Solution. Assignment 1

1. Let $D = \mathbf{R} \times (0, \infty)$ and let $X : D \to \mathbf{R}$ be given by

$$X(t, y) = \frac{\cosh(t)\sqrt{|y^2 - 1|}}{y}.$$

Let $D_1 = (0, \infty) \times (0, \infty)$ and let X_1 be the restriction of X to D_1.

(a) Give the maximal solution to the initial value problem

$$\frac{dy}{dt} = X_1(y, t), \quad y(1) = \cosh(1).$$

SOLUTION: When $y > 0$ and $y \neq 1$ we can apply the method of separation of variables. We find

$$\frac{y}{\sqrt{|y^2 - 1|}} \frac{dy}{dt} = \cosh(t)$$

and hence

$$\int \frac{y}{\sqrt{|y^2 - 1|}}\, dy = \int \cosh(t)\, dt.$$

We substitute $u = y^2 - 1$ and obtain on the left side

$$\frac{1}{2} \int |u|^{-\frac{1}{2}}\, du = \operatorname{sign}(u)|u|^{1/2},$$

and on the other side

$$\int \cosh(t)\, dt = \sinh(t),$$

with addition of arbitrary constants. Hence $u = \pm(\sinh(t)+c)^2$ and $y = y_1$ or $y = y_2$ where

$$y_1 = \sqrt{1 + (\sinh(t) + c)^2}, \quad y_2 = \sqrt{1 - (\sinh(t) + c)^2}.$$

The initial condition $y(1) = \cosh(1) > 1$ implies that we need the solution y_1 with $\sqrt{1 + (\sinh(1) + c)^2} = \cosh(1)$, and from the relation $\cosh^2 t - \sinh^2 t = 1$ we deduce $c = 0$ and

$$y_1 = \cosh(t).$$

We observe that $y_1 > 1$ and hence the steps performed above are valid for all $t > 0$. It follows that $((0, \infty), y_1)$ is a solution. It satisfies the initial condition and it is obviously maximal in D_1.

(b) Show that

$$(\mathbf{R}, t \mapsto 1)$$
$$\left(\mathbf{R}, t \mapsto \begin{cases} 1 & t \le 0 \\ y_1(t) & t > 0 \end{cases}\right)$$
$$\left((-\text{arsinh}(1), \infty), t \mapsto \begin{cases} \sqrt{1 - \sinh^2(t)} & t \le 0 \\ y_1(t) & t > 0 \end{cases}\right)$$

are maximal solutions of $\dfrac{dy}{dt} = X(y, t)$. (Recall that arsinh is the inverse function of $\sinh$.) Conclude that the initial value problem

$$\frac{dy}{dt} = X(t, y), \quad y(0) = 1$$

does not have a unique solution.

SOLUTION: All the mentioned functions are defined on $\mathbf{R}$, so if they solve the equation they are maximal solutions.

It is clear that the constant function 1 solves the equation.

The function $y_1(t) = \cosh(t)$ is a solution to $\dot{y} = X(t, y)$ also when its domain of definition is extended to $[0, \infty)$, since its right derivative at $t = 0$ is 0 which equals $X(0, \cosh(0))$. The gluing lemma implies that the second mentioned function is a solution.

The function $y_2(t) = \sqrt{1 - \sinh^2(t)}$ is defined for $|\sinh(t)| < 1$. Since

$$\dot{y}_2(t) = \frac{-\cosh(t)\sinh(t)}{\sqrt{1 - \sinh^2(t)}}$$

while

$$X(t, y_2) = \frac{\cosh(t)|\sinh(t)|}{\sqrt{1 - \sinh^2(t)}},$$

it solves the equation when in addition $t \le 0$. Again the gluing lemma implies that the third mentioned function is a solution.

As all three solutions satisfy the initial condition $y(0) = 1$, there is not a unique maximal solution.

2. Let $D = (0, \infty) \times \mathbf{R}$ and let $X : D \to \mathbf{R}$ be given by

$$X(t, y) = \frac{2y}{\sinh(2t)} + \sinh(t)$$

(a) Show that

$$\frac{\tanh'(t)}{\tanh(t)} = \frac{2}{\sinh(2t)}.$$

(b) Give all maximal solutions of

$$\frac{dy}{dt} = X(y,t).$$

SOLUTION

(a) follows from the facts that $\tanh'(t) = \frac{1}{\cosh^2(t)}$ and $\sinh(2t) = 2\sinh(t)\cosh(t)$.

(b) The equation is linear, and hence we can solve by the formula

$$y = e^{G(t)} \int e^{-G(s)} b(s)\, ds$$

where $G(t)$ is a primitive of $\frac{2}{\sinh(2t)}$. From (a) we find

$$G(t) = \int \frac{\tanh'(t)}{\tanh(t)}\, dt = \ln(\tanh(t))$$

and hence

$$\begin{aligned} y &= e^{\ln(\tanh(t))} \int e^{-\ln(\tanh(s))} \sinh(s)\, ds \\ &= \tanh(t) \int \frac{\sinh(s)}{\tanh(s)}\, ds \\ &= \tanh(t) \int \cosh(s)\, ds = \tanh(t)(\sinh(t) + c) \end{aligned}$$

with $t \in \mathbf{R}$.