Note. Regular Borel measures. MAT 4AN, E2004

Let X be a locally compact Hausdorff space. A subset $A \subset X$ is called a *Borel set* if it belongs to the *Borel algebra* $B(X)$, which by definition is the smallest σ-algebra containing all open subsets of X (Meise and Vogt, p. 412). A *Borel measure* on X is a measure which is defined on $B(X)$. Such a measure μ is called *regular* if it satisfies the following three properties:

(1) $\forall K \subset X$ compact: $\mu(K) < \infty$

(2) $\forall G \subset X$ open: $\mu(G) = \sup\{\mu(K) \mid K \subset G, K \text{ compact}\}$

(3) $\forall A \in B(X)$: $\mu(A) = \inf\{\mu(G) \mid G \supset A, G \text{ open}\}$.

Properties (2) and (3) are called, respectively, *inner* and *outer* regularity. For example, Lebesgue measure on $\mathbb{R}^n$ is a regular Borel measure (see below).

On certain rather complicated locally compact Hausdorff spaces there exist Borel measures which satisfy (1) but not (2) or (3). However, it is a theorem (Rudin, Real and Complex Analysis, Thm. 2.18) that if X is a locally compact metric space which is separable (that is, it has a countable dense subset Q), then (1) implies (2) and (3) for all Borel measures on X. This theorem implies the statement made above about Lebesgue measure, since it is clear that (1) is satisfied.

The following result is now a generalisation of Prop. A 10. It is needed later in the book.

Theorem. *Let μ be a σ-finite regular Borel measure on a locally compact space X. Then $C_c(X)$ is dense in $L^p(X, \mu)$ for $1 \leq p < \infty$.*

Proof. For $p = 1$ see M&V, Lemma A.4 (the definition of L^1 agrees with the present, by M&V, Prop. A.11). For general p, the proof can be found in Rudin, Real and Complex Analysis, Thm. 3.14.

The density of $C_c(X)$ is easily seen to be false for $p = \infty$, for example in the case of Lebesgue measure.

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