\text{im } f}$. Thus $\phi$ induces a map $\phi' : R/\text{im } f \rightarrow R/\text{im } f$ given by $\phi'(r + \text{im } f) = \phi(r) + \text{im } f$ for all $r \in R$. Note that $\text{im } f = \ker g$. By the first isomorphism theorem, $g$ naturally induces an isomorphism $\tilde{g} : R/\text{im } f \rightarrow M$.

Since $\phi \cdot (f, g) = (f, g)$ and $\phi$ is an isomorphism, we have $g \phi = g$. Thus $\tilde{g} \phi' = \tilde{g}$, and since $\tilde{g}$ is an isomorphism, it follows that $\phi' = id|_{R/\text{im } f}$.

Let $\eta = \phi - id_R$, where $id_R$ is the identity on $R$. Then $\eta$ is uniquely determined by $\phi$. For all $r \in \text{im } f$, $\eta(r) = \phi(r) - r = r - r = 0$. Thus $\eta|_{\text{im } f} = 0$, and $\eta$ also descends to a map $\eta' : R/\text{im } f \rightarrow R/\text{im } f$, given by $\eta'(r) = \eta(r) + \text{im } f$ for all $r \in R$. From what we have shown about $\phi'$, it follows that $\eta' = 0$. Thus for all $r \in R$, we have $\eta(r) \in \text{im } f$. This shows that $\eta \in \text{Hom}(R, \text{im } f)$.

Conversely, suppose that $\eta \in \text{Hom}(R, \text{im } f)$ such that $\eta|_{\text{im } f} = 0$. Let $\phi = id_R + \eta$. Then $\phi$ is uniquely determined by $\eta$. If $\phi(r) = 0$, then $r = -\eta(r) \in \text{im } f$, hence $0 = \phi(r) = r + 0$ and so $\phi$ is injective. Moreover, note that $\eta^2 = 0$, and so for $r \in R$, $\phi(r - \eta(r)) = r - \eta(r) + \eta(r - \eta(r)) = r$, showing that $\phi$ is surjective, hence $\phi \in \text{Aut}(R)$. For all $r \in R$, $(\phi f)(r) = f(r) + \eta(f(r)) = f(r)$, and $(g \phi)(r) = g(r + \eta(r)) = g(r) + g(\eta(r)) = g(r)$ since $\text{im } f = \ker g$, and so $\phi \cdot (f, g) = (f, g)$.
Since the map $\eta$ in the above steps can be identified with an element in $\text{Hom}(R/\text{im } f, \text{im } f) \cong \text{Hom}(M, N)$, we conclude that $|\text{Stab}_{(f,g)}| = |\text{Hom}(M, N)|$.

Therefore by Burnside’s lemma:

$$|\{ \xi \in \text{Ext}^1(M, N) : X_\xi \cong R \}| = \frac{|\text{Hom}(M, N)|}{a_R} P_{M,N}.$$

This, along with the fact that $|\text{Hom}(M, N)|/|\text{Ext}^1(M, N)| = \langle M, N \rangle_2^m$ (since $\text{gldim}(A) \leq 1$), proves the lemma.

\textbf{Remark 4.2.4.}

1. $H_A$ is naturally graded by the Grothendieck group:

$$H_A = \bigoplus_{\alpha \in K(A)} H_A(\alpha),$$

where:

$$H_A(\alpha) = \bigoplus_{M = \alpha} \mathbb{C}[M]$$

and $\overline{M}$ denotes the class of $M$ in $K(A)$.

2. The product in the Hall algebra encapsulates information on the different ways that two objects can be put together to construct extensions. This is reflected by the Hall numbers.

3. Under certain finitary conditions (which are satisfied for example by the category of representations of a quiver over a finite field), it is possible to define a dual of the product, which intuitively splits one object into two in all possible ways. This operation is called a \textit{coproduct} (cf. Green’s coproduct, [Sch06]).

4. The product and coproduct can have compatibility conditions that turn the algebra into a \textit{Hopf algebra}.

\textbf{4.3 The Harder-Narasimhan recursion}

In this section, we consider a finite field $k = \mathbb{F}_q$, and choose $v \in \mathbb{C}$ such that $v^2 = q$. Let $Q$ be a quiver with no oriented cycles, and consider the quiver variety $R_d$ of representations of $Q$ over $k$ of dimension type $d \in \mathbb{Z}^{Q_0}$, and let $G_d$ be the corresponding base change group. In [Rei03, 4.1], Reineke defines the Hall algebra of $Q$ as follows.
**Definition 4.3.1.** For a dimension type \( d \in \mathbb{Z}^{Q_0} \), let \( \mathcal{H}_d \) be the set of arbitrary \( \mathbb{C} \)-valued function on \( R_d \) which are \( G_d \)-invariant (that is, they can be defined on the isomorphism classes of \( \text{mod}_k Q \)). Let \( \mathcal{H} = \bigoplus_{d \in \mathbb{Z}^{Q_0}} \mathcal{H}_d \) be the \( \mathbb{C} \)-vector space graded by \( \mathbb{Z}^{Q_0} \) and with multiplication:

\[
(f \ast g)(X) = v^{\langle d, e \rangle} \sum_{U \subset X} f(U)g(X/U)
\]

for \( X \in R_{d+e} \) where \( f \in \mathcal{H}_d \) and \( g \in \mathcal{H}_e \).

With the considerations of the previous section (Remark 4.1.6, Proposition 4.2.1), we see that Reineke’s definition of the Hall algebra of a quiver coincides with that in [Sch06].

The definition immediately suggests a link between iterating the product of the Hall algebra and filtrations, given by the following lemma.

**Lemma 4.3.2.** (cf. [Rei03, 4.2]) For \( f_k \in \mathcal{H}_{d_k}, k = 1, \cdots, s \), and \( X \in R_{d_1 + \cdots + d_s} \), we have:

\[
(f_1 \ast \cdots \ast f_s)(X) = v^{\langle d^* \rangle} \sum_{0 = X_0 \subset \cdots \subset X_s = X} f_1(X_1/X_0) \cdots f_s(X_s/X_{s-1})
\]

where:

\[
\langle d^* \rangle = \sum_{k<l} \langle d_k, d_l \rangle
\]

To establish a link between the HN filtration and the Hall algebra, we make the following definition.

**Definition 4.3.3.** For the dimension type \( d \in \mathbb{Z}^{Q_0} \), define:

\[
\chi_d := \chi_{R_d} \in \mathcal{H}_d,
\]

the characteristic function of the variety \( R_d \). For the HN type \( d^* = (d_1, \cdots, d_s) \), of weight \( d \), let:

\[
\chi_{d^*}^{HN} := \chi_{R_{d^*}^{HN}} \in \mathcal{H}_d
\]

be the characteristic function of the HN stratum for \( d^* \). In particular, let \( \chi_d^{ss} := \chi_{(d)}^{HN} \). Thus \( \chi_d^{ss} \) is the characteristic function of \( R_d^{ss} \).

The HN stratification (Proposition 3.3.7) is then reflected in the Hall algebra by the following lemma.

---

*Note however that the order inside the sum in the definition of the product is the opposite to Schiffmann’s.*
Lemma 4.3.4. For HN type \( d^* = (d^1, \ldots, d^s) \) we have:
\[
\chi_{d^*}^{HN} = v^{-(d^*)} \chi_{d^1}^{ss} \cdots \chi_{d^s}^{ss}
\]

Proof. We use Lemma 4.3.2 to evaluate the product \( \chi_{d^1}^{ss} \cdots \chi_{d^s}^{ss} \). The only non-zero terms involved in the sum correspond by definition to a HN filtration, but there exists one and only one such filtration by Proposition 3.3.4. Thus
\[
\chi_{d^1}^{ss} \cdots \chi_{d^s}^{ss} = v^{(d^*)} \chi_{d^*}^{HN}.
\]

We thus obtain the following HN recursion.

Proposition 4.3.5. For all \( d \in \mathbb{Z}^{Q_0} \),
\[
\chi_d^{ss} = \chi_d - \sum_{d^*} v^{-(d^*)} \chi_{d^1}^{ss} \cdots \chi_{d^s}^{ss}
\]
where the sum is over HN types \( d^* \neq (d) \) of weight \( d \). We have \( \chi_d^{ss} = \chi_d \) if and only if \( \Theta \) is constant on \( \text{supp} \ d := \{ i \in Q_0 : d_i \neq 0 \} \).

Proof. If \( \Theta \) is constant on \( \text{supp} \ d \) then any representation of dimension type \( d \) is semistable (cf. Example 3.2.8). In this case, \( \chi_d = \chi_d^{ss} \). Otherwise, by the HN stratification (Proposition 3.3.7), we have a partition of \( R_d \) into a disjoint union:
\[
R_d = \bigsqcup_{d^*} R_{d^*}^{HN}
\]
over all HN types \( d^* \) with weight \( d \). In terms of characteristic functions, the HN stratification can be written as:
\[
\chi_d = \sum_{d^*} \chi_{d^*}^{HN}
\]
from which the HN recursion immediately follows.

Suppose that \( \Theta \) is not constant on \( \text{supp} \ d \), but that \( \chi_d = \chi_d^{ss} \). Then there exist \( i, j \in Q_0 \) such that \( \Theta(i) > \Theta(j) \), so that \( \mu(S(i)) > \mu(S(j)) \); and from the HN recursion, we must have \( R_{d^*}^{HN} = \emptyset \) for all HN types \( d^* \neq (d) \) with weight \( d \). Consider the representation \( S(i) \oplus S(j) \). Then this representation has HN filtration with HN type \( (e_i, e_j) \neq (d) \), where \( e_i = \dim(S(i)) \) and \( e_j = \dim(S(j)) \), which yields a contradiction. Thus \( \chi_d = \chi_d^{ss} \) if and only if \( \Theta \) is constant on \( \text{supp} \ d \).

Remark 4.3.6. Define \( C \) to be the \( \mathbb{C} \)-algebra generated by \( \chi_i \) for \( i \in Q_0 \) (i.e. corresponding to the simple representations). Then \( C \) is called the composition
algebra. Using the fact that $Q$ has no oriented cycles, it can be shown ([Rei03, 4.4]) that $\chi_d \in C$ for all dimension types $d \in \mathbb{Z}^Q_0$. By a theorem of Green, $C$ is isomorphic to the quantized enveloping algebra $U_q(n^+)$ of the positive part of the symmetric Kac-Moody algebra corresponding to the quiver $Q$, specialized at $v$.

**Example 4.3.7.** As an example, consider the $A_2$ quiver:

$$
\begin{array}{c}
i \\
\longrightarrow \\
j
\end{array}
$$

with $\Theta = -i^* - 2j^*$. Then $\chi_{(1,0)}^{ss} = \chi_{(1,0)}$, and $\chi_{(0,1)}^{ss} = \chi_{(0,1)}$. For dimension type $d = (1,1)$, $\mu(d) = -\frac{3}{2}$, and we only have two tuples $\neq (d)$ of weight $d$, which are $[(1,0),(0,1)]$ and $[(0,1),(1,0)]$. We quickly see that only the first one satisfies the conditions of an HN type. Now for $d^1 = (1,0)$ and $d^2 = (0,1)$ we have $v(d^1) = v(d^2,d^1) = 1$, and for all $X \in R_{(1,1)}$:

$$(\chi_{d^1}^{ss} \ast \chi_{d^2}^{ss})(X) = \sum_{X_1 \subset X} \chi_{d^1}(X_1) \chi_{d^2}(X/X_1) = \begin{cases} 
1 & \text{if } \exists X_1 \subset X : \dim(X_1) = (1,0) \\
0 & \text{otherwise}
\end{cases}$$

Thus by the HN recursion, we find that for a representation $X$ of dimension type $d = (1,1)$:

$$
\chi_d^{ss}(X) = \begin{cases} 
1 & \text{if } \exists X = k \xrightarrow{\alpha \neq 0} k \\
0 & \text{if } X = k \xrightarrow{0} k
\end{cases}
$$

which agrees with results from previous examples (cf. Example 3.2.8, 4).
Chapter 5

Cohomology of Quiver Moduli

5.1 Resolving the HN recursion

In this we discuss the resolution of the HN recursion, and obtain an explicit form for the solution. To do this we need to discuss some combinatorial aspects of the HN strata.

Definition 5.1.1. Let \( d^* = (d^1, \cdots, d^s) \) be a tuple of dimension types in \( \mathbb{N}^{Q_0} \).

1. The polygon in \( \mathbb{N}^2 \) associated to \( d^* \) consists of vertices

\[
\left( \sum_{l=1}^{k} \dim d^l, \sum_{l=1}^{k} \Theta(d^l) \right)
\]

for \( k = 1, \ldots, s \), connected in increasing order, starting with the point \( (0,0) \) and return to it (check the picture).
This polygon is denoted $P(d^*)$. We say that $d^*$ is convex if $P(d^*)$ is convex. We define a partial order on the set of tuples: for $d^*, e^*$, we say that $d^* \leq e^*$ (resp $< e^*$) if $P(d^*)$ lies on or below (resp strictly below) $P(e^*)$. Note that the consecutive slopes of the edges of the polygon obtained are $\mu(d^1), \ldots, \mu(d^n)$. Thus HN types are convex by definition.

2. For a subset $I = \{s_1, s_2, \cdots, s_k\} \subset \{1, 2, \cdots, s-1\}$, the $I$-coarsening of $d^*$ is defined as:

$$c_I^*(d^*) = (d^1 + \cdots + d^{s_1}, d^{s_1+1} + \cdots + d^{s_2}, \cdots, d^{s_k+1} + \cdots + d^s).$$

Pictorially, $P(c_I^*(d^*))$ is obtained from $P(d^*)$, by considering only the vertices corresponding to elements of $I$, disregarding the rest. In the following picture, the dashed lines correspond to the coarsening:

3. The set $I$ is called $d^*$-admissible if:

a) $c_I^*(d^*)$ is convex,

b) for all $i = 0, \cdots, k$, we have $(d^{s_i+1}, \cdots, d^{s_{i+1}}) \geq (d^{s_{i+1}+1} + \cdots + d^{s_{i+1}}) = c_I^*(d^*)$. That is, $P(c_I^*(d^*)) \leq P(d^*)$.

We denote by $A(d^*)$ the set of all $d^*$-admissible subsets of $\{1, \cdots, s-1\}$.

We need the following definition from combinatorics.

**Definition 5.1.2.** A simplicial complex is a non-empty family $K$ of finite sets which is closed under taking subsets, i.e. for any $A \in K$ and $B \subset A$, we have $B \in K$.

*The precise name here is abstract simplicial complex, while the term simplicial complex is reserved for the geometric gluing of polyhedra. There is a relation, however, between the two concepts: to any abstract simplicial complex $K$ we can associate a geometric simplicial complex $|K|$ whose set of vertices is (or can be represented by) $K$. We will not need to distinguish the two concepts.*

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Lemma 5.1.3. For any tuple $d^*$, the set $\mathcal{A}(d^*)$ is a simplicial complex.

Proof. Let $I \in \mathcal{A}(d^*)$, and $J \subset I$. Then immediately from the definition we find that $c_I^\star(d^*) = c_J^\star(c_I^\prime(d^*))$. Since $I$ is $d^*$-admissible, $c_I^\prime(d^*)$ is convex, therefore $c_J^\star(d^*)$ is also convex. Also, $c_J^\star(d^*)$ lies on or below $(c_I^\star(d^*))$ which in turns lies on or below $d^*$. Thus $J$ is also $d^*$-admissible.

Lemma 5.1.4. For any tuple $d^*$ of weight $d$ such that $d^* = (d)$ or $d^* > (d)$, we have:

$$\sum_{I \in \mathcal{A}(d^*)} (-1)^{|I|} = \begin{cases} 0 & \text{if } d^* \neq (d), \\ 1 & \text{if } d^* = (d) \end{cases}$$

Proof. We proceed by induction on $s = l(d^*)$. If $s = 1$, there is nothing to prove. Otherwise let $d^1$ and $d^2$ be the first two entries in $d^*$. If $\mu(d^1) < \mu(d^2)$, let $I_0 \subset \{1, \ldots, s - 1\}$ be the subset of all $k$ such that $\mu(d^1 + \cdots + d^k) \leq \mu(d^1)$. Then it follows from the definition (and it can be easily seen pictorially) that a coarsening of $d^*$ is admissible if and only if it is a coarsening of $c_{I_0}^\star(d^*)$. Therefore $\mathcal{A}(d^*) = \mathcal{A}(c_{I_0}^\star(d^*))$. But $2 \notin I_0$, hence $l(c_{I_0}^\star(d^*)) < l(d^*)$ and the result holds by induction.

Suppose that $\mu(d^1) \geq \mu(d^2)$. Let $d^* = (d^2, \ldots, d^s)$ and $d''^* = (d^1 + d^2, \ldots, d^s)$. Let $I \in \mathcal{A}(d^*)$. If $I$ is not also in $\mathcal{A}(d''^*)$, then $I$ must be of the form $\{1, s_1 + 1, \ldots, s_k + 1\}$ for some $\{s_1, \ldots, s_k\} \in \mathcal{A}(d^*)$. Thus we have a disjoint union:

$$\mathcal{A}(d^*) = \mathcal{A}(d''^*) \cup (\{1\} \cup (\mathcal{A}(d''^*) + 1))$$

which translates into the equation:

$$\sum_{I \in \mathcal{A}(d^*)} (-1)^{|I|} = \sum_{I \in \mathcal{A}(d''^*)} (-1)^{|I|} + \sum_{I \in \mathcal{A}(d''^*)} (-1)^{|I|+1}$$

which equals zero by induction.

We now come to the resolution of the HN recursion. We have the following result:

Theorem 5.1.5. For all dimension types $d \in \mathbb{Z}^{Q_0}$,

$$\chi_d^s = \sum_{d^*} (-1)^{s-1} v^{-(d^*)} \chi_{d^1} \ast \cdots \ast \chi_{d^s}$$

where the sum runs over all tuples of non-zero dimension types $d^* = (d^1, \ldots, d^s)$ of weight $d$ such that $d^* = (d)$ or $d^* > (d)$, i.e. $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1 \cdots s - 1$. 43
Proof. If $\Theta$ is constant on $\text{supp } d$, then by the HN recursion (Proposition 4.3.5), $\chi_d^{ss} = \chi_d$. Also, for any dimension type $e \leq d$, we have $\mu(e) = \mu(d)$. Thus the sum in the theorem sums over the single tuple $(d)$, and there is nothing further to show.

Otherwise, consider the HN recursion:

$$
\chi_d = \sum_{d^*} v^{-(d^*)} \chi_{d^1} * \cdots * \chi_{d^s}.
$$

If we replace each $\chi_d^{ss}$ by the claimed formula, we obtain the expression:

$$
\sum_{d^*} \sum_{d^1, \ldots, d^s, t} v^{-(d^*)} (-1)^{\sum_{i=1}^s (t_i-1)} v^{\sum_{i=1}^s (d^*)} \chi_{d^1,1} * \cdots * \chi_{d^s,1} * \cdots * \chi_{d^s,t} \tag{†}
$$

where the outer sum runs over all HN types $d^*$ of weight $d$, and the inner sums run over all sequences $(d^1, \ldots, d^s, t)$ of tuples $d^i$ of weight $d^i$ and length $t_i$ such that $d^i_\infty = (d^i)$ or $d^i_\infty > d^i$.

To exchange the order of summation, let $e^*$ denote the concatenation $e^* = (d^{1,1}, \ldots, d^{s,t_s})$ of the tuples $d^i$. Then the obtained tuples $e^*$ run over all tuples of weight $d$ such that $e^* = (d)$ or $e^* > (d)$. Thus, each HN type $d^*$ appearing in the outer sum is an admissible coarsening of $e^*$, and so the sum (†) can be written as:

$$
\sum_{e^*} (-1)^{|e^*|-1} v^{-(e^*)} \sum_{d^*} (-1)^{l(d^*)-1} \chi_{e^1} * \cdots * \chi_{e^l(e)} \tag{†}
$$

where $e^*$ runs over all tuples $e^*$ of weight $d$ such that $e^* = (d)$ or $e^* > (d)$, and $d^*$ runs over all admissible coarsenings of $e^*$. By Lemma 5.1.4, this sum reduces to simply $\chi_d$, and the theorem is established.

5.2 The Betti numbers of quiver moduli

We can now put all the results together to obtain an explicit formula for the Poincaré polynomial of the quiver moduli. First, we’ll briefly explain what the Betti numbers are, and outline the strategy of the proof, mainly following [Rei08, §8].

We’re interested in projective varieties $X$ over $\mathbb{C}$, with the topology induced from the natural $\mathbb{C}$-topology on complex projective spaces (that is, induced by the usual cell decomposition of $\mathbb{C}P^n$ into affine cells; in other words, it’s the topology of $X$ as a projective complex manifold). We then consider the singular cohomology
with coefficients in \( \mathbb{Q} \). We denote by \( b_i(X) \), or simply \( b_i = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \), the \( i \)th Betti number of \( X \).

Now \( X \) can be embedded as a closed subset of some projective space \( \mathbb{P}^N(\mathbb{C}) \), and is cut out by certain polynomial equations:

\[
X = \{ P \in \mathbb{P}^N(\mathbb{C}) : f_i(P) = 0 \}.
\]

A fact from algebraic geometry is that \( X \) is defined over a finitely generated subring \( R \) of \( \mathbb{C} \) ([Kir84, §15]). For any non-zero prime \( p \subset R \), \( R/p \) is a finite field \( k \). We can thus “reduce” \( X \) to \( k \), i.e. consider the defining equations of \( X \) modulo \( p \) and obtain a locally closed \( X(k) \subset \mathbb{P}^N(k) \), which is a finite set. We can then study \( |X(k)| \) and how it depends on \( |k| = q \).

We assume that \( Q \) is a quiver without oriented cycles, so that \( \mathcal{M}^ss_d \) is a projective variety for all dimension types \( d \). We assume a further condition on the dimension type \( d \) (coprime, see below) to ensure that \( \mathcal{M}^ss_d \) is smooth (See Remark 3.4.5), and this will enable us to call upon the Weil conjectures. The idea of the proof is to find a formula for \( \mathcal{M}^ss_d(k) \) in terms of \( q \), showing that \( |\mathcal{M}^ss_d(k)| \in \mathbb{Q}(q) \). It then follows from this by application of the Weil conjectures ([Goe94, Remark 1.2.2]) that:

\[
|\mathcal{M}^ss_d(k)| = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{Q}} H^i(\mathcal{M}^ss_d(\mathbb{C}), \mathbb{Q}) v^i = P(v),
\]

the Poincaré polynomial of \( \mathcal{M}^ss_d(\mathbb{C}) \).

**Lemma 5.2.1.** Let \( \mathcal{H} \) be the Hall algebra associated to the quiver \( Q \) over \( k \). The evaluation map \( ev : \mathcal{H} \to \mathbb{C} \) defined by:

\[
ev(f) = \frac{1}{|G_d|} \sum_{X \in R_d} f(X)
\]

for \( f \in \mathcal{H}_d \) satisfies the identity:

\[
ev(f * g) = v^{-(e,d)} ev(f)ev(g)
\]

for \( f \in \mathcal{H}_d, g \in \mathcal{H}_e \).

**Proof.** (cf. [Rei03, 6.1]) Without loss of generality, assume \( f \) and \( g \) are the characteristic functions of the orbits \( O_M \) and \( O_N \), respectively, for some \( M \in R_d \) and \( N \in R_e \). By definition of \( ev \), and using Lemma 3.1.2, (3), we have \( ev(\chi_{O_M}) = |\text{Stab}(M)|^{-1} = |\text{Aut}(M)|^{-1} \); and it follows immediately from Definition 4.3.1 that \( \chi_{O_M} \ast \chi_{O_N} = v^{(e,d)} \sum_{[X]} F^X_{N,M} \chi_{O_X} \), where \( F^X_{N,M} \) denotes the number of
subrepresentations of $X$ which are isomorphic to $M$, with quotient isomorphic to $N$. That is:

$$F_{N,M}^X = |\{L \subseteq X : L \cong M \text{ and } X/L \cong N\}|.$$

By Lemma 4.2.2:

$$F_{N,M}^X = \frac{1}{|\text{Aut}(M)||\text{Aut}(N)|} P_{N,M}^X.$$ By Lemma 4.2.3 it then follows that

$$F_{N,M}^X = \frac{|\text{Aut}(X)|}{|\text{Aut}(M)||\text{Aut}(N)|} \frac{1}{|\text{Ext}^1(N,M)_X|} (M,N)_{m}^{-2}$$

$$= \frac{|\text{Aut}(X)|}{|\text{Aut}(M)||\text{Aut}(N)|} \frac{|\text{Ext}^1(N,M)|}{|\text{Ext}^1(N,M)_X|}$$

$$= v^{-2\text{hom}(N,M)} \frac{|\text{Aut}(X)|}{|\text{Aut}(M)||\text{Aut}(N)|} |\text{Ext}^1(N,M)_X|$$

where $\text{hom}(N,M) = \dim \text{Hom}(N,M)$, and $\text{Ext}^1(N,M)_X$ denotes the set of extension classes with middle term isomorphic to $X$. Then it follows by easy calculation:

$$ev(\chi_{\mathcal{O}_M} * \chi_{\mathcal{O}_N}) = \frac{1}{|G_{d+c}|} \sum_X (\chi_{\mathcal{O}_M} * \chi_{\mathcal{O}_N})(X)$$

$$= \frac{1}{|G_{d+c}|} \sum_X v^{(e,d)} \sum_{Y} F_{N,M}^X \chi_{\mathcal{O}_Y}(X)$$

$$= \frac{v^{(e,d)}}{|G_{d+c}|} \sum_{Y} \sum_{X \subseteq Y} F_{N,M}^X$$

$$= \frac{v^{(e,d)}}{|G_{d+c}|} \sum_{Y} \frac{1}{|\text{Aut}(Y)|} F_{N,M}^Y$$

$$= \frac{v^{(e,d)}}{|\text{Aut}(N)||\text{Aut}(M)|} v^{-2\text{hom}(N,M)} |\text{Ext}^1(N,M)_Y|$$

$$= \frac{v^{(e,d)}}{|\text{Aut}(N)||\text{Aut}(M)|} \sum_{Y} |\text{Ext}^1(N,M)_Y|$$

$$= \frac{v^{(e,d)}}{|\text{Aut}(N)||\text{Aut}(M)|} |\text{Ext}^1(N,M)|$$

$$= \frac{v^{(e,d)}}{|\text{Aut}(N)||\text{Aut}(M)|}$$

$$= v^{(e,d)} ev(\chi_{\mathcal{O}_M}) ev(\chi_{\mathcal{O}_N})$$

$\square$

---

1 In [Rei03, 6.1], Reineke cites [Rei04] for this formula. However I have provided a proof based on the cited lemmas from [Sch06], which I found easier.
Corollary 5.2.2. For all dimension types $d \in \mathbb{Z}^{Q_0}$, we have:

$$\frac{|R^ss_d|}{|G_d|} = \sum_{d^*} (-1)^{s-1} v^{-\langle d^* \rangle} \prod_{k=1}^s \frac{|R^s_{dk}|}{|G_{dk}|}$$

where the sum runs over all tuples $d^*$ of weight $d$ such that $\mu(\sum_{l=1}^k d^l) > \mu(d)$ for $k = 1, \cdots, s - 1$.

Proof. Apply the evaluation map $ev$ to both sides of the formula in Theorem 5.1.5, and use Lemma 5.2.1. \qed

Definition 5.2.3. A dimension type $d \in \mathbb{Z}^{Q_0}$ is said to be coprime if the integers $\dim d$ and $\Theta(d)$ are coprime.

Lemma 5.2.4. For coprime dimension $d$, we have $R^s_{ss} = R^s_{d}$, and $\text{End}_{kQ}(X) \cong k$ for all $X \in R^s_{ss}$.

Proof. Let $X \in R^s_{ss}$. If $X$ were not stable, then there would exist a proper subrepresentation $U \subset X$ such that $\Theta(U) = \Theta(X)$, which contradicts coprimality. Consider the extension of scalars $\overline{X} = \overline{k} \otimes_k X$ to the algebraic closure $\overline{k}$ of $k$.

If we represent the components of $\overline{X}$ (which are linear maps) as matrices, then the Frobenius map $a \mapsto a^q$ acts on $\overline{X}$ by acting on these matrices entry-wise. Since the HN filtration of $\overline{X}$ is unique, it is fixed by the Frobenius map, and hence it must descend to a HN filtration of $X$, and so must be trivial since $X$ is semistable. Thus $\overline{X}$ is also semistable, having trivial HN filtration. By the first part of the lemma, $\overline{X}$ is also stable.

Let $Y \in R^s_{ss}$, so that the above also applies to $\overline{Y}$ constructed similarly, and $\mu(\overline{X}) = \mu(\overline{Y}) = \mu$. Let $f \in \text{Hom}_{kQ}(\overline{X}, \overline{Y})$. By Lemma 3.2.7, $\ker f$ and $\text{im} f$ both have slope $\mu$, and therefore by stability of $\overline{X}$ and $\overline{Y}$, either $f = 0$ or $f$ is an isomorphism. Consequently⁴, $\text{End}_{kQ}(\overline{X}) \cong \overline{k}$, and thus $\text{End}_{kQ}(X) \cong k$. \qed

Lemma 5.2.5. For coprime $d$, the action of $PG_d = G_d/k^*$ on $R^s_{ss}$ is free in the sense of Mumford (cf. [MFK94, 0.8, iv]), that is, the map:

$$\Psi : PG_d \times R^s_{ss} \rightarrow R^s_{ss} \times R^s_{ss}$$

$$(g, x) \mapsto (x, g \cdot x)$$

is injective and a closed immersion.

⁴This is a Schur’s Lemma type argument. Since $\overline{k}$ is algebraically closed, endomorphisms will have eigenvalues.
Proof. \( \text{PG}_d \) acts set-theoretically free on \( R^{ss}_d \) by Lemma 5.2.4, so \( \Psi \) is injective. Let \( E_d = \bigoplus_{i \in Q_0} \text{End}_k(k^{d_i}) \supset G_d \). Consider the map \( \Phi : R^{ss}_d \times R^{ss}_d \to \text{Hom}_k(E_d, R^d) \) given by:

\[
\Phi(X, Y)(\bigoplus_{i \in Q_0} \phi_i) = \bigoplus_{Q_0 \ni \rho \rightarrow j} (\phi_j X_\rho - Y_\rho \phi_i).
\]

Note that \( \ker \Phi(X, Y) \) represents precisely \( \text{Hom}_{kQ}(X, Y) \), which by (the proof of) Lemma 5.2.4 is non-trivial if and only if \( X \cong Y \), i.e. \( X = g \cdot Y \) for some \( g \in \text{PG}_d \). Thus the image \( Z \) of \( \Psi \) consists of precisely the pairs \((X, Y)\) for which \( \Phi(X, Y) \) has a fixed rank \( r \). On the open subset of \( Z \) where a fixed \( r \times r \) minor is non-vanishing, we can recover from the pair \((X, Y)\) a non-zero matrix in \( E_d \), thus we can recover the unique element of \( \text{PG}_d \) mapping \( X \) to \( Y \). This gives us a local inverse morphism \( \Psi^{-1} : Z \to \text{PG}_d \times R^{ss}_d \), showing that \( \Psi \) is a closed immersion.

**Proposition 5.2.6.** For coprime \( d \), we have the following formula for the number of \( k \)-rational points:

\[
|\mathcal{M}^{ss}_d| = \frac{|R^{ss}_d|}{|\text{PG}_d|}
\]

**Proof.** This is intuitively obvious, since \( \mathcal{M}^{ss}_d \) is the quotient of \( R^{ss}_d \) by the action of \( \text{PG}_d \), which acts freely. The detailed proof is in [Rei03, 6.6], and uses Mumford's GIT.

Finally, all the above considerations lead to our desired theorem:

**Theorem 5.2.7.** For coprime \( d \),

\[
\sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}} H^i(\mathcal{M}^{ss}_d(C)) v^i = (v^2 - 1) \sum_{d^*} (-1)^{s-1} v^{-2(d^*)} \prod_{k=1}^{s} \frac{|R_{d^*}^k|}{|G_{d^*}|}
\]

where the sum runs over all tuples \( d^* = (d^1, \ldots, d^s) \) of non-zero dimension types and weight \( d \), such that \( \mu(\sum_{i=1}^{k} d^i) > \mu(d) \) for all \( k = 1, \ldots, s - 1 \).

**Proof.** By Corollary 5.2.2, the right hand side equals \( \frac{|R^{ss}_d|}{|\text{PG}_d|} \), which by Proposition 5.2.6 equals \( |\mathcal{M}^{ss}_d(k)| \). This is obviously in \( \mathbb{Q}(q) \), and so as explained in the outline at the beginning of this section (cf. [Goe94, Remark 1.2.2]), we conclude the formula for the Poincaré polynomial.

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Chapter 6

Epilogue

6.1 Further examples

To conclude this thesis, we will use the main result obtained in the last chapter to calculate the Betti numbers for some well known projective varieties, by realizing them as moduli of carefully chosen quivers and stabilities. These examples are based on [Cra11].

6.1.1 Projective Space: $\mathbb{P}^n$

Consider the $(n+1)$-Kronecker quiver, and representations of dimension type $(1,1)$:

\[ k \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_n} k \]

with the arrows going from vertex $i$ to vertex $j$. Let $\Theta = i^*$. It is easy to see that such a representation is semistable if and only if $0 \neq (\alpha_0, \ldots, \alpha_n) \in \mathbb{A}^{n+1}$, and that $G_d$ acts on $R_{d^*}^{ss}$ by:

\[ (\lambda, \mu) \cdot (\alpha_0, \cdots, \alpha_n) = \frac{\mu}{\lambda} (\alpha_0, \cdots, \alpha_n) \]

so that $\mathcal{M}_d^{ss} = \mathbb{P}^n$.

The only admissible tuples $d^*$ are $d_1^* = [(1, 1)]$ and $d_2^* = [(1, 0), (0, 1)]$, and $\langle d_2^* \rangle = 0$. We have $|R_{(1,1)}| = q^{n+1}$, $G_{1,1} = (q - 1)^2$, $|R_{(1,0)}| = |R_{(0,1)}| = 1$, $|G_{(1,0)}| = 49$. 
\[ |G_{(0,1)}| = q - 1, \text{ and hence:} \]
\[ P(q) = (q - 1) \left( \frac{q^{n+1}}{(q-1)^2} - \frac{1}{(q-1)^2} \right) = 1 + q + \cdots + q^n. \]

Thus \( b_0 = b_2 = \cdots = b_{2n} = 1 \) and \( b_i = 0 \) otherwise.

### 6.1.2 Grassmannian: \( G(r, n) \)

Consider the \( n \)-Kronecker quiver, and representations of dimension type \( d = (1, r) \):

\[
\begin{array}{ccc}
  k & \xrightarrow{\alpha_1} & k^r \\
  & \xleftarrow{\alpha_n} & \\
\end{array}
\]

with \( \Theta = i^* \). Thus \( R_d = M_{r \times n} \), and such a representation is semistable if and only if the matrix:

\[
M = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
\]

has full rank. Such matrix with full rank determines a surjective linear map \( T : k^n \to k^r \), and this in turn determines an \( r \)-dimensional subspace of \( k^n \), by taking the inverse image. Two such linear maps \( T \) and \( T' \) determine the same subspace if and only if \( T = g \cdot T' \) for some \( g \in GL(k^r) \), which can be identified with \( G_d \). Thus the quiver moduli in this case parametrizes \( r \)-dimensional subspaces of \( k^n \), i.e. it corresponds to the Grassmannian \( G(r, n) \).

The only admissible dimension types in this case are \( d^* = [(1, r_1), (0, r_2), \cdots, (0, r_s)] \) for any partition \( r_1 + r_2 + \cdots + r_s = r \) such that \( 0 \not\in \{r_2, \cdots, r_s\} \).

As a simple example, we can work out the first interesting Grassmannian, \( G(2, 4) \).

The admissible tuples are \([(1, 2)], [(1, 0), (0, 1)], [(1, 1), (0, 1)], \) and \([(1, 0), (0, 1), (0, 1)]\).

We obtain:

\[
P(q) = 1 + q + 2q^2 + q^3 + q^4
\]

thus \( b_0 = b_2 = b_6 = b_8 = 1, b_4 = 2, \) and \( b_i = 0 \) otherwise.

### 6.1.3 Flag variety: \( Fl(n, r_1, \cdots, r_s) \)

Consider the following quiver:

\[
\begin{array}{c}
i \\
\xrightarrow{\alpha_1} & \xrightarrow{\beta_1} & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{s-1}} & j_s \\
& \xleftarrow{\alpha_n} & \xleftarrow{j_1} & \xleftarrow{j_2} & \cdots & \xleftarrow{j_{s-1}}
\end{array}
\]
with representations of dimension type \( d = (1, r_1, \cdots, r_s) \), \( r_1 > r_2 > \cdots > r_s \), and with \( \Theta = i^\ast \). Then a representation is semistable if and only if the matrix:

\[
M = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n
\end{pmatrix}
\]

has full rank and each \( \beta_i \) is surjective, \( i = 1, \cdots, s - 1 \). Therefore we have surjective maps \( k^n \rightarrow k^{r_1} \rightarrow \cdots \rightarrow k^{r_s} \). Such a representation determines a flag in \( k^n \) by taking successive inverse images, going from \( j_k \) back to \( k^n \). By an argument similar to that in the previous section, we see that the moduli space in this case is the partial flag variety \( Fl(n, r_1, \cdots, r_s) \).

As the calculation becomes too tedious to be carried out by hand, we have supplied a Mathematica program that performs the calculation, with inputs: the graph of the quiver, the slope function, and the dimension type. See Appendix. Using this program we find that for the complete flag variety \( Fl(3, 2, 1) \):

\[
P(q) = 1 + 2q + 2q^2 + q^3.
\]
Appendices
Appendix A

More on quiver representations

In this appendix, we present a brief exposition, mostly without proofs, of further facts about the representations and combinatorics of quivers. We follow §4 and §5 of [CB92], except where otherwise indicated.

A.1 Roots and the Weyl group

Let $Q$ be a quiver. Recall the definition of the Tits form:

$$q(d) = \langle d, d \rangle = \sum_{i \in Q_0} d_i^2 - \sum_{\rho \in Q_1} d_{s(\rho)}d_{t(\rho)}$$

for $d \in \mathbb{Z}Q_0$.

**Definition A.1.1.** Let $\Delta = \{ \alpha \in \mathbb{Z}Q_0 : q(\alpha) \leq 1 \}$. Elements of $\Delta$ are called roots. A root $\alpha$ is real if $q(\alpha) = 1$, and imaginary if $q(\alpha) = 0$.

We list some of the properties associated to roots.

**Lemma A.1.2.**

1. The dimension types corresponding $e_i$ to the simple representations $S(i)$ are roots. They are referred to as simple roots.

2. For each $e_i$, define the simple reflection operator:

$$r_i : \mathbb{Z}Q_0 \to \mathbb{Z}Q_0$$
\[ d \mapsto d - \frac{2(d, e_i)}{(e_i, e_i)}. \]

Let \( W \) be the subgroup of \( \text{Aut}(\mathbb{Z}^{Q_0}) \) generated by the \( r_i \) operators. Then elements of \( W \) map roots to roots. The group \( W \) is called the \textit{Weyl group}.

Let \( \Pi \) be the set of simple roots. Then the set \( \bigcup_{w \in W} (w \cdot \Pi) \) is exactly the set of real roots ([Hub04, p. 82]).

## A.2 Dynkin and Euclidean quivers

**Definition A.2.1.** Suppose \( Q \) is a connected quiver.

1. \( Q \) is said to be Dynkin if the underlying graph is in one of the following families of diagrams:

   \[ \begin{array}{ccc}
   A_n & B_n & C_n & D_n \\
   \end{array} \]

   \[ \begin{array}{ccc}
   E_6 & E_7 & E_8 \\
   \end{array} \]

2. \( Q \) is said to be Euclidean if the underlying graph is in one of the following families of diagrams:

   \[ \begin{array}{ccc}
   \tilde{A}_n & \tilde{B}_n & \tilde{C}_n \\\n   \end{array} \]

   \[ \begin{array}{ccc}
   \tilde{E}_6 & \tilde{E}_7 & \tilde{E}_8 \\
   \end{array} \]
Lemma A.2.2. If $Q$ is Dynkin, then $q$ is positive definite. If $Q$ is Euclidean, then $q$ is positive semidefinite. If $Q$ is Dynkin then $\Delta$ is finite.

A.3 Gabriel’s and Kac’s theorems

In this section, $k$ is an algebraically closed field. Then Gabriel’s theorem classifies the quivers with finitely many isomorphism classes of indecomposables ([Sav06, 4.5]).

Theorem A.3.1. (Gabriel’s theorem)

1. Suppose $Q$ is a Dynkin quiver. Then there exists a one to one correspondence between the isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of $q$.

2. If $Q$ is a connected quiver, then $Q$ has finitely many isomorphism classes of indecomposable representations if and only if it is Dynkin.

In the general case, we have a theorem by Kac ([Sav06, 4.6]):

Theorem A.3.2. (Kac’s theorem) Let $Q$ be an arbitrary quiver The dimension vectors of indecomposable representations are correspond to the positive roots of $q$ (and so they are independent of the orientation of the arrows). If $Q$ is a connected quiver, then $Q$ has finitely many isomorphism classes of indecomposable representations if and only if it is Dynkin.

A.4 Reflection Functors

Let $Q$ be a quiver. Recall the definitions of sink:

\[
\begin{array}{c}
| \\
\end{array}
\]

and source:

\[
\begin{array}{c}
| \\
\end{array}
\]
Suppose a vertex $i$ is a sink (resp. a source). Then we can obtain a new quiver $Q'$ by inverting the direction of each arrow point towards (resp. away from) $i$. Moreover, given a representation $X$ of $Q$, we can define a systematic way of obtaining from it a representation $X'$ of $Q'$. This is described by the \textit{reflection functors} $F_i^+$ and $F_i^-$. 

\textbf{Definition A.4.1.} 

1. \textit{i is a sink.} Define the functor $F_i^+$ by: 
   
a) $Q' = F_i^+(Q)$ is the quiver obtained from $Q$ by inverting all the arrows pointing at $i$,
   
b) $X' = F_i^+(X)$ is the representation of $Q'$ given by $X'_j = X_j$ if $j \neq i$, and $X'_i = \ker h$ where 
      \[
      h : \bigoplus_{\rho : t(\rho) = i} X_{s(\rho)} \to X_i
      \]
      \[
      h = \sum_{\rho : t(\rho) = i} X_{s(\rho)}.
      \]
      The linear maps of $X'$ corresponding to the inverted arrows are $X'_\rho : X'_i \to X'_{s(\rho)}$ given by the natural projection of $\bigoplus_{\rho : t(\rho) = i} X_{s(\rho)}$ on $X_{s(\rho)}$.

2. \textit{i is a source.} Define the functor $F_i^-$ by: 
   
a) $Q' = F_i^-(Q)$ is the quiver obtained from $Q$ by inverting all the arrows starting at $i$,
   
b) $X' = F_i^-(X)$ is the representation of $Q'$ given by $X'_j = X_j$ if $j \neq i$, and $X'_i = \coker \tilde{h}$ where 
      \[
      \tilde{h} : X_i \to \bigoplus_{\rho : s(\rho) = i} X_{t(\rho)}
      \]
      \[
      \tilde{h} = \bigoplus_{\rho : s(\rho) = i} X_{t(\rho)}
      \]
      The linear maps of $X'$ corresponding to the inverted arrows are $X'_\rho : X'_t(\rho) \to X'_i$ given by the restriction of the natural projection of $\bigoplus_{\rho : s(\rho) = i} X_{t(\rho)}$ onto $\coker \tilde{h}$. 

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Proposition A.4.2. (cf. [EGH+11, §5.6])

1. $F_i^+$ is left exact and $F_i^-$ is right exact.

2. If $i$ is a sink, then for any indecomposable representation $X$ of $Q$, $F_i^+(X) = 0$ if and only if $X \cong S(i)$. Otherwise, $F_i^+(X)$ is an indecomposable representation, and $F_i^-F_i^+(X) \cong X$.

3. If $i$ is a source, then for any indecomposable representation $X$ of $Q$, $F_i^-(X) = 0$ if and only if $X \cong S(i)$. Otherwise, $F_i^-F_i^-(X) \cong X$.

4. If $X$ is indecomposable, then $\dim(F_i^\pm(X)) = r_i(\dim(X))$, where $r_i$ is the reflection operator defined above.

A.5 Example

In this section, consider the Dynkin quiver $D_4$:

```
4
/  \
2 - 1 - 3
```

We can use the Weyl group, along with Gabriel’s theorem, to calculate the dimension types of the indecomposable representations of $D_4$. The simple roots are $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,1)$. We calculate the matrices corresponding to the reflection operators, and we find:

- $r_1 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$

By using these matrices to calculate the orbit of the simple roots, we find that there are 12 positive roots:

$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,0,1)$,
By Gabriel’s theorem, these must be the dimension types of the indecomposable representations.

We can use the reflection functors to calculate some of these indecomposable representations. Suppose we want to find the indecomposable representation of dimension type \((2, 1, 1, 1)\). We start with a representation \(X\):

\[
\begin{array}{c}
\pi \\
k \xrightarrow{\pi} k^2 \xrightarrow{\pi} k
\end{array}
\]

\[
\pi : (x, y) \mapsto x
\]

This representation is clearly decomposable. However, by Proposition A.4.2, we can hope that by applying the reflection functors \(F^+_1\) and \(F^-_1\) we can get rid of simple components and arrive at an indecomposable representation. For \(Y = F^-_1(X)\) we get the following representation:

\[
\begin{array}{c}
k \\
k \xrightarrow{\alpha_2} k^2 \xrightarrow{\alpha_3} k
\end{array}
\]

\[
\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},
\]

which we can check is indecomposable. Note that \(F^+_1(Y)\) gives:

\[
\begin{array}{c}
k \\
k \xleftarrow{1} k \xrightarrow{1} k
\end{array}
\]

which is indecomposable but not isomorphic to \(X\), but that \(F^-_1 F^+_1(Y) \cong Y\).
Appendix B

Mathematica code

We provide below the Mathematica code to calculate the Poincaré polynomial of a quiver, given its graph \( g \), the slope \( m \), and the dimension type \( d \).

\[
(* \text{list all dimension types, with bound on entries} *)
\]
\[
\text{Needs["GraphUtilities"]}
\]
\[
\text{vectors}[d_] := \text{Module}[\{i, l\}, (* \text{return vectors } d^* \text{ of } d *)]
\]
\[
l = \text{Table}[\text{Table}[k, \{k, 0, d[[i]]\}], \{i, 1, \text{Length}[d]\}];
l = \text{RandomSample}[l];
\text{Return}[\text{Delete}[l, 1]];\]

\[
\text{weight}[ds_] := \text{Total}[ds]
\]
\[
\text{AdmissibleQ}[ds_, d_, m_] := \text{Module}[\{i, t, b\}, (* \text{check if } d^* \text{ is good} *)]
\]
\[
\text{If}[\text{weight}[ds]d, \text{Return}[\text{False}]];
t = \text{FoldList}[\text{Plus}, 0, ds];
t = \text{Map}[m, \text{Delete}[t, \{\{1\}, \{\text{Length}[t]\}]\}] ;
t = \text{Map}[\#1 > m[d] & , t];
b = \text{Apply}[\text{And}, t];
\text{Return}[b];\]

\[
\text{AdmissibleVectorsN}[d_, m_, n_] := \text{Module}[\{i, vl, l, t\}, (* \text{ret good } d^* \text{ of length } n *)]
\]
\[
vl = \text{RandomSample}[\text{Tuples}[\text{vectors}[d], n]];\]
l={};
admiq[dv_]:=AdmissibleQ[dv,d,m];
t=Map[admiq,vl];
l=Position[t,False];
Return[Delete[vl,l]];
]

AdmissibleVectors[d_,m_:]=Module[{i,lt,t},
(*returns a list of all good d^* types to sum over *)
lt=Table[i,{i,1,weight[d]}];
admivec[n_:]=AdmissibleVectorsN[d,m,n];
t=Map[admivec,lt];
Return[Flatten[t,1]];
]

(* determine the Euler form from the adjacency matrix *)
EulerForm[d_,e_,M_:]=d.(IdentityMatrix[Length[M[[1]]]]-M).e

(* determine product of |R_dk| given adjacency matrix *)
Rd[d_,M_:]=Product[v^((2*M[[i]])[[j]]*d[[i]]*d[[j]])),
{i,1,Length[M[[1]]]},
{j,1,Length[M[[1]]]}]
R[ds_,M_:]=Product[Rd[ds[[i]],M],{i,1,Length[ds]}]

(* determine product of |G_dk| *)
Gd[d_:]=Product[Product[v^((2*d[[i]])-(2*j)),{j,0,d[[i]]-1}],
{i,1,Length[d]}]
G[ds_:]=Product[Gd[ds[[i]]],{i,1,Length[ds]}]

(* Euler form sum over d^* type *)
Brak[ds_,M_:]=Sum[Sum[EulerForm[ds[[l]],ds[[k]],M],{k,1,l-1}],
{l,1,Length[ds]}]

(* calculate the Poincar polynomial of a quiver moduli given coprime dimension vector d, quiver graph g, slope m *)
Poincare[g_,d_,m_:]= Module[{advect,p,M},

M = AdjacencyMatrix[g];
advect = AdmissibleVectors[d, m];
p = (v^2 - 1) * Sum[(-1)^(Length[advect[[i]]] - 1) * v^(-2*Brak[advect[[i]], M]) * R[advect[[i]], M]/G[advect[[i]]], {i, 1, Length[advect]}];
Return[Simplify[p]];
]
(*Example:*)
g := {1 -> 2, 1 -> 2}
m[d_] := d[[1]]/Total[d]
Poincare[g, {1, 1}, m]
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